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Some Results on Additive Number Theory III

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§ 1. The main theorem.

Let k be an integer >1 and l be a positive integer. Let $\{P_{ij}: i=1, \cdots, k; j=1, \cdots, l\}$ be a given family of sets, each consisting of prime numbers, subject to the following conditions:

(C_1) For each $i=1, \cdots, k$, the sets $P_{ij}(j=1, \cdots, l)$ are pairwise disjoint;

(C_2) As $x \rightarrow \infty$,

$$\sum_{p \leq x, p \in P_{ij}} \frac{1}{p} = \lambda_{ij} \log \log x + o(\sqrt{\log \log x})$$

with positive constants λ_{ij} for $i=1, \cdots, k; j=1, \cdots, l$.

The sets P_{ij} with distinct i's need not be disjoint, and $P_{i1} \cup \cdots \cup P_{il}$ may not contain all primes.

Throughout the paper, without repeated comment, the double subscripts ij will always run through the kl pairs of integers $i=1, \cdots, k$; $j=1, \cdots, l$.

Let $\omega_{ij}(n)$ denote the number of distinct prime factors of a positive integer n, which belong to the set P_{ij}:

$$\omega_{ij}(n) = \sum_{p|n, p \in P_{ij}} 1 .$$

THEOREM 1. *Let E be a Jordan-measurable set, bounded or unbounded in the space R^{kl} of kl dimensions. For sufficiently large integer N, let $A(N; E)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that, if we put*

$$x_{ij} = \frac{\omega_{ij}(n_i) - \lambda_{ij} \log \log N}{\sqrt{\lambda_{ij} \log \log N}},$$

the point $(x_{11}, \cdots, x_{1l}, \cdots, x_{k1}, \cdots, x_{kl})$ belongs to the set E. Then, as $N \rightarrow \infty$, we have

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$$A(N;E) \sim \frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\int_E \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{l} x_{ij}^2\right)dx_{11}\cdots dx_{kl} .$$

This theorem was announced in [5] without proof, and the outline of the proof of a very special case of the theorem was sketched in [6]. We assume here for simplicity that l_i's in [5] are equal to l. We shall, in Section 2, prove the theorem. The paper could somewhat be shortened by omitting some parts of the proof and making references to author's previous papers [1], [2], [3], and [4], but, for the reader's convenience, we shall give here the complete proof so that this will be read as a self-contained paper. In Section 3, we shall refer to some special cases of the theorem.

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§ 2. Proof of the theorem.

We first give a lemma concerning the number of solutions of a linear Diophantine equation in positive integers.

LEMMA 1. *Let $a_1, \cdots, a_t (t>1)$, and b be positive integers such that the greatest common divisor $(a_1, \cdots, a_t)$ divides b. Let $S_t = S_t(a_1, \cdots, a_t, b)$ denote the number of solutions of the Diophantine eqution*

(1) $$a_1x_1 + \cdots + a_tx_t = b$$

in positive integers, then we have

(2) $$\left| S_t - \frac{(a_1, \cdots, a_t)b^{t-1}}{(t-1)!\, a_1\cdots a_t} \right| < C_t b^{t-2} ,$$

where C_t is a suitable positive number dependent only on t, and independent of $a_1, \cdots, a_t$, and b.

PROOF. We shall prove the lemma by induction on t beginning with $t=2$. The case when $t=2$, our Diophantine equation is $a_1x_1 + a_2x_2 = b$ with $(a_1, a_2)|b$, and, from the well-known property of this equation we can easily see that

$$\left| S_2 - \frac{(a_1, a_2)b}{a_1a_2} \right| < C_2 ,$$

where C_2 is independent of a_1, a_2, and b.

Next we assume that (2) holds for one value of t, and consider $S_{t+1} = S_{t+1}(a_1, \cdots, a_{t+1}, b)$, the number of solutions of the Diophantine equation

$$a_1x_1+\cdots+a_{t+1}x_{t+1}=b$$

or

$$(3)\qquad a_1x_1+\cdots+a_tx_t=b-a_{t+1}x_{t+1}$$

with $(a_1,\cdots,a_{t+1})|b$ in positive integers. Let $(x_{10},\cdots,x_{t+1,0})$ be an integral solution, not necessarily positive, of (3), then, as is easily seen, for any positive integral solution, if it exists, we can put

$$x_{t+1}=x_{t+1,0}+\frac{(a_1,\cdots,a_t)}{(a_1,\cdots,a_{t+1})}u\ ,$$

where u is an integer for which $0<b-a_{t+1}x_{t+1}<b$. For each of such u, if we denote for brevity by $S_t^*(u)$ the number of solutions of (3) in positive integers $x_1,\cdots,x_t$, then, as (2) is assumed to be valid for the equation (1), replacing b by

$$b-a_{t+1}x_{t+1}=b-a_{t+1}x_{t+1,0}-\frac{(a_1,\cdots,a_t)a_{t+1}}{(a_1,\cdots,a_{t+1})}u\ ,$$

we can write

$$\left|S_t^*(u)-\frac{(a_1,\cdots,a_t)}{(t-1)!\ a_1\cdots a_t}\left\{b-a_{t+1}x_{t+1,0}-\frac{(a_1,\cdots,a_t)a_{t+1}}{(a_1,\cdots,a_{t+1})}u\right\}^{t-1}\right|<C_tb^{t-2}\ .$$

Now, since

$$S_{t+1}=\sum_u S_t^*(u)\ ,$$

where u runs through the integers with above-mentioned condition, and the number of integers admissible for u is less than b, we obtain

$$\left|S_{t+1}-\frac{(a_1,\cdots,a_t)}{(t-1)!\ a_1\cdots a_t}\sum_u\left\{b-a_{t+1}x_{t+1,0}-\frac{(a_1,\cdots,a_t)a_{t+1}}{(a_1,\cdots,a_{t+1})}u\right\}^{t-1}\right|<C_tb^{t-1}\ .$$

Also, approximating the summation by an appropriate integral, we can easily obtain

$$\frac{(a_1,\cdots,a_t)}{(t-1)!\ a_1\cdots a_t}\sum_u\left\{b-a_{t+1}x_{t+1,0}-\frac{(a_1,\cdots,a_t)a_{t+1}}{(a_1,\cdots,a_{t+1})}u\right\}^{t-1}$$
$$=\frac{(a_1,\cdots,a_{t+1})b^t}{t!\ a_1\cdots a_{t+1}}+O(b^{t-1})\ ,$$

where the constant implied in O-symbol is independent of $a_1,\cdots,a_{t+1}$.

Thus we see that an inequality obtained by replacing t by $t+1$ in (2) also holds with C_{t+1} independent of $a_1, \cdots, a_{t+1}$, and b. Our induction is now completed.

Next we give two simple lemmas concerning binomial coefficients.

LEMMA 2. *Let a and b be non-negative integers, then*

$$\sum_{c=0}^{b} (-1)^c \binom{a}{c} \begin{cases} =1, \text{ when } a=0, \\ \geqq 0, \text{ when } a>0 \text{ and } b \text{ is even}, \\ \leqq 0, \text{ when } a>0 \text{ and } b \text{ is odd}. \end{cases}$$

We omit the proof.

LEMMA 3. *Let $a_1, \cdots, a_t$ and $b_1, \cdots, b_t$ be non-negative integers. If we put*

$$\begin{aligned}\gamma &= \gamma(a_1, \cdots, a_t; b_1, \cdots, b_t) \\ &= \sum_{s=1}^{t} \left\{ \sum_{c_s=0}^{2b_s+1} (-1)^{c_s} \binom{a_s}{c_s} \cdot \prod_{\substack{r=1 \\ r\neq s}}^{t} \sum_{c_r=0}^{2b_r} (-1)^{c_r} \binom{a_r}{c_r} \right\} - (t-1) \prod_{s=1}^{t} \sum_{c_s=0}^{2b_s} (-1)^{c_s} \binom{a_s}{c_s},\end{aligned}$$

then we have

$$\gamma \begin{cases} =1, \text{ when } a_1 = \cdots = a_t = 0, \\ \leqq 0, \text{ when at least one of the } a\text{'s is positive}. \end{cases}$$

PROOF. The case $a_1 = \cdots = a_t = 0$ is trivial. Suppose that at least one of the a's is positive. Without loss of generality, we can assume that $a_s > 0$ for $s=1, \cdots, t_1$, and $a_s = 0$ for $s = t_1+1, \cdots, t$. Then we easily have

$$\gamma = \sum_{s=1}^{t_1} \left\{ \sum_{c_s=0}^{2b_s+1} (-1)^{c_s} \binom{a_s}{c_s} \cdot \prod_{\substack{r=1 \\ r\neq s}}^{t_1} \sum_{c_r=0}^{2b_r} (-1)^{c_r} \binom{a_r}{c_r} \right\} - (t_1-1) \prod_{s=1}^{t_1} \sum_{c_s=0}^{2b_s} (-1)^{c_s} \binom{a_s}{c_s},$$

from which, applying the case $a>0$ of Lemma 2, we see that $\gamma \leqq 0$. Thus the lemma is proved.

Now we define some functions and sets which will be used in the sequel. The positive integer N will be assumed to be sufficiently large as occasion demands.

We put

$$y_{ij}(N) = \sum_{p \leq N, p \in P_{ij}} \frac{1}{p},$$

then, by (C_2) in Section 1,

(4) $$y_{ij}(N)=\lambda_{ij}\log\log N+o(\sqrt{\log\log N}) .$$

We define the sets Q_{ijN} as

(5) $$Q_{ijN}=\{p\colon p\in P_{ij},\ e^{\{y_{ij}(N)\}^2}<p<N^{\{y_{ij}(N)\}^{-2}}\} .$$

We introduce these sets obtained from P_{ij} by omitting comparatively small and large primes in analogy to the truncation method used in probability theory.

We put

$$z_{ij}(N)=\sum_{p\in Q_{ijN}}\frac{1}{p} .$$

Then obviously $z_{ij}(N)\leqq y_{ij}(N)$. Also we have

LEMMA 4. $z_{ij}(N)=\lambda_{ij}\log\log N+o(\sqrt{\log\log N})$.

PROOF. As is well-known

$$\sum_{p\leqq x}\frac{1}{p}=\log\log x+O(1) .$$

Now we can write

$$y_{ij}(N)-z_{ij}(N)\leqq\Sigma_1+\Sigma_2 ;$$

in Σ_1, p runs through primes $\leqq e^{\{y_{ij}(N)\}^2}$ and, in Σ_2, p runs through primes satisfying $N^{\{y_{ij}(N)\}^{-2}}\leqq p\leqq N$. Hence we have

$$\Sigma_1=2\log y_{ij}(N)+O(1) ,$$

and

$$\Sigma_2=\log\log N-\log\frac{\log N}{\{y_{ij}(N)\}^2}+O(1)=2\log y_{ij}(N)+O(1) .$$

Hence the lemma follows from (4).

Now we continue defining some further functions.

We denote by $\omega_{ijN}(n)$ the number of distinct prime factors of a positive integer n, which belong to the set Q_{ijN}:

(6) $$\omega_{ijN}(n)=\sum_{p\mid n,\, p\in Q_{ijN}}1 .$$

For any positive integer t, we define the sets $M_{ijN}(t)$ consisting of positive integers as

(7) $$M_{ijN}(t)=\{m\colon m \text{ is squarefree} ;$$

m has t prime factors ;
m is composed only of primes $\in Q_{ijN}\}$.

We put for convenience $M_{ijN}(0)=\{1\}$.

For any kl positive integers t_{ij}, we denote by $F(N; t_{11}, \cdots, t_{kl})$ the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that $\omega_{ijN}(n_i)=t_{ij}$ simultaneously.

For any kl positive integers m_{ij} such that $m_{ij}\in M_{ijN}(t_{ij})$ with some positive integers t_{ij}, we denote by $G(N; m_{11}, \cdots, m_{kl})$ the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that

$$\prod_{p|n_i,\, p\in Q_{ijN}} p=m_{ij} \tag{8}$$

simultaneously.

We obviously have

$$F(N; t_{11}, \cdots, t_{kl})=\sum_{m_{11}\in M_{11N}(t_{11})}\cdots\sum_{m_{kl}\in M_{klN}(t_{kl})} G(N; m_{11}, \cdots, m_{kl})\ , \tag{9}$$

where the summation symbols $\sum$ are repeated kl times.

For any $2kl$ positive integers t_{ij} and T_{ij}, we put

$$\begin{aligned} &\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) \\ &\quad =\sum_{m_{11}\in M_{11N}(t_{11})}\cdots\sum_{m_{kl}\in M_{klN}(t_{kl})} \mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl})\ , \\ &\mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) \\ &\quad =\sum_{\tau_{11}=0}^{2T_{11}}\cdots\sum_{\tau_{kl}=0}^{2T_{kl}} (-1)^{\tau_{11}+\cdots+\tau_{kl}} \mathscr{L}(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})\ , \\ &\mathscr{L}(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})=\sum_{\substack{\mu_{11}\in M_{11N}(\tau_{11})\\ (\mu_{11}, m_{11})=1}}\cdots\sum_{\substack{\mu_{kl}\in M_{klN}(\tau_{kl})\\ (\mu_{kl}, m_{kl})=1}}\ \sum_{\substack{n_1+\cdots+n_k=N\\ m_{ij}\mu_{ij}|n_i}} 1\ . \end{aligned} \tag{10}$$

In the sum defining $\mathscr{L}(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$, the kl summation-variables μ_{ij} run through positive integers satisfying the assigned conditions, and, for each of the systems of such μ_{ij}, the innermost sum means the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that $m_{ij}\mu_{ij}|n_i$ simultaneously. Similarly we put

$$\begin{aligned} &\mathscr{H}^{(ij)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) \\ &\quad =\sum_{m_{11}\in M_{11N}(t_{11})}\cdots\sum_{m_{kl}\in M_{klN}(t_{kl})} \mathscr{K}^{(ij)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl})\ , \\ &\mathscr{K}^{(ij)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) \\ &\quad =\sum_{\tau_{11}=0}^{2T_{11}}\cdots\sum_{\tau_{ij}=0}^{2T_{ij}+1}\cdots\sum_{\tau_{kl}=0}^{2T_{kl}} (-1)^{\tau_{11}+\cdots+\tau_{kl}} \mathscr{L}(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})\ , \end{aligned}$$

where the summation-variable τ_{ij} runs through the integers $0, \cdots, 2T_{ij}+1$ and other τ's, in number $kl-1$, run through the same integers as in the definition of $\mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl})$ respectively.

LEMMA 5. *For any $2kl$ positive integers t_{ij} and T_{ij}, we have*

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\mathscr{H}^{(ij)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) \\ -(kl-1)\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) \\ \leqq F(N; t_{11}, \cdots, t_{kl}) \leqq \mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) .$$

PROOF. Because of the assumption (C_1), we can change the order of summations defining $\mathscr{L}(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$ as follows:

$$\begin{aligned} &\mathscr{L}(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl}) \\ &= \sum_{\substack{n_1+\cdots+n_k=N \\ m_{ij}|n_i}} \sum_{\substack{\mu_{11}\in M_{11N}(\tau_{11}) \\ (\mu_{11}, m_{11})=1 \\ \mu_{11}|n_1}} \cdots \sum_{\substack{\mu_{kl}\in M_{klN}(\tau_{kl}) \\ (\mu_{kl}, m_{kl})=1 \\ \mu_{kl}|n_k}} 1 \\ &= \sum_{\substack{n_1+\cdots+n_k=N \\ m_{ij}|n_i}} \prod_{i=1}^{k}\prod_{j=1}^{l} \sum_{\substack{\mu_{ij}\in M_{ijN}(\tau_{ij}) \\ (\mu_{ij}, m_{ij})=1 \\ \mu_{ij}|n_i}} 1 \\ &= \sum_{\substack{n_1+\cdots+n_k=N \\ m_{ij}|n_i}} \prod_{i=1}^{k}\prod_{j=1}^{l} \binom{\omega_{ijN}(n_i)-t_{ij}}{\tau_{ij}} . \end{aligned}$$

Hence we can put

$$\mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) = \sum_{\substack{n_1+\cdots+n_k=N \\ m_{ij}|n_i}} \delta(n_1, \cdots, n_k) ,$$

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\mathscr{K}^{(ij)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) \\ -(kl-1)\mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) = \sum_{\substack{n_1+\cdots+n_k=N \\ m_{ij}|n_i}} \delta'(n_1, \cdots, n_k) ,$$

where

$$\delta(n_1, \cdots, n_k) = \prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{\tau_{ij}=0}^{2T_{ij}}(-1)^{\tau_{ij}}\binom{\omega_{ijN}(n_i)-t_{ij}}{\tau_{ij}} ,$$

$$\begin{aligned} &\delta'(n_1, \cdots, n_k) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{l}\left\{\sum_{\tau_{ij}=0}^{2T_{ij}+1}(-1)^{\tau_{ij}}\binom{\omega_{ijN}(n_i)-t_{ij}}{\tau_{ij}} \cdot \prod_{\substack{r=1 \\ (r,s)\neq(i,j)}}^{k}\prod_{s=1}^{l}\sum_{\tau_{rs}=0}^{2T_{rs}}(-1)^{\tau_{rs}}\binom{\omega_{rsN}(n_r)-t_{rs}}{\tau_{rs}}\right\} \\ &\quad -(kl-1)\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{\tau_{ij}=0}^{2T_{ij}}(-1)^{\tau_{ij}}\binom{\omega_{ijN}(n_i)-t_{ij}}{\tau_{ij}} . \end{aligned}$$

Now, owing to Lemma 3, $\delta(n_1, \cdots, n_k)=\delta'(n_1, \cdots, n_k)=1$ when $\omega_{ijN}(n_i)=t_{ij}$

simultaneously; and $\delta(n_1, \cdots, n_k) \geqq 0$, $\delta'(n_1, \cdots, n_k) \leqq 0$ when at least one inequality $\omega_{ijN}(n_i) > t_{ij}$ holds. Hence, recalling (8) in the definition of $G(N; m_{11}, \cdots, m_{kl})$, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{l} \mathscr{K}^{(ij)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) - (kl-1)\mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) \leqq G(N; m_{11}, \cdots, m_{kl}) \leqq \mathscr{K}^{(0)}(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) .$$

Now by (9) and (10) we obtain the lemma.

This lemma enables us to obtain a certain asymptotic formula for $F(N; t_{11}, \cdots, t_{kl})$ proving the easier ones for $\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$ and $\mathscr{H}^{(ij)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$ giving $T_{11}, \cdots, T_{kl}$ appropriate values. This procedure might be said to be a type of sieve method. Again $\mathscr{H}^{(ij)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$ can be dealt with in almost the same way as $\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$, and so we shall for the present be concerned with $\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$. For this purpose we introduce some more functions. We put

$$H(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) = \sum_{m_{11} \in M_{11N}(t_{11})} \cdots \sum_{m_{kl} \in M_{klN}(t_{kl})} K(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) ,$$

$$K(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) = \sum_{\tau_{11}=0}^{2T_{11}} \cdots \sum_{\tau_{kl}=0}^{2T_{kl}} (-1)^{\tau_{11}+\cdots+\tau_{kl}} L(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl}) ,$$

$$L(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl}) = \sum_{\substack{\mu_{11} \in M_{11N}(\tau_{11}) \\ (\mu_{11}, m_{11})=1}} \cdots \sum_{\substack{\mu_{kl} \in M_{klN}(\tau_{kl}) \\ (\mu_{kl}, m_{kl})=1 \\ (m_1\mu_1, \cdots, m_k\mu_k) | N}} \frac{(m_1\mu_1, \cdots, m_k\mu_k)}{m_1\mu_1 \cdots m_k\mu_k} ,$$

where we have put for brevity

$$m_i = \prod_{j=1}^{l} m_{ij} , \quad \mu_i = \prod_{j=1}^{l} \mu_{ij} . \tag{11}$$

We put further

$$H_1(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) = \sum_{m_{11} \in M_{11N}(t_{11})} \cdots \sum_{m_{kl} \in M_{klN}(t_{kl})} K_1(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) ,$$

$$K_1(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) = \sum_{\tau_{11}=0}^{2T_{11}} \cdots \sum_{\tau_{kl}=0}^{2T_{kl}} (-1)^{\tau_{11}+\cdots+\tau_{kl}} L_1(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$$

$$L_1(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$$

$$= \sum_{\substack{\mu_{11} \in M_{11N}(\tau_{11}) \\ (\mu_{11}, m_{11})=1}} \cdots \sum_{\substack{\mu_{kl} \in M_{klN}(\tau_{kl}) \\ (\mu_{kl}, m_{kl})=1 \\ (m_1\mu_1, \cdots, m_k\mu_k)=1}} \frac{1}{m_1\mu_1 \cdots m_k\mu_k} ;$$

$$H_2(N; t_{11}, \cdots t_{kl}; T_{11}, \cdots, T_{kl})$$
$$= \sum_{m_{11} \in M_{11N}(t_{11})} \cdots \sum_{m_{kl} \in M_{klN}(t_{kl})} K_2(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl}) ,$$

$$K_2(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl})$$
$$= \sum_{\tau_{11}=0}^{2T_{11}} \cdots \sum_{\tau_{kl}=0}^{2T_{kl}} (-1)^{\tau_{11}+\cdots+\tau_{kl}} L_2(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$$

$$L_2(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$$
$$= \sum_{\substack{\mu_{11} \in M_{11N}(\tau_{11}) \\ (\mu_{11}, m_{11})=1}} \cdots \sum_{\substack{\mu_{kl} \in M_{klN}(\tau_{kl}) \\ (\mu_{kl}, m_{kl})=1 \\ (m_1\mu_1, \cdots, m_k\mu_k)>1 \\ (m_1\mu_1, \cdots, m_k\mu_k) \mid N}} \frac{(m_1\mu_1, \cdots, m_k\mu_k)}{m_1\mu_1 \cdots m_k\mu_k} ;$$

$$H_3(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$$
$$= \sum_{m_{11} \in M_{11N}(t_{11})} \cdots \sum_{m_{kl} \in M_{klN}(t_{kl})} K_3(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl})$$

$$K_3(N; m_{11}, \cdots, m_{kl}; T_{11}, \cdots, T_{kl})$$
$$= \sum_{\tau_{11}=0}^{2T_{11}} \cdots \sum_{\tau_{kl}=0}^{2T_{kl}} (-1)^{\tau_{11}+\cdots+\tau_{kl}} L_3(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$$

$$L_3(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$$
$$= \sum_{\substack{\mu_{11} \in M_{11N}(\tau_{11}) \\ (\mu_{11}, m_{11})=1}} \cdots \sum_{\substack{\mu_{kl} \in M_{klN}(\tau_{kl}) \\ (\mu_{kl}, m_{kl})=1 \\ (m_1\mu_1, \cdots, m_k\mu_k)>1}} \frac{1}{m_1\mu_1 \cdots m_k\mu_k} .$$

The apparent complexity of introducing such similar but slightly different expressions would rather facilitate the subsequent arguments. Now, from the above definitions, we at once have

$$(12) \quad H(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$$
$$= H_1(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) + H_2(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) ,$$

$$(13) \quad H_1(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl}) + H_3(N; t_{11}, \cdots, t_{kl}; T_{11}, \cdots, T_{kl})$$
$$= \prod_{i=1}^{k} \prod_{j=1}^{l} \sum_{m_{ij} \in M_{ijN}(t_{ij})} \frac{1}{m_{ij}} \sum_{\tau_{ij}=0}^{2T_{ij}} (-1)^{\tau_{ij}} \sum_{\substack{\mu_{ij} \in M_{ijN}(\tau_{ij}) \\ (\mu_{ij}, m_{ij})=1}} \frac{1}{\mu_{ij}} .$$

LEMMA 6. *Let* $T_{ijN} = [5z_{ij}(N)]$. *Then, as* $N \to \infty$,

$$H_1(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN}) + H_3(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$$
$$= \frac{\{z_{11}(N)\}^{t_{11}} \cdots \{z_{kl}(N)\}^{t_{kl}} e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}! \cdots t_{kl}!} \{1+o(1)\}$$

uniformly in t_{ij} with $t_{ij}<2z_{ij}(N)$ simultaneously.

PROOF. In view of (13), we are allowed to consider the kl expressions

$$\sum_{m_{ij}\in M_{ijN}(t_{ij})} \frac{1}{m_{ij}} \sum_{\tau_{ij}=0}^{2T_{ijN}} (-1)^{\tau_{ij}} \sum_{\substack{\mu_{ij}\in M_{ijN}(\tau_{ij}) \\ (\mu_{ij},m_{ij})=1}} \frac{1}{\mu_{ij}}$$

separately. We shall for simplicity omit the subscripts ij, and for a while deal with the expression

$$\sum_{m\in M_N(t)} \frac{1}{m} \sum_{\tau=0}^{2T_N} (-1)^{\tau} \sum_{\substack{\mu\in M_N(\tau) \\ (\mu,m)=1}} \frac{1}{\mu}$$

under the condition that $t<2z(N)$.

Now, by the definition of the set $M_N(\tau)$, we have

$$\sum_{\tau=0}^{\infty} (-1)^{\tau} \sum_{\substack{\mu\in M_N(\tau) \\ (\mu,m)=1}} \frac{1}{\mu} = \prod_{\substack{p\in Q_N \\ p\nmid m}} \left(1-\frac{1}{p}\right),$$

where the left-hand side is essentially a finite sum. From this we can write

$$\left|\sum_{\tau=0}^{2T_N} (-1)^{\tau} \sum_{\substack{\mu\in M_N(\tau) \\ (\mu,m)=1}} \frac{1}{\mu} - \prod_{\substack{p\in Q_N \\ p\nmid m}} \left(1-\frac{1}{p}\right)\right| \leqq \sum_{\tau=2T_N+1}^{\infty} \sum_{\mu\in M_N(\tau)} \frac{1}{\mu}. \tag{14}$$

We shall first estimate the right-hand member of (14). From the definitions of $z(N)$ and $M_N(\tau)$, we have

$$\sum_{\mu\in M_N(\tau)} \frac{1}{\mu} \leqq \frac{\{z(N)\}^{\tau}}{\tau!},$$

which gives

$$\sum_{\tau=2T_N+1}^{\infty} \sum_{\mu\in M_N(\tau)} \frac{1}{\mu} \leqq \sum_{\tau=2T_N+1}^{\infty} \frac{\{z(N)\}^{\tau}}{\tau!} < \frac{\{z(N)\}^{2T_N+1} e^{z(N)}}{(2T_N+1)!} < \left(\frac{ez(N)}{2T_N+1}\right)^{2T_N+1} e^{z(N)},$$

where the last step is due to the inequality $(2T_N+1)^{2T_N+1}<(2T_N+1)!\,e^{2T_N+1}$. Also $2T_N+1>9z(N)$, so that

$$\left(\frac{ez(N)}{2T_N+1}\right)^{2T_N+1} < \left(\frac{e}{q}\right)^{9z(N)} < e^{-9z(N)}.$$

Thus we have

(15)
$$\sum_{\tau=2T_N+1}^{\infty} \sum_{\mu \in M_N(\tau)} \frac{1}{\mu} = O(e^{-9z(N)}) = o(e^{-z(N)}) .$$

On the other hand, since the primes contained in Q_N are greater than $e^{\{y(N)\}^2}$ by (5), we have

$$\sum_{p \in Q_N} \frac{1}{p^2} = o(1) ,$$

so that

$$\prod_{p \in Q_N} \left(1-\frac{1}{p}\right) = \exp\left\{\sum_{p \in Q_N} \log\left(1-\frac{1}{p}\right)\right\} = \exp\left\{-\sum_{p \in Q_N} \frac{1}{p} + O\left(\sum_{p \in Q_N} \frac{1}{p^2}\right)\right\}$$
$$= \exp\{-z(N)+o(1)\} = e^{-z(N)}\{1+o(1)\} .$$

Also, since $m \in M_N(t)$ with $t<2z(N)$, the number of prime factors of m is less than $2y(N)$, and each of the prime factors is $>e^{\{y(N)\}^2}$ by (5), we can deduce that

$$1 < \prod_{p|m} \left(1-\frac{1}{p}\right)^{-1} < \prod_{p|m} \left(1+\frac{2}{p}\right) < (1+2e^{-\{y(N)\}^2})^{2y(N)} = 1+o(1) .$$

Thus we have

(16)
$$\prod_{\substack{p \in Q_N \\ p \nmid m}} \left(1-\frac{1}{p}\right) = e^{-z(N)}\{1+o(1)\} ,$$

and this holds uniformly in m with $m \in M_N(t)$, $t<2z(N)$.

It follows from (14), (15), and (16) that

$$\sum_{\tau=0}^{2T_N} (-1)^\tau \sum_{\substack{\mu \in M_N(\tau) \\ (\mu, m)=1}} \frac{1}{\mu} = e^{-z(N)}\{1+o(1)\}$$

uniformly in the above-mentioned sense, and hence

(17)
$$\sum_{m \in M_N(t)} \frac{1}{m} \sum_{\tau=0}^{2T_N} (-1)^\tau \sum_{\substack{\mu \in M_N(\tau) \\ (\mu, m)=1}} \frac{1}{\mu} = \{1+o(1)\} e^{-z(N)} \sum_{m \in M_N(t)} \frac{1}{m}$$

uniformly in t with $t<2z(N)$.

It follows from the multinomial theorem that

(18)
$$\sum_{m \in M_N(t)} \frac{1}{m} \leq \frac{\{z(N)\}^t}{t!} \leq \sum_{m \in M_N(t)} \frac{1}{m} + \sum_{w} \frac{1}{w} ,$$

where the summation-variable w runs through positive integers which are not squarefree and composed of t primes $\in Q_N$. For each of these

w, we can uniquely put $w=d^2q$ with squarefree q. Since d is composed only of primes $\in Q_N$ and $d>1$, it follows that $d>e^{\{y(N)\}^2}$ by (5). Thus we can write

$$\sum \frac{1}{w} \leqq \sum_d \frac{1}{d^2} \sum_q \frac{1}{q},$$

where

$$\sum_d \frac{1}{d^2} = O(e^{-\{y(N)\}^2}) = O(e^{-\{z(N)\}^2}) .$$

As for the sum with the summation-variable q, we have

$$\sum_q \frac{1}{q} < 1 + z(N) + \frac{\{z(N)\}^2}{2!} + \cdots = e^{z(N)},$$

thus is follows that

$$\sum_w \frac{1}{w} = O(e^{z(N)-\{z(N)\}^2}) .$$

Also, since we assume that $t<2z(N)$, we have

$$\frac{\{z(N)\}^t}{t!} > \left(\frac{t}{2}\right)^t \cdot \frac{1}{t^t} = 2^{-t} > e^{-2z(N)} .$$

Hence we can write

$$\sum_w \frac{1}{w} = O\left(\frac{\{z(N)\}^t}{t!} e^{3z(N)-\{z(N)\}^2}\right),$$

which implies

$$\sum_w \frac{1}{w} = o\left(\frac{\{z(N)\}^t}{t!}\right) .$$

Now, by this and (18), we have

$$\sum_{m \in MN(t)} \frac{1}{m} = \frac{\{z(N)\}^t}{t!}\{1+o(1)\},$$

and the above deduction shows that this holds uniformly in t with $t<2z(N)$.

It follows from this and (17) that

$$\sum_{m \in M_N(t)} \frac{1}{m} \sum_{\tau=0}^{2T_N} (-1)^{\tau} \sum_{\substack{\mu \in M_{N'}(\tau) \\ (\mu, m)=1}} \frac{1}{\mu} = \frac{\{z(N)\}^t e^{-z(N)}}{t!} \{1+o(1)\}$$

uniformly in t with $t<2z(N)$, or, attaching now the subscripts ij

$$\sum_{m_{ij}\in M_{ijN}(t_{ij})}\frac{1}{m_{ij}}\sum_{\tau_{ij}=0}^{2T_{ijN}}(-1)^{\tau_{ij}}\sum_{\substack{\mu_{ij}\in M_{ijN}(\tau_{ij})\\(\mu_{ij},m_{ij})=1}}\frac{1}{\mu_{ij}}=\frac{\{z_{ij}(N)\}^{t_{ij}}e^{-z_{ij}(N)}}{t_{ij}!}\{1+o(1)\}$$

uniformly in t_{ij} with $t_{ij}<2z_{ij}(N)$. Finally the lemma follows on multiplying thus obtained kl equalities.

LEMMA 7. *Let* $T_{ijN}=[5z_{ij}(N)]$. *Then, as* $N\to\infty$,

$$H_2(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})=o\left(\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}\right)$$

uniformly in t_{ij} *with* $t_{ij}<2z_{ij}(N)$.

PROOF. For each summand of the sum defining $L_2(N;m_{11},\cdots,m_{kl};\tau_{11},\cdots,\tau_{kl})$ we put $d=(m_1\mu_1,\cdots,m_k\mu_k)$. Then we can obtain the positive integers m_i', μ_i', and m_{ij}', μ_{ij}' with $m_{ij}'\in M_{ijN}(t_{ij}')$, $t_{ij}'\leqq t_{ij}$, $\mu_{ij}'\in M_{ijN}(\tau_{ij}')$, $\tau_{ij}'\leqq\tau_{ij}$ such that $m_i'|m_i$, $\mu_i'|\mu_i$, $m_i\mu_i=m_i'\mu_i'd$, and $m_{ij}'|m_{ij}$, $m_i'=m_{i1}'\cdots m_{il}'$, $\mu_{ij}'|\mu_{ij}$, $\mu_i'=\mu_{i1}'\cdots\mu_{il}'$, and, since the number of prime factors of $m_{ij}\mu_{ij}$ is less than $12z_{ij}(N)$, it is easily seen that at most $2^{12\{z_{11}(N)+\cdots+z_{kl}(N)\}}$ distinct summands of $L_2(N;m_{11},\cdots,m_{kl};\tau_{11},\cdots,\tau_{kl})$ go to the same system m_{ij}', μ_{ij}' and d.

Now we have

$$\begin{aligned}&|H_2(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})|\\&\leqq\sum_{m_{11}\in M_{11N}(t_{11})}\cdots\sum_{m_{kl}\in M_{klN}(t_{kl})}\sum_{\tau_{11}=0}^{2T_{11N}}\cdots\sum_{\tau_{kl}=0}^{2T_{klN}}\\&\qquad\sum_{\substack{\mu_{11}\in M_{11N}(\tau_{11})\\(\mu_{11},m_{11})=1}}\cdots\sum_{\substack{\mu_{kl}\in M_{klN}(\tau_{kl})\\(\mu_{kl},m_{kl})=1\\(m_1\mu_1,\cdots,m_k\mu_k)>1\\(m_1\mu_1,\cdots,m_k\mu_k)|N}}\frac{(m_1\mu_1,\cdots,m_k\mu_k)}{m_1\mu_1\cdots m_k\mu_k},\end{aligned}$$

and, applying the above considerations to each of the summands, we can write

$$\begin{aligned}&|H_2(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})|\leqq2^{12\{z_{11}(N)+\cdots+z_{kl}(N)\}}\\&\quad\cdot\sum_d\frac{1}{d}\cdot\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{t_{ij}'=0}^{t_{ij}}\sum_{m_{ij}'\in M_{ijN}(t_{ij}')}\frac{1}{m_{ij}'}\cdot\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{\tau_{ij}'=0}^{2T_{ijN}}\sum_{\mu_{ij}'\in M_{ijN}(\tau_{ij}')}\frac{1}{\mu_{ij}'},\end{aligned}$$

a fortiori

$$\text{(19)}\quad\begin{aligned}&|H_2(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})|\\&\leqq2^{12\{z_{11}(N)+\cdots+z_{kl}(N)\}}\sum_d\frac{1}{d}\cdot\left(\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{t_{ij}=0}^{\infty}\sum_{m_{ij}\in M_{ijN}(t_{ij})}\frac{1}{m_{ij}}\right)^2.\end{aligned}$$

If we put for convienence

$$y_*(N)=\min_{\substack{1\leqq i\leqq k\\1\leqq j\leqq l}} y_{ij}(N)\ ,\quad y^*(N)=\max_{\substack{1\leqq i\leqq k\\1\leqq j\leqq l}} y_{ij}(N)\ ,$$

then, by (C_2) in Section 1,

$$y_*(N)=\lambda_*\log\log N+o(\sqrt{\log\log N})\ ,$$
$$y^*(N)=\lambda^*\log\log N+o(\sqrt{\log\log N})\ ,$$

with

$$\lambda_*=\min_{\substack{1\leqq i\leqq k\\1\leqq j\leqq l}} \lambda_{ij}\ ,\quad \lambda^*=\max_{\substack{1\leqq i\leqq k\\1\leqq j\leqq l}} \lambda_{ij}\ ,$$

and for the validity of (19), it will be sufficient to make d run through integers such that the number of prime factors of d is less than $12y^*(N)$ and each of them is greater than $e^{\{y_*(N)\}^2}$. Hence we can write

$$\sum_d \frac{1}{d}<(1+e^{-\{y_*(N)\}^2})^{12y^*(N)}-1\ ,$$

which gives

$$\sum_d \frac{1}{d}=O(e^{-\{y_*(N)\}^2}y^*(N))\ . \tag{20}$$

Again, as in the proof of Lemma 6, we have

$$\sum_{m_{ij}\in M_{ijN}(t_{ij})} \frac{1}{m_{ij}}\leqq\frac{\{z_{ij}(N)\}^{t_{ij}}}{t_{ij}!}\ ,$$

which implies

$$\sum_{t_{ij}=0}^{\infty}\ \sum_{m_{ij}\in M_{ijN}(t_{ij})} \frac{1}{m_{ij}}=O(e^{z_{ij}(N)})\ . \tag{21}$$

It follows from (19), (20), and (21) that

$$H_2(N; t_{11},\cdots, t_{kl}; T_{11N},\cdots, T_{klN})=O(e^{14\{z_{11}(N)+\cdots+z_{kl}(N)\}-\{y_*(N)\}^2}y^*(N))\ .$$

On the other hand, since $t_{ij}<2z_{ij}(N)$, we can argue

$$\begin{aligned}&\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}\\&>\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{\{2z_{11}(N)\}^{t_{11}}\cdots\{2z_{kl}(N)\}^{t_{kl}}}=2^{-(t_{11}+\cdots+t_{kl})}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}\\&>e^{-3\{z_{11}(N)+\cdots+z_{kl}(N)\}}\ ,\end{aligned} \tag{22}$$

and hence we can write

$$H_2(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$$
$$=\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}O(e^{17\{z_{11}(N)+\cdots+z_{kl}(N)\}-\{y_*(N)\}^2}y^*(N)) .$$

Now, by Lemma 4 and the above formulas for $y_*(N)$ and $y^*(N)$, we obtain the lemma.

LEMMA 8. *Let* $T_{ijN}=[5z_{ij}(N)]$. *Then, as* $N\to\infty$,

$$H_3(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN}) =o\left(\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}\right)$$

uniformly in t_{ij} *with* $t_{ij}<2z_{ij}(N)$.

PROOF. For each summand of the sum defining $L_3(N; m_{11}, \cdots, m_{kl}; \tau_{11}, \cdots, \tau_{kl})$ we put $d=(m_1\mu_1, \cdots, m_k\mu_k)$, then, as in the proof of Lemma 7, we can obtain

$$|H_3(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})|$$
$$\leqq 2^{12\{z_{11}(N)+\cdots+z_{kl}(N)\}}\sum_d \frac{1}{d^2}\cdot\left(\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{t_{ij}=0}^{\infty}\sum_{m_{ij}\in M_{ijN}(t_{ij})}\frac{1}{m_{ij}}\right)^2 ,$$

and this inequality is valid, if we make d run through the positive integers greater than $e^{\{y_*(N)\}^2}$, so that

$$\sum_d \frac{1}{d^2}=O(e^{-\{y^*(N)\}^2}) .$$

The remaining part of the proof can also be performed as in the proof of Lemma 7.

LEMMA 9. *Let* $T_{ijN}=[5z_{ij}(N)]$. *Then, as* $N\to\infty$,

$$H(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$$
$$=\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}\{1+o(1)\}$$

uniformly in t_{ij} *with* $t_{ij}<2z_{ij}(N)$.

PROOF. The lemma follows from (12) and Lemmas 6, 7, and 8.

LEMMA 10. *Let* $T_{ijN}=[5z_{ij}(N)]$. *Then, as* $N\to\infty$,

$$\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})-\frac{N^{k-1}}{(k-1)!}H(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$$

$$=o\left(\frac{N^{k-1}\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}\right)$$

uniformly in t_{ij} *with* $t_{ij}<2z_{ij}(N)$.

PROOF. From Lemma 1, we have

$$(23)\quad \mathscr{H}^{(0)}(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})-\frac{N^{k-1}}{(k-1)!}$$
$$\cdot H(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})=O\Big(N^{k-2}\sum_{m_{11}\in M_{11N}(t_{11})}\cdots\sum_{m_{kl}\in M_{klN}(t_{kl})}$$
$$\sum_{\tau_{11}=0}^{2T_{11N}}\cdots\sum_{\tau_{kl}=0}^{2T_{klN}}\sum_{\mu_{11}\in M_{11N}(\tau_{11})}\cdots\sum_{\mu_{kl}\in M_{klN}(\tau_{kl})}1\Big)$$
$$=O\left\{N^{k-2}\Big(\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{t_{ij}=0}^{2T_{ijN}}\sum_{m_{ij}\in M_{ijN}(t_{ij})}1\Big)^{2}\right\}$$

since $t_{ij}<T_{ijN}$ by the assumptions.

Now, denoting by $|Q_{ijN}|$ the number of elements of the set Q_{ijN}, we have

$$\sum_{m_{ij}\in M_{ijN}(t_{ij})}1=\binom{|Q_{ijN}|}{t_{ij}}\leqq\frac{|Q_{ijN}|^{t_{ij}}}{t_{ij}!},$$

and hence

$$\sum_{t_{ij}=0}^{2T_{ijN}}\sum_{m_{ij}\in M_{ijN}(t_{ij})}1\leqq\sum_{t_{ij}=0}^{2T_{ijN}}\frac{|Q_{ijN}|^{t_{ij}}}{t_{ij}!}<e|Q_{ijN}|^{2T_{ijN}}.$$

Since $T_{ijN}\leqq 5z_{ij}(N)\leqq 5y_{ij}(N)$ by assumption, and $|Q_{ijN}|<N^{\{y_{ij}(N)\}^{-2}}$ by (5), it follows that

$$\sum_{t_{ij}=0}^{2T_{ijN}}\sum_{m_{ij}\in M_{ijN}(t_{ij})}1=O(N^{10\{y_{ij}(N)\}^{-1}})=O(N^{10\{y_*(N)\}^{-1}}).$$

From this and (23), we obtain

$$\mathscr{H}^{(0)}(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})-\frac{N^{k-1}}{(k-1)!}$$
$$\cdot H(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})=O(N^{k-2+20kl\{y_*(N)\}^{-1}}).$$

Again, from this and (22), we can write

$$\mathscr{H}^{(0)}(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})-\frac{N^{k-1}}{(k-1)!}$$
$$\cdot H(N;t_{11},\cdots,t_{kl};T_{11N},\cdots,T_{klN})=\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}$$

$$\cdot O(e^{3\{z_{11}(N)+\cdots+z_{kl}(N)\}}N^{k-2+20kl\{y_*(N)\}^{-1}}) .$$

Now the lemma follows from Lemma 4 and the formula for $y_*(N)$.

LEMMA 11. *Let* $T_{ijN}=[5z_{ij}(N)]$. *Then, as* $N\to\infty$,

$$\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN}) = \frac{N^{k-1}\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{(k-1)!\ t_{11}!\cdots t_{kl}!}\{1+o(1)\}$$

uniformly in t_{ij} *with* $t_{ij}<2z_{ij}(N)$, *and the same formulas also hold for* $\mathscr{H}^{(ij)}(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$.

PROOF. The asymptotic formula for $\mathscr{H}^{(0)}(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$ follows from Lemmas 9 and 10. $\mathscr{H}^{(ij)}(N; t_{11}, \cdots, t_{kl}; T_{11N}, \cdots, T_{klN})$ can be treated in much the same way, as is easily seen, from their definitions.

LEMMA 12. *As* $N\to\infty$,

$$F(N; t_{11}, \cdots, t_{kl}) = \frac{N^{k-1}\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{(k-1)!\ t_{11}!\cdots t_{kl}!}\{1+o(1)\}$$

uniformly in t_{ij} *with* $t_{ij}<2z_{ij}(N)$.

PROOF. The lemma follows from Lemmas 5 and 11.

LEMMA 13. *Let* $\alpha_{ij}<\beta_{ij}$. *Let* t_{ij} *be positive integers such that* $t_{ij}=z_{ij}(N)+x_{ij}\sqrt{z_{ij}(N)}$ *with* $\alpha_{ij}<x_{ij}<\beta_{ij}$. *Then, as* $N\to\infty$,

$$F(N; t_{11}, \cdots, t_{kl}) = \frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\{z_{11}(N)\cdots z_{kl}(N)\}^{-1/2}e^{-(x_{11}^2+\cdots+x_{kl}^2)/2}\{1+o(1)\}$$

uniformly in t_{ij} *with above-mentioned restrictions.*

PROOF. We use the Stirling formula

$$t! = \sqrt{2\pi}t^{t+1/2}e^{-t}\left\{1+O\left(\frac{1}{t}\right)\right\} .$$

We put $t=t_{ij}=z_{ij}(N)+x_{ij}\sqrt{z_{ij}(N)}$, and consider large N, letting x_{ij} be bounded, then easy calculations give

$$t_{ij}! = \sqrt{2\pi}\{z_{ij}(N)\}^{z_{ij}(N)+x_{ij}\sqrt{z_{ij}(N)}+1/2}e^{-z_{ij}(N)+x_{ij}^2/2}\left\{1+O\left(\frac{1}{\sqrt{z_{ij}(N)}}\right)\right\} ,$$

or

$$\frac{\{z_{ij}(N)\}^{t_{ij}}e^{-z_{ij}(N)}}{t_{ij}!}=\frac{e^{-x_{ij}^2/2}}{\sqrt{2\pi z_{ij}(N)}}\left\{1+O\left(\frac{1}{\sqrt{z_{ij}(N)}}\right)\right\} .$$

Multiplying thus obtained kl formulas, we obtain

$$\frac{\{z_{11}(N)\}^{t_{11}}\cdots\{z_{kl}(N)\}^{t_{kl}}e^{-\{z_{11}(N)+\cdots+z_{kl}(N)\}}}{t_{11}!\cdots t_{kl}!}$$
$$=(2\pi)^{-kl/2}\{z_{11}(N)\cdots z_{kl}(N)\}^{-1/2}e^{-(x_{11}^2+\cdots+x_{kl}^2)/2}\{1+o(1)\} .$$

Since $t_{ij}<2z_{ij}(N)$ for large N, the lemma follows from this and Lemma 12 including the enunciated uniformity.

LEMMA 14. *Let $\alpha_{ij}<\beta_{ij}$, and let $A^{**}(N)=A^{**}(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that*

$$z_{ij}(N)+\alpha_{ij}\sqrt{z_{ij}(N)}<\omega_{ijN}(n_i)<z_{ij}(N)+\beta_{ij}\sqrt{z_{ij}(N)}$$

simultaneously. Then, as $N\to\infty$, we have

$$A^{**}(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}}^{\beta_{ij}}e^{-x_{ij}^2/2}dx_{ij} .$$

PROOF. By the definition of $F(N;t_{11},\cdots,t_{kl})$, we can write

$$A^{**}(N)=\sum_{t_{ij}}F(N;t_{11},\cdots,t_{kl}) ,$$

the summation extending over the systems of kl positive integers t_{ij} such that

$$z_{ij}(N)+\alpha_{ij}\sqrt{z_{ij}(N)}<t_{ij}<z_{ij}(N)+\beta_{ij}\sqrt{z_{ij}(N)}$$

simultaneously. Now let these values of t_{ij} be $t_{ij\nu}$, and let $t_{ij\nu}=z_{ij}(N)+x_{ij\nu}\sqrt{z_{ij}(N)}$ with $\nu=1,\cdots,s_{ij}$. Then

$$x_{ij,\nu+1}-x_{ij\nu}=\{z_{ij}(N)\}^{-1/2}\quad(\nu=1,\cdots,s_{ij}-1) ,$$

and hence from Lemma 13, we obtain

$$A^{**}(N)=\{1+o(1)\}\frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\sum_{\nu=1}^{s_{ij}-1}e^{-x_{ij\nu}^2/2}(x_{ij,\nu+1}-x_{ij\nu}) .$$

The lemma follows by making $N\to\infty$ in this formula.

LEMMA 15. *Let $\alpha_{ij}<\beta_{ij}$, and let $A^*(N)=A^*(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that*

$$z_{ij}(N)+\alpha_{ij}\sqrt{z_{ij}(N)}<\omega_{ij}(n_i)<z_{ij}(N)+\beta_{ij}\sqrt{z_{ij}(N)}$$

simultaneously. Then, as $N\to\infty$, we have

$$A^*(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}}^{\beta_{ij}}e^{-x_{ij}^2/2}dx_{ij}\ .$$

PROOF. We shall estimate the sum

$$\sum_{n_1+\cdots++n_k=N}\{\omega_{ij}(n_i)-\omega_{ijN}(n_i)\}$$

extended over the systems of positive integers $n_1,\cdots,n_k$ such that $n_1+\cdots+n_k=N$. Transforming the summation to the form

$$\sum_{n_i<N}\{\omega_{ij}(n_i)-\omega_{ijN}(n_i)\}\sum_{n_1+\cdots+n_{i-1}+n_{i+1}+\cdots+n_k=N-n_i}1\ ,$$

and estimating the inner sum trivially as $<N^{k-2}$, we have

$$\sum_{n_1+\cdots+n_k=N}\{\omega_{ij}(n_i)-\omega_{ijN}(n_i)\}\leqq N^{k-2}\sum_{n_i\leqq N}\{\omega_{ij}(n_i)-\omega_{ijN}(n_i)\}\ .$$

Also, by (4) and Lemma 4,

$$\sum_{n_i\leqq N}\{\omega_{ij}(n_i)-\omega_{ijN}(n_i)\}=\sum_{n_i\leqq N}\ \sum_{p|n_i,p\in P_{ij}-Q_{ijN}}1=\sum_{p\leqq N,p\in P_{ij}-Q_{ijN}}\left[\frac{N}{p}\right]$$
$$\leqq N\sum_{p\leqq N,p\in P_{ij}-Q_{ijN}}\frac{1}{p}=N\{y_{ij}(N)-z_{ij}(N)\}=o(N\sqrt{z_{ij}(N)})\ .$$

Hence

$$\sum_{n_1+\cdots+n_k=N}\{\omega_{ij}(n_i)-\omega_{ijN}(n_i)\}=o(N^{k-1}\sqrt{z_{ij}(N)})\ .$$

Now it follows that, for any given $\varepsilon>0$, we can take $N_1=N_1(\varepsilon)$ so large that, when $N>N_1$, the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$, such that at least one of the kl inequalities $\omega_{ij}(n_i)-\omega_{ijN}(n_i)>\varepsilon\sqrt{z_{ij}(N)}$ holds, is less than εN^{k-1}. Then, for $N>N_1$,

$$A^{**}(N;\alpha_{11},\beta_{11}-\varepsilon,\cdots,\alpha_{kl},\beta_{kl}-\varepsilon)-\varepsilon N^{k-1}$$
$$<A^*(N;\alpha_{11},\beta_{11},\cdots,\alpha_{jl},\beta_{kl})$$
$$<A^{**}(N;\alpha_{11}-\varepsilon,\beta_{11},\cdots,\alpha_{kl}-\varepsilon,\beta_{kl})+\varepsilon N^{k-1}\ .$$

From this and Lemma 14, we conclude that

$$\frac{1}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}}^{\beta_{ij}-\varepsilon}e^{-x_{ij}^2/2}dx_{ij}-\varepsilon$$
$$\leqq \liminf_{N\to\infty}\frac{A^*(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})}{N^{k-1}}$$
$$\leqq \limsup_{N\to\infty}\frac{A^*(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})}{N^{k-1}}$$
$$\leqq \frac{1}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}-\varepsilon}^{\beta_{ij}}e^{-x_{ij}^2/2}dx_{ij}+\varepsilon,$$

which gives the lemma.

Lemma 16. *Let* $\alpha_{ij}<\beta_{ij}$, *and let* $A(N)=A(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})$ *denote the number of representations of N as the sum of k positive integers:* $N=n_1+\cdots+n_k$ *such that*

$$\lambda_{ij}\log\log N+\alpha_{ij}\sqrt{\lambda_{ij}\log\log N}<\omega_{ij}(n_i)<\lambda_{ij}\log\log N+\beta_{ij}\sqrt{\lambda_{ij}\log\log N}$$

simultaneously. Then, as $N\to\infty$, we have

$$A(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}}^{\beta_{ij}}e^{-x_{ij}^2/2}dx_{ij}.$$

Proof. It follows from Lemma 4 that for any given $\varepsilon>0$, we can take $N_2=N_2(\varepsilon)$ so large that, when $N>N_2$,

$$z_{ij}(N)+(\alpha_{ij}-\varepsilon)\sqrt{z_{ij}(N)}<\lambda_{ij}\log\log N$$
$$+\alpha_{ij}\sqrt{\lambda_{ij}\log\log N}<z_{ij}(N)+(\alpha_{ij}+\varepsilon)\sqrt{z_{ij}(N)},$$
$$z_{ij}(N)+(\beta_{ij}-\varepsilon)\sqrt{z_{ij}(N)}<\lambda_{ij}\log\log N$$
$$+\beta_{ij}\sqrt{\lambda_{ij}\log\log N}<z_{ij}(N)+(\beta_{ij}+\varepsilon)\sqrt{z_{ij}(N)},$$

so that

$$A^*(N;\alpha_{11}+\varepsilon,\beta_{11}-\varepsilon,\cdots,\alpha_{kl}+\varepsilon,\beta_{kl}-\varepsilon)$$
$$\leqq A(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})$$
$$\leqq A^*(N;\alpha_{11}-\varepsilon,\beta_{11}+\varepsilon,\cdots,\alpha_{kl}-\varepsilon,\beta_{kl}+\varepsilon).$$

From this and Lemma 15, we conclude that

$$\frac{1}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}+\varepsilon}^{\beta_{ij}-\varepsilon}e^{-x_{ij}^2/2}dx_{ij}$$

$$\leqq \liminf_{N\to\infty} \frac{A(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})}{N^{k-1}}$$
$$\leqq \limsup_{N\to\infty} \frac{A(N;\alpha_{11},\beta_{11},\cdots,\alpha_{kl},\beta_{kl})}{N^{k-1}}$$
$$\leqq \frac{1}{(k-1)!}\,(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}-\varepsilon}^{\beta_{ij}+\varepsilon} e^{-x_{ij}^2/2}dx_{ij}\,,$$

which gives the lemma.

Lemma 16 is the special case of Theorem 1, when the set E is an interval.

THE PROOF OF THEOREM 1. First we consider the case when the set E in R^{kl} is bounded. We take two systems of intervals I_ν, I'_ν ($\nu=1, 2, \cdots$), finite in number, such that

$$\bigcup_\nu I_\nu \subset E \subset \bigcup_\nu I'_\nu$$

and any two of the intervals I_ν do not overlap. Then we have

$$\sum_\nu A(N; I_\nu) \leqq A(N; E) \leqq \sum_\nu A(N; I'_\nu)\,.$$

On applying Lemma 16 to the intervals I_ν, I'_ν, we obtain

$$\frac{1}{(k-1)!}\,(2\pi)^{-kl/2}\sum_\nu \int_{I_\nu} \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{l}x_{ij}^2\right)dx_{11}\cdots dx_{kl}$$
$$\leqq \liminf_{N\to\infty}\frac{A(N;E)}{N^{k-1}} \leqq \limsup_{N\to\infty}\frac{A(N;E)}{N^{k-1}}$$
$$\leqq \frac{1}{(k-1)!}\,(2\pi)^{-kl/2}\sum_\nu \int_{I'_\nu} \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{l}x_{ij}^2\right)dx_{11}\cdots dx_{kl}\,.$$

Now, since E is supposed to be Jordan-measurable, we can take, for any positive ε, the intervals I_ν, I'_ν such that

$$\int_E - \varepsilon < \sum_\nu \int_{I_\nu} \leqq \sum_\nu \int_{I'_\nu} < \int_E + \varepsilon$$

omitting the common integrand

$$\frac{1}{(k-1)!}\,(2\pi)^{-kl/2}\exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{l}x_{ij}^2\right),$$

and, combining thus obtained inequalities, we obtain

$$\int_E -\varepsilon < \liminf_{N\to\infty} \frac{A(N; E)}{N^{k-1}} \leqq \limsup_{N\to\infty} \frac{A(N; E)}{N^{k-1}} < \int_E + \varepsilon$$

which gives

$$\lim_{N\to\infty} \frac{A(N; E)}{N^{k-1}} = \int_E .$$

Next we consider the case when the set E is not bounded. For any given $\varepsilon > 0$, we can take an interval I so large that

$$\lim_{N\to\infty} \frac{A(N; I)}{N^{k-1}} = \int_I > \frac{1}{(k-1)!} - \varepsilon ,$$

or

$$\lim_{N\to\infty} \frac{A(N; I^c)}{N^{k-1}} = \int_{I^c} < \varepsilon ,$$

which implies

$$\limsup_{N\to\infty} \frac{A(N; E \cap I^c)}{N^{k-1}} < \varepsilon .$$

Also, since the set $E \cap I$ is bounded, as is already proved,

$$\lim_{N\to\infty} \frac{A(N; E \cap I)}{N^{k-1}} = \int_{E\cap I} .$$

Thus we have

$$\liminf_{N\to\infty} \frac{A(N; E)}{N^{k-1}} \geqq \lim_{N\to\infty} \frac{A(N; E \cap I)}{N^{k}} = \int_{E\cap I} > \int_E - \varepsilon ,$$

$$\limsup_{N\to\infty} \frac{A(N; E)}{N^{k-1}} = \lim_{N\to\infty} \frac{A(N; E \cap I)}{N^{k-1}} + \limsup_{N\to\infty} \frac{A(N; E \cap I^c)}{N^{k-1}} < \int_E + \varepsilon ,$$

which gives

$$\lim_{N\to\infty} \frac{A(N; E)}{N^{k-1}} = \int_E ,$$

and the proof of Theorem 1 is completed.

§ 3. Some special cases.

Let m be a positive integer, and put $l = \varphi(m)$. For each i, let $r_{i1}, \cdots, r_{il}$ be a reduced system of residues with respect to the modulus

m in an arbitrary order. Let $\omega_{ij}(n)$ denote the number of distinct prime factors of n which are congruent to r_{ij} to modulus m. In this case $\lambda_{ij}=1/l$ and so we have

THEOREM 2. *Let $\alpha_{ij}<\beta_{ij}$, and let $A(N)=A(N; \alpha_{11}, \beta_{11}, \cdots, \alpha_{kl}, \beta_{kl})$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that*

$$\frac{1}{l}\log\log N+\frac{\alpha_{ij}}{\sqrt{l}}\sqrt{\log\log N}<\omega_{ij}(n_i)<\frac{1}{l}\log\log N+\frac{\beta_{ij}}{\sqrt{l}}\sqrt{\log\log N}$$

simultaneously. Then, as $N\to\infty$, we have

$$A(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-kl/2}\prod_{i=1}^{k}\prod_{j=1}^{l}\int_{\alpha_{ij}}^{\beta_{ij}}e^{-x_{ij}^2/2}dx_{ij}\,.$$

THEOREM 3. *Let $A(N)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that*

$$\omega_{i1}(n_i)<\omega_{i2}(n_i)<\cdots<\omega_{il}(n_i)$$

simultaneously. Then, as $N\to\infty$, we have

$$A(N)\sim\frac{N^{k-1}}{(k-1)(l!)^k}\,.$$

Let $\omega(n)$ without subscript denote the number of distinct prime factors of n. It would be not so easy to prove the following theorem independently of this paper.

THEOREM 4. *Let $A(N)$ denote the number of representations of N as the sum of two positive integers: $N=n_1+n_2$ such that $\omega(n_1)=\omega(n_2)$. Then, as $n\to\infty$, we have*

$$A(N)=o(N)\,.$$

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