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A Necessary Condition for Hypoellipticity of Degenerate Elliptic-Parabolic Operators

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Introduction

The aim of this paper is to study hypoellipticity of degenerate elliptic-parabolic operators from the view point of the control theory. Hörmander and Oleĭnik-Radkevič proved (see [4]) that the degenerate elliptic-parabolic operator

$$L=\sum_{i,j=1}^{d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(x)\frac{\partial}{\partial x_i}+c(x) \tag{1}$$

in an open set M in $\boldsymbol{R}^d$ with real C^∞-smooth coefficients is hypoelliptic if $\dim\mathscr{L}(X_0, X_1, \cdots, X_d)\equiv d$ (for the notation, see §1), where

$$\begin{aligned} X_0&=\sum_{i=1}^{d}\Big(b_i-\sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial x_j}\Big)\frac{\partial}{\partial x_i}\ , \\ X_i&=\sum_{j=1}^{d}a_{ij}\frac{\partial}{\partial x_j}\ , \qquad 1\leqq i\leqq d\ , \end{aligned} \tag{2}$$

and conversely, when the coefficients are real analytic, $\dim\mathscr{L}(X_0, X_1, \cdots, X_d)\equiv d$ if the operator L is hypoelliptic. Chow and Nagano proved (see [7]) that for a set of C^∞-smooth vector fields $\{X_0, X_1, \cdots, X_d\}$ the system

$$\dot{x}=\sum_{i=0}^{d}\xi_i X_i(x)\ , \qquad \xi_i\in\boldsymbol{R}^1 \tag{3}$$

is controllable in every subdomain in M if $\dim\mathscr{L}(X_0, X_1, \cdots, X_d)\equiv d$, and proved that the converse proposition holds when the vector fields are real analytic. Thus we are led naturally to the following problems:

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Is the system (3) controllable in every subdomain in M if the operator L is hypoelliptic?; conversely, when the vector fields $X_0, X_1, \cdots, X_d$ are real analytic, is the operator L hypoelliptic if the system (3) is controllable in every subdomain in M? We give an affirmative answer for the former problem (Theorem 2). In view of Hörmander-Oleĭnik-Radkevič's and Nagano's results the latter problem is trivial, so we modify the question: When the vector fields $X_0, X_1, \cdots, X_d$ are merely C^∞-smooth, does the answer for the latter problem remain affirmative? The answer is negative in general, but we can show that there is a closed set F in M such that $\mathring{F}=\emptyset$, $F\subset\{x\in M;\ \dim\mathscr{L}(X_0, X_1, \cdots, X_d)<d \text{ at } x\}$ and such that L is hypoelliptic in $M\backslash F$ if the system (3) is controllable in every subdomain in M (Theorem 1 and Remark 1).

In Section 1, Theorem 1 is proved. In Section 2, Theorem 2 is reduced to a certain proposition which is proved in Section 4 by using some probabilistic lemmas prepared in Section 3. We can see easily that the whole statement of this paper remains true when M is a C^∞-manifold.

NOTATIONS:
$C^k(V)$ is the set of all C^k functions defined in V.
$C_0^k(V)$ is the set of all functions in $C^k(V)$ with compact support in V.
$\mathscr{D}'(V)$ is the set of all distributions in V.

§ 1. Proof of Theorem 1.

Let M be an open set in $\boldsymbol{R}^d$ and let L be a differential operator in M of the form (1) with real C^∞-smooth coefficients. Throughout this paper we assume that $(a_{ij}(x))$ is a nonnegative symmetric $d\times d$ matrix for every x in M, that is, L is the degenerate elliptic-parabolic operator in M. Furthermore we assume that the second order terms and the first order ones of L never vanish simultaneously, i.e.,

$$\sum_{i,j=1}^{d}|a_{ij}(x)|+\sum_{i=1}^{d}|b_i(x)|\neq 0 \tag{4}$$

for all x in M. $X_0, X_1, \cdots, X_d$ will denote the vector fields defined by (2) and $\mathscr{L}(X_0, X_1, \cdots, X_d)$ will denote the Lie algebra generated over $\boldsymbol{R}$ by the vector fields $X_0, X_1, \cdots, X_d$.

It is easy to show the following lemma which will be used in the proof of Theorem 1.

LEMMA 1. *If* $\dim\mathscr{L}(X_0, X_1, \cdots, X_d)\leqq r$ *in an open set* U *in* M, *then the set* $\{x\in U;\ \dim\mathscr{L}(X_0, X_1, \cdots, X_d)=r \text{ at } x\}$ *is open.*

THEOREM 1. *If the system* (3) *is controllable in every subdomain in* M, *then the set* $\{x \in M;\ \dim \mathscr{L}(X_0, X_1, \cdots, X_d) < d \text{ at } x\}$ *is closed in* M *and has no interior.*

PROOF. The closedness follows immediately from Lemma 1. If there is an open set U in which $\dim \mathscr{L}(X_0, X_1, \cdots, X_d) < d$, then

$$\max_{x \in U} \dim \mathscr{L}(X_0, X_1, \cdots, X_d) = r < d .$$

Lemma 1 shows that $\dim \mathscr{L}(X_0, X_1, \cdots, X_d) \equiv r$ in some non-empty domain, say V, contained in U, so $\mathscr{L}(X_0, X_1, \cdots, X_d)$ is an r-dimensional involutive distribution in V. By the Frobenius' theorem, V is able to be cut into slices of r-dimensional integral manifolds of the distribution $\mathscr{L}(X_0, X_1, \cdots, X_d)$. Hence the system (3) is not controllable in V; this is a contradiction.

REMARK 1. By combining Theorem 1 with Hörmander-Oleĭnik-Radkevič's (see [4]) and Nagano's results (see [7]) we obtain the following: If the system (3) is controllable in every subdomain in M, then there is a closed set F in M such that $\mathring{F} = \emptyset$, $F \subset \{x \in M;\ \dim \mathscr{L}(X_0, X_1, \cdots, X_d) < d \text{ at } x\}$ and such that the operator L is hypoelliptic in $M \backslash F$. In particular, when the coefficients a_{ij} and b_i are real analytic, we have $F = \emptyset$. It is generally not possible to show $F = \emptyset$, although Fediĭ [1] actually proved this for a certain kind of infinitely degenerate elliptic-parabolic operators.

§ 2. Proof of Theorem 2 (Part 1).

We summarize Sussmann's results ([7]) which are necessary in proving Theorem 2. Let D be the set of vector fields $\{X_0, X_1, \cdots, X_d\}$ and let Δ_D be the distribution spanned by D. For an open set U in M, $G_D(U)$ will denote the group of local C^∞-diffeomorphisms on U generated by $D|_U$ (cf. [7]). Sussmann's distribution $S_D(U)$ is the smallest $G_D(U)$-invariant distribution on U which contains $\Delta_D|_U$, that is, the space $S_D(U)(x)$ is the linear hull of all the vectors $v \in T_x U$ such that $v \in \Delta_D(x)$ or $v = d\varphi(w)$, where $\varphi \in G_D(U)$ and, for some $y \in U$, $x = \varphi(y)$ and $w \in \Delta_D(y)$. The distribution $S_D(U)$ has the maximal integral manifolds property in the sense of [7] and further, the system (3) is controllable in U if and only if $\dim S_D(U) \equiv d$.

THEOREM 2. *If the operator* L *is hypoelliptic in* M, *then the system* (3) *is controllable in every subdomain in* M.

PROOF (Part 1). Assume that there is a subdomain U in M such that the system (3) is not controllable in U. By the Sussmann's result this means that $\dim S_D(U)<d$ at some point p in U. Since $S_D(U)$ has the maximal integral manifolds property, there passes a maximal integral manifold of $S_D(U)$ through the point p. So it is easy to show, by the inverse function theorem, that there passes a regular maximal integral manifold, say N, of $S_D(U)$ through p. By (4) and $\dim S_D(U)(p)<d$, we have easily $1\leqq \dim N<d$.

Since N is regular and since hypoellipticity is a local property, we may suppose by performing a suitable change of local coordinates that

$$N=\{x\in R^d;\ x_{r+1}=\cdots=x_d=0\} \tag{5}$$

in a neighborhood of p, where $r=\dim N$; furthermore, by considering ψL instead of L (where $0\leqq\psi\leqq 1$ is a C^∞-smooth function in R^d such that $\psi\equiv 0$ outside a small neighborhood of p and that $\psi\equiv 1$ in a smaller neighborhood of p) we may suppose that

$$a\in C^\infty_{bdd}(R^d, S_d)\ ,\qquad b\in C^\infty_{bdd}(R^d, R^d)\ ,\qquad c\in C^\infty_{bdd}(R^d, R^1)\ . \tag{6}$$

Here a is the $d\times d$ matrix (a_{ij}), b is the vector $(b_1, \cdots, b_d)$ and S_d denotes the class of symmetric nonnegative matrices.

Now it will suffice to show the following: There is an open neighborhood V of the point p and a locally integrable function u defined in V such that $N\cap V\subset \text{sing supp}\, u$ and $Lu=0$ in V.

Theorem 2 will be proved completely at the end of Section 4 after a preliminary study in §3.

§3. Probabilistic lemmas.

In this section we assume the conditions (6). $\sigma(x)=(\sigma_{ij}(x))$ denotes a symmetric nonnegative $d\times d$ matrix such that $a(x)=(1/2)\sigma^2(x)$. By (6) each $\sigma_{ij}(x)$ is Lipschitz continuous in R^d, so for any x in R^d there exists a unique solution, say $x^x(t)$, with

$$dx(t)=\sigma(x(t))dw(t)+b(x(t))dt\ ,\qquad x(0)=x\quad \text{a.s.} \tag{7}$$

in $M^2_w[0, T]$, $T>0$. Here $w(t)$, $t\geqq 0$, is a d-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$ and $M^2_w[0, T]$ denotes a set of all non-anticipative functions $f(t)$ satisfying

$$E\left[\int_0^T |f(t)|^2 dt\right]<\infty\ .$$

$x(t)$, $t\geqq 0$, will denote the time-homogeneous diffusion process that is the solution of the stochastic differential equation (7).

DEFINITION ([3]). If N is a subset in R^d such that $P_x[x(t) \notin N$ for all $t \geqq 0] = 1$ whenever $x \notin N$, then we say that N is *nonattainable* by the process $x(t)$.

LEMMA 2. *Let N be closed in R^d and nonattainable by the process $x(t)$ and let $v \in C^2(R^d \backslash N)$. Then*

$$d\left[v(x(t)) \exp\left\{\int_0^t c(x(s))ds\right\}\right]$$
$$= \nabla v(x(t)) \exp\left\{\int_0^t c(x(s))\,ds\right\}\sigma(x(t))dw(t)$$
$$+ Lv(x(t)) \exp\left\{\int_0^t c(x(s))\,ds\right\}dt$$

P_x-a.s. *for all $x \notin N$.*

This is a generalization of Itô's formula. For the proof we have only to approximate v by suitable functions in $C^2(R^d)$ and use Ito's formula.

Let V be an open set in R^d with C^∞-smooth boundary. The exit time τ of $\bar{V}$ is defined by

$$\tau = \inf\{t \geqq 0;\ x(t) \notin \bar{V}\} .$$

Γ and Σ are the subsets on the boundary ∂V of V defined by

$$\Gamma = \{x \in \partial V;\ P_x(\tau > 0) = 0\}$$

and

$$\Sigma = \{x \in \partial V;\ \langle \nu, a(x)\nu \rangle > 0 \quad \text{or} \quad \langle \nu, X_0(x) \rangle < 0\} ,$$

where $\langle\ ,\ \rangle$ denotes the inner product in R^d and ν is the inward normal vector on ∂V. For brevity we set

$$C = \sup_{x \in V} c(x) \vee 0 .$$

Then we have the following lemma.

LEMMA 3. *Assume that*

$$\sup_{x \in \bar{V}} E_x[(1+\tau)e^{C\tau}] < \infty .$$

For given $f \in L^\infty(V)$ and $g \in L^\infty(\Gamma) \cap C(\Sigma)$, the function

$$u(x)=E_x\left[g(x(\tau))\exp\left\{\int_0^\tau c(x(s))ds\right\}-\int_0^\tau f(x(t))\exp\left\{\int_0^t c(x(s))ds\right\}dt\right]$$

is a unique solution of the Dirichlet problem

$$(8)\qquad \begin{cases} Lu=f \quad in \quad V \\ \operatorname{ess}\lim\limits_{\substack{x\to a\\ x\in V}} u(x)=g(a)\,, \qquad a\in\Sigma \end{cases}$$

in $L^\infty(V)$.

Here (8) means that u can be changed on a set of Lebesgue measure zero so that the following relations may be satisfied;

$$\begin{cases} \int uL^*\varphi dx=\int f\varphi dx\,, \qquad \varphi\in C_0^\infty(V)\,, \\ \lim\limits_{\substack{x\to a\\ x\in V}} u(x)=g(a)\,, \qquad a\in\Sigma\,. \end{cases}$$

Lemma 3 is a generalization of Strook-Varadhan's theorem ([6]). The proof is parallel to that of Strook-Varadhan's theorem but we will have to use Lemma 2.1 in [5] instead of Lemma 4.1 in [6].

LEMMA 4. *Let V_ρ be an open neighborhood of a fixed point p in R^d, with diameter $V_\rho=\rho$, and let τ_ρ be the exit time of $\bar V_\rho$. If the condition* (4) *is satisfied at the point p, then we obtain*

$$\overline{\lim_{\rho\downarrow 0}}\sup_{x\in\bar V_\rho} E_x[e^{C\tau_\rho}]<\infty$$

and

$$\lim_{\rho\downarrow 0}\sup_{x\in\bar V_\rho} E_x[(\tau_\rho e^{C\tau_\rho})^k]=0$$

for any constant $C\geqq 0$ and any $k=1, 2, \cdots$.

PROOF. According to Freĭdlin [2] there are small positive constants ε and δ independent of all sufficiently small $\rho>0$ such that

$$P_x\left[\tau_\rho<\frac{\rho}{\delta}\right]>\varepsilon^{\rho/\delta}$$

for all $x\in\bar V_\rho$. The Markov property gives

$$P_x\left[\tau_\rho\geqq n\frac{\rho}{\delta}\right]\leqq(1-\varepsilon^{\rho/\delta})^n$$

for all $x \in \bar{V}_\rho$ and all $n=0, 1, 2, \cdots$; so we have

$$\begin{aligned}
&E_x[e^{C\tau_\rho}] \\
&\quad = \sum_{n=0}^{\infty} E_x[\chi_{n\rho/\delta \leqq \tau_\rho < (n+1)\rho/\delta} e^{C\tau_\rho}] \\
&\quad \leqq \sum_{n=0}^{\infty} e^{C(n+1)\rho/\delta} P_x\left[\tau_\rho \geqq n\frac{\rho}{\delta}\right] \\
&\quad \leqq e^{C\rho/\delta} \sum_{n=0}^{\infty} \{e^{C\rho/\delta}(1-\varepsilon^{\rho/\delta})\}^n .
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&E_x[(\tau_\rho e^{C\tau_\rho})^k] \\
&\quad \leqq \left(\frac{\rho}{\delta} e^{C\rho/\delta}\right)^k \sum_{n=0}^{\infty} (n+1)^k \{e^{kC\rho/\delta}(1-\varepsilon^{\rho/\delta})\}^n .
\end{aligned}$$

This completes the proof.

REMARK 2. It follows immediately from Lemmas 3 and 4 that if the condition (4) is satisfied at a point p in M, then the equation $Lu=f$ is locally solvable at p, i.e., there is an open neighborhood V of p such that

$$L\mathscr{D}'(V) \supset C_0^\infty(V) .$$

It is to be noted that the leading symbol of L does not always admit an expression in the form of a sum of squares of symbols of principal type.

LEMMA 5. *Let τ be a stopping time such that* $0 \leqq \tau < \infty$ a.s. *and let $f(t)$ belong to $M_w^2[0, T]$ for each $T>0$. Then*

$$E\left[\sup_{0\leqq t\leqq \tau}\left|\int_0^t f(s)dw(s)\right|^2\right] \leqq 4E\left[\int_0^\tau |f(s)|^2 ds\right] .$$

PROOF. For any $T>0$ we easily have

$$\begin{aligned}
&E\left[\sup_{0\leqq t\leqq \tau\wedge T}\left|\int_0^t f(s)dw(s)\right|^2\right] \\
&\quad = E\left[\sup_{0\leqq t\leqq \tau\wedge T}\left|\int_0^{\tau\wedge t} f(s)dw(s)\right|^2\right] \\
&\quad \leqq E\left[\sup_{0\leqq t\leqq T}\left|\int_0^{\tau\wedge t} f(s)dw(s)\right|^2\right] \\
&\quad \leqq 4E\left[\int_0^{\tau\wedge T} |f(s)|^2 ds\right]
\end{aligned}$$

by the martingale inequality. Letting $T\to\infty$, the desired inequality follows.

§ 4. Proof of Theorem 2 (Part 2).

PROOF OF THEOREM 2 (Part 2). Let us take a sufficiently small open neighborhood V of the point p, with C^{∞}-smooth boundary. Then one of the desired functions in Part 1 will be given by

$$u(x)=E_x\left[g(x(\tau))\exp\left\{\int_0^{\tau}c(x(s))ds\right\}\right],$$

where $g(x)=\log(\sum_{i=r+1}^{d}x_i^2)^{-1/2}$. Here $x(t)$ is the time-homogeneous diffusion process constructed from the solutions of the stochastic differential equation (7) and τ is the exit time of $\bar{V}$.

Since the vector fields X_i, $0\leqq i\leqq d$, are all tangential to N at each point on N, N is nonattainable by the process $x(t)$ (see [3, Section 9.4]). $(L-c)g(x)$ and $\nabla g(x)\sigma(x)$ are bounded functions in $V\backslash N$. In fact, by (5) and (6), $a_{ij}(x)=O(\sum_{i=r+1}^{d}x_i^2)$, $b_i(x)=O(\sum_{i=r+1}^{d}x_i^2)^{1/2}$ and $\sigma_{ij}(x)=O(\sum_{i=r+1}^{d}x_i^2)^{1/2}$ as $d(x, N)\to 0$ for $r+1\leqq i$, $j\leqq d$.

We first show that $u\in L_1^{\mathrm{loc}}(V)$ and $u(x)\to\infty$ as $d(x, N)\to 0$, i.e., $N\cap V\subset\operatorname{sing\,supp} u$. Since

$$\nabla g(x^x(t))\exp\left\{\int_0^t c(x^x(x))ds\right\}\sigma(x^x(t))\in M_w^2[0, T]$$

for any $T>0$ if $x\in V\backslash N$, Lemma 2 gives

$$\begin{aligned}u(x)&=E_x\left[g(x(\tau))\exp\left\{\int_0^{\tau}c(x(s))ds\right\}\right]\\&=g(x)+E_x\left[\int_0^{\tau}(L-c)g(x(t))\exp\left\{\int_0^t c(x(s))ds\right\}dt\right]\\&\quad+E_x\left[\int_0^{\tau}c(x(t))g(x(t))\exp\left\{\int_0^t c(x(s))ds\right\}dt\right]\\&=g(x)+I_1+I_2\end{aligned}$$

for all $x\in V\backslash N$ and clearly

$$|I_1|\leqq C_1E_x[\tau e^{C\tau}],$$
$$|I_2|\leqq CE_x\left[e^{C\tau}\int_0^{\tau}g(x(t))dt\right],$$

where $C=\sup_{x\in V}c(x)\vee 0$ and $C_1=\sup_{x\in V\backslash N}|(L-c)g(x)|$. Furthermore, by Lemma 2,

$$
\begin{aligned}
&CE_x\left[e^{C\tau}\int_0^\tau g(x(t))dt\right] \\
&\quad = CE_x\left[e^{C\tau}\int_0^\tau\left\{g(x)+\int_0^t \nabla g(x(s))\sigma(x(s))dw(s)\right.\right. \\
&\qquad \left.\left. +\int_0^t (L-c)g(x(s))ds\right\}dt\right] \\
&\quad \leqq CE_x[\tau e^{C\tau}]g(x)+C(E_x[(\tau e^{C\tau})^2])^{1/2} \\
&\qquad \times\left(E_x\left[\sup_{0\leqq t\leqq\tau}\left|\int_0^t \nabla g(x(s))\sigma(x(s))dw(s)\right|^2\right]\right)^{1/2} \\
&\qquad + CC_1E_x\left[\frac{1}{2}\tau^2e^{C\tau}\right]
\end{aligned}
$$

and, by Lemma 5,

$$
\leqq CE_x[\tau e^{C\tau}]g(x)+4CC_2(E_x[(\tau e^{C\tau})^2])^{1/2}(E_x[\tau])^{1/2}+CC_1E_x\left[\frac{1}{2}\tau^2e^{C\tau}\right],
$$

where $C_2=\sup_{x\in V\setminus N}|\nabla g(x)\sigma(x)|$, and so

$$
\leqq CE_x[\tau e^{C\tau}]g(x)+2C(C_1+C_2)\{E_x[\tau e^{C\tau}]+E_x[(\tau e^{C\tau})^2]\}\,.
$$

Therefore, it will suffice to take diameter V so small that $C\sup_{x\in\bar V}E_x[\tau e^{C\tau}]<1$ and $\sup_{x\in\bar V}E_x[(\tau e^{C\tau})^2]<\infty$. Here we have used Lemma 4.

We next show that $Lu=0$ in V. Set

$$
u_n(x)=E_x\left[g_n(x(\tau))\exp\left\{\int_0^\tau c(x(s))ds\right\}\right],
$$

where $g_n(x)=g(x)\wedge n$ and $n=1, 2, \cdots$. Let us take diameter V sufficiently small so that $u\in L_1^{\mathrm{loc}}(V)$ and $\sup_{x\in\bar V}E_x[(1+\tau)e^{C\tau}]<\infty$. Then Lemma 3 shows

$$
\int u_nL^*\varphi dx=0\,,\quad \varphi\in C_0^\infty(V)\,;
$$

this easily gives, by letting $n\to\infty$,

$$
\int uL^*\varphi dx=0\,,\qquad \varphi\in C_0^\infty(V)\,.
$$

The proof of Theorem 2 is now complete.

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