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On Generalized Fourier-Stieltjes Transforms

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SUMMARY. The generalized Fourier-Stieltjes transform of order k is defined by

$$f(t)=\int_{|x|\leq c}\left(e^{itx}-\sum_{j=0}^{k-1}\frac{(itx)^j}{j!}\right)\frac{dF(x)}{x^k}+\int_{|x|>c}e^{itx}\frac{dF(x)}{x^k}$$

for some constant $c>0$ up to a polynomial of degree at most $k-1$, where $F(x)$ is supposed to be of bounded variation in every finite interval and to satisfy the condition $\int_{-\infty}^{\infty}|dF(x)|/(1+|x|^k)<\infty$.

An analogue of Khinchine's "unimodal theorem" for generalized Fourier-Stieltjes transforms is proved, and it is applied to obtain the representation

$$f(t)\overset{k-2}{=\!=}(it)^{k-2}\log\xi(t)+\log\eta(t)\ ,$$

where $\xi(t)$ and $\eta(t)$ are "infinitely divisible Fourier-Stieltjes transforms".

Introduction

Bochner [2] introduced the integral transform $\phi(t)$ of $\Phi(x)$ defined by

$$2\pi\phi(t)=\int_{|x|\leq 1}\left(e^{-itx}-\sum_{j=0}^{k-1}\frac{(-itx)^j}{j!}\right)\frac{\Phi(x)}{(-ix)^k}dx+\int_{|x|>1}e^{-itx}\frac{\Phi(x)}{(-ix)^k}dx$$

up to a polynomial of degree at most $k-1$, where $\Phi(x)$ is a function with $\int_{-\infty}^{\infty}|\Phi(x)|/(1+|x|^k)dx<\infty$. The function $\phi(t)$ is called the k-transform of $\Phi(x)$.

In connection with the consideration of the Lévy canonical representation of an infinitely divisible characteristic function, we define a slightly more general transform of Fourier-Stieltjes type.

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Let k be a nonnegative integer and write

$$E_k(t, x)=\begin{cases} e^{itx}-\sum_{j=0}^{k-1}\frac{(itx)^j}{j!}, & k=1, 2, \cdots, \\ e^{itx}, & k=0. \end{cases}$$

The generalized Fourier-Stieltjes transform $f(t)$ of $F(x)$ of order k is defined by

$$f(t)=\int_{|x|\leqq c} E_k(t, x)\frac{dF(x)}{x^k}+\int_{|x|>c} e^{itx}\frac{dF(x)}{x^k}+P_{k-1}(it), \tag{0.1}$$

where $F(x)$ is a given function on $(-\infty, \infty)$ which is of bounded variation in every finite interval and satisfies the condition

$$\int_{-\infty}^{\infty}\frac{|dF(x)|}{1+|x|^k}<\infty, \tag{0.2}$$

in which $c>0$ is an arbitrary but fixed continuity point of $F(x)$ and $P_{k-1}(\cdot)$ denotes an arbitrary polynomial with real coefficients of degree at most $k-1$ when $k\geqq 1$ and $p_{-1}(\cdot)$ denotes the zero function. Furthermore the integrand $E_k(t, x)/x^k$ is assumed to be continuous at $x=0$, i.e.,

$$\left.\frac{E_k(t, x)}{x^k}\right|_{x=0}=\frac{(it)^k}{k!}.$$

Following Bochner, we write equation (0.1) as

$$f(t)\overset{k}{=}\int_{|x|\leqq c} E_k(t, x)\frac{dF(x)}{x^k}+\int_{|x|>c} e^{itx}\frac{dF(x)}{x^k}. \tag{0.3}$$

That is, the symbol $\overset{k}{=}$ means that the difference of the both sides of it is equal to some polynomial $P_{k-1}(it)$.

In this paper, we shall simply call the generalized Fourier-Stieltjes transform $f(t)$ defined by (0.1) or (0.3) the k-transform of $F(x)$.

The class of all left continuous functions of bounded variation in every finite interval, satisfying the condition (0.2) is denoted by V_k and the subclass of V_k consisting of all nondecreasing functions is denoted by D_k. Moreover let $\hat{V}_k$ and $\hat{D}_k$ be the totalities of k-transforms of functions belonging to V_k and belonging to D_k, respectively.

In Section 1, we prove an analogue (Theorem 1) of the well known "unimodal theorem" of Khinchine. This implies as a particular case a recent result of Alf and O'Connor [1] on the unimodality of the Lévy spectral function in the Lévy canonical representation of an infinitely divisible characteristic function. In Section 2, we state our main theorem (Theorem 2) and prove some lemmas needed for the proof of the main theorem. One of the lemmas (Lemma 1) gives the relationship between the classes $\hat{D}_k$ and $\hat{D}_{k-2}$. The main theorem deals with the representation of any $f(t) \in \hat{D}_k (k \geqq 2)$ as

$$f(t) \overset{k-2}{=\!=} (it)^{k-2} \log \xi(t) + \log \eta(t) ,$$

where $\xi(t)$ and $\eta(t)$ are certain "infinitely divisible Fourier-Stieltjes transforms". When $k=2$, this relation tells us that the 2-transform is the logarithm of an infinitely divisible Fourier-Stieltjes tranform, which is of course an implication of the Lévy formula. Actually the 2-transform relates to the logarithm of an infinitely divisible characteristic function in the following way.

Write $f(t) \in \hat{D}_2$ as

$$f(t) = \int_{|x| \leq c} (e^{itx} - 1 - itx) \frac{dF(x)}{x^2} + \int_{|x|>c} e^{itx} \frac{dF(x)}{x^2} + iat + b . \tag{0.4}$$

This can also be written as

$$f(t) - f(0) = i\alpha t - \frac{\sigma^2}{2} t^2 + \int_{|x|>0} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dM(x) , \tag{0.5}$$

where α real, $\sigma^2 \geqq 0$ and $M(x)$ is a nondecreasing function over each of $(-\infty, 0)$ and $(0, \infty)$ vanishing at $\mp\infty$, with $\int_{|x|\leq\varepsilon} x^2 dM(x) < \infty$ for every $\varepsilon > 0$. (Just the Lévy formula.) The relationship between $(M(x), \alpha, \sigma^2)$ in (0.5) and $(F(x), a, b)$ in (0, 4) is given by

$$F(x) = \begin{cases} \int_{0+}^{x-} y^2 dM(y) + \sigma^2 , & x > 0 \\ -\int_{x}^{0-} y^2 dM(y) , & x < 0 \end{cases}$$

$$a = \alpha + \int_{|x| \leq c} \frac{x^3}{1+x^2} dM(x) - \int_{|x|>c} \frac{x}{1+x^2} dM(x)$$

anb b is an arbitrary real number. From the above we may claim that *a function* $\phi(t)$ *is an infinitely divisible characteristic function if and only if there exists a 2-transform* $f(t) \in \hat{D}_2$ *with* $\log \phi(t) = f(t) - f(0)$.

Section 3 is devoted to the proof of Theorem 2.

§ 1. A basic property of the k-transform.

It is well known (Khinchine) that a distribution function $G(x)$ is unimodal with vertex at the origin, i.e., $G(x)$ is convex over $(-\infty, 0)$ and is concave over $(0, \infty)$ if and only if its characteristic function $g(t)$ can be written in the form

$$\frac{1}{t}\int_0^t f(u)du = g(t) ,$$

where $f(u)$ is a characteristic function (see [3], p. 92). Alf and O'Connor [1] obtained a similar result for canonical representation of infinitely divisible characteristic functions. Theorem 1 below is a generalization of these two results. Actually, Khinchine's result is the particular case of the theorem with $k=0$ and $n=m=1$, and the result of Alf and O'Connor is the case with $k=2$ and $n=m=1$. The theorem also plays an important role in the theory of the k-transforms.

THEOREM 1. *Let $k \geqq 0$, $n \geqq 1$ and $m (1 \leqq m \leqq n)$ be integers. Then* (i) *for every $F(x) \in D_k$, there exists a $G(x) \in D_k$ which satisfies*

$$\frac{1}{(it)^m}\left[\int_{|x|\leqq c} E_{k+n}(t, x)\frac{dF(x)}{x^{k+n}} + \int_{|x|>c} E_n(t, x)\frac{dF(x)}{x^{k+n}}\right]$$
$$= \int_{|x|\leqq c} E_{k+n-m}(t, x)\frac{dG(x)}{x^{k+n-m}} + \int_{|x|>c} E_{n-m}(t, x)\frac{dG(x)}{x^{k+n-m}} \tag{1.1}$$

for any $c>0$, provided that $\pm c$ are continuity points of $F(x)$. $G(x)$ has the properties that (a):

$$\Phi(x) = \begin{cases} -\displaystyle\int_x^\infty \frac{dG(y)}{y^{k+n-m}} , & x>0 \\ \displaystyle\int_{-\infty}^x \frac{dG(y)}{|y|^{k+n-m}} , & x<0 \end{cases}$$

is $m-1$ times differentiable except at the origin, and its j-th derivatives $\Phi^{(j)}(x)$, $j=0, 1, 2, \cdots, m-1$, vanish at $\mp\infty$ and are absolutely continuous over every finite closed interval contained in either $(-\infty, 0)$ or $(0, \infty)$. (b): *$\Phi^{(m-1)}(x)$ is nondecreasing and convex over $(-\infty, 0)$ and $(-1)^{m-1}\Phi^{(m-1)}(x)$ is nondecreasing and concave over $(0, \infty)$.*

(ii) *Conversely, if $G(x) \in D_k$ has the properties* (a) *and* (b), *then there exists a $F(x) \in D_k$ satisfying* (1.1) *for some $c>0$.*

PROOF. (i) Since $F(u)$ belongs to D_k, we can define $\Psi_{k,n,m}(y) = \Psi(y)$ for $n \geqq m$ by

$$\Psi(y) = -\frac{1}{m!}\int_y^\infty (u-y)^m \frac{dF(u)}{u^{k+n}} , \quad y>0 .$$

As is easily seen, $\Psi(y)$ is nondecreasing and $m-1$ times differentiable over $(0, \infty)$, and its j-th derivative $\Psi^{(j)}(y)$, $j=0, 1, 2, \cdots, m-1$, is given by

$$\Psi^{(j)}(y)=-\frac{(-1)^j}{(m-j)!}\int_y^\infty (u-y)^{m-j}\frac{dF(u)}{u^{k+n}}, \qquad y>0 .$$

We have

$$D^+\Psi^{(m-1)}(y)=-(-1)^m\int_{y+}^\infty \frac{dF(u)}{u^{k+n}}, \qquad y>0$$

and

$$D^-\Psi^{(m-1)}(y)=-(-1)^m\int_y^\infty \frac{dF(u)}{u^{k+n}}, \qquad y>0 .$$

We then see

$$\Psi^{(j)}(y)=-\int_y^\infty \Psi^{(j+1)}(u)du, \qquad y>0$$

for $j=0, 1, 2, \cdots, m-1$, where $\Psi^{(m)}(y)$ denotes either $D^+\Psi^{(m-1)}(y)$ or $D^-\Psi^{(m-1)}(y)$. From now on, just for definiteness we take

$$\Psi^{(m)}(y)=D^-\Psi^{(m-1)}(y), \qquad y>0 . \tag{1.2}$$

Thus $\Psi^{(j)}(y)$, $j=0, 1, 2, \cdots, m-1$, vanish at ∞ and are absolutely continuous over every finite closed interval contained in $(0, \infty)$, and $(-1)^{m-1}\Psi^{(m-1)}(y)$ is nondecreasing and concave over $(0, \infty)$.

Obviously, we have for sufficiently small $y>0$ and $j=1, 2, \cdots, m$,

$$y^{k+n-m+j}\int_y^\infty (u-y)^{m-j}\frac{dF(u)}{u^{k+n}} \leqq \int_{0+}^1 H_y(u)dF(u)+y^{k+n-m+j}\int_{1+}^\infty \frac{dF(u)}{u^k},$$

where $H_y(u)=0$ $(0<u<y)$, $H_y(u)=(y/u)^{k+n-m+j}$ $(y\leqq u\leqq 1)$. The first integral of the right hand side tends to 0 as $y\to 0+$ by the bounded convergence theorem and the second summand clearly converges to zero. Also we have that

$$y^{n-m+j}\int_y^\infty (u-y)^{m-j}\frac{dF(u)}{u^{k+n}} \leqq \int_y^\infty \frac{dF(u)}{u^k} \longrightarrow 0 \quad \text{as} \quad y\longrightarrow\infty .$$

Consequently

$$\Psi^{(j)}(y)=o(y^{-(k+n-m+j)}) \quad \text{as} \quad y\longrightarrow 0+ \tag{1.3}$$

$$\Psi^{(j)}(y)=o(y^{-(n-m+j)}) \quad \text{as} \quad y\longrightarrow\infty \tag{1.4}$$

for $j=1, 2, \cdots, m$.

Using now integration by parts, we obtain for $0<a<c$

$$\frac{1}{(k+n-m)!}\int_a^c y^{k+n-m}d\Psi(y)$$
$$=\frac{1}{(k+n)!}\int_a^c dF(y)+\sum_{j=1}^m \frac{(-1)^{j-1}}{(k+n-m+j)!}[y^{k+n-m+j}\Psi^{(j)}(y)]_a^c$$

and for $0<c<b$

$$\frac{1}{(n-m)!}\int_c^b y^{n-m}d\Psi(y)$$
$$=\frac{1}{n!}\int_c^b \frac{dF(y)}{y^k}+\sum_{j=1}^m \frac{(-1)^{j-1}}{(n-m+j)!}[y^{n-m+j}\Psi^{(j)}(y)]_c^b .$$

Let $a\to 0+$ and $b\to\infty$. Then it follows from the relations (1.3) and (1.4) that $\int_{0+}^c y^{k+n-m}d\Psi(y)$ and $\int_c^\infty y^{n-m}d\Psi(y)$ are finite for $c>0$.

Hence

$$\tilde{G}(x)=\int_{0+}^x y^{k+n-m}d\Psi(y) , \qquad x>0 \tag{1.5}$$

belongs to D_k on the positive axis. Applying integration by parts to $(it)^m\int_a^c E_{k+n-m}(t, x)(d\tilde{G}(x)/x^{k+n-m})$ and to $(it)^m\int_c^b E_{n-m}(t, x)(d\tilde{G}(x)/x^{k+n-m})$ we easily see that

$$(it)^m\int_{0+}^c E_{k+n-m}(t, x)\frac{d\tilde{G}(x)}{x^{k+n-m}}$$
$$=\int_{0+}^c E_{k+n}(t, x)\frac{dF(x)}{x^{k+n}}-\sum_{j=1}^m (-1)^j\Psi^{(j)}(c)(it)^{m-j}E_{k+n-m+j}(t, c)$$

and

$$(it)^m\int_c^\infty E_{n-m}(t, x)\frac{d\tilde{G}(x)}{x^{k+n-m}}$$
$$=\int_c^\infty E_n(t, x)\frac{dF(x)}{x^{k+n}}+\sum_{j=1}^m(-1)^j\Psi^{(j)}(c)(it)^{m-j}E_{n-m+j}(t, c) .$$

Hence

$$(it)^m\left[\int_{0+}^c E_{k+n-m}(t, x)\frac{d\tilde{G}(x)}{x^{k+n-m}}+\int_c^\infty E_{n-m}(t, x)\frac{d\tilde{G}(x)}{x^{k+n-m}}\right] \tag{1.6}$$
$$=\int_{0+}^c E_{k+n}(t, x)\frac{dF(x)}{x^{k+n}}+\int_c^\infty E_n(t, x)\frac{dF(x)}{x^{k+n}}+\sum_{j=n}^{k+n-1}\lambda_j(it)^j$$

for any continuity point $c>0$ of $\Psi^{(m)}(x)$, and hence for continuity point $c>0$ of $F(x)$. Here all the coefficients λ_j are real.

Similar results are obtained on the negative axis for

$$\Psi(y)=\frac{1}{m!}\int_{-\infty}^{y}(y-u)^m\frac{dF(u)}{|u|^{k+n}}\,, \qquad y<0$$

$$\tilde{G}(x)=-\int_{x}^{0-}|y|^{k+n-m}d\Psi(y)\,, \qquad x<0\,.$$

Now define

$$G(x)=\begin{cases}\tilde{G}(x)+\dfrac{(k+n-m)!}{(k+n)!}(F(0+)-F(0))\,, & x>0\\ 0\,, & x=0\\ \tilde{G}(x)\,, & x<0\,.\end{cases}$$

Then $G(x)$ belongs to D_k and satisfies (1.1). Moreover $\Phi(x)=\Psi(x)$, $x\neq 0$, holds, so that $G(x)$ has the asserted properties.

(ii) Since $\Phi^{(m-1)}(\infty)=0$ and $(-1)^{m-1}\Phi^{(m-1)}(y)$ is nondecreasing, concave and absolutely continuous over any finite closed interval contained in $(0, \infty)$, $\Phi^{(m-1)}(y)$ is represented as

$$\Phi^{(m-1)}(y)=-\int_{y}^{\infty}\Phi^{(m)}(u)du\,, \qquad y>0\,,$$

where $\Phi^{(m)}(u)$ is such that $(-1)^m\Phi^{(m)}(u)$ is nonpositive and nondecreasing over $(0, \infty)$. This fact and the assumption (a) imply that $\Phi^{(j)}(y)$, $j=1, 2, \cdots, m$, are either nonnegative or nonpositive over $(0, \infty)$ and that $|\Phi^{(j)}(y)|$, $j=1, 2, \cdots, m$, are nonincreasing over $(0, \infty)$. Hence we have, for $y>0$

$$\begin{aligned}y^{k+n-m+j}|\Phi^{(j)}(2y)| &\leqq y^{k+n-m+j-1}\int_{y}^{2y}|\Phi^{(j)}(u)|du\\ &=y^{k+n-m+j-1}|\Phi^{(j-1)}(2y)-\Phi^{(j-1)}(y)|\,,\end{aligned}$$

and similarly

$$y^{n-m+j}|\Phi^{(j)}(2y)|\leqq y^{n-m+j-1}|\Phi^{(j-1)}(2y)-\Phi^{(j-1)}(y)|\,.$$

Consider first the case $j=1$. We then have

$$y^{k+n-m+1}|\Phi^{(1)}(2y)|\leqq y^{k+n-m}\int_{y}^{2y}\frac{dG(u)}{u^{k+n-m}}\leqq\int_{y}^{2y}dG(u)\longrightarrow 0 \quad \text{as} \quad y\longrightarrow 0+$$

$$y^{n-m+1}|\Phi^{(1)}(2y)|\leqq y^{n-m}\int_{y}^{2y}\frac{dG(u)}{u^{k+n-m}}\leqq\int_{y}^{2y}\frac{dG(u)}{u^{k}}\longrightarrow 0 \quad \text{as} \quad y\longrightarrow\infty\,.$$

In general we can conclude by the mathematical induction that

$$\Phi^{(j)}(y)=o(y^{-(k+n-m+j)}) \quad \text{as} \quad y\longrightarrow 0+ \tag{1.7}$$

$$\Phi^{(j)}(y)=o(y^{-(n-m+j)}) \quad \text{as} \quad y\longrightarrow \infty \tag{1.8}$$

for $j=1, 2, \cdots, m$. Again by integration by parts and the relations (1.7) and (1.8) we can see that writing

$$\tilde{F}(x)=(-1)^m\int_{0+}^{x-} y^{k+n}d\Phi^{(m)}(y) , \qquad x>0 ,$$

$\tilde{F}(x)$ is a function of D_k on the positive axis. Note that

$$\Phi^{(m)}(y-)=-(-1)^m\int_y^\infty \frac{d\tilde{F}(u)}{u^{k+n}} , \qquad y>0$$

$$G(x)=\int_{0+}^x y^{k+n-m}d\Phi(y)+G(0+) , \qquad x>0 .$$

These are the relations same as the equations (1.2) and (1.5), if $\tilde{F}(u)$,$G(x)$ and $\Phi(y)$ are replaced by $F(u)$, $\tilde{G}(x)$, and $\Psi(y)$, respectively. Hence, from the équation (1.6)

$$(it)^m\left[\int_{0+}^c E_{k+n-m}(t, x)\frac{dG(x)}{x^{k+n-m}}+\int_c^\infty E_{n-m}(t, x)\frac{dG(x)}{x^{k+n-m}}\right]$$
$$=\int_0^c E_{k+n}(t, x)\frac{d\tilde{F}(x)}{x^{k+n}}+\int_c^\infty E_n(t, x)\frac{d\tilde{F}(x)}{x^{k+n}}+\sum_{j=n}^{k+n-1}\mu_j(it)^j .$$

For the function

$$\tilde{F}(x)=-\int_x^{0-}|y|^{k+n}d\Phi^{(m)}(y) , \qquad x<0 ,$$

a similar argument is applied, where $\Phi^{(m)}(y)$, $y<0$, is the function determined by

$$\Phi^{(m-1)}(y)=\int_{-\infty}^y \Phi^{(m)}(u)du , \qquad y<0 .$$

Therefore

$$F(x)=\begin{cases}\tilde{F}(x)+\dfrac{(k+n)!}{(k+n-m)!}(G(0+)-G(0)) , & x>0\\ 0 , & x=0\\ \tilde{F}(x) , & x<0\end{cases}$$

is the function with the desired properties. This completes the proof.

§ 2. The main theorem and some lemmas.

THEOREM 2. *Every k-transform* $f(t) \in \hat{D}_k$ $(k \geqq 2)$ *is represented as*

$$f(t) \overset{k-2}{=\!=} (it)^{k-2} \log \xi(t) + \log \eta(t) , \tag{2.1}$$

where $\xi(t)$ *and* $\eta(t)$ *are such that for every positive integer* n

$$[\xi(t)]^{1/n} \in \hat{D}_0 \tag{2.2}$$

and

$$[\eta(t)]^{1/n} \in \hat{D}_0 \text{ (k: even) and } [\eta(t)]^{1/n} \in \hat{V}_0 \text{ (k: odd)} . \tag{2.3}$$

In order to prove the theorem, we need some lemmas, and we give in this section the proof of them. The proof of Theorem 2 will be given in the following Section 3.

LEMMA 1. *For an arbitrary integer* $k \geqq 2$ *and for every* $f(t) \in \hat{D}_k$, *there exist* $f_n(t) \in \hat{D}_{k-2}$ $(n=1, 2, \cdots)$ *such that*

$$f(t) = \lim_{n\to\infty} \left[f_n(t) + (it)^{k-2} \left(i\alpha_n t - \frac{\sigma^2}{2} t^2 + \beta_n \right) \right] \tag{2.4}$$

where α_n, β_n $(n=1, 2, \cdots)$ *and* $\sigma^2 \geqq 0$ *are real numbers.*

PROOF. For a $F(x) \in D_k$ and for real numbers c_1 and c_2, we write

$$\begin{aligned} f(t) &\overset{k-2}{=\!=} \int_{|x|\leqq c} E_k(t, x) \frac{dF(x)}{x^k} + \int_{|x|>c} e^{itx} \frac{dF(x)}{x^k} + c_1(it)^{k-1} + c_2(it)^{k-2} \\ &= g(t) + h(t) + c_1(it)^{k-1} + c_2(it)^{k-2} , \qquad \text{say} . \end{aligned} \tag{2.5}$$

Define $G_n(x) \in D_{k-2}$, $n=1, 2, \cdots$, by

$$G_n(x) = \begin{cases} \displaystyle\int_{1/n}^{x-} \frac{dF(y)}{y^2} , & x > 1/n \\ 0 , & -1/n < x \leqq 1/n \\ \displaystyle -\int_{x}^{-1/n} \frac{dF(y)}{y^2} , & x \leqq -1/n . \end{cases}$$

Then we have

$$(2.6)\quad \begin{aligned} g(t)&+c_1(it)^{k-1}+c_2(it)^{k-2}\\ &=\lim_{n\to\infty}\Big[\int_{1/n\leq|x|\leq c}E_{k-2}(t,x)\frac{dF(x)}{x^k}-\frac{(it)^{k-1}}{(k-1)!}\int_{1/n\leq|x|\leq c}\frac{dF(x)}{x}\\ &\quad-\frac{(it)^{k-2}}{(k-2)!}\int_{1/n\leq|x|\leq c}\frac{dF(x)}{x^2}\Big]\\ &\quad+\frac{F(0+)-F(0)}{k!}(it)^k+c_1(it)^{k-1}+c_2(it)^{k-2}\\ &=\lim_{n\to\infty}\Big[g_n(t)+(it)^{k-2}\Big(i\alpha_n t-\frac{\sigma^2}{2}t^2+\beta_n\Big)\Big],\end{aligned}$$

where

$$\alpha_n=c_1-\frac{1}{(k-1)!}\int_{|x|\leq c}xdG_n(x),\ \beta_n=c_2-\frac{1}{(k-2)!}\int_{|x|\leq c}x^2dG_n(x)$$

$$\frac{\sigma^2}{2}=\frac{F(0+)-F(0)}{k!},\qquad g_n(t)=\int_{|x|\leq c}E_{k-2}(t,x)\frac{dG_n(x)}{x^{k-2}}.$$

Hence we have

$$(2.7)\qquad f(t)\overset{k-2}{=\!=}\lim_{n\to\infty}\Big[g_n(t)+(it)^{k-2}\Big(i\alpha_n t-\frac{\sigma^2}{2}t^2+\beta_n\Big)\Big]+h(t),$$

where $h(t)$ is $\int_{|x|>c}e^{itx}(dF(x)/x^k)$ which is identical with $\int_{|x|>c}e^{itx}(dG_n(x)/x^{k-2})$ for all large n, and hence (2.4) is shown to hold with $f_n(t)\overset{k-2}{=\!=}g_n(t)+h(t)$.

LEMMA 2. *Suppose that $f(t)$ is the Fourier-Stieltjes transform of $F(x)$ with finite total variation K on $(-\infty,\infty)$ and that $\Lambda(z)$ is an analytic function in $|z|<M$, where $K<M\leqq\infty$. Then $\Lambda(f(t))$ is also the Fourier-Stieltjes transform of a function $G(x)$ of bounded variation. If $F(x)$ is a bounded nondecreasing function and the coefficients of the Taylar expansion of $\Lambda(z)$ are all nonnegative, then $G(x)$ is a bounded nondecreasing function.*

This is well known (cf. [3], pp. 318–319), so that we omit the proof here.

LEMMA 3. *For every integer $n\geqq0$ and for every $F(x)\in D_0$, there exists a $G(x)\in D_0$ such that*

$$\frac{1}{(it)^n}\int_{-\infty}^{\infty}E_n(t,x)\frac{dF(x)}{x^n}=\int_{-\infty}^{\infty}e^{itx}dG(x).$$

PROOF. Although this is the special case of Theorem 1 with $k=0$

and $m=n$, we give a proof which is independent of that theorem. From Theorem 12.1.1 in [3] and the relationship

$$\frac{E_k(t, x)}{(itx)^k}=\begin{cases}\displaystyle\int_0^x \frac{E_{k-1}(t,u)}{(itu)^{k-1}}d\Big(\frac{1}{k}\Big(\frac{u}{x}\Big)^k\Big), & x\neq 0\\ \displaystyle\frac{1}{k!}, & x=0\end{cases}$$

for $k\geqq 1$, it follows that $E_n(t, x)/(itx)^n \in \hat{D}_0$ for each x. Therefore again from Theorem 12.1.1, we have $\int_{-\infty}^{\infty}(E_n(t, x)/(itx)^n)dF(x) \in \hat{D}_0$.

§ 3. The proof of the main theorem.

Now we go back to the proof of Theorem 2.

PROOF OF THEOREM 2. We shall use the notations in the proof of Lemma 1. By (2.5) $f(t)$ is rewritten as

$$f(t)\overset{k-2}{=\!=}(it)^{k-2}\Big(\frac{g(t)}{(it)^{k-2}}+ic_1t+c_2\Big)+h(t) ,$$

while we know from (2.7)

$$f(t)\overset{k-2}{=\!=}(it)^{k-2}\lim_{n\to\infty}\Big(\frac{g_n(t)}{(it)^{k-2}}+i\alpha_nt-\frac{\sigma^2}{2}t^2+\beta_n\Big)+h(t) ,$$

where $g(t)/(it)^{k-2}$ and $g_n(t)/(it)^{k-2}$ are defined for $t=0$ by continuity. That is

$$\frac{g(t)}{(it)^{k-2}}\Big|_{t=0}=0 , \quad \frac{g_n(t)}{(it)^{k-2}}\Big|_{t=0}=c_2-\beta_n .$$

Recall that $g_n(t)$ is of the form $\int_{|x|\leqq c}E_{k-2}(t, x)(dG_n(x)/x^{k-2})$. It follows from Lemma 3 and Lemma 2 that

$$\exp\Big(\frac{g_n(t)}{(it)^{k-2}}\Big)\in \hat{D}_0 .$$

Hence

$$\xi_n(t)=\exp\Big(\frac{g_n(t)}{(it)^{k-2}}+i\alpha_nt-\frac{\sigma^2}{2}t^2+\beta_n\Big)$$

belong to $\hat{D}_0$ for all n. Furthermore it follows from (2.6) that $\xi_n(t)$ converges to the continuous function

$$\xi(t)=\exp\left(\frac{g(t)}{(it)^{k-2}}+ic_1t+c_2\right)$$

for all t. Therefore by the continuity theorem, $\xi(t)\in\hat{D}_0$.

On the other hand, $h(t)$ is thought of as a Fourier-Stieltjes transform, and then by Lemma 2

$$\eta(t)=\exp(h(t))$$

belongs to $\hat{D}_0$ or $\hat{V}_0$ according as the case when k is an even or an odd integer. Thus the equation (2.1) is proved.

The properties (2.2) and (2.3) are verified by repeating the same argument for the k-transform $f(t)/n$.

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