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An Asymptotic Formula for Cayley's Double Partition Function $p(2; n)$

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Introduction

A *double partition* of a positive integer n is a representation of n as a double sum

$$\begin{aligned} n &= a_1 + \cdots\cdots + a_k ;\\ a_1 &= b_{11} + \cdots\cdots + b_{1m_1} ,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ a_i &= b_{i1} + \cdots\cdots + b_{im_i} ,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ a_k &= b_{k1} + \cdots\cdots + b_{km_k} , \end{aligned}$$

where the *rough parts* $a_1, \cdots, a_k$ and all *fine parts* $b_{i1}, \cdots, b_{im_i}$ are positive integers. The order of each of rough parts and fine parts is disregarded. The numbers $k, m_1, \cdots, m_k$ are arbitrary positive integers. We may express this double partition in the form

$$n = (b_{11} + \cdots + b_{1m_1}) + \cdots + (b_{k1} + \cdots + b_{km_k}) .$$

Thus

$$3 = (3) = (2+1) = (1+1+1) = (2) + (1) = (1+1) + (1) = (1) + (1) + (1)$$

has 6 double partitions.

Let us denote the numbers of ordinary and double partitions of n by $p(n)$ and $p(2; n)$ respectively. The functions $p(n)$ and $p(2; n)$ have the relation

$$(0.1) \qquad p(2; n) = \sum_{\substack{s_1, s_2, \cdots \geq 0 \\ 1\cdot s_1 + 2\cdot s_2 + \cdots = n}} \prod_{i=1}^{n} {}_{p(i)}\mathrm{H}_{s_i} ,$$

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where ${}_{p(i)}\mathrm{H}_{s_i}$ is the number of repeated combinations of choosing s_i objects from a collection of $p(i)$ distinct objects, namely

$${}_{p(i)}\mathrm{H}_{s_i}=\frac{(p(i)+s_i-1)!}{s_i!\,(p(i)-1)!}\ .$$

The relation (0.1) is shown as follows. Let $n=a_1+\cdots+a_k$ be an ordinary partition of n, and let

$$s_i=\sum_{i=a_j(1\leq j\leq k)} 1\ .$$

Then

$$n=1\cdot s_1+2\cdot s_2+\cdots\ .$$

This is another expression of a partition of n. The number of the ways of partitioning s_i rough parts $i,\cdots,i$ into fine parts is ${}_{p(i)}\mathrm{H}_{s_i}$. So the number of the ways of partitioning every rough parts $a_1,\cdots,a_k$ into fine

TABLE OF $p(2;\,n)$

n	$p(2;\,n)$	n	$p(2;\,n)$	n	$p(2;\,n)$
1	1	26	8619551	51	2423003050071
2	3	27	14722230	52	3892531857557
3	6	28	25057499	53	6243041523533
4	14	29	42494975	54	9996843883291
5	27	30	71832114	55	15982620492896
6	58	31	121024876	56	25513313739945
7	111	32	203286806	57	40666250753266
8	223	33	340435588	58	64723795589093
9	424	34	568496753	59	102865540395574
10	817	35	946695386	60	163254461674365
11	1527	36	1572318704	61	258739042857968
12	2870	37	2604620337	62	409517817903949
13	5279	38	4303967991	63	647305414472473
14	9710	39	7094812475	64	1021839273593070
15	17622	40	11668071461	65	1611030410069440
16	31877	41	19145625665	66	2536782457453554
17	57100	42	31346243117	67	3989617115561102
18	101887	43	51211991765	68	6266969254661484
19	180406	44	83494293481	69	9832677052500390
20	318106	45	135851400237	70	15409305679317382
21	557453	46	220606815930	71	24121280926956918
22	972796	47	357554969083	72	37716580361109313
23	1688797	48	578440031745	73	58909762678433168
24	2920123	49	943083024470	74	91912246311214836
25	5026410	50	1505721089508	75	143251278581721424

n	$p(2; n)$	n	$p(2; n)$
76	223033602180779324	101	10625999485553889933793
77	346893510149724456	102	16179184801946827857949
78	538993270493029627	103	24616977325542965865767
79	836640343271910272	104	37428964189359689250714
80	1297386451788210261	105	56869518597233685716440
81	2009933200312154854	106	86348250925220887503827
82	3110871214093056439	107	131018704502650557435922
83	4810351817700282148	108	198665410843796875868654
84	7431425693793647923	109	301039712138051727038741
85	11470275706825584508	110	455870186958818086950184
86	17688368032373092724	111	689886577043140083439589
87	27253324613423500684	112	1043365491835182992315747
88	41954109377728309487	113	1576959776253614829623365
89	64529478726248551908	114	2381952946807720108159469
90	99168934646008215424	115	3595647739004779607125639
91	152276208829194456178	116	5424446359910459530030051
92	233632003770322699661	117	8178447855477605909522360
93	358163490674033022989	118	12323281254197305810563680
94	548635256692817087165	119	18557718918459568001676537
95	839738635347008546374	120	27929829980951484755907602
96	1284301754941134768291	121	42010719110833409517337077
97	1962713579957981264686	122	63154262851246486627350516
98	2997215564669187986751	123	94885208353108255653834861
99	4573560453099579148954	124	142478772151894818857110559
100	6973813984312291101797	125	213825788478399740335375184

parts is

$$\prod_{i=1}^{n} {}_{p(i)}\mathrm{H}_{s_i} .$$

Whence we get (0.1).

The generating function of $p(2; n)$, which was found by Cayley [1], is

$$f(2; u)=1+\sum_{n=1}^{\infty} p(2; n)u^n=\prod_{m=1}^{\infty}(1-u^m)^{-p(m)} , \tag{0.2}$$

where u is a complex variable with $|u|<1$. More generally, we can derive the following

PROPOSITION. *Let $a(1)$, $a(2)$, $\cdots$ be a sequence of positive integers. Assume that the generating function $\sum a(n)u^n$ has a positive radius R of convergence. If $b(n)$ is given by*

$$b(n)=\sum_{\substack{s_1, s_2, \cdots \geqq 0 \\ 1\cdot s_1+2\cdot s_2+\cdots=n}} \prod_{i=1}^{n} {}_{a(i)}\mathrm{H}_{s_i} , \tag{0.3}$$

then $b(n)$ *has the generating function*

$$(0.4)\qquad 1+\sum_{n=1}^{\infty}b(n)u^{n}=\prod_{m=1}^{\infty}(1-u^{m})^{-a(m)}\ ,\qquad |u|<\min(R,1)\ .$$

In the cases $a(n)=1$ and $a(n)=p(n)$, (0.4) gives the generating functions of $p(n)$ and $p(2;\,n)$ respectively.

The proof of Proposition is similar to the case $a(n)=1$ (Hardy and Wright [2], 275-276). Here we may use the identity

$$\prod_{m=1}^{r}(1-u^{m})^{-a(m)}=1+\sum_{n=1}^{r}b(n)u^{n}+\sum_{n=r+1}^{\infty}b_{r}(n)u^{n}\ ,$$

where

$$b_{r}(n)=\sum_{\substack{s_1,\cdots,s_r\geq 0\\ 1\cdot s_1+\cdots+r\cdot s_r=n}}\ \prod_{i=1}^{r}{}_{a(i)}\mathrm{H}_{s_i}\ .$$

We get further the recurrence formula

$$(0.5)\qquad b(n)=\frac{1}{n}\sum_{k=1}^{n}\sigma(k)\cdot b(n-k)\ ,$$

where

$$\sigma(k)=\sum_{d|k}d\cdot a(d)$$

and $b(0)=1$. This is derived from the logarithmic derivative of (0.4).

The purpose of this paper is to prove the following

THEOREM. *The function* $p(2;\,n)$ *has the asymptotic formula*

$$(0.6)\qquad p(2;\,n)=(2\pi)^{-3/2}v''(x_0)^{-1/2}x_0e^{v(x_0)+nx_0+\kappa}(1+O(\log^{-1}n))\ ,$$

where

$$v(x)=(x/2\pi)^{1/2}\sum_{m=1}^{\infty}m^{-1/2}\exp\left(\frac{\pi^2}{6mx}-\frac{mx}{24}\right),\quad x>0\ ,$$

x_0 *is the root of the equation* $v'(x)+n=0$ *and* κ *is a constant given by*

$$\kappa=\int_0^{\infty}(f(e^{-2\pi t})-\sqrt{t}\,e^{\pi(1/t-t)/12}+e^{-t}-1)\frac{dt}{t}=-1.12893\cdots$$

in which $f(u)=1+\sum_{m=1}^{\infty}p(m)u^{m}$ *is the generating function of* $p(m)$.

As for the order of magnitudes of $x_0(n)$, $v(x_0)$ and $v''(x_0)$ in (0.6), we have the estimates:

(0.7) $$x_0(n)=\frac{\pi^2}{6l}\left(1+O\left(\frac{\log\log n}{\log^2 n}\right)\right),$$

where

$$l=\log n-\frac{3}{2}\log\log n+\frac{1}{2}\log\frac{\pi^3}{3},$$

(0.8) $$v(x_0)=\frac{\pi^2 n}{6\log^2 n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right),$$

and

(0.9) $$v''(x_0)=\frac{6}{\pi^2}n\log^2 n\left(1+O\left(\frac{\log\log n}{\log n}\right)\right).$$

From the theorem and (0.7)$\sim$(0.9), we can easily obtain the following

COROLLARY.

(0.10) $$p(2;n)=(\pi/12)^{3/2}n^{-1/2}\log^{-2}n\cdot e^{v(x_0)+nx_0+\kappa}(1+O(\log^{-1}n))$$

and

(0.11) $$\log p(2;n) = \frac{\pi^2}{6}n(l^{-1}+\log^{-2}n)+O\left(\frac{n\log\log n}{\log^3 n}\right) \sim \frac{\pi^2 n}{6\log n}.$$

Our proof of the theorem is carried out in substantially the same way as that of the elementary proof of

$$p(n)=\frac{1}{4\sqrt{3}\,n}e^{\sqrt{2n/3}}(1+O(n^{-(1/4)+\varepsilon})), \quad \varepsilon>0,$$

due to Postonikov [3], but we need some more elaborate estimates. It depends on the saddle point method with the aid of the asymptotic behavior of $\log f(2;\exp(-z))$ which we shall give in Section 1. We shall prove the theorem as well as (0.7)$\sim$(0.9) in Section 2. In what follows we shall denote by ε or ε' some arbitrary positive numbers.

§ 1. Asymptotic behavior of $\log f(2;e^{-z})$.

From (0.4), we obtain

(1.1) $$\log\left(1+\sum_{n=1}^{\infty}b(n)u^n\right)=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}a(m)u^{mn}.$$

Since $a(n)=p(n)$ in our case,

$$\log f(2;\, u) = \sum_{n=1}^{\infty} \frac{1}{n}(f(u^n)-1)\,. \tag{1.2}$$

Now we put $u=e^{-2\pi z}$, then $f(u)$ and $f(2;\, u)$ are regular functions of z for $\operatorname{Re} z>0$. The function $f(u)$ satisfies the equation

$$f(e^{-2\pi z}) = \sqrt{z}\, e^{\pi/12((1/z)-z)} f(e^{2\pi/z})\,, \qquad \operatorname{Re} z>0 \tag{1.3}$$

which is equivalent to the well-known formula

$$\eta(-1/\tau) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)\,,$$

where $\eta(\tau)$ is the Dedekind η-function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{m=1}^{\infty} (1-e^{2\pi i m\tau})\,, \qquad \operatorname{Im} \tau>0\,.$$

We introduce a function

$$V(z) = \sqrt{z} \sum_{m=1}^{\infty} m^{-1/2} \exp\left(\frac{\pi}{12}\left(\frac{1}{mz}-mz\right)\right)\,, \qquad \operatorname{Re} z>0\,, \tag{1.4}$$

which is uniformly convergent in every compact subset of the domain $\operatorname{Re} z>0$. Thus $V(z)$ is regular for $\operatorname{Re} z>0$. Therefore from (1.2), (1.4) and

$$\begin{aligned}\log z &= \log(1-e^{-z}) + \log\frac{z}{1-\exp(-z)} \\ &= -z\sum_{m=1}^{\infty}\frac{e^{-mz}}{mz} + O(|z|)\,, \qquad |z|\to 0\,,\end{aligned}$$

we obtain

$$\log f(2;\, e^{-2\pi z}) = V(z) + \log z + z\cdot\sum_{n=1}^{\infty}\varphi(nz) + O(|z|),\ |z|\to 0\,, \tag{1.5}$$

where

$$\varphi(z) = \frac{1}{z}\left(f(e^{-2\pi z}) - 1 - \sqrt{z}\exp\left(\frac{\pi}{12}\left(\frac{1}{z}-z\right)\right) + e^{-z}\right)\,.$$

Now we shall prove the following asymptotic formula for $\log f(2;\, e^{-2\pi z})$ which is stated as

LEMMA. *In* $|\arg z| \leqq \alpha_0 < \pi/2$, *we have*

$$\log f(2;\, e^{2-\pi z}) = V(z) + \log z + \kappa + O(|z|)\,,$$

where

$$\kappa=\int_0^\infty \varphi(x)dx .$$

Equivalently, in the same domain,

$$\log f(2; e^{-z})=v(z)+\log\left(\frac{z}{2\pi}\right)+\kappa+O(|z|) , \tag{1.6}$$

where

$$v(z)=V\left(\frac{z}{2\pi}\right)=\left(\frac{z}{2\pi}\right)^{1/2}\sum_{m=1}^{\infty} m^{-1/2}\exp\left(\frac{\pi^2}{6mz}-\frac{mz}{24}\right) .$$

PROOF. From (1.5), we may show that in $|\arg z|\leqq\alpha_0<\pi/2$,

$$z\sum_{n=1}^{\infty}\varphi(nz)=\int_0^\infty \varphi(x)dx+O(|z|) , \qquad |z|\to 0 . \tag{1.7}$$

By (1.3) and the definition of $\varphi(z)$, the integral

$$\int_0^\infty \varphi(x)dx$$

is convergent. Since $\varphi(w)$ is regular for $\mathrm{Re}\, w>0$, we have

$$\int_\Gamma \varphi(w)dw=0 ,$$

where Γ is a curve joining Γ_1, Γ_2, Γ_3, and Γ_4 as is indicated in Fig. 1.

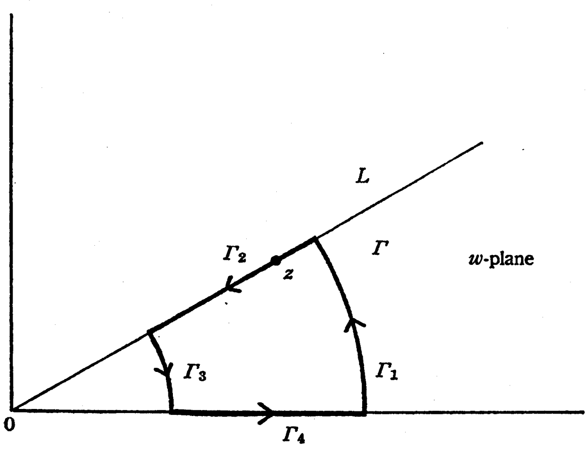

FIGURE 1

Two integrals along the circular arcs Γ_1 and Γ_3 with center $w=0$ tend to zero as their radii tend to infinity and zero respectively. So we have

$$\int_0^\infty \varphi(x)dx=\int_L \varphi(w)dw=\kappa\ , \quad \text{say}\ ,$$

where $L=\{w=tz;\ 0\leqq t\}$. Then we get

$$\begin{aligned}(1.8)\qquad \kappa-z\sum_{n=1}^{\infty}\varphi(nz)&=\sum_{n=1}^{\infty}\left(\int_{L_n}\varphi(w)dw-z\varphi(nz)\right)\\ &=-\sum_{n=1}^{\infty}\int_{L_n}(w-(n-1)z)\varphi'(w)dw\ ,\end{aligned}$$

where $L_n=\{w=tz;\ n-1\leqq t\leqq n\}$.

Since

$$\begin{aligned}\varphi'(w)=&-\sqrt{w}\,e^{\pi((1/w)-w)/12}\{f(e^{-2\pi/w})-1\}\{w^{-2}/2+\pi(w^{-3}+w^{-1})/12\}\\ &+2\pi\sqrt{w}\,w^{-3}\exp\left(-\frac{\pi}{12}\left(\frac{23}{w}+w\right)\right)f'(e^{-2\pi/w})-(e^{-w}-1)/w^2-e^{-w}/w\ ,\end{aligned}$$

we find

$$\lim_{w\to 0}\varphi'(w)=-\lim_{w\to 0}((e^{-w}-1)/w^2+e^{-w}/w)=1/2\ ,$$

in $|\arg w|\leqq\alpha_0$. Say $\varphi'(0)=1/2$, then $\varphi'(w)$ is a continuous function in the domain $\{0\}\cup\{w;\ |\arg w|\leqq\alpha_0\}$ *a fortiori* in L_n.

From (1.8), we have

$$\left|\kappa-z\sum_{n=1}^{\infty}\varphi(nz)\right|\leqq|z|^2\sum_{n=1}^{\infty}\max_{w\in L_n}|\varphi'(w)|\ ,$$

where

$$\begin{aligned}\sum_{n=1}^{\infty}\max_{w\in L_n}|\varphi'(w)|=&\sum_{n\leqq 1+(\lambda|z|)^{-1}}\max_{w\in L_n}|\varphi'(w)|\\ &+\sum_{n>1+(\lambda|z|)^{-1}}\max_{w\in L_n}|\varphi'(w)|\\ =&A_1(z)+A_2(z),\ \text{say},\end{aligned}$$

where λ is a constant such that $0<\lambda<\cos\alpha_0$.

We estimate the first sum $A_1(z)$. If $w\in L_n$ and $n\leqq 1+(\lambda|z|)^{-1}$, then

$$|w|\leqq n|z|\leqq\lambda^{-1}+|z|\leqq\lambda^{-1}+\varepsilon\ ,$$

for small $|z|$. So we have

$$\begin{aligned}A_1(z) &\leqq \sum_{n\leqq 1+(\lambda|z|)^{-1}} \max_{w\in L,|w|\leqq\lambda^{-1}+\varepsilon} |\varphi'(w)| \\ &\leqq \sum_{n\leqq 1+(\lambda|z|)^{-1}} \max_{w\in D} |\varphi'(w)| \\ &\leqq (1+(\lambda|z|^{-1}) \max_{w\in D} |\varphi'(w)| \\ &\leqq O(|z|^{-1}) ,\end{aligned}$$

where D is the compact set

$$\{0\}\cup\{w;\ |w|\leqq\lambda^{-1}+\varepsilon,\ |\arg w|\leqq\alpha_0\} .$$

It remains to prove that $A_2(z)=O(|z|^{-1})$. Using

$$\begin{aligned}\varphi'(w)=\sqrt{w}\exp\left(\frac{\pi}{12}\left(\frac{1}{w}-w\right)\right)(w^{-2}/2+(w^{-3}+w^{-1})/12) \\ -w^{-2}f(e^{-2\pi w})-2\pi w^{-1}e^{-2\pi w}f'(e^{-2\pi w})-(e^{-w}-1)/w^2-e^{-w}/w\end{aligned}$$

with the inequality

$$1<1/\lambda<(n-1)|z|\leqq|w|$$

for $w\in L_n$ and $n>1+(\lambda|z|)^{-1}$, we have

$$\begin{aligned}A_2(z) &= O\Big(\sum_{n>1+(\lambda|z|)^{-1}} \max_{w\in L_n}(e^{-\pi \mathrm{Re}\, w/12}+|w|^{-2})\Big) \\ &= O\Big(\sum_{n>1+(\lambda|z|)^{-1}} e^{-\pi\lambda(n-1)|z|/12}+(n-1)^{-2}|z|^{-2}\Big) \\ &= O\Big(\Big(1-\exp\Big(-\frac{\pi}{12}\lambda|z|\Big)\Big)^{-1}+|z|^{-2}\int_{(\lambda|z|)^{-1}}^{\infty} x^{-2}dx\Big) \\ &= O(|z|^{-1}) .\end{aligned}$$

This completes the proof of lemma.

§ 2. Proof of the theorem.

For the proof of the theorem, we define a function

$$v_\sigma(z)=(z/2\pi)^{1/2}\sum_{m=1}^{\infty} m^{-\sigma}\exp\left(\frac{\pi^2}{6mz}-\frac{mz}{24}\right) , \tag{2.1}$$

where σ is any real number. This infinite series converges uniformly in every compact subset of the domain $\mathrm{Re}\, z>0$. Note that $v_{1/2}(z)=v(z)$ (see (1.6)). We have the r-th derived function

$$v_\sigma^{(r)}(z)=\sum_{k=-r}^{r}(2\pi)^{-k}C_{r,k}z^{-r+k}v_{\sigma-k}(z) , \tag{2.2}$$

where $C_{0,0}=1$ and for $r>0$,

$$C_{r,k}=\begin{cases}0\ , & \text{if}\quad |k|>r\ ,\\ \left(k-r+\frac{3}{2}\right)C_{r-1,k}-\frac{\pi}{12}(C_{r-1,k+1}+C_{r-1,k-1})\ , & \text{if}\quad |k|\leqq r\ ,\end{cases}$$

and especially,

$$C_{r,\pm r}=(-\pi/12)^r\ ,\qquad r\geqq 0\ .$$

Obviously, we get

$$v_o(x)\sim(x/2\pi)^{1/2}\exp(\pi^2/6x)\ ,\qquad x\to 0\ , \tag{2.3}$$

where x is a positive real variable. (2.2) and (2.3) together lead to

$$v^{(r)}(x)\sim(-\pi^2/6)^r x^{-2r}(x/2\pi)^{1/2}\exp(\pi^2/6x)\ . \tag{2.4}$$

This implies that $v'(x)$ is monotone increasing for $0<x\leqq T$ with a sufficiently small T, and $v'(x)$ tends to negative infinity as x tends to zero. Thus for any integer $n\geqq -v'(T)$, the equation

$$v'(x_0)+n=0 \tag{2.5}$$

has a unique solution $x_0=x_0(n)$ in $0<x_0\leqq T$. From (2.4) and (2.5), we have

$$n\sim(\pi^2/6)x_0^{-2}(x_0/2\pi)^{1/2}\exp(\pi^2/6x_0)\ , \tag{2.6}$$

so that

$$\log n\sim\pi^2/6x_0\ ,$$

namely

$$x_0\sim\frac{\pi^2}{6\log n}\ , \tag{2.7}$$

which together with (2.4) and (2.6) leads to

$$v^{(r)}(x_0)\sim(-1)^r(\pi^2/6)^{1-r}n\log^{2r-2}n\ . \tag{2.8}$$

Using Cauchy's theorem, we have

$$\begin{aligned} p(2;\,n)&=\frac{1}{2\pi i}\int_{|u|=e^{-x_0}}\frac{f(2;\,u)}{u^{n+1}}du \\ &=\frac{1}{2\pi i}\int_{x_0-\pi i}^{x_0+\pi i}f(2;\,\exp(-z))e^{nz}dz \\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(2;\,\exp(-x_0-iy))e^{n(x_0+iy)}dy\ . \end{aligned} \tag{2.9}$$

Let $y_0=y_0(n)=n^{-\mu}$ and

$$I=\frac{1}{2\pi}\int_{-y_0}^{y_0} f(2;\ \exp(-x_0-iy))e^{n(x_0+iy)}dy\ ,$$

where μ is a constant such that

$$1/3<\mu<1/2\ .$$

$p(2;\ n)$ is expressed now by

$$p(2;\ n)=I+E+\bar{E}\ ,$$

where

$$E=\frac{1}{2\pi}\int_{y_0}^{\pi} f(2;\ \exp(-x_0-iy))e^{n(x_0+iy)}dy$$

and $\bar{E}$ is the complex conjugate of E.

We shall estimate first, the leading term I of $p(2;\ n)$. Suppose $|y|\leqq y_0$. Applying Taylor's theorem to the function $v(x_0+iy)$ of y, we have

$$v(x_0+iy)=v(x_0)+iyv'(x_0)-\frac{y^2}{2}v''(x_0)$$
$$+\frac{y^3}{6}(\mathrm{Im}\ v^{(3)}(x_0+i\theta_1 y)-i\ \mathrm{Re}\ v^{(3)}(x_0+i\theta_2 y))\ ,$$

where $0<\theta_1<1$, $0<\theta_2<1$. Let $\zeta=x_0+i\theta y(0<\theta<1)$. Taking absolute values in (2.1), we get

$$|v_\sigma(\zeta)|\leqq(|\zeta|/2\pi)^{1/2}\sum_{m=1}^{\infty} m^{-\sigma}\exp\left(\frac{\pi^2 x_0}{6m(x_0^2+\theta^2 y^2)}-\frac{mx_0}{24}\right)\ .$$

By (2.7), we see $y_0=o(x_0)$, so that

$$|\zeta|=O(x_0)\ ,$$
$$|v_\sigma(\zeta)|=O(v_\sigma(x_0))=O(v(x_0))\ .$$

Hence, by (2.4), (2.7) and (2.8),

$$|y^3 v^{(3)}(\zeta)|=O(y_0^3 x_0^{-6} v(x_0))=O(n^{1-3\mu}\log^4 n)\ ,$$

which implies

$$v(x_0+iy)=v(x_0)+iyv'(x_0)-\frac{y^2}{2}v''(x_0)+O(n^{1-3\mu}\log^4 n)\ .$$

Using (1.6) with

$$\log(x_0+iy)=\log x_0+O\left(\frac{y_0}{x_0}\right)=\log x_0+O(n^{-\mu}\log n)$$

and

$$|x_0+iy|=O(x_0)=O(\log^{-1} n)\ ,$$

we have

$$\log f(2;\exp(-x_0-iy))=v(x_0)-iny-\frac{y^2}{2}v''(x_0)+\log\frac{x_0}{2\pi}+\kappa+O(\log^{-1} n)\ .$$

Hence

$$I=\frac{x_0\cdot J}{(2\pi)^2}\exp(v(x_0)+nx_0+\kappa+O(\log^{-1}n))\ ,$$

where

$$J=\int_{-y_0}^{y_0}\exp(-y^2v''(x_0)/2)dy\ .$$

Putting $t=y\sqrt{v''(x_0)}$, we get

$$J=(2\pi/v''(x_0))^{1/2}\left(1+O\left(\int_{y_0\sqrt{v''(x_0)}}^{\infty}e^{-t^2/2}dt\right)\right)\ .$$

Since

$$y_0\sqrt{v''(x_0)}\sim\sqrt{6}\,\pi^{-1}n^{1/2-\mu}\log n\to\infty\ ,$$

we have

$$\begin{aligned}\int_{y_0\sqrt{v''(x_0)}}^{\infty}e^{-t^2/2}dt&=O\left(\frac{\exp(-y_0^2v''(x_0)/2)}{y_0\sqrt{v''(x_0)}}\right)\\&=O\left(\frac{\exp(-2^{-1}n^{1-2\mu}\log^2 n)}{n^{1/2-\mu}\log n}\right)\ .\end{aligned}$$

This is much smaller than the order of $\log^{-1}n$. Therefore

$$I=(2\pi)^{-3/2}v''(x_0)^{-1/2}x_0e^{v(x_0)+nx_0+\kappa}(1+O(\log^{-1} n))\ . \tag{2.10}$$

We have to estimate the term E. To find a good path of integration, we consider

$$\operatorname{Re} v(z)=(|z|/2\pi)^{1/2}\sum_{m=1}^{\infty} m^{-1/2}\exp\left(\frac{\pi^2}{6m}\operatorname{Re}\frac{1}{z}-\frac{m}{24}\operatorname{Re} z\right)$$
$$\times\cos\left(\frac{\pi^2}{6m}\operatorname{Im}\frac{1}{z}-\frac{m}{24}\operatorname{Im} z+\frac{1}{2}\arg z\right).$$

The first term of this infinite series vanishes, if

$$\cos\left(\frac{\pi^2}{6}\operatorname{Im}\frac{1}{z}-\frac{1}{24}\operatorname{Im} z+\frac{1}{2}\arg z\right)=0. \tag{2.11}$$

Write $\alpha=\arg z$, $r=|z|$ and

$$\beta=\frac{\pi^2}{6}\operatorname{Im}\frac{1}{z}-\frac{1}{24}\operatorname{Im} z+\frac{1}{2}\arg z.$$

We have the relation

$$\frac{\alpha/2-\beta}{\sin\alpha}=\frac{\pi}{12}\left(\frac{2\pi}{r}+\frac{r}{2\pi}\right). \tag{2.12}$$

When β is a fixed real number, this expresses a curve in z-plane. Moreover, if

$$\beta=(m-1/2)\pi,$$

where m is an integer independent of z, the condition (2.11) is satisfied on the curve (2.12).

Let $y_1=\xi\log^{-2}n$, where ξ is a variable ranging over $0<\xi<M$, and M is a sufficiently large constant. From (2.7), we have

$$y_1=o(x_0), \qquad n\to\infty.$$

If $z_1=x_0+iy_1$, then

$$\operatorname{Im}\frac{1}{z_1}=-y_1x_0^{-2}(1+O(y_1^2/x_0^2))\sim-(\pi^2/6)^{-2}\xi,$$

and

$$\operatorname{Im} z_1\to 0, \qquad \arg z_1\to 0.$$

Now

$$\beta_1=\frac{\pi^2}{6}\operatorname{Im}\frac{1}{z_1}-\frac{1}{24}\operatorname{Im} z_1+\frac{1}{2}\arg z_1$$

is a continuous function of ξ for each fixed n. For sufficiently large n, there is a value $\xi(n)$ of ξ such that

$$\beta_1 = -\pi/2 .$$

Then, we have

$$\xi(n) \sim \pi^3/12 , \qquad y_1(n) \sim \frac{\pi^3}{12} \log^{-2} n . \tag{2.13}$$

The point $z_1(n) = x_0 + iy_1(n)$ is on the curve

$$\frac{\alpha+\pi}{2 \sin \alpha} = \frac{\pi}{12}\left(\frac{2\pi}{r} + \frac{r}{2\pi}\right) . \tag{2.14}$$

The right hand side of (2.14) is monotone decreasing for $0<r<2\pi$ and the left hand side is monotone decreasing for $0<\alpha<\pi/4$. The equation (2.14) has a solution

$$\begin{cases} r = |z_1(n)| , \\ \alpha = \arg z_1(n) . \end{cases}$$

When we let r increase from $|z_1|$ to $2|z_1|$, α increases from $\arg z_1$ to a value α_1', say, of α. Let $z_1' = x_1' + iy_1' = 2|z_1| \exp i\alpha_1'$. Then from (2.14), we have

$$\begin{aligned} &\sin \alpha_1' \sim 2 \sin \alpha_1 , \\ &y_1' \sim 4y_1 \sim (\pi^3/3) \log^{-2} n , \\ &x_1' \sim 2x_0 \sim (\pi^2/3) \log^{-1} n . \end{aligned} \tag{2.15}$$

We denote by γ_1, the curve (2.14) from z_1 to z_1'.

Let $y_2 = \log^{-\nu} n$, $z_2 = x_0 + iy_2$, where $1+\varepsilon<\nu<3/2-\varepsilon$. Since

$$\operatorname{Im} \frac{1}{z_2} \sim -y_2 x_0^{-2} \sim -(36/\pi^4) \log^{2-\nu} n \to -\infty ,$$

and

$$\operatorname{Im} z_2 \to 0 , \qquad \arg z_2 \to 0 ,$$

we can choose $\nu = \nu(n)$ such that

$$\cos \beta_2 = 0 ,$$

where

$$\beta_2 = \frac{\pi^2}{6} \operatorname{Im} \frac{1}{z_2} - \frac{1}{24} \operatorname{Im} z_2 + \frac{1}{2} \arg z_2 .$$

So the equation

$$\frac{\alpha/2-\beta_2(n)}{\sin\alpha}=\frac{\pi}{12}\left(\frac{2\pi}{r}+\frac{r}{2\pi}\right) \tag{2.16}$$

has a solution

$$\begin{cases} r=|z_2(n)|\ , \\ \alpha=\arg z_2(n)\ . \end{cases}$$

When we let r increase from $|z_2|$ to $2|z_2|$, α increases from $\arg z_2$ to a value α_2', say, of α. Let $z_2'=x_2'+iy_2'=2|z_2|\exp i\alpha_2'$. We have

$$\begin{cases} y_2'\sim 4y_2=4(\log n)^{-\nu(n)}=O(\log^{-1-\varepsilon} n)\ , \\ x_2'\sim 2x_0\sim(\pi^2/3)\log^{-1} n\ . \end{cases} \tag{2.17}$$

We denote by γ_2 and γ_3, the segment from z_1' to z_2' and the curve (2.16) from z_2' to z_2 respectively.

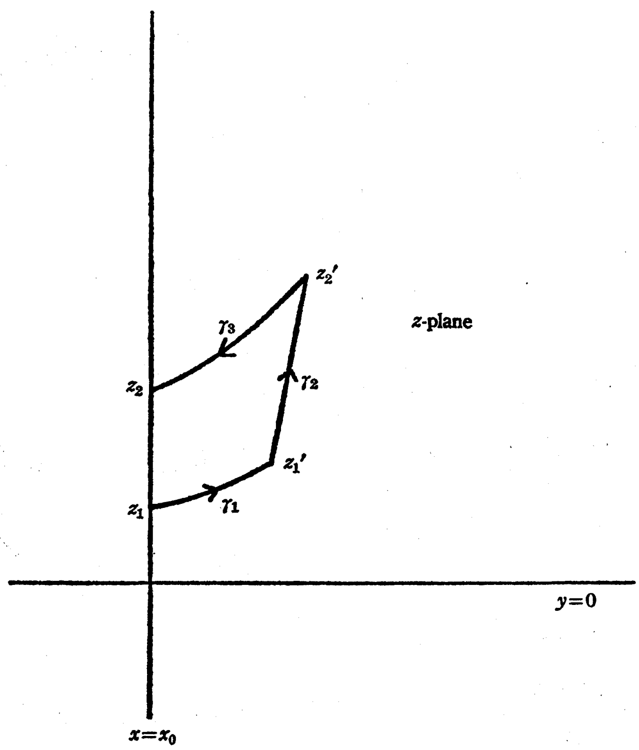

FIGURE 2

We divide the integral E as follows:

$$\begin{aligned}E=&\frac{1}{2\pi}\int_{y_0}^{y_1}f(2;\ e^{-x_0-iy})e^{n(x_0+iy)}dy\\&+\frac{1}{2\pi i}\int_{\gamma_1}f(2;\ e^{-z})e^{nz}dz\\&+\frac{1}{2\pi i}\int_{\gamma_2}f(2;\ e^{-z})e^{nz}dz\\&+\frac{1}{2\pi i}\int_{\gamma_3}f(2;\ e^{-z})e^{nz}dz\\&+\frac{1}{2\pi}\int_{y_2}^{Cx_0}f(2;\ e^{-x_0-iy})e^{n(x_0+iy)}dy\\&+\frac{1}{2\pi}\int_{Cx_0}^{\pi}f(2;\ e^{-x_0-iy})e^{n(x_0+iy)}dy\ ,\\=&E_1+E_{\gamma_1}+E_{\gamma_2}+E_{\gamma_3}+E_2+E_3\ ,\quad \text{say}\ ,\end{aligned}$$

where C is a positive constant. We now give in what follows the estimate of each E_j and each $E_{\gamma_j}(j=1, 2, 3)$.

(i) Estimation of $|E_1|$. Let $z=x_0+iy$ $(y_0\leqq y\leqq y_1)$. By Taylor's theorem,

$$\begin{aligned}\operatorname{Re} v(z)&=v(x_0)-\frac{v''(x_0)}{2}y^2+\frac{\operatorname{Re} v^{(4)}(x_0+i\theta y)}{24}y^4\\&=v(x_0)-\left(\frac{v''(x_0)}{2}-\frac{|v^{(4)}(x_0+i\theta y)|}{24}y_1^2\right)y^2\ ,\end{aligned}$$

where $0<\theta<1$. Let K be a constant such that

$$1<\sqrt{K}<48/\pi^2\ .$$

From (2.1), (2.2), (2.7), (2.8) and (2.13), we have

$$\begin{aligned}\frac{y_1^2}{24}\left|v^{(4)}(x_0+i\theta y)\right|&\leqq\frac{y_1^2}{24}\sqrt{K}\sum_{k=-4}^{4}(2\pi)^{-k}|C_{4,k}|x_0^{-4+k}v_{1/2-k}(x_0)\\&\sim\frac{y_1^2}{24}\sqrt{K}(2\pi)^4\left(\frac{\pi}{12}\right)^4x_0^{-8}v(x_0)\sim\frac{1}{16}\sqrt{K}\,n\log^2 n\ ,\end{aligned}$$

for sufficiently large n. Let

$$D(n)=\frac{v''(x_0)}{2}-\frac{y_1^2}{24}\sqrt{K}\sum_{k=-4}^{4}(2\pi)^{-k}|C_{4,k}|x_0^{-4+k}v_{1/2-k}(x_0)\ .$$

We then have

$$\operatorname{Re} v(z)\leqq v(x_0)-D(n)y^2$$

and

$$D(n)\sim\left(\frac{3}{\pi}-\frac{1}{16}\sqrt{K}\right)n\log^2 n\rightarrow+\infty\ .$$

In view of (2.7) and (2.13) and using Lemma,

$$(2.18)\qquad \begin{aligned}|E_1|&=O\left(\int_{y_0}^{y_1}\exp(\operatorname{Re}v(z)+\log|z|+nx_0)dy\right)\\ &=O\left(x_0e^{v(x_0)+nx_0}\int_{y_0}^{\infty}e^{-D(n)y^2}dy\right)\\ &=O\left(\frac{x_0}{D(n)y_0}\exp(v(x_0)+nx_0-y_0{}^2D(n))\right)\\ &=O(n^{-1/2}\log^{-3}n\ \exp(v(x_0)+nx_0))\ .\end{aligned}$$

(ii) Estimation of $|E_{\gamma_1}+E_{\gamma_3}|$. From the definitions of γ_1 and γ_3, we have

$$\arg z_1\leqq\arg z\leqq\arg z_1'\ ,\qquad \text{if}\quad z\quad\text{is on}\quad\gamma_1\ ,$$

and

$$\arg z_2\leqq\arg z\leqq\arg z_2'\ ,\qquad \text{if}\quad z\quad\text{is on}\quad\gamma_3\ .$$

From (2.7), (2.15), (2.17) and Lemma,

$$|E_{\gamma_1}+E_{\gamma_3}|=O\left(x_0e^{nx_0}\int_{\gamma_1+\gamma_3}e^{\operatorname{Re}v(z)}|dz|\right)\ ,$$

and for z on γ_1 or γ_3,

$$\begin{aligned}\operatorname{Re}v(z)&=O\left(\sqrt{x_0}\sum_{m=2}^{\infty}m^{-1/2}\exp\left(\frac{\pi^2}{6mx_0}-\frac{mx_0}{24}\right)\right)\\ &=O\left(\sqrt{x_0}\exp\left(\frac{x_0}{12x_0}\right)\right)\ ,\qquad n\rightarrow\infty\ .\end{aligned}$$

Thus we get

$$(2.19)\qquad |E_{\gamma_1}+E_{\gamma_3}|\ll x_0e^{nx_0+O(\sqrt{x_0}\exp(\pi^2/12x_0))}\ ,$$

where "$A\ll B$" means "$A=O(B)$".

(iii) Estimation of $|E_{\gamma_2}|$. From (2.7), (2.15), (2.17) and Lemma, we get

$$(2.20)\qquad |E_{\gamma_2}|=O(x_0e^{nx_0+v((2-\varepsilon)x_0)})\ .$$

(iv) Estimation of $|E_2|$. By Lemma, we have

$$|E_2|=O\Big(x_0 e^{nx_0}\int_{y_\varepsilon}^{Cx_0} e^{\mathrm{Re}\,v(z)}\,dy\Big)\,,$$

where $y_\varepsilon=(\log n)^{-3/2+\varepsilon}<y_2$ and $z=x_0+iy$ ($y_\varepsilon\leqq y\leqq Cx_0$). Now from the relations

$$\mathrm{Re}\,v(z)=O\Big(\sqrt{x_0}\sum_{m=1}^{\infty}m^{-1/2}\exp\Big(\frac{\pi^2}{6m}\mathrm{Re}\frac{1}{z}-\frac{mx_0}{24}\Big)\Big)\,,$$

$$\exp\Big(\frac{\pi^2}{6}\mathrm{Re}\frac{1}{z}-\frac{x_0}{24}\Big)\leqq\exp\Big(\Big(\frac{\pi^2}{6x_0}-\frac{x_0}{24}\Big)-\frac{\pi^2}{6x_0}\Big(1-\frac{x_0^2}{x_0^2+y_\varepsilon^2}\Big)\Big)\,,$$

$$\frac{\pi^2}{6x_0}\Big(1-\frac{x_0^2}{x_0^2+y_\varepsilon^2}\Big)\sim\frac{\pi^2 y_\varepsilon^2}{6x_0^2}\sim(\pi^2/6)^{-2}\log^{2\varepsilon}n\,,$$

it is shown that

$$\begin{aligned}\mathrm{Re}\,v(z)&=O\Big(\sqrt{x_0}\exp\Big(\frac{\pi^2}{6x_0}-\frac{x_0}{24}-K_1\log^{2\varepsilon}n\Big)\Big)\\&\quad+O\Big(\sqrt{x_0}\sum_{m=2}^{\infty}m^{-1/2}\exp\Big(\frac{\pi^2}{6mx_0}-\frac{mx_0}{24}\Big)\Big)\\&=O(v(x_0)\exp(-K_1\log^{2\varepsilon}n))+O\Big(\sqrt{x_0}\exp\frac{\pi^2}{12x_0}\Big)\,,\end{aligned}$$

where K_1 is a constant such that $0<K_1<(\pi^2/6)^{-2}$. Thus we have obtained the estimate

$$|E_2|\ll x_0\exp\{nx_0+O(v(x_0)e^{-K_1\log^{2\varepsilon}n})+O(\sqrt{x_0}e^{\pi^2/12x_0})\}\,. \tag{2.21}$$

(v) Estimation of $|E_3|$. In this case, Lemma can not be applied, and we proceed as follows. We see

$$|E_3|\leqq e^{nx_0}\int_{Cx_0}^{\pi}\exp|\log f(2;\,e^{-z})|\,dy\,,$$

where $z=x_0+iy$ ($Cx_0\leqq y\leqq\pi$), and

$$|\log f(2;\,e^{-z})|\leqq|f(e^{-z})-1|+\sum_{m=2}^{\infty}\frac{1}{m}(f(e^{-mx_0})-1)\,,$$

as is obtained from (1.2). We shall give the estimate of each term on the right hand side of the last inequality.

First, applying (1.1) to the ordinary partition case $a(n)=1$,

$$f(e^{-mx_0})=\exp\sum_{k=1}^{\infty}k^{-1}(e^{kmx_0}-1)^{-1}$$

$$\leqq \exp \sum_{k=1}^{\infty} k^{-2}(mx_0)^{-1}$$
$$= \exp(\pi^2/6mx_0) .$$

Let $s = \pi^2/6x_0$. We have

$$\sum_{m=2}^{\infty} \frac{1}{m}(f(e^{-mx_0})-1) \leqq \sum_{m=2}^{\infty} \frac{1}{m}(e^{s/m}-1)$$
$$= \sum_{m=2}^{\infty} m^{-2}\left(s + \frac{s^2}{2!m} + \frac{s^3}{3!m} + \cdots\right)$$
$$\leqq 2\left(\frac{\pi^2}{6}-1\right)(e^{s/2}-1)$$
$$= O(n^{(1+\varepsilon)/2})$$

for sufficient large n since $s < (1+\varepsilon)\log n$ (using (2.7)).

Next, we have

$$|f(e^{-z})-1| \leqq 1 + \exp\left(|e^z-1|^{-1} + \sum_{m=2}^{\infty} m^{-1}|e^{mz}-1|^{-1}\right) .$$

Since

$$|e^z - 1| = \{(e^{x_0}-1)^2 + 4e^{x_0}\sin(y/2)\}^{1/2}$$
$$\geqq 2|\sin(y/2)| \geqq 2|y|/\pi \geqq 2Cx_0/\pi$$

and

$$|e^{mz}-1| \geqq e^{mx_0}-1 \geqq mx_0 ,$$

we get

$$|f(e^{-z})-1| \leqq 1 + \exp(C_1/x_0) ,$$

where $C_1 = (\pi/2C) + (\pi^2/6) - 1$.

Thus we have

$$|\log f(2;\, e^{-z})| = O(e^{C_1/x_0}) + O(n^{(1+\varepsilon)/2})$$
$$= O(n^{6C_1/\pi^2(1-\varepsilon)}) + O(n^{(1+\varepsilon)/2}) .$$

Now we take the constant C to satisfy

$$C > \left(\frac{2}{\pi} - \frac{\pi}{3}\varepsilon\right)^{-1} .$$

This choice leads to

$$6C_1/\pi^2(1-\varepsilon)<1$$

and

$$|\log f(2;\, e^{-s})|=O(n^{1-\varepsilon'})\ .$$

Using the above two estimates together, we have

(2.22) $$|E_3|\ll e^{nx_0+O(n^{1-\varepsilon'})}\ .$$

From (2.7), (2.8), (2.10), (2.18) and (2.22), we get (0.6). This completes the proof of Theorem.

Now in order to show Corollary it remains to show (0.7), (0.8) and (0.9). From (2.2), we have

$$v'(x)=-(x/2\pi)^{1/2}\sum_{m=1}^{\infty}\left(\frac{\pi^2}{6x^2}m^{-3/2}-\frac{m^{-1/2}}{2x}+\frac{m^{1/2}}{24}\right)\exp\left(\frac{\pi^2}{6mx}-\frac{mx}{24}\right)\ .$$

Let

$$w(x)=\frac{\pi^2}{6x^2}(x/2\pi)^{1/2}\exp(\pi^2/6x)\ .$$

Then

$$\begin{aligned} n&=-v'(x_0)\\ &=w(x_0)\exp(-x_0/24)+O(x_0^{-1/2}\exp(\pi^2/6x_0))\\ &=w(x_0)\exp(-x_0/24)+O(n\log^{-1}n)\ . \end{aligned}$$

Let x_0^* be such that $n=w(x_0^*)$. Then it follows that

$$(x_0/x_0^*)^{-3/2}\exp\left(\frac{\pi^2}{6}\left(\frac{1}{x_0}-\frac{1}{x_0^*}\right)-\frac{x_0}{24}\right)=1+O(\log^{-1}n)$$

and

$$-\frac{3}{2}(\log x_0-\log x_0^*)+\frac{\pi^2}{6}\left(\frac{1}{x_0}-\frac{1}{x_0^*}\right)=O(\log^{-1}n)\ .$$

Putting $s=\pi^2/6x_0$ and $s^*=\pi^2/6x_0^*$, we have

$$\begin{aligned} s&=s^*+\frac{3}{2}(\log x_0-\log x_0^*)+O(\log^{-1}n)\\ &=s^*-\frac{3}{2}(\log s-\log s^*)+O(\log^{-1}n)\\ &=s^*-\frac{3}{2}\log\left(1+\frac{3}{2s^*}\log\frac{x_0}{x_0^*}+O\left(\frac{1}{s^*\log n}\right)\right)+O(\log^{-1}n)\ . \end{aligned}$$

Since $s \sim s^* \sim \log n$, we have

(2.23) $$s = s^* + O(\log^{-1} n) .$$

From the definitions of x_0^* and s^*,

$$s^* = \log n - \frac{3}{2}\log s^* + \frac{1}{2}\log\frac{\pi^3}{3} = \log n + O(\log\log n) ,$$

so that

$$\log s^* = \log\log n + O\left(\frac{\log\log n}{\log n}\right) .$$

Therefore we get

(2.24) $$s^* = l + O\left(\frac{\log\log n}{\log n}\right) ,$$

where

$$l = \log n - \frac{3}{2}\log\log n + \frac{1}{2}\log(\pi^3/3) .$$

From (2.23) and (2.24),

(2.25) $$s = l + O\left(\frac{\log\log n}{\log n}\right) ,$$

$$x_0 = \frac{\pi^2}{6}\Big/ l\left(1 + O\left(\frac{\log\log n}{l \cdot \log n}\right)\right)$$
$$= \frac{\pi^2}{6l}\left(1 + O\left(\frac{\log\log n}{\log^2 n}\right)\right) .$$

Thus (0.7) is shown. Especially, we have

(2.26) $$x_0 = \frac{\pi^2}{6\log n}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right) .$$

(0.8) follows from (2.25), (2.26) and the fact that

$$v(x_0) = (x_0/2\pi)^{1/2} e^s (e^{-x_0/24} + O(e^{-\pi^2/12x_0}))$$
$$= (x_0/2\pi)^{1/2} e^s (1 + O(x_0)) .$$

Finally, (2.25), (2.26) and the estimate

$$v''(x_0) = (\pi^2/6)^2 x_0^{-4} (x_0/2\pi)^{1/2} \exp\left(s - \frac{x_0}{24}\right) + O(x_0^{-5/2} e^s)$$
$$= (\pi^2/6)^2 x_0^{-4} (x_0/2\pi)^{1/2} e^s (1 + O(\log^{-1} n)) + O(n \log n)$$

lead to (0.9).

REMARK 1. We may think of the more general "r-fold partitions of n" the definition of which will obviously be made. Let $p(r; n)$ be the number of the r-fold partitions of n. From proposition, we get the generating function

$$1+\sum_{n=1}^{\infty} p(r; n)u^n = \prod_{m=1}^{\infty} (1-u^m)^{-p(r-1;m)}$$

of $p(r; n)$, where $p(0; m)=1$. Asymptotic formulae for $p(r; n)$ with fixed $r\geqq 3$ have not however been obtained.

REMARK 2. The table of $p(2; n)$ in the Introduction was taken from the one made by Mr. Syukai Akiyama under his permission.

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