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On the Duality Mapping of l^∞

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This paper is concerned with a measure theoretic characterization of the duality mapping of the space l^∞ of bounded sequences of real numbers. The duality mapping of a Banach space X is a possibly multi-valued mapping F from X into its dual space X^* which assigns to each $u \in X$ a subset of X^* defined by

$$F(u)=\{f \in X^*: \langle u, f\rangle=\|u\|^2=\|f\|^2\} ,$$

where $\langle u, f\rangle$ stands for the value of $f \in X^*$ at the point $u \in X$. The mapping F is well-defined on all of X by means of the Hahn-Banach theorem, and it is well-known ([1], [4], [9]) that $F(u)$ is weakly-star compact and convex for each $u \in X$; and F is weakly-star demi-closed in the sense that if u_n converges strongly to u in X, $f_n \in F(u_n)$, and f is a weak-star cluster point of the sequence $\{f_n: n\uparrow\infty\}$, then $f \in F(u)$. The space l^∞ is one of the typical non-reflexive classical Banach spaces in the sense that it is a Banach lattice with respect to the usual ordering and every separable Banach space can be embedded isometrically and isomorphically in l^∞. Accordingly, the duality mapping of l^∞ is a prototype of the duality mappings of general non-reflexive Banach spaces.

Here we investigate the structure and topological properties of the duality mapping F of l^∞. This problem was arised both in the study of generalized derivatives of strongly absolutely continuous functions which take values in non-reflexive Banach spaces and in the investigation of nonlinear dissipative operators. The results obtained in this paper will suggest not only typical properties possessed by the duality mapping of a general nonreflexive Banach space but also counterexamples concerning generalized derivatives and nonlinear dissipative operators.

Our work is mainly devoted to two problems: The first aim is to investigate the structure of the values $F(u)$, $u \in l^\infty$; and the second

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purpose is to give some topological properties of the multi-valued mapping $F: l^\infty \to (l^\infty)^*$. Since the dual space $(l^\infty)^*$ is identified with the space ba of bounded, finitely additive measures on the power set Σ of the set N of all positive integers, we shall fully apply the theory of integration with respect to finitely additive measures and characterize $F(u)$ in terms of the finitely additive measure theory.

In this paper we shall employ three means to investigate the structure of the mapping F. The first means is the Jordan decomposition of measures in ba. In fact, a measure λ in $F(u)$ is represented as $\lambda = \|\lambda^+\|\nu^+ - \|\lambda^-\|\nu^-$, where $\lambda = \lambda^+ - \lambda^-$ is the Jordan decomposition of λ, and ν^+, ν^- are positive measures such that if $u^+ = u \vee 0$ and $u^- = (-u) \vee 0$ then $\|u^+\|\nu^+ \in F(u^+)$ and $\|u^-\|\nu^- \in F(u^-)$, respectively. Hence our problem is reduced to the considerations of the values of F for positive elements $u \geqq 0$. The second means is the Yosida-Hewitt decomposition. That is, we shall employ the fact that every λ in ba is decomposed as the sum of a countably additive measure λ_c and a purely finitely additive measure λ_p. The Yosida-Hewitt decomposition is equivalent to the Dixmier decomposition, since ba is the third conjugate of the space c_0 of sequences converging to 0, the λ_c is regarded as an element of the space l^1 of absolutely convergent sequences, and the λ_p is regarded as an annihilator of the closed subspace c_0 of l^∞. Now by means of this decomposition, detailed properties of measures in $F(u)$ can be discussed along with various types of bounded sequences u in l^∞. The third means is the use of 0-1 measures. A 0-1 measure is a measure which assumes only the values 0 and 1, and such a measure is either countably additive or purely finitely additive. Now extremal points of the weakly-star compact and convex set $F(u)$ are characterized as 0-1 measures and the set of all extremal points of $F(u)$ is described in terms of those of $F(u^+)$ and $F(u^-)$. Accordingly, it turns out that the structure of $F(u)$ is determined through Krein-Milman's theorem by the 0-1 measures belonging to the $F(u)$.

Applying the results concerning the above-mentioned facts, precise structures of the unit balls in l^∞ and ba are obtained. We shall divide the surface S of the unit ball in l^∞ into five zones and find a partition of the surface S^* of the unit ball in ba which is associated through the mapping F with this partition of S. In fact, it will be shown that S^* shapes a "cylinder" in the space ba and is divided into three zones. Besides, the range of F will be considered with the aid of Bishop-Phelps' theorem and James' theorem. Moreover, the application of our results enables us to characterize extremal points and smooth points of S; and it is interesting to note that the set of smooth points of S, sm S, is open-dense in S. These characterizations will play important roles to discuss the topological properties of F.

Finally, topological properties of the mapping F will be investigated by restricting it on the surface S of the unit ball in l^∞. After some aspects of the weak-star demi-closedness of F are given, it will be shown that F is single-valued and norm continuous on the open-dense subset $\mathrm{sm}\, S$ of S. F is genuinely multi-valued on $S-\mathrm{sm}\, S$, the boundary of $\mathrm{sm}\, S$. Now the value $F(v)$ of F at each boundary point $v \in S-\mathrm{sm}\, S$ can be viewed as a "boundary value" of the single-valued mapping F restricted on $\mathrm{sm}\, S$, since for every extremal point λ of $F(v)$ there exists a sequence $\{v_n\}$ in $\mathrm{sm}\, S$ such that v is a strong cluster point of the sequence $\{v_n : n\uparrow\infty\}$ and λ is a weak-star cluster point of the sequence $\{F(v_n) : n\uparrow\infty\}$ in S^*.

Section 1 contains some basic facts on finitely additive measures belonging to ba. In this section we shall briefly review Yosida-Hewitt's theory. In Section 2 we discuss 0-1 measures in connection with the duality mapping of l^∞. Section 3 concerns a general representation of measures in $F(u)$ in terms of the Jordan decomposition. Section 4 treats the characterization of $F(u)$ in terms of its extremal points. In this section 0-1 measures will play an essential role. In Section 5 the structure of $F(u)$ will be discussed from the point of view of the Yosida-Hewitt decomposition. In this section we shall give a complete relationship between Yosida-Hewitt's decomposition theorem and Dixmier's decomposition theorem. Section 6 concerns geometrical interpretations of our results obtained in the previous sections. Moreover, in this section, extremal points and smooth points of the unit sphere in l^∞ will be discussed. Finally, Section 7 treats topological properties of the duality mapping F.

§1. Basic facts on the dual space $(l^\infty)^*$.

Let N be the set of all positive integers, Σ the power set of N, and let $\mu(E)$ be the cardinality of $E \in \Sigma$. Then l^∞ is regarded as the Lebesgue space $L^\infty(N, \Sigma, \mu)$ and elements of l^∞ are understood to be real-valued function on N; the s-th element of the sequence $u \in l^\infty$ is denoted by $u(s)$. The norm of l^∞ is denoted by $\|\cdot\|$. By $(l^\infty)^+$ we mean the positive cone $\{u \in l^\infty : u(s) \geqq 0$ for all $s \in N\}$. Every element u in l^∞ can be decomposed as $u=u^+-u^-$, where $u^+=u\vee 0$ and $u^-=(-u)\vee 0$. In this paper S and S^* denote the surfaces of the closed unit balls of l^∞ and $(l^\infty)^*$, respectively. By the definition of duality mapping F, $F(0)$ is simply a singleton set consisting of the null functional 0 on l^∞ and nothing interesting happens. Accordingly, in what follows, we shall treat only the case $u \neq 0$ and restrict ourselves to the investigation of the *normalized duality mapping* F_0 defined by

$$F_0(u) = \{\lambda \in (l^\infty)^*: \langle u, \lambda \rangle = \|u\|, \|\lambda\| = 1\}, \quad u \neq 0 \tag{1.1}$$

instead of F. For a given $K \subset l^\infty$, $F_0(K)$ denotes the union $\bigcup\{F_0(u): u \in K\}$.

As is well-known, $(l^\infty)^*$ is isometrically isomorphic to the space $ba \equiv ba(N, \Sigma, \mu)$ of bounded, finitely additive measures on Σ; hence the natural pairing between l^∞ and ba is represented as

$$\langle u, \lambda \rangle = \int_N u(s)\lambda(ds), \quad u \in l^\infty, \quad \lambda \in ba. \tag{1.2}$$

For the terminology and fundamental facts on the integration of $u \in l^\infty$ with respect to $\lambda \in ba$, we refer to the treatise of Dunford-Schwartz [7], Chapters 3 and 4.

Let $\lambda \in ba$. We write $\lambda \geqq 0$ when $\lambda(E) \geqq 0$ for $E \in \Sigma$; and for $\lambda, \nu \in ba$, we write $\lambda \geqq \nu$ provided $\lambda - \nu \geqq 0$. ba forms a vector lattice with respect to this ordering. In fact, for every pair λ, ν in ba define the meet $\lambda \wedge \nu$ and the join $\lambda \vee \nu$ by

$$(\lambda \wedge \nu)(E) = \inf\{\lambda(T) + \nu(E - T): T \subset E\}, \quad E \in \Sigma$$

and $\lambda \vee \nu = -((-\lambda) \wedge (-\nu))$, respectively; then $\lambda \wedge \nu$, $\lambda \vee \nu$ belong to ba and give the greatest lower bound and the least upper bound of λ, ν, respectively. We shall use in later arguments the following simple fact:

(1.3) If $\lambda, \nu \in ba^+$ and $\lambda \wedge \nu = 0$, then $\alpha\lambda \wedge \beta\nu = 0$ for $\alpha, \beta \geqq 0$;

hence $\lambda \wedge \nu = 0$ iff $\alpha\lambda \wedge \beta\nu = 0$ for some $\alpha, \beta > 0$.

In this paper, we denote by ba^+ the positive cone $\{\lambda \in ba: \lambda \geqq 0\}$ of this vector lattice. For a given $\lambda \in ba$, the representation $\lambda = \lambda^+ - \lambda^-$ means the Jordan decomposition of λ, where λ^+ and λ^- stand respectively for the positive and negative variations of λ, i.e., $\lambda^+ = \lambda \vee 0$ and $\lambda^- = (-\lambda) \vee 0$. Note that $\lambda^+ \wedge \lambda^- = 0$. For a given $E \in \Sigma$, $v(\lambda, E)$ denotes the total variation of λ on E; hence $v(\lambda, E) = \lambda^+(E) + \lambda^-(E)$. The norm of λ is then defined by $\|\lambda\| = v(\lambda, N)$. Also, the relation

$$\lambda + \nu = (\lambda \vee \nu) + (\lambda \wedge \nu) \tag{1.4}$$

holds for $\lambda, \nu \in ba$. Now suppose that $\lambda, \nu, \gamma \in ba$, $\lambda \wedge \gamma = \lambda \wedge \nu$ and $\lambda \vee \gamma = \lambda \vee \nu$; then the application of (1.4) yields $\gamma = \nu$. From this we infer with the aid of Bergmann's theorem ([2], p. 134) that ba forms a distributive lattice. In fact, ba forms a Banach lattice. For the detailed arguments, see Birkhoff [2] and Yosida [12].

Let $\lambda \in ba$. If every countably additive measure ν in ba such that $0 \leqq \nu \leqq v(\lambda, \cdot)$ is identically zero, then λ is said to be *purely finitely additive* (cf. [11], Theorem 1.17). We sometimes permit ourselves the common abbreviations, c.a. measure and p.f.a. measure, in referring respectively to the countably additive and purely finitely additive measures. The following Yosida-Hewitt's decomposition theorem plays an important role in this paper:

THEOREM 1.1 (Yosida-Hewitt). *Let $\lambda \in ba$. Then λ is uniquely decomposed as the sum of a* c.a. *measure λ_c and a* p.f.a. *measure λ_p,* i.e., $\lambda = \lambda_c + \lambda_p$. *If in particular, $\lambda \geqq 0$, then $\lambda_p \geqq 0$ and $\lambda_c \geqq 0$.*

The following lemma is also useful for our later arguments:

LEMMA 1.2. *Let $\lambda \in ba$ and suppose that λ is written as $\lambda = \lambda_1 - \lambda_2$, where $\lambda_i \in ba^+$, $i = 1, 2$. If $\lambda_1 \wedge \lambda_2 = 0$, then this representation gives the Jordan decomposition of λ,* i.e., $\lambda_1 = \lambda^+$ *and* $\lambda_2 = \lambda^-$.

PROOF. The application of (1.4) yields $\lambda^+ = \lambda \vee 0 = (\lambda_1 - \lambda_2) \vee 0 = (\lambda_1 \vee \lambda_2) - \lambda_2 = \lambda_1 - (\lambda_1 \wedge \lambda_2) = \lambda_1 - 0 = \lambda_1$; and $\lambda^- = \lambda_2$ in a similar way.

q.e.d.

By means of this lemma, the variation of λ is also decomposed in accordance with the Yosida-Hewitt decomposition:

PROPOSITION 1.3. *Let $\lambda \in ba$ and let $\lambda = \lambda_c + \lambda_p$ be the Yosida-Hewitt decomposition of λ. Then we have* $||\lambda|| = ||\lambda_c|| + ||\lambda_p||$.

PROOF. Consider the Jordan decomposition $\lambda = \lambda^+ - \lambda^-$ and apply Theorem 1.1 to get the decompositions $\lambda^+ = \lambda_c^+ + \lambda_p^+$ and $\lambda^- = \lambda_c^- + \lambda_p^-$. Then λ can be written as $\lambda = (\lambda_c^+ - \lambda_c^-) + (\lambda_p^+ - \lambda_p^-)$. Hence, if we set $\lambda_c = \lambda_c^+ - \lambda_c^-$ and $\lambda_p = \lambda_p^+ - \lambda_p^-$, then λ_c and λ_p are respectively c.a. and p.f.a. ([11], Theorems 1.14 and 1.17). Moreover these two expressions give the Jordan decompositions of λ_c and λ_p, respectively. In fact, noting that $0 \leqq \lambda_c^+ \leqq \lambda^+$ and $0 \leqq \lambda_c^- \leqq \lambda^-$, we have $0 \leqq \lambda_c^+ \wedge \lambda_c^- \leqq \lambda^+ \wedge \lambda^- = 0$, and so $\lambda_c^+ \wedge \lambda_c^- = 0$. From this and Lemma 1.2 we see that $\lambda_c = \lambda_c^+ - \lambda_c^-$ gives the Jordan decomposition of λ_c. Similarly, $\lambda_p = \lambda_p^+ - \lambda_p^-$ gives that of λ_p. Therefore, we have $||\lambda|| = ||\lambda^+|| + ||\lambda^-|| = ||\lambda_c^+|| + ||\lambda_c^-|| + ||\lambda_p^+|| + ||\lambda_p^-|| = ||\lambda_c|| + ||\lambda_p||$.

q.e.d.

We shall use the following notation: For a given $E \in \Sigma$, χ_E denotes the characteristic function of E; and χ_E is regarded as an element of l^∞ in the sense that it defines a sequence $\{\chi_E(n)\}$ such that $\chi_E(n)=1$ for $n \in E$ and $=0$ for $n \in E^c$. We then write

$$\langle u\chi_E, \lambda\rangle = \int_E u(s)\lambda(ds) , \quad \text{and}$$
$$(1.5) \qquad \langle |u|\chi_E, v(\lambda, \cdot)\rangle = \int_E |u(s)| v(\lambda, ds) .$$

Accordingly, $\lambda(E)=\langle \chi_E, \lambda\rangle$, $v(\lambda, E)=\langle \chi_E, v(\lambda, \cdot)\rangle$, and the Lebesgue dominated convergence theorem may be restated as follows:

THE DOMINATED CONVERGENCE THEOREM. *Let $\lambda \in ba$ and let $\{u_n\}$ be a sequence in l^∞ such that $\|u_n\| \leqq M$ for $n \geqq 1$ and u_n converges to $u \in l^\infty$ in λ-measure,* i.e., $\lim v(\lambda; \{s: |u_n(s)-u(s)|>\varepsilon\})=0$ *for $\varepsilon>0$. Then we have the convergence*

$$\lim \langle u_n, \lambda\rangle = \langle u, \lambda\rangle .$$

We shall also use the following fact: Let $E \in \Sigma$, $u \in l^\infty$ and let $\lambda \in ba$. Then we have

$$(1.6) \qquad |\langle u\chi_E, \lambda\rangle| \leqq \langle |u|\chi_E, v(\lambda, \cdot)\rangle \leqq \sup_{s \in E} |u(s)| v(\lambda, E) \leqq |u| v(\lambda, E) .$$

Finally, we shall frequently use extremal points and smooth points of subset of ba as well as l^∞. For a given a set F in l^∞ (or in ba), ext F will denote the set of all extremal points of F and sm F will stand for the set of all smooth points of F.

§2. 0–1 measures.

In this section we study 0–1 measures in ba and give a method for computing the values of the integrals of elements in l^∞ with respect to such measures.

To discuss the structure of the values $F_0(u)$, we need a notion of 0–1 measure introduced by Yosida and Hewitt [11]. Let $\alpha=1$ or -1. By a 0–α *measure* on Σ we mean a nonzero element $\lambda \in ba$ which assumes only the values 0 and α. If λ in ba is a 0–1 measure, it has the following properties:

(i) If $\lambda(E)=1$ and $E \subset M$, then $\lambda(M)=1$.

(ii) If $\lambda(E)=0$ and $M \subset E$, then $\lambda(M)=0$.

(iii) $\|\lambda\| = \lambda(N) = 1$.

(iv) $\lambda(E) = 1$ iff $\lambda(E^c) = 0$.

(v) If $\lambda(M) = \lambda(E) = 1$, then $M \cap E \neq \varnothing$, $\lambda(M \cap E) = 1$ and $\lambda(M \triangle E) = 0$, where $M \triangle E$ means the symmetric difference of M and E.

A typical example of 0–1 measures is the so-called *point mass*: Let $k \in N$ and define $\delta_k: \Sigma \to \{0, 1\}$ by setting $\delta_k(E) = 1$ if $k \in E$ and $\delta_k(E) = 0$ if $k \notin E$. Then $\delta_k \in ba$ in the sense that $\langle u, \delta_k \rangle = u(k)$ for $u \in l^\infty$ and it is a 0–1 measure on Σ. Note that δ_k is countably additive. A general argument for the construction of 0–1 measures is given in [11], Theorem 4.1. However for the sake of later arguments we here attempt to construct such measures by means of ultrafilters on the set N. In fact, as suggested by properties (i) through (v) mentioned above, one may obtain a one-to-one correspondence between the class of all ultrafilters on N and that of 0–1 measures:

PROPOSITION 2.1. (a) *For a given* 0–1 *measure* λ *in* ba, *let* $\mathscr{F} = \{E \in \Sigma: \lambda(E) = 1\}$. *Then* $\mathscr{F}$ *is an ultrafilter on* N. (b) *Conversely, for every ultrafilter* $\mathscr{F}$ *on* N, *define* $\lambda: \Sigma \to \{0, 1\}$ *by setting* $\lambda(E) = 1$ *if* $E \in \mathscr{F}$ *and* $\lambda(E) = 0$ *if* $E \notin \mathscr{F}$. *Then* λ *is a* 0–1 *measure in* ba.

Let $\mathscr{A}$ be any nonempty family of nonempty subsets of N such that the intersection of any two sets, belonging to $\mathscr{A}$, contains a set which belongs to $\mathscr{A}$. Then Proposition 2.1 enables us to construct a 0–1 measure λ such that $\lambda(E) = 1$ for all $E \in \mathscr{A}$, since there is at least one ultrafilter which is finer than the filter generated by $\mathscr{A}$.

0–1 measures are classified into two types: 0–1 measures of the first type are point masses, δ_k, $k \in N$, and these are all countably additive. 0–1 measures of the second type are p.f.a. 0–1 measures. To describe this, we introduce two kinds of ultrafilters on N: An ultrafilter $\mathscr{F}$ on N is said to be principal (resp. nonprincipal) if $\bigcap \mathscr{F} \neq \varnothing$ (resp. $\bigcap \mathscr{F} = \varnothing$). If $\mathscr{F}$ is a principal ultrafilter on N, then there is one and only one point $p \in N$ and $\mathscr{F}$ is written as $\mathscr{F} = \{E \in \Sigma: p \in E\}$. Thus, there is a one-to-one correspondence between the class of point masses δ_k, $k \in N$, and that of principal ultrafilters.

As compared with principal ultrafilters, any nonprincipal ultrafilter $\mathscr{F}$ has the property that it contains no finite subsets in N; and this property characterizes non-principal ultrafilters. More precisely, given ultrafilter $\mathscr{F}$ the following conditions are equivalent:

(F1) $\mathscr{F}$ is nonprincipal.

(F2) $\mathscr{F}$ contains the filter $\{E \in \Sigma: E^c$ is finite$\}$.

(F3) $\mathscr{F}$ contains no finite subsets of N.

There are uncountably many nonprincipal ultrafilters on N. A typical example of non-principal ultrafilters is an ultrafilter $\mathscr{F}$ which contains $\mathscr{F}_0=\{N-\{1, 2, \cdots, n\}: n\geqq 1\}$.

Now p.f.a. 0-1 measures are associated with non-principal ultrafilters on N:

PROPOSITION 2.2. *If λ is a* p.f.a. 0-1 *measure on Σ, then $\mathscr{F}=\{E\in\Sigma: \lambda(E)=1\}$ is non-principal. Conversely for every non-principal ultrafilter $\mathscr{F}$, define a measure λ in the same way as in Proposition* 2.1; *then λ is* p.f.a.

PROOF. Let λ be a p.f.a. 0-1 measure on Σ. Then $\lambda\wedge\delta_k=0$ for $k\in N$, by [11], Theorem 1.16, where δ_k is the point mass concentrated at k. Hence in particular, $(\lambda\wedge\delta_k)(\{k\})=\min\{\lambda(\{k\}), 1\}=0$ or $\lambda(\{k\})=0$ for $k\in N$. Thus, $\lambda(F)=0$ and $F\notin\mathscr{F}$ for every finite subset F of N. This means that $\mathscr{F}$ satisfies condition (F3), so $\mathscr{F}$ is non-principal. Conversely, let $\mathscr{F}$ be any non-principal ultrafilter and λ a 0-1 measure defined as in Proposition 2.1. Then, every finite set F in N does not belong to $\mathscr{F}$ by (F3); hence $\lambda(F)=0$ by definition. Now let ν be any c.a. measure satisfying $0\leqq\nu\leqq\lambda$. Then, $0\leqq\nu(F)\leqq\lambda(F)=0$ for every finite set F in N. Since ν is c.a., $\|\nu\|=\nu(N)=\sum_{k=1}^{\infty}\nu(\{k\})=0$. This means that λ is p.f.a. q.e.d.

Therefore, any 0-1 measure is either c.a. or p.f.a.

In the remainder part of this section we discuss the integration of elements of l^∞ with respect to 0-1 measures.

First the value of the integral of any element v of l^∞ with respect to a point mass λ is simply given by

$$\langle v, \lambda\rangle=v(k)\ , \quad \text{provided that} \quad \lambda=\delta_k\ . \tag{2.1}$$

Next, by connecting non-principal ultrafilters on N with the Bolzano-Weierstrass property of bounded sequences in R, we can characterize the values of integrals of elements in l^∞ with respect to p.f.a. 0-1 measures in terms of the filter theory.

Let $\mathscr{F}$ be a nonprincipal ultrafilter and let v be a fixed element of l^∞. We recall that every E in $\mathscr{F}$ is an infinite set. Let $v(N)$ denote the range of v and let $\mathscr{S}_v=\{S: S\subset v(N), v^{-1}(S)\in\mathscr{F}\}$. Then $\mathscr{S}_v$ forms an ultrafilter on the set $v(N)$.

Let then $\overline{v(N)}$ be the closure of $v(N)$ in R; hence $\overline{v(N)}\subset[-\|v\|, \|v\|]$ and $\mathscr{S}_v$ is a base for a filter on $\overline{v(N)}$. Let $\overline{\mathscr{S}_v}$ be the filter generated on $\overline{v(N)}$ by $\mathscr{S}_v$. Then $\overline{\mathscr{S}_v}$ forms an ultrafilter on $\overline{v(N)}$ in accordance with the following proposition:

LEMMA 2.3. *Let X be a nonvoid set and let Y be any nonvoid subset of X. If $\mathscr{F}$ is a filter on Y, then $\mathscr{F}$ is a base for a filter on X. Let $\mathscr{G}$ be the filter generated on X by $\mathscr{F}$. Then we have:*

(a) *For every $A \subset X$ with $A \neq \emptyset$, $A \cap Y \in \mathscr{F}$ iff $A \in \mathscr{G}$.*

(b) *If $\mathscr{F}$ is an ultrafilter on Y, then $\mathscr{G}$ is an ultrafilter on X.*

Now since $\overline{v(N)}$ is compact, $\overline{\mathscr{S}_v}$ converges to some element α in $\overline{v(N)}$; and the limit α is unique as $\overline{v(N)}$ is a metric space. Moreover, we have just shown that given a non-principal ultrafilter $\mathscr{F}$ on N and an element v of l^∞, the family $\mathscr{S}_v$, and consequently $\overline{\mathscr{S}_v}$, was uniquely determined. Hence we conclude that to every $\mathscr{F}$ and v there corresponds a unique real number α in $\overline{v(N)}$. We then consider the p.f.a. 0-1 measure λ associated through Proposition 2.2 with $\mathscr{F}$ and characterize the value of the integral of v with respect to λ.

PROPOSITION 2.4. *Let $v \in l^\infty$, λ any p.f.a. 0-1 measure, $\mathscr{F}$ the associated non-principal ultrafilter on N in the sense of Proposition 2.2, and let $\overline{\mathscr{S}_v}$ be the ultrafilter on the compact set $\overline{v(N)}$ specified as above. Then, the value $\langle v, \lambda\rangle$ is given as the limit of $\overline{\mathscr{S}_v}$ and $\langle v, \lambda\rangle \in \overline{v(N)}$.*

PROOF. That $\overline{\mathscr{S}_v}$ converges to the limit α means that $U \cap \overline{v(N)} \in \overline{\mathscr{S}_v}$ for every neighborhood U of α. Hence by Lemma 2.3, $U \cap v(N) \in \mathscr{S}_v$ for every neighborhood U of α. We then set $U_n = \{\xi \in \boldsymbol{R}: |\xi - \alpha| < 1/n\}$, $S_n = U_n \cap v(N)$, and $E'_n = v^{-1}(S_n)$. Then $S_n \in \mathscr{S}_v$ (and hence $v(N) - S_n \notin \mathscr{S}_v$). So, $E'_n \in \mathscr{F}$ and $N - E'_n \notin \mathscr{F}$. Since each E'_n is an infinite set, one can choose an infinite sequence $\{k_n\}$ such that $k_n \in E'_n$ and $k_n \geqq k_{n-1}+1$ and $v(k_n) \to \alpha$ as $n \to \infty$. Let $E_n = E'_n - \{1, 2, \cdots, k_n - 1\}$ for $n \geqq 1$. Then $k_n = \min E_n$ and $E_n \in \mathscr{F}$ for all $n \geqq 1$. Next, define a sequence $\{v^n\}$ of simple functions on N by setting $v^n = v(k_n)\chi_{E_n}$; and set $M_n^\varepsilon = \{s \in N: |v^n(s) - v(s)| > \varepsilon\}$ for $\varepsilon > 0$ and $n \geqq 1$. Then noting that $\lambda(E_n^c) = 0$ and $|v^n(s) - v(s)| \leqq |v(k_n) - \alpha| + |\alpha - v(s)| < 2/n$ for $s \in E_n (\subset E'_n)$, we infer that $2/n < \varepsilon$ implies

$$v(\lambda, M_n^\varepsilon) = v(\lambda, M_n^\varepsilon \cap E_n^c) + v(\lambda, M_n^\varepsilon \cap E_n) = 0 ,$$

which means that v^n converges to v in λ-measure. Since $\|v^n\| \leqq \|v\|$ for $n \geqq 1$, the dominated convergence theorem yields

$$\langle v, \lambda\rangle = \lim \langle v^n, \lambda\rangle = \lim v(k_n) = \alpha \in \overline{v(N)} . \qquad \text{q.e.d.}$$

Finally, we give the following useful result as an application of Proposition 2.4.

PROPOSITION 2.5. *Let $v \in l^\infty$. Given $\varepsilon>0$ let $E_\varepsilon=v^{-1}(U_\varepsilon(\|v\|))$, where $U_\varepsilon(\|v\|)$ denotes the ε-neighborhood in $\boldsymbol{R}$ of $\|v\|$. Then for every* p.f.a. 0-1 *measure λ belonging to $F_0(v)$, we have $\lambda(E_\varepsilon)=1$ for $\varepsilon>0$.*

PROOF. Let $\overline{\mathscr{S}_v}$ be the ultrafilter on the compact set $\overline{v(N)}$ specified as before. Then $\overline{\mathscr{S}_v}$ converges to the value $\alpha=\|v\|$ since $\langle v, \lambda\rangle=\|v\|$ for every p.f.a. 0-1 measure in $F_0(v)$. Hence it is seen from the proof of Proposition 2.4 that $\lambda(E_\varepsilon)=1$ for $\varepsilon>0$. q.e.d.

§3. Representation of measures in $F_0(u)$.

In this section we first establish two decomposition theorems for the scalar products $\langle u, \lambda\rangle$, $\lambda \in F_0(u)$, and then give general representations of measures in $F_0(u)$ in terms of the measures which belong to $F_0(u^+)$ and $F_0(u^-)$. We start with the following

LEMMA 3.1. *Let $u \in l^\infty-\{0\}$, $\lambda \in F_0(u)$, and let $E \in \Sigma$. Then we have $\langle u\chi_E, \lambda\rangle=\langle |u|\chi_E, v(\lambda, \cdot)\rangle=\|u\|v(\lambda, E)$.*

PROOF. The desired relation is obtained by comparing the corresponding terms in the estimate:

$$\begin{aligned}\|u\|=\langle u\chi_E, \lambda\rangle+\langle u\chi_{N-E}, \lambda\rangle &\leqq \langle |u|\chi_E, v(\lambda, \cdot)\rangle \\ +\langle |u|\chi_{N-E}, v(\lambda, \cdot)\rangle &\leqq \|u\|v(\lambda, E)+\|u\|v(\lambda, N-E)=\|u\| . \quad \text{q.e.d.}\end{aligned}$$

The first decomposition theorem for the scalar product $\langle u, \lambda\rangle$ is given in terms of the Jordan decompositions of u and λ.

PROPOSITION 3.2. *Let $u \in l^\infty-\{0\}$, $\lambda \in F_0(u)$, and let $E \in \Sigma$. Let $u=u^+-u^-$ and $\lambda=\lambda^+-\lambda^-$. Then we have $\langle u\chi_E, \lambda\rangle=\langle u^+\chi_E, \lambda^+\rangle+\langle u^-\chi_E, \lambda^-\rangle$. Moreover, if $E=N$ then each term on the right side of this relation can be written as*

$$\langle u^+, \lambda^+\rangle=\|u^+\|\,\|\lambda^+\|=\|u\|\,\|\lambda^+\|$$

and

$$\langle u^-, \lambda^-\rangle=\|u^-\|\,\|\lambda^-\|=\|u\|\,\|\lambda^-\| ,$$

where we understand that $\|u^\pm\|<\|u\|$ implies $\lambda^\pm=0$, respectively.

PROOF. First we infer that

$$\begin{aligned}\langle u\chi_E, \lambda\rangle=\langle u^+\chi_E, \lambda^+\rangle-\langle u^+\chi_E, \lambda^-\rangle \\ -\langle u^-\chi_E, \lambda^+\rangle+\langle u^-\chi_E, \lambda^-\rangle .\end{aligned} \tag{3.1}$$

On the other hand, we have

$$(3.2)\qquad \langle |u|\chi_E, v(\lambda, \cdot)\rangle = \langle u^+\chi_E, \lambda^+\rangle + \langle u^+\chi_E, \lambda^-\rangle + \langle u^-\chi_E, \lambda^+\rangle + \langle u^-\chi_E, \lambda^-\rangle .$$

But, the left sides of (3.1) and (3.2) are equal by Lemma 3.1; hence $\langle u^+\chi_E, \lambda^-\rangle + \langle u^-\chi_E, \lambda^+\rangle = 0$ from which, together with (3.1), we obtain the first relation in the statement. To get the last assertion, apply the first relation just obtained; then we have $||u|| = \langle u^+, \lambda^+\rangle + \langle u^-, \lambda^-\rangle \leqq ||u^+|| \, ||\lambda^+|| + ||u^-|| \, ||\lambda^-|| \leqq ||u|| \, ||\lambda^+|| + ||u|| \, ||\lambda^-|| = ||u||$ since $||\lambda^+|| + ||\lambda^-|| = ||\lambda|| = 1$. Comparing the corresponding term, we get the last two relations in the statement. Finally, the above estimate also means that if $||u^\pm|| < ||u||$ then $\lambda^\pm$ must be identically zero, respectively. q.e.d.

The following is an immediate consequence of Proposition 3.2:

COROLLARY 3.3. *Let* $u \in l^\infty - \{0\}$ *and let* $\lambda \in F_0(u)$. *If* $u \in (l^\infty)^+$, *then* $\lambda \geqq 0$; *and if* $-u \in (l^\infty)^+$, *then* $\lambda \leqq 0$.

This result also states that the duality mapping F_0 is order-preserving in the sense that $u_2 - u_1 \in (l^\infty)^+$ implies $F_0(u_2 - u_1) \subset ba^+$.

The second decomposition theorem for the scalar product $\langle u, \lambda\rangle$ is described in terms of the Yosida-Hewitt decomposition.

PROPOSITION 3.4. *Let* $u \in l^\infty - \{0\}$, $\lambda \in F_0(u)$, *and let* $\lambda = \lambda_c + \lambda_p$. *Then we have* $||u|| = \langle u, \lambda_c\rangle + \langle u, \lambda_p\rangle$,

$$\langle u, \lambda_c\rangle = \langle |u|, v(\lambda_c, \cdot)\rangle = ||u|| \, ||\lambda_c|| , \quad \textit{and}$$
$$\langle u, \lambda_p\rangle = \langle |u|, v(\lambda_p, \cdot)\rangle = ||u|| \, ||\lambda_p|| .$$

PROOF. Employing the same idea as in the proof of Proposition 3.2, the desired equalities are obtained by comparing the corresponding terms in the estimate

$$||u|| = \langle u, \lambda_c\rangle + \langle u, \lambda_p\rangle \leqq \langle |u|, v(\lambda_c, \cdot)\rangle + \langle |u|, v(\lambda_p, \cdot)\rangle \leqq ||u|| \, ||\lambda_c|| + ||u|| \, ||\lambda_p|| = ||u|| ,$$

where we used Proposition 1.3. q.e.d.

We are now in a position to state the main theorem of this setion.

THEOREM 3.5. *Let* $u \in l^\infty - \{0\}$, $\lambda \in F_0(u)$, *and let* $\lambda = \lambda^+ - \lambda^-$. *Then* λ *is written as* $\lambda = ||\lambda^+||\nu^+ - ||\lambda^-||\nu^-$, *where* $\nu^+ \in F_0(u^+)$, $\nu^- \in F_0(u^-)$ *and* $||\lambda^+||\nu^+ \wedge ||\lambda^-||\nu^- = 0$.

PROOF. First suppose that $\lambda^+ = 0$. Then $||\lambda^-|| = 1$ and $\langle u^-, \lambda^-\rangle = ||u^-||$

by Proposition 3.2, i.e., $\lambda^- \in F_0(u^-)$. Therefore, letting $\nu^- = \lambda^-$ and ν^+ be any element of $F_0(u^+)$ yields the desired representation for λ. Similarly, in case of $\lambda^- = 0$, we obtain the representation by taking $\nu^+ = \lambda^+$ and an arbitrary element ν^- of $F_0(u^-)$. Finally, assume that both λ^+ and λ^- are nonzero. In this case, let $\nu^+ = \lambda^+/||\lambda^+||$ and $\nu^- = \lambda^-/||\lambda^-||$. Then we infer with the aid of Proposition 3.2 that $\langle u^\pm, \nu^\pm \rangle = ||u^\pm||$ and $\nu^\pm \in F_0(u^\pm)$, respectively. It is now clear that the representation is valid for these measures ν^+ and ν^-. q.e.d.

§4. Structure of the convex set $F_0(u)$.

In this section we discuss the structure of the convex sets $F_0(u)$ in terms of 0-1 measures. Since each of $F_0(u)$, $u \in l^\infty$, is weakly-star compact in *ba*, the structure of $F_0(u)$ is determined through Krein-Milman's theorem by its extremal points. We first investigate the extremal points of $F_0(u)$ in case of $u \geqq 0$ and then discuss the general case.

THEOREM 4.1. *Let* $u \in (l^\infty)^+ - \{0\}$ *and let* $\lambda \in F_0(u)$. *Then,* λ *is an extremal point of* $F_0(u)$ iff *it is a* 0-1 *measure.*

PROOF. Suppose first that λ is a 0-1 measure. Let $\alpha, \beta > 0$, $\alpha + \beta = 1$, $\lambda_0, \lambda_1 \in F_0(u)$, and let $\lambda = \alpha\lambda_0 + \beta\lambda_1$. We here note that $\lambda_0 \geqq 0$ and $\lambda_1 \geqq 0$ by Corollary 3.3. Now let E be an arbitrary element of Σ. If $\lambda(E) = 0$, then $\lambda_0(E) = \lambda_1(E) = 0$. Assume that $\lambda(E) = 1$. If $0 \leqq \lambda_0(E) < 1$, then $||\lambda_1|| \geqq \lambda_1(E) = \beta^{-1}(1 - \alpha\lambda_0(E)) > \beta^{-1}(1 - \alpha) = 1$, which contradicts to the fact that $||\lambda_1|| = 1$. Hence, $\lambda_0(E)$ must be 1; and $\lambda_1(E) = 1$ in a similar way. This means that $\lambda = \lambda_0 = \lambda_1$, i.e., λ is an extremal point of $F_0(u)$. Conversely, let λ be an extremal point of $F_0(u)$ and assume that $0 < \lambda(E_0) < 1$ for some $E_0 \in \Sigma$. We then define two bounded additive set functions λ_1 and λ_2 on Σ by setting $\lambda_1(E) = \lambda(E \cap E_0)$ and $\lambda_2(E) = \lambda(E \cap E_0^C)$ for $E \in \Sigma$. Then, $\lambda_i \geqq 0$ and $||\lambda_i|| > 0$ since $\lambda \geqq 0$ by Corollary 3.3. Moreover, noting that $\lambda(E) = \lambda_1(E) + \lambda_2(E)$ for $E \in \Sigma$, we have $||\lambda|| = ||\lambda_1|| + ||\lambda_2|| = 1$. Now define $\nu_i = \lambda_i/||\lambda_i||$ for $i = 1, 2$. Then we have $\nu_i \geqq 0$, $||\nu_i|| = 1$, and

$$\lambda = ||\lambda_1||\nu_1 + ||\lambda_2||\nu_2 \,. \tag{4.1}$$

We then demonstrate that $\nu_i \in F_0(u)$, $i = 1, 2$. Since $||u|| = \langle u, \lambda_1 \rangle + \langle u, \lambda_2 \rangle \leqq ||u|| \, ||\lambda_1|| + ||u|| \, ||\lambda_2|| = ||u||$, we have $||\lambda_i||^{-1}\langle u, \lambda_i \rangle = ||u||$, i.e., $\langle u, \nu_i \rangle = ||u||$, from which it follows that $\nu_i \in F_0(u)$. But, λ is an extremal point of $F_0(u)$, hence (4.1) implies that $\nu_1 = \nu_2 = \lambda$. Therefore, we have $0 < \lambda(E_0) =$

$\nu_2(E_0)=\|\lambda_2\|^{-1}\lambda(E_0\cap E_0^C)=0$, a contradiction. This means that λ can not take values between 0 and 1, i.e., λ is a 0–1 measure. q.e.d.

The above theorem states that if $u\in(l^\infty)^+-\{0\}$, $\operatorname{ext}F_0(u)$ consists of only 0–1 measures. Since each of $F_0(u)$ in a convex and weakly-star compact subset of ba, the application of Krein-Milman's theorem yields the following characterization of the convex set $F_0(u)$ in terms of 0–1 measures.

THEOREM 4.2. *If $u\in(l^\infty)^+-\{0\}$, then $F_0(u)$ contains at least one 0–1 measure, and $F_0(u)$ is a weakly-star closed convex hull of 0–1 measures in $F_0(u)$.*

Next, let us consider the general case. Let $u\in l^\infty$, $u=u^+-u^-$, and assume that $\|u^+\|>0$ and $\|u^-\|>0$. Moreover, let

$$E_0^+=\{s: u(s)>0\}\ ,\quad E_0^-=\{s: u(s)<0\}\ ,$$
$$E^+=\{s: u(s)\geqq 0\}\ ,\quad E^-=\{s: u(s)\leqq 0\}\ . \tag{4.2}$$

Clearly, E_0^+ and E_0^- are disjoint. Employing these sets, we have:

LEMMA 4.3. *If $\nu^+\in F_0(u^+)$ and $\nu^-\in F_0(u^-)$, then $\nu^+(E_0^+)=\nu^-(E_0^-)=1$ and $\nu^+(E^-)=\nu^-(E^+)=0$.*

PROOF. First, $\phi^+(E_0^+)=\phi^-(E_0^-)=1$ for $\phi^+\in\operatorname{ext}F_0(u^+)$ and $\phi^-\in\operatorname{ext}F_0(u^-)$. For if $\phi^+(E_0^+)=0$, then $\|u^+\|=\langle u^+,\phi^+\rangle=\langle u^+\chi_{E^-},\phi^+\rangle=0$ and we have a contradiction; furthermore, it is impossible to assume $\phi^-(E_0^-)=0$ by the same reason. This fact also means that $\phi^+(E^+)=\phi^-(E^-)=1$ and $\phi^+(E^-)=\phi^-(E^+)=0$. Now let $\nu^+\in F_0(u^+)$ and $\nu^-\in F_0(u^-)$. Then by Theorem 4.2, there exist generalized sequences $\{\phi_\alpha^+\}$ and $\{\phi_\beta^-\}$ such that $\phi_\alpha^+\in\operatorname{co}[\operatorname{ext}F_0(u^+)]$, $\phi_\beta^-\in\operatorname{co}[\operatorname{ext}F_0(u^-)]$ and $\{\phi_\alpha^+\}$ and $\{\phi_\beta^-\}$ converge respectively to ν^+ and ν^- in the weak-star topology of ba. Hence we have $\langle\chi_{E_0^+},\phi_\alpha^+\rangle=\phi_\alpha^+(E_0^+)=1$, $\langle\chi_{E_0^-},\phi_\beta^-\rangle=\phi_\beta^-(E_0^-)=1$, and consequently, $\nu^+(E_0^+)=\langle\chi_{E_0^+},\nu^+\rangle=\lim_\alpha\langle\chi_{E_0^+},\nu^+\rangle=1$ and $\nu^-(E_0^-)=\lim_\beta\phi_\beta^-(E_0^-)=1$. Thus, the first assertion is obtained. The last assertion is now evident from the additivity of $\nu^\pm$ and the fact that $\nu^+(N)=\nu^-(N)=1$. q.e.d.

PROPOSITION 4.4. *Let $u\in l^\infty$ be such that $u^\pm\neq 0$. If $\nu^+\in F_0(u^+)$ and $\nu^-\in F_0(u^-)$, then $\nu^+\wedge\nu^-=0$ and $\langle u^+,\nu^-\rangle=\langle u^-,\nu^+\rangle=0$.*

PROOF. For $E\in\Sigma$, the application of Lemma 4.3 yields

$$(\nu^+\wedge\nu^-)(E)=\inf_{T\subset E}\{\nu^+(T\cap E_0^+)+\nu^-(T^C\cap E\cap E_0^-)\}\quad(\geqq 0)\ .$$

But, the right side turns to be 0 if we take $T=E\cap E_0^-$. Thus the first assertion is obtained. The last assertion follows from Lemma 4.3 with the aid of the relations

$$\langle u^+, \nu^- \rangle = \langle u^+ \chi_{E^-}, \nu^- \rangle = 0 \tag{4.3}$$

and (4.3) with u^+ and ν^- replaced respectively by u^- and ν^+. q.e.d.

Using the results mentioned above, we obtain a converse of Theorem 3.5.

PROPOSITION 4.5. *Let $u\in l^\infty-\{0\}$, $u=u^+-u^-$, $\nu^+\in F_0(u^+)$, and $\nu^-\in F_0(u^-)$. Let α, β be any non-negative numbers satisfying $\alpha+\beta=1$ and $\alpha\|u^+\|+\beta\|u^-\|=\|u\|$, and define $\lambda=\alpha\nu^+-\beta\nu^-$. Then $\lambda\in F_0(u)$, and in this case, $\lambda^+=\alpha\nu^+$ and $\lambda^-=\beta\nu^-$.*

PROOF. It follows from Proposition 4.4, (1.3) and Lemma 1.2 that $\alpha\nu^+-\beta\nu^-$ gives the Jordan decomposition of λ, i.e., $\lambda^+=\alpha\nu^+$ and $\lambda^-=\beta\nu^-$. Hence, $\|\lambda\|=\alpha\|\nu^+\|+\alpha\|\nu^-\|=1$. On the other hand, we see from Proposition 4.4 and the restrictions on α, β that $\langle u, \lambda\rangle=\alpha\langle u^+, \nu^+\rangle+\beta\langle u^-, \nu^-\rangle=\|u\|$. Thus, $\lambda\in F_0(u)$, and the proof is complete.

Now combining Proposition 4.5, with Theorem 3.5, we give the main result of this section:

THEOREM 4.6. *For $u\in l^\infty-\{0\}$, we have*

$$F_0(u)=\bigcup_{\alpha,\beta}[\alpha F_0(u^+)+\beta F_0(-u^-)]\,, \tag{4.4}$$

where the union is taken over all $\alpha, \beta\geqq 0$ satisfying $\alpha+\beta=1$ and $\alpha\|u^+\|+\beta\|u^-\|=\|u\|$. Therefore we have:

(i) *If $\|u^-\|<\|u\|$ then $F_0(u)=F_0(u^+)$.*
(ii) *If $\|u^+\|<\|u\|$ then $F_0(u)=F_0(-u^-)$.*
(iii) *If $\|u^+\|=\|u^-\|=\|u\|$, then $F_0(u)=\mathrm{co}\,[F_0(u^+)\cup F_0(-u^-)]$ and*

$$\mathrm{ext}\, F_0(u)=\mathrm{ext}\, F_0(u^+)\cup \mathrm{ext}\, F_0(-u^-)\,.$$

PROOF. Theorem 3.5 states that every element λ of $F_0(u)$ belongs to the set $\|\lambda^+\|F_0(u^+)+\|\lambda^-\|F_0(-u^-)$, and so $F_0(u)$ is contained in the right side of (4.4). The converse inclusion follows from Proposition 4.5. We now prove (i) through (iii). If $\|u^-\|<\|u\|$, then only $\alpha=1$ and $\beta=0$ must be taken; hence $F_0(u)$ coincides with $F_0(u^+)$. Similarly, if $\|u^+\|<\|u\|$ then $F_0(u)=-F_0(u^-)=F_0(-u^-)$. However in case of $\|u^+\|=\|u^-\|=\|u\|$, we can take any non-negative numbers α, β with $\alpha+\beta=1$. This means that $F_0(u)=\mathrm{co}\,[F_0(u^+)\cup F_0(-u^-)]$. To get the last assertion of (iii) we first observe that the set of extremal points of the set $W \equiv F_0(u^+) \cup F_0(-u^-)$ is exactly the set of those of $F_0(u^+)$ and $F_0(-u^-)$, i.e.,

$$\text{ext } W = \text{ext } F_0(u^+) \cup \text{ext } F_0(-u^-) . \tag{4.5}$$

In fact, it is clear that $\text{ext } W \subset \text{ext } F_0(u^+) \cup \text{ext } F_0(-u^-)$. Conversely, suppose that ϕ is an extremal point of, say, $F_0(u^+)$. Assume then that ϕ is written as $\phi = \alpha\lambda + \beta\nu$ for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and some $\lambda, \nu \in W$. First of all, both λ and ν can not belong to $F_0(-u^-)$. Also let $\lambda \in F_0(u^+)$, $\nu \in F_0(-u^-)$, and let $E_0^{\pm}$ be the sets specified as in (4.2); then $(\alpha\lambda + \beta\nu)(E_0^-) = -\beta < 0$ by Lemma 4.3. This contradicts the fact that ϕ is 0-1 measure. Consequently, both ν and λ must belong to $F_0(u^+)$. But, in this case $\lambda = \nu = \phi$ since $\phi \in \text{ext } F_0(u^+)$. Thus, $\text{ext } F_0(u^+) \subset \text{ext } W$. Similarly, $\text{ext } F_0(-u^-) \subset \text{ext } W$; and so we have (4.5). We then show that $\text{ext } W = \text{ext } [\text{co } W]$. Since both the set W and its weakly-star closed convex hull are weakly-star compact, the only extremal points in $\text{co}[W]$ are points in W by [7], Lemma v. 8.5, p. 440. From this we see that $\text{ext}[\text{co } W] \subset \text{ext } W$. Conversely, let $\lambda \in \text{ext } W$. Then (4.5) states that λ belongs to $\text{ext } F_0(u^+)$ or $\text{ext } F_0(u^-)$; we may assume without loss of generality that $\lambda \in F_0(u^+)$. Suppose now that $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2$ for some $\alpha \in (0, 1)$ and some $\lambda_1, \lambda_2 \in \text{co } W$. Then we must have $\lambda_1 \in F_0(u^+)$. In fact, if $\lambda_1 \notin F_0(u^+)$, then $\lambda_1 = \alpha_1\mu_1 + (1-\alpha_1)\nu_1$ for some $\alpha_1 \in [0, 1)$ and $\mu_1 \in F_0(u^+)$ and some $\nu_1 \in F_0(-u^-)$, while $\lambda_2 = \alpha_2\mu_2 + (1-\alpha_2)\nu_2$ for some $\alpha_2 \in [0, 1]$, $\mu_2 \in F_0(u^+)$ and $\nu_2 \in F_0(-u^-)$. Let E_0^- be the set specified as in (4.2), then Lemma 4.3 yields that $\lambda(E_0^-) = -\alpha(1-\alpha_1) - (1-\alpha)(1-\alpha_2) < 0$. This contradicts the assumption that $\lambda \in F_0(u^+)$. Therefore, $\lambda_1 \in F_0(u^+)$. Similarly, we have $\lambda_2 \in F_0(u^+)$. But, $\lambda \in \text{ext } F_0(u^+)$; hence it follows that $\lambda = \lambda_1 = \lambda_2$. This means that $\lambda \in \text{ext } [\text{co } W]$. Consequently, combining the above-mentioned yields the last assertion of (iii). q.e.d.

§5. The Dixmier decomposition of *ba*.

In this section we first show that in the space *ba*, the Yosida-Hewitt decomposition is equivalent to the Dixmier decomposition since *ba* is the third conjugate of the space c_0 of sequences converging to 0. We then discuss the structure of $F_0(u)$ from the point of view of the Yosida-Hewitt decomposition.

LEMMA 5.1. *Let $\lambda \in ba$ and let $\lambda = \lambda_c + \lambda_p$ be the Yosida-Hewitt decomposition of λ. Then $v(\lambda_p, F) = 0$ for every finite subset F of N.*

PROOF. We employ the same technique as in the proof of Proposition 2.2. Let $\lambda_p=\lambda_p^+-\lambda_p^-$ be the Jordan decomposition of λ_p; then λ_p^+ and λ_p^- are non-negative, p.f.a. measures by definition. Now let $k\in N$ and let δ_k be the point mass concentrated at k. Then $\lambda_p^+\wedge\delta_k=\lambda_p^-\wedge\delta_k=0$ by [11], Theorem 1.16. Hence in particular, we have $0=(\lambda_p^+\wedge\delta_k)(\{k\})=\min\{\lambda_p^+(\{k\}), 1\}$, or $\lambda_p^+(\{k\})=0$; and $\lambda_p^-(\{k\})=0$ in a similar way. Now the finite additivity of λ_p^+ and λ_p^- implies the assertion. q.e.d.

In the following let l^1 be the usual space of absolutely convergent sequences with norm $\|\cdot\|_1$.

PROPOSITION 5.2. *Let $\lambda\in ba$ and let $\lambda=\lambda_c+\lambda_p$ denote the Yosida-Hewitt decomposition.* (a) *Define a sequence $f=\{f(k)\}$ by setting $f(k)=\lambda_c(\{k\})$ for $k\in N$. Then $f\in l^1$, $\langle u, f\rangle=\langle u, \lambda_c\rangle$ for all $u\in l^\infty$, and $\|\lambda_c\|=\|f\|_1=\sum_k|f(k)|$. Therefore, $\lambda_c\in l^1$ in the sense of the natural embedding.* (b) *$\langle u, \lambda_p\rangle=0$ for all $u\in c_0$.*

PROOF. (a) Since λ_c is c.a., $\|\lambda_c\|=\sum_k v(\lambda_c, \{k\})=\sum_k|\lambda_c(\{k\})|=\|f\|_1$; and so $f\in l^1$. Moreover, $\langle u, f\rangle=\sum_k u(k)f(k)=\langle u, \lambda_c\rangle$, and so f is identified in ba with λ_c in the sense of the natural embedding of l^1 in ba. (b) Given $n\in N$, let $F_n=\{1, 2, 3, \cdots, n\}$. Then for every $u\in c_0$ the application of Lemma 5.1 yields the estimate $|\langle u, \lambda_p\rangle|\leqq\langle|u|\chi_{N-F_n}, v(\lambda_p, \cdot)\rangle\leqq(\sup_{s\geqq n}|u(s)|)\|\lambda_p\|$ for $n\in N$. Since the extreme right side goes to 0 as $n\to\infty$ ($u\in c_0$), we have $\langle u, \lambda_p\rangle=0$. Thus (b) is obtained. q.e.d.

Dixmier's decomposition theorem [6] states that if X is a Banach space then the third conjugate X^{***} is decomposed as the direct sum of X^* and the closed subspace $X^\perp$ consisting of the functionals vanishing on X. Accordingly, ba is decomposed as the direct sum $ba=l^1+c_0^\perp$, $c_0^\perp=\{\lambda\in ba: \langle u, \lambda\rangle=0$ for all $u\in c_0\}$. Thus combining this with Proposition 5.2, we have:

THEOREM 5.3. *In the space ba, the Yosida-Hewitt decomposition is equivalent to the Dixmier decomposition.*

Now in the remainder part of this section we discuss the structure of $F_0(u)$ from the point of view of Theorem 5.3. First of all, we consider two extreme cases.

PROPOSITION 5.4. *Let $u\in l^\infty-\{0\}$. Then $F_0(u)\subset l^1$ iff $\limsup_{k\to\infty}|u(k)|<\|u\|$. Moreover in this case, $F_0(u)$ is the convex closure of a finite number of* c.a. *measures of the form δ_k or $-\delta_k$.*

PROOF. Suppose that $\alpha=\limsup|u(k)|<\|u\|$, and let $E^*=\{k\in N: |u(k)|=\|u\|\}$. Then $E^*\neq\emptyset$ and E^* is a finite set. We then write $E^*=$

$\{k_1, \cdots, k_l\}$. Then $\langle u, \operatorname{sgn}(u(k_i))\delta_{k_i}\rangle = |u(k_i)| = ||u||$ and so $\operatorname{sgn}(u(k_i))\delta_{k_i}$, $1 \leqq i \leqq l$, belong to $\operatorname{ext} F_0(u)$ by Theorem 4.6. Now $F_0(u)$ has no other extremal points. For, suppose that λ is an extremal point of $F_0(u)$, different from $\operatorname{sgn}(u(k_i))\delta_{k_i}$, $1 \leqq i \leqq l$; then it follows from Theorem 4.6 that either λ or $-\lambda$ is a 0-1 measure; and either $\lambda \in l^1$ or $\lambda \in c_0^\perp$. But, if $\lambda \in l^1$, then $\lambda = \delta_k$ for some $k \notin E^*$, which contradicts the definition of E^*. Thus, λ must belong to $c_0^\perp$. Let $\varepsilon = (||u|| - \alpha)/2$. Then there is an n_ε such that $n_\varepsilon \geqq \max\{k_i: 1 \leqq i \leqq l\}$ and $n \geqq n_\varepsilon$ implies $|u(n)| \leqq \alpha + \varepsilon$. Since λ is now p.f.a., one may find an $E \in \Sigma$ such that $E \subset N - \{1, 2, \cdots, n_\varepsilon\}$ and $|\lambda(E)| = v(\lambda, E) = 1$. Hence writing F_ε for the set $\{1, 2, \cdots, n_\varepsilon\}$, we see with the aid of Lemma 3.1 that $||u|| = \langle |u| \chi_{N-F_\varepsilon}, \ v(\lambda, \cdot)\rangle \leqq (\alpha + \varepsilon) v(\lambda, N - F_\varepsilon) \leqq \alpha + \varepsilon < ||u||$. This contradiction shows that $F_0(u)$ has no other extremal points than $\operatorname{sgn}(u(k_i))\delta_{k_i}$, $1 \leqq i \leqq l$, and consequently, $F_0(u)$ is the convex closure of these countably additive measure. Conversely, assume that $F_0(u) \subset l^1$ and $\limsup |u(k)| = ||u||$. Then, there exists a subsequence $\{k_j\}$ such that $\lim |u(k_j)| = ||u||$; one may assume without loss of generality that $u(k_j) \geqq 0$ and $\lim u(k_j) = ||u||$. Let $E = \{k_j: j \geqq 1\}$ and let $\mathscr{A} = \{E - F: F = \varnothing$ or $\operatorname{card}(F) < \infty\}$. Then there exists a nonprincipal ultrafilter $\mathscr{F}$ which contains $\mathscr{A}$ as its subfamily. Let λ be the 0-1 measure associated with this ultrafilter $\mathscr{F}$. Then we infer with the aid of Propositions 2.2 and 5.2 that $\lambda \in c_0^\perp$ and $v(\lambda, E^c \cup F) = 0$ for all finite set F in N. Now for a given $n \in N$, define a simple function u^n in l^∞ by setting $u^n = u(k_n)\chi_{E_n}$ and $E_n = \{k_j: j \geqq n\}$; note that $\lambda(E_n) = 1$ for $n \geqq 1$. Then for every n, $||u^n|| \leqq ||u||$ and for a given $\varepsilon > 0$ the set $\{k \in E: |u^n(k) - u(k)| > \varepsilon\}$ contains at most a finite number of k_j's. Hence noting that $v(\lambda, N - E) = 0$ and using Lemma 5.1, we infer that $\lim v(\lambda, \{k: |u^n(k) - u(k)| > \varepsilon\}) = 0$ for every $\varepsilon > 0$. So, u^n converges to u in λ-measure and the dominated convergence theorem yields $\langle u, \lambda\rangle = \lim \langle u^n, \lambda\rangle = \lim u(k_n)\lambda(E_n) = ||u||$. This means that $\lambda \in F_0(u)$ and contradicts the assumption that $F_0(u) \subset l^1$. Therefore, we conclude that $\limsup |u(k)| < ||u||$. q.e.d.

PROPOSITION 5.5. *Let* $u \in l^\infty - \{0\}$. *Then,* $F_0(u) \subset c_0^\perp$ iff $|u(k)| < ||u||$ *for all* $k \in N$.

PROOF. Suppose first that $|u(k)| < ||u||$ for all $k \in N$. Let $\lambda \in F_0(u)$ and let $\lambda = \lambda_c + \lambda_p$ be the Yosida-Hewitt decomposition. Then by Proposition 3.4, we have the relation

$$\int_N |u(s)| \, v(\lambda_c, ds) = ||u|| \, ||\lambda_c|| = \int_N ||u|| \, v(\lambda_c, ds) \ ,$$

and so the countable additivity of λ_c yields

$$0=\int_N (\|u\|-|u(s)|)v(\lambda_c, ds)=\sum_{k=1}^{\infty}(\|u\|-|u(k)|)|\lambda_c(\{k\})| .$$

But, $\|u\|-|u(k)|>0$ for every k; hence $\lambda_c(\{k\})=0$ for $k\in N$ and this fact implies that $\lambda_c=0$, i.e., λ is p.f.a. Thus, $\lambda\in c_0^\perp$ by Proposition 5.2. To get the converse, assume that $|u(k)|=\|u\|$ for some $k\in N$. Then we have $\langle |u|, \delta_k\rangle=|u(k)|=\|u\|$ for the point mass δ_k, i.e., $\operatorname{sgn}(u(k))\in F_0(u)$. Since $\operatorname{sgn}(u(k))\delta_k$ is c.a., this contradicts the assumption that every element of $F_0(u)$ is p.f.a. q.e.d.

REMARK. If $|u(\cdot)|$ attains $\|u\|$ at infinitely many points, say k_i, $i\geqq 1$, then $F_0(u)$ contains infinitely many c.a. 0-1 (or 0-(−1)) measures since for each i either δ_{k_i} or $-\delta_{k_i}$ is in $\operatorname{ext} F_0(u)$. In this case $F_0(u)$ must also have at least one p.f.a. 0-1 (or 0-(−1)) measure. In fact, suppose that $u(k_i)=\|u\|$ for $i\geqq 1$ (we choose a subsequence of $\{k_i\}$ if necessary) and let $E=\{k_i: i\geqq 1\}$. Since the family $\mathscr{B}=\{E-F: F$ is finite$\}$ forms a base for a filter on N, we may take a nonprincipal ultrafilter which is finer than the filter generated by $\mathscr{B}$. Then the p.f.a. 0-1 measure associated with this ultrafilter is in $F_0(u)$. If $u(k_i)=-\|u\|$ for $i\geqq 1$, we get a p.f.a. 0-(−1) measure in a similar way.

We now consider the general case. The convex set $F_0(u)$ is in general a weakly-star closed convex hull of a disjoint union of a subset of l^1 and that of $c_0^\perp$ (cf. Theorem 5.3).

THEOREM 5.6. *Let $u\in l^\infty-\{0\}$. Then $F_0(u)$ is written as $F_0(u)=\overline{\mathrm{co}}^{\sigma(ba,l^\infty)}[C\cup P]$, where C is the set of all* c.a. 0-1 *or* 0-(−1) *measures in $F_0(u)$, P the set of all* p.f.a. 0-1 *or* 0-(−1) *measures in $F_0(u)$, and $\overline{\mathrm{co}}^\sigma$ means the weakly-star closed convex hull of $C\cup P$.*

PROOF. Theorem 4.6 states that either $\operatorname{ext} F_0(u)=\operatorname{ext} F_0(u^+)$ or $\operatorname{ext} F_0(u)=\operatorname{ext} F_0(-u^-)$ or $\operatorname{ext} F_0(u)=\operatorname{ext} F_0(u^+)\cup\operatorname{ext} F_0(-u^-)$. Now Theorem 4.1 says that $\operatorname{ext} F_0(u^+)$ consists of 0-1 measures, while $\operatorname{ext} F_0(-u^-)$ consists of 0-(−1) measures. Thus $\operatorname{ext} F_0(u)\subset C\cup P\subset F_0(u)$, so that $F_0(u)=\overline{\mathrm{co}}^\sigma[C\cup P]$ by Theorem 4.2. q.e.d.

§6. Geometrical interpretations.

In this section we give some geometrical interpretations of our results established so far in connection with the structures of the unit balls in l^∞ and ba. Moreover, the characterizations of extremal points and smooth points of the unit ball in l^∞ will be given as applications of our results.

We first divide the surface $S=\{u\in l^\infty: \|u\|=1\}$ of the unit ball in l^∞ into the following five zones:

$A_+=\{u\in l^\infty\colon \|u\|=1,\ u\geqq 0\}$,
$T_+=\{u\in l^\infty\colon u=u^+-u^-,\ 0<\|u^-\|<\|u^+\|=1\}$,
$T_0=\{u\in l^\infty\colon u=u^+-u^-,\ \|u^+\|=\|u^-\|=1\}$,
$T_-=\{u\in l^\infty\colon u=u^+-u^-,\ 0<\|u^+\|<\|u^-\|=1\}$,
$A_-=\{u\in l^\infty\colon \|u\|=1,\ u\leqq 0\}$.

We wish to consider the partition of the surface $S^*=\{\lambda\in ba\colon \|\lambda\|=1\}$ of the unit ball in ba, which is associated through the duality mapping F_0 with the above-mentioned heuristic partition of S. The S^* may be divided into three zones which are defined as;

$A_+^*=ba^+\cap S^*=\{\lambda\in ba\colon \lambda\in ba^+,\ \|\lambda\|=1\}$,
$T_0^*=\{\lambda\in ba\colon \lambda$ satisfies condition $(C)\}$,
$A_-^*=-A_+^*=\{\lambda\in ba\colon -\lambda\in ba^+,\ \|\lambda\|=1\}$,

where λ is said to satisfy condition (C), if it is written in the form $\lambda=\alpha\nu_1-\beta\nu_2$ for some ν_1, ν_2 in A_+^* with $\nu_1\wedge\nu_2=0$ and some $\alpha,\beta\in(0,1)$ with $\alpha+\beta=1$; note that $\alpha\nu_1=\lambda^+$, $\beta\nu_2=\lambda^-$ and $\|\lambda\|=1$ by (1.3) and Lemma 1.2. Observe that T_0^* consists of proper convex combinations of A_+^* and those of A_-^*. Also, we have $S^*=A_+^*\cup T_0^*\cup A_-^*$. In fact, let $\lambda\in S^*$ and let $\lambda=\lambda^+-\lambda^-$ be the Jordan decomposition of λ. If any one of λ^+ and λ^- is a zero measure, we have either $\lambda\in A_+^*$ or $\lambda\in A_-^*$. If $\|\lambda^+\|\|\lambda^-\|>0$, then $\lambda^+\wedge\lambda^-=0$ and $\|\lambda^+\|+\|\lambda^-\|=\|\lambda\|=1$. So, if we set $\nu_1=\lambda^+/\|\lambda^+\|$ and $\nu_2=\lambda^-/\|\lambda^-\|$, then $\lambda=\|\lambda^+\|\nu_1-\|\lambda^-\|\nu_2\in T_0^*$.

We now demonstrate that this partition is the desired one for S^*. First, Theorem 4.6 states that F_0 maps $A_+\cup T_+$ into A_+^*, and $A_-\cup T_-$ into A_-^*; and secondly, F_0 maps T_0 into T_0^* by Proposition 4.4 and Theorem 4.6. That $F_0(A_+\cup T_+)=A_+^*$ and $F_0(A_-\cup T_-)=A_-^*$ hold follows from the facts $F_0(\chi_N)=A_+^*$ and $F_0(-\chi_N)=A_-^*$. Each of $F_0(u)$, $u\in S$, forms a "flat" part of the unit surface S^* in the sense that it forms a part of S^* and is a weakly-star closed convex hull of 0-1 (or 0-(−1)) measures. (In this sense each of $F_0(u)$ is called a *face* of S, see Phelps [10].) The above facts, together with Theorem 4.2, state that A_+^* and A_-^* are flat on S^*. Therefore, extremal points of S^* are all on the "edges" of the closed convex sets A_+^* and A_-^*. This means that S^* shapes a "cylinder" in the space ba, and T_0^* turns to be a rich and complicated zone, in contrast to the "thin" zone T_0.

According to James' theorem ([5], p. 12, Theorem 3), the range of F_0, $R(F_0)=\cup\{F_0(u)\colon u\in S\}$, is a proper subset of S^*. Hence, F_0 does not map T_0 onto T_0^*. On the other hand, Bishop-Phelps' theorem [3] states that $R(F_0)$ is norm-dense in the surface S^*. (In fact, the subreflexivity of l^∞ is equivalent to the norm-denseness of $F_0(S)$ in S^*.) Moreover, $l^1\cap S^*$ lies in $R(F_0)$ since if $\lambda\in l^1\cap S^*$ then $1=||\lambda||=\max\{|\langle u, \lambda\rangle|: ||u||=1\}=\langle u_0, \lambda\rangle=||u_0||$ for some $u_0\in S$ by the Hahn-Banach theorem. Therefore, we can say that the surfaces S^* is (norm-) densely patched by the faces $F_0(u)$, $u\in l^\infty$, in such a way that $S^*-R(F_0)\subset T_0^*\cap c_0^\perp$. Although $F_0(T_0)\subsetneqq T_0^*$, we can show that $F_0(T_0)$ covers T_0^* essentially:

PROPOSITION 6.1. *Let ν_1, ν_2 be any 0-1 measures, and let $\alpha, \beta>0$ and $\alpha+\beta=1$. If $\nu_1\wedge\nu_2=0$, then $\lambda=\alpha\nu_1-\beta\nu_2\in F_0(T_0)$.*

PROOF. First we note that $\lambda\in T_0^*$. If $\nu_1, \nu_2\in l^1$ then $\nu_1=\delta_j$ and $\nu_2=\delta_k$ for same $j, k\in N$ with $j\neq k$. Hence, if u is a simple function $u=\chi_{\{j\}}-\chi_{\{k\}}$, then we have $u\in T_0$ and $\langle u, \lambda\rangle=1$, i.e., $\lambda\in F_0(T_0)$. If $\nu_1\in l^\infty$ and $\nu_2\in c_0^\perp$, then $\nu_1=\delta_j$ for some $j\in N$ and $\nu_2(E)=1$ for some $E\in\Sigma$ with $j\notin E$. Hence in this case we take a simple function $u=\chi_{\{j\}}-\chi_E$; then $u\in T_0$ and $\langle u, \lambda\rangle=1$, which means that $\lambda\in F_0(T_0)$. Similarly we also have $\lambda\in F_0(T_0)$ if $\nu_1\in c_0^\perp$ and $\nu_2\in l^1$. Suppose now that $\nu_1, \nu_2\in c_0^\perp$. Since $\nu_1, \nu_2\in A_+^*$, there exist $u_1', u_2'\in A_+$ such that $\nu_i\in F_0(u_i')$, $i=1, 2$. We may assume that $\limsup_{k\to\infty} u_i'(k)=1$ for $i=1, 2$, for otherwise, ν_i must belong to l^1 by Proposition 5.4. Let $0<\varepsilon<\min\{\alpha, \beta\}$ and set $E_i'=\{s\in N: |u_i'(s)-1|<\varepsilon\}$ for $i=1, 2$. Then we see from Proposition 2.5 that $\nu_1(E_1')=1$. Noting that $(\alpha\nu_1\wedge\beta\nu_2)(E_1')=0$, one can find a $T\in\Sigma$ such that $T\subset E_1'$ and $\alpha\nu_1(T)+\beta\nu_2(E_1'-T)<\varepsilon$. But, $\nu_1(T)=\nu_2(E_1'-T)=0$ since ν_1 and ν_2 are 0-1 measures. Thus, $v_1(E_1'-T)=1$ and $v_2(E_1'-T)=0$. We then set $E_1=E_1'-T$ and $E_2=(E_1'-T)^c\cap E_2'$. Then $E_1\cap E_2=\varnothing$ and $\nu_i(E_i)=1$ for $i=1, 2$. So, if we set $u_i=\chi_{E_i}u_i'$ for $i=1, 2$ and $u=u_1-u_2$, then $u^+=u_1$, $u^-=u_2$, $u\in T_0$ and $\nu_i\in F_0(u_i)$ for $i=1, 2$. Moreover, $\lambda\in F_0(u)\subset F_0(T_0)$. q.e.d.

Now in the remainder of this section we discuss extremal points and smooth points of the unit sphere S. First, we characterize extremal points of S.

PROPOSITION 6.2. *An element $u\in l^\infty$ is in* ext S iff *$|u(s)|=1$ for all $s\in N$.*

PROOF. Suppose that $u\in l^\infty$ and $|u(s)|=1$ for $s\in N$. Assume then that there exist $u_1, u_2\in S$ and $\alpha\in(0, 1)$ such that $u=u_1+(1-\alpha)u_2$. Let $u(s)=1$. Then we have $u_1(s)=u_2(s)=1$, for if $u_1(s)<1$ then we get a contradiction that $1=u(s)=\alpha u_1(s)+(1-\alpha)u_2(s)<\alpha+1-\alpha=1$. Similarly, if $u(s)=-1$, it is shown that $u(s)=u_1(s)=u_2(s)=-1$. Thus, we have $u=u_1=u_2$ and this means that $u\in\operatorname{ext} S$. Conversely, suppose that $u\in\operatorname{ext} S$. Assume that $|u(k)|<1$ for some $k\in N$ and define u_1, u_2, and α by $u_1(s)=1$

if $s=k$, $u_1(s)=u(s)$ if $s\neq k$; $u_2(s)=-1$ if $s=k$, and $u_2(s)=u(s)$ if $s\neq k$; and $\alpha=(u(k)+1)/2$. Then $u_1\neq u_2$, $u_1, u_2\in S$, $\alpha\in(0,1)$, and $u=\alpha u_1+(1-\alpha)u_2$. This contradicts the assumption that $u\in \operatorname{ext} S$. q.e.d.

Next, we prepare the following lemma to characterize smooth points of S.

LEMMA 6.3. *Let $u\in l^\infty-\{0\}$. If $F_0(u)\cap c_0^\perp\neq\emptyset$, then $F_0(u)$ contains at least one* p.f.a. 0-1 (or 0-(−1)) *measure; and in this case, $F_0(u)$ is an infinite set.*

PROOF. The first assertion is evident from Theorem 5.6. To get the last assertion we may assume without loss of generality that $\|u\|=1$ and $F_0(u)$ contains a p.f.a. 0-1 measure ϕ. Then $\phi\in F_0(u^+)$ and $\|u^+\|=1$ by Theorem 4.6. Let $\mathscr{F}=\{E\in\Sigma\colon \phi(E)=1\}$ be the non-principal ultrafilter associated with ϕ and let $E_n=\{s\colon 1-1/n\leqq u^+(s)\leqq 1\}$ for $n\geqq 1$. Then, $E_n\in\mathscr{F}$ for all n. For otherwise, $\phi(E_n)=0$ and so we get a contradiction that

$$1=\int_N u^+(s)\phi(ds)\leqq\int_{N-E_n}\left(1-\frac{1}{n}\right)\phi(ds)=1-\frac{1}{n}<1\ .$$

Hence, each E_n is an infinite set, and a sequence $\{s_n\}$ of positive integers can be chosen so that $s_n\in E_n$ and $s_{n+1}>s_n$ for $n\geqq 1$. Let $E_0=\{s_n\colon n\geqq 1\}$, $F_n=E_0\cap E_n=\{s_k\colon k\geqq n\}$ and define

$$F_n^1=\{s_k\colon k\geqq n,\ k \text{ is odd}\}\ ,$$
$$F_n^2=\{s_k\colon k\geqq n,\ k \text{ is even}\}\ .$$

Clearly, $F_n^1\cap F_n^2=\emptyset$ and $F_n=F_n^1\cup F_n^2$ for $n\geqq 1$. Now both of the sequences $\{F_n^1\}$ and $\{F_n^2\}$ are monotone decreasing sequences of nonempty sets, and so they form bases of filters on N. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be any ultrafilters which are finer than the filters generated by $\{F_n^1\}$ and $\{F_n^2\}$, respectively. Then $\mathscr{F}_1\neq\mathscr{F}_2$ and $\mathscr{F}_1$, $\mathscr{F}_2$ are non-principal. Hence, to the $\mathscr{F}_1$ and $\mathscr{F}_2$ there correspond p.f.a. 0-1 measures ϕ_1 and ϕ_2, respectively. We then have $\phi_1, \phi_2\in F_0(u^+)$. In fact, since $F_n^1\subset E_n$ for $n\geqq 1$,

$$1\geqq\int_N u^+(s)\phi_1(ds)=\int_{F_n^1} u^+(s)\phi_1(ds)\geqq\int_{F_n^1}\left(1-\frac{1}{n}\right)\phi_1(ds)=1-\frac{1}{n}$$

for all $n\geqq 1$, which means that $\langle u^+, \phi_1\rangle=1$ and $\phi_1\in F_0(u^+)$. Similarly, $\phi_2\in F_0(u^+)$. Consequently, Theorem 4.6 yields that $\phi_1, \phi_2\in F_0(u)$. Now the last assertion follows from the convexity of $F_0(u)$. q.e.d.

THEOREM 6.4. *Let* $u \in S$. $F_0(u)$ *is a singleton set* iff *there exists a* $k_0 \in N$ *such that* $|u(k_0)|=1$, $|u(s)|<1$ *for* $s \neq k_0$ *and* $\limsup_{s\to\infty}|u(s)|<1$.

PROOF. Assume that $F_0(u)$ is a singleton set $\{\phi\}$. Since ϕ is an extremal point of $F_0(u)$, Theorem 4.6 implies that ϕ is a 0-1 (or 0-(-1)) measure. We may suppose that ϕ is a 0-1 measure. Now from Lemma 6.3 we see that ϕ can not be p.f.a. and hence ϕ is a c.a. 0-1 measure. Thus, Proposition 5.4 yields that a unique k_0 can be found such that $\phi=\delta_{k_0}$, $|u(s)|<1$ for $s\neq k_0$ and $\limsup_{s\to\infty}|u(s)|<1$. The converse is evident from Propositions 5.4 and 5.5. q.e.d.

The above theorem can be rewritten in the following form.

COROLLARY 6.5. *A point u on S is a smooth point*, i.e., $u \in \mathrm{sm}\, S$ iff $\limsup_{s\to\infty}|u(s)|<1$ *and* $|u(\cdot)|$ *attains* 1 *at only one point* $k_0 \in N$.

COROLLARY 6.6. $\mathrm{sm}\, S$ *is open-dense in S.*

PROOF. First we show that $\mathrm{sm}\, S$ is dense in S. Let $u \in S$ and $\varepsilon>0$. Let $E_\varepsilon=\{k\colon |u(k)|>1-\varepsilon\}$; then $E_\varepsilon \neq \emptyset$. Fix any $k_\varepsilon \in E_\varepsilon$ and define u_ε by

$$u_\varepsilon(k)=\begin{cases}\mathrm{sgn}\, u(k) & k=k_\varepsilon\ ,\\ (1-\varepsilon)\,\mathrm{sgn}\, u(k) & k \in E_\varepsilon-\{k_\varepsilon\}\ ,\\ u(k) & k \notin E_\varepsilon\ .\end{cases}$$

Then $\limsup_{k\to\infty}|u_\varepsilon(k)|\leqq 1-\varepsilon<1$ and $|u_\varepsilon(\cdot)|$ attains 1 only at k_ε. $u_\varepsilon \in \mathrm{sm}\, S$ by Corollary 6.5. Also, it is clear from the definition of u_ε that $\|u_\varepsilon-u\|<\varepsilon$. This means that $\mathrm{sm}\, S$ is norm-dense in S. Next, we show that $\mathrm{sm}\, S$ is open in S. Let $u_0 \in \mathrm{sm}\, S$. Then there exists a k_0 such that $|u_0(k_0)|=1$, $|u_0(k)|<1$ for $k\neq k_0$ and $\alpha=\limsup|u_0(k)|<1$. So, there is a k_1 such that $|u_0(k)|<\alpha+(1-\alpha)/2=(1+\alpha)/2<1$ for $k\geqq k_1$. Let

$$\varepsilon=\frac{1}{2}\min\{1-|u_0(k)|\,(k\neq k_0,\ 1\leqq k<k_1),\ (1-\alpha)/2\}(>0)\ ,$$

and let $\|u-u_0\|<\varepsilon$. If $1\leqq k<k_0$, and $k\neq k_0$, then $|u(k)|<|u_0(k)|+\varepsilon\leqq (1+|u_0(k)|)/2<1$; and if $k\geqq k_1$, then $|u(k)|<(1+\alpha)/2+(1-\alpha)/4=(3+\alpha)/4<1$ and $\limsup|u(k)|\leqq(3+\alpha)/4<1$. If in addition $u \in S$, then $|u(k_0)|=1$. This means that $B_\varepsilon(u_0)\cap S\subset \mathrm{sm}\, S$, where $B_\varepsilon(u_0)$ denotes the ε-spherical neighborhood of u_0. q.e.d.

PROPOSITION 6.7. $\mathrm{sm}\, S$ *consists of a countable number of connected components* $C_k^{\pm}=\{u \in \mathrm{sm}\, S\colon \pm u(k)=1\}$, $k\geqq 1$.

PROOF. First we see from Corollary 6.5 that $\text{sm}\, S=\bigcup_{k=1}^{\infty}(C_k^+ \cup C_k^-)$. Each of C_k^+ and C_k^-, $k\geqq 1$, is convex and open by Corollary 6.6. Also, these convex open sets are pairwise disjoint. q.e.d.

§7. Topological properties of F_0.

In this section we discuss topological properties of the duality mapping F_0. We start with the following

LEMMA 7.1. *Let λ and ν be two distinct* 0-1 *measures. Then* $\lambda\wedge\nu=0$ *and* $\|\lambda\pm\nu\|=2$.

PROOF. Let $\mathscr{F}$ and $\mathscr{K}$ be the ultrafilters on N associated through Proposition 2.1 with λ and ν, respectively. Since $\mathscr{F}\neq\mathscr{K}$, there exists an $E_0\in\mathscr{F}-\mathscr{K}$. Hence $E_0^c\in\mathscr{K}-\mathscr{F}$; so $\lambda(E_0)=\nu(E_0^c)=1$. Since $\lambda(T)=\lambda(T\cap E_0)$ and $\nu(T)=\nu(T\cap E_0^c)$ for every $T\in\Sigma$, we infer that $\lambda(T)+\nu(E-T)=\lambda(T\cap E)+\nu(E\cap T^c\cap E_0^c)$ for $T\subset E$. Hence if we take $T=E\cap E_0^c$, then $\lambda(T)+\nu(E-T)=0$. This means that $\lambda\wedge\nu=0$. The last assertion follows from the estimate $2=\lambda(E_0)+\nu(E_0^c)\leqq|(\lambda\pm\nu)(E_0)|+|(\lambda\pm\nu)(E_0^c)|\leqq(\lambda\pm\nu, N)=\|\lambda\pm\nu\|<\|\lambda\|+\|\nu\|=2$. q.e.d.

LEMMA 7.2. *Let $\{\lambda_n\}$ be a sequence of* 0-1 (*resp.* 0-(−1)) *measures, and let λ be a weak-star cluster point of the sequence $\{\lambda_n: n\uparrow\infty\}$. Then λ is also a* 0-1 (*resp.* 0-(−1)) *measure.*

PROOF. For every $E\in\Sigma$, $\varepsilon>0$ and $p\in N$, there exists an n such that $n\geqq p$ and $|\lambda_n(E)-\lambda(E)|<\varepsilon$. Hence, if $0<\lambda(E)<1$ and $\varepsilon=\min\{\lambda(E), 1-\lambda(E)\}(>0)$, then $|\lambda_n(E)-\lambda(E)|<\varepsilon<1$. But, $\lambda_n(E)$ is either 1 or 0, we get a contradiction. Thus, $\lambda(E)$ is either 1 or 0. q.e.d.

Lemma 7.2 states that a weak-star cluster point of a net consisting of extremal points of S^* is always an extremal point of S^*. Now as mentioned in the introduction, F_0 is weakly-star demi-closed in the sense that if $v_n\in S$, $\|v_n-v\|\to 0$, $\lambda_n\in F_0(v_n)$ and if λ is a weak-star cluster point of the net $\{\lambda_n: n\uparrow\infty\}$, then $v\in S$ and $\lambda\in F_0(v)$. The following result gives another aspect of the weak-star demi-closedness of F_0.

PROPOSITION 7.3. *Let $\{v_n\}$ be a sequence contained in S such that $\|v_n-v\|\to 0$. Let $\lambda_n\in \text{ext}\, F_0(v_n)$, $n\geqq 1$, and let λ be any weak-star cluster point of the sequence $\{\lambda_n: n\uparrow\infty\}$. Then $\lambda\in\text{ext}\, F_0(v)$. If the sequence $\{\lambda_n\}$ contains infinitely many* 0-1 (*resp.* 0-(−1)) *measures, then* $\text{ext}\, F_0(v)$ *contains at least one* 0-1 (*resp.* 0-(−1)) *measure. If the $\{\lambda_n\}$ consists of distinct elements, then it constains no strongly convergent subsequences.*

PROOF. The first two assertions follow from the weak-star demi-closedness of F_0 and Lemma 7.2; and the last assertion is evident from Lemma 7.1. q.e.d.

It is well-known ([5], p. 22) that any single-valued selection of F_0 is norm to weak-star continuous from S into S^* at every smooth point of S. But we have the following stronger result which is a direct consequence of Theorem 6.4 and Proposition 6.7.

PROPOSITION 7.4. *Let C_k^+ and C_k^-, $k \geqq 1$, be the connected components of* sm S *mentioned as in Proposition* 6.7. *Then, F_0 is single-valued and is constant on each of C_k^+ and C_k^- in such a way that $F_0(u)=\{\delta_k\}$ for $u \in C_k^+$ and $F_0(u)=\{-\delta_k\}$ for $u \in C_k^-$, $k \geqq 1$. Therefore, F_0 restricted on* sm S *is norm-to-norm continuous from S to S^*.*

Corollary 6.5 states that F_0 is multi-valued on $S-\text{sm}\,S$. We then show with the aid of Corollary 6.6 that the values of F_0 on the set $S-\text{sm}\,S$ can be viewed as boundary values of the restriction of F_0 on the open set sm S.

THEOREM 7.5. *Let $v \in S-\text{sm}\,S$.* (1) *If $\lambda \in \text{ext}\,F_0(v) \cap l^1$, then there exists a sequence $\{v_n\}$ in* sm S *such that $\|v_n - v\| \to 0$ and $F_0(v_n)=\{\lambda\}$.* (2) *If $\lambda \in \text{ext}\,F_0(v) \cap c_0^\perp$, then there exists a sequence $\{v_n\}$ in* sm S *such that for every $\varepsilon > 0$, there is a subsequence $\{v_{\varepsilon,n}\}$ of $\{v_n\}$ with the following properties:*

(a) *$\|v_{\varepsilon,n} - v\| \leqq \varepsilon$ for all n; and* (b) *λ is a weak-star cluster point of the sequence $\{\lambda_{\varepsilon,n}: n \uparrow \infty\}$, where $\lambda_{\varepsilon,n}=F_0(v_{\varepsilon,n})$ for $n \geqq 1$.*

PROOF. (1): Let $\lambda \in \text{ext}\,F_0(v) \cap l^1$. Then λ is a signed point mass, so that we may assume without loss of generality that $\lambda=\delta_{s_0}$ for some $s_0 \in N$. Note that in this case, $v(s_0)=\langle v, \lambda \rangle = 1$. Let $\{\varepsilon_n\}$ be any null sequence contained in $(0, 1/2]$, and let $\{v_n\}$ be a sequence in S such $v_n(s_0)=1$, $|v_n(s)| \leqq 1-\varepsilon_n$ for $s \neq s_0$ and $|v_n(s)-v(s)| \leqq \varepsilon_n$ for all s. (We choose for instance $\{v_n\}$ defined by setting $v_n(s)=v(s)-\varepsilon_n \operatorname{sgn} v(s)$ for $s \neq s_0$ and $v_n(s_0)=1$.) Then, $\limsup_{s \to \infty} |v_n(s)| \leqq 1-\varepsilon_n$, $v_n \in \text{sm}\,S$ and $F_0(v_n)=\delta_{s_0}$ for all n. Therefore, $\|v_n - v\| \to 0$ as $n \to \infty$ and $\{\lambda\}=\{\delta_{s_0}\}=F_0(v_n)$ for $n \geqq 1$.

(2): Let $\lambda \in \text{ext}\,F_0(v) \cap c_0^\perp$. We shall give the proof of Assertion (2) under the assumption that $\lambda \geqq 0$, since the proof for the negative case is similar. Since $\langle v, \lambda \rangle = 1$ and λ is a p.f.a. 0-1 measure, each of the sets $E_\varepsilon = v^{-1}(U_\varepsilon(1))$, $\varepsilon > 0$, has λ-measure 1 by Proposition 2.5, where $U_\varepsilon(1)$ denotes the ε-spherical neighborhoood in R of 1. Take any null sequence $\{\varepsilon_p\}$ contained in $(0, 1/2]$ and put $\hat{E}_1 = E_{\varepsilon_1}$ and $\hat{E}_p = E_{\varepsilon_p} -$

$\{1, 2, \cdots, \min \hat{E}_{p-1}\}$ for $p \geqq 2$. Since E_i's are infinite sets, $\{\hat{E}_p\}$ forms a strictly monotone decreasing sequence $\{\hat{E}_p\}$ in Σ such that $\lambda(\hat{E}_p)=1$ for $p \geqq 1$ and $\bigcap_{p\geqq 1} \hat{E}_p=\varnothing$. We then define a family $\{H_p\}$ of pairwise disjoint elements of Σ by setting $H_p=\hat{E}_p-\hat{E}_{p+1}$ for $p \geqq 1$. Note that $H_p \neq \varnothing$, $\lambda(H_p)=0$, and $\hat{E}_p=\bigcup_{i\geqq p} H_i$ for $p \geqq 1$. Let $\{s_n\}$ be the increasing sequence of natural numbers such that $\hat{E}_1=\{s_n: n \geqq 1\}$; and for s_n with $s_n \in H_p$, choose an element $v_n \in l^\infty$ so that $v_n(s_n)=1$, $|v_n(s)|<1-\varepsilon_p$ for $s \neq s_n$ and $|v_n(s)-v(s)| \leqq \varepsilon_p$ for all s. (For instance, we can take v_n satisfying $v_n(s_n)=1$ and $v_n(s)=v(s)-\varepsilon_p \operatorname{sgn} v(s)$ for $s \neq s_n$.) Then $v_n \in \operatorname{sm} S$, $F_0(v_n)=\delta_{s_n}$ for $n \geqq 1$, and $\|v_n-v\| \leqq \varepsilon_p$ for $s_n \in \hat{E}_p$ and $p \geqq 1$. We now demonstrate that this sequence $\{v_n\}$ is the desired sequence. Let $\varepsilon>0$ and choose an ε_p such that $\varepsilon_p<\varepsilon$. Then, $\{v_{s_n}: s_n \in \hat{E}_p\}$ is viewed as a subsequence of $\{v_n\}$ by enumerating the suffices of the elements in order; we denote this subsequence by $\{v_{\varepsilon,k}\}$. First, it is clear that $\|v_{\varepsilon,k}-v\|<\varepsilon$ for all $k \geqq 1$. Next, for $k \geqq 1$, let $\lambda_{\varepsilon,k}$ denote the element of the singleton set $F_0(v_{\varepsilon,k})$; then λ becomes a weak-star cluster point of the net $\{\lambda_{\varepsilon,k}: k \uparrow \infty\}$. To show this, let $\mathscr{F}$ be the ultrafilter on N associated with λ, $u \in l^\infty$, and let $\bar{\mathscr{F}}_u$ be the non-principal ultrafilter on the compact set $\overline{u(N)}$ specified as in Proposition 2.4; hence the value $\alpha \equiv \langle u, \lambda \rangle$ is given as the limit of $\bar{\mathscr{F}}_u$. Now recalling the proof of Proposition 2.4, we set $U_i=\{\xi \in \boldsymbol{R}: |\xi-\alpha|<1/i\}$, $S_i=U_i \cap u(N)$, and $E_i'=u^{-1}(S_i)$ for $i \geqq p$. Then $E_i' \cap \hat{E}_i \in \mathscr{F}$ for $i \geqq p$ and each $E_i' \cap \hat{E}_i$ is an infinite set, so that there is a sequence $\{\hat{s}_i\}$ such that $\hat{s}_i \in E_i \cap \hat{E}_i$, $\hat{s}_i > \hat{s}_{i-1}$ for $i \geqq p+1$, and $u(\hat{s}_i) \to \alpha$ as $i \to \infty$. Set $E_i=E_i' \cap \hat{E}_i-\{1, 2, \cdots, \hat{s}_i-1\}$ for $i \geqq p$ (hence $\hat{s}_i=\min E_i$ and $\lambda(E_i)=1$ for $i \geqq p$) and define a sequence $\{u^i\}$ of simple functions on N by $u^i=u(s_i)\chi_{E_i}$ $(i \geqq p)$. Moreover, put $M_i^\delta=\{s \in N: |u^i(s)-u(s)|>\delta\}$ for $\delta>0$ and $i \geqq p$. Then $\lambda(E_i^c)=0$ and $|u^i(s)-u(s)|<2/i$ for $s \in E_i$ by the same reason as in the proof of Proposition 2.4. Therefore, if $2/i<\delta$ then $v(\lambda, E_i^c)=v(\lambda, M_i^\delta \cap F_i^c)+v(\lambda, M_i^\delta \cap E_i)=0$. That is, u^i converges to u in λ-measure and $\langle u, \lambda \rangle=\lim u(\hat{s}_i)=\lim \langle u, \delta_{\hat{s}_i} \rangle$. Since u was arbitrary in l^∞ and each $\hat{s}_i$ belongs to the set $\hat{E}_p$, it follows that λ is a weak-star cluster point of the net $\{\lambda_{\varepsilon,n}: n \uparrow \infty\}$. q.e.d.

REMARK. Assertion (2) of the above theorem states that v is only a strong cluster point of the net $\{v_n: n \uparrow \infty\}$. However, it is desirable to choose a sequence $\{v_n\}$ in $\operatorname{sm} S$ so that v is the limit of $\{v_n\}$. Although the authors do not know at this moment whether or not this is possible in general, they are able to give a necessary and sufficient condition for given $v \in l^\infty$ and $\lambda \in \operatorname{ext} F_0(v) \cap c_0^\perp$ to admit such a sequence $\{v_n\}$.

PROPOSITION 7.6. *Let* $v \in S-\operatorname{sm} S$, $\lambda \in \operatorname{ext} F_0(v) \cap c_0^\perp$, *and let* $\lambda \geqq 0$. *Then the following are equivalent*:

(1) *There exists a sequence* $\{v_n\}$ *in* sm S *such that* $\|v_n - v\| \to 0$ *as* $n \to \infty$ *and* λ *is a weak-star cluster point of the sequence* $\{\lambda_n : n \uparrow \infty\}$, *where* $\lambda_n = F_0(v_n)$ *and* $\lambda_n \geqq 0$ *for* $n \geqq 1$.

(2) *There exists a sequence* $\{s_n\}$ *in* N *such that* $v(s_n) \to 1$ *as* $n \to \infty$ *and the set* $\{s_n : n \geqq 1\}$ *has* λ*-measure* 1.

PROOF. (2)$\Rightarrow$(1): Let $E = \{s_n : n \geqq 1\}$; then $\lambda(E) = 1$. So, if we replace E_{ε_p} $(p \geqq 1)$ in the proof of Assertion (2) of Theorem 7.5 by $E \cap E_{\varepsilon_p}$ $(p \geqq 1)$, then each of the sets H_p $(p \geqq 1)$ becomes a finite set, and consequently, we can conclude that $\|v_n - v\| \to 0$ as $n \to \infty$ and λ is then a weak-star cluster point of the net $\{\lambda_n : n \uparrow \infty\}$.

(1)$\Rightarrow$(2): Given n, let s_n be a point in N such that $v_n(s_n) = 1$. Set $E_0 = \{s_n : n \geqq 1\}$. Then $\lambda_n (= F_0(v_n))$ is regarded as point mass δ_{s_n}. First, we have that $v(s_n) \to 1$ as $n \to \infty$ since $|v(s_n) - 1| = |v(s_n) - v_n(s_n)| \leqq \|v - v_n\| \to 0$. Next we show that $\lambda(E_0) = 1$. Let $\mathscr{F}$ be the non-principal ultrafilter on N associated with λ, and E any element of $\mathscr{F}$. Then $\langle \chi_E, \lambda \rangle = \lambda(E) = 1$; and for every $\varepsilon \in (0, 1/2)$ and n, there exists an m such that $m \geqq n$ and $|\langle \chi_E, \delta_{s_n} \rangle - 1| = |\langle \chi_E, \lambda_m - \lambda \rangle| < \varepsilon < 1/2$. This means that $s_m \in \chi_E^{-1}(U_\varepsilon(1)) \cap E_0 \subset E \cap E_0 \neq \varnothing$. Since E was arbitrary and $\mathscr{F}$ is an ultrafilter on N, it $E \cap E_0$, i.e., follows that $E_0 \in \mathscr{F}$ and $\lambda(E_0) = 1$. q.e.d.

Particular examples will be useful to illustrate the above result. First, let $v = \chi_N$ and $\{v_n\}$ a sequence in S such that $\|v_n - v\| \to 0$ and $F_0(v_n) = \delta_n$ for $n \geqq 1$ (e.g., we choose $\{v_n\}$ defined as $v_n(k) = 1$ for $k = n$ and $v_n(k) = 1 - 1/n$ for $k \neq n$). Observe that $v(k) \to 1$ as $k \to \infty$ and Condition (2) of Theorem 7.6 is satisfied. Then ext $F_0(v)$ is the set of all 0-1 measures on Σ and ext $F_0(v) \cap c_0^\perp$ coinsides with the set of all weak-star cluster point of the sequence $\{\delta_n : n \uparrow \infty\}$. Second, if $v \in S$ and $\{v(k)\}$ is a strictly monotone increasing, non-negative sequence convering to 1 (hence Theorem 7.6 (2) holds), then a sequence $\{v_n\}$ can be found in sm S so that ext $F_0(v)$ $(\subset c_0^\perp$ by Proposition 5.5) is the set of all weak-star cluster points of the sequence $\{\delta_n : n \uparrow \infty\}$, where $\lambda_n = F_0(v_n)$, $n \geqq 1$. In fact, let H_p, $p \geqq 1$, be specified as in the proof of Assertion (2) of Theorem 7.5. Then, H_p's are all finite sets and $N - \hat{E}_p (= N - \bigcup_{i \geq p} H_i)$ is also a finite set. Hence, if $\{v_n\}$ is determined just in the same way as in the proof of Theorem 7.5 (2), then $\|v_n - v\| \to 0$ and every element of ext $F_0(v)$ is a weak-star cluster point of the sequence $\{\lambda_n : n \uparrow \infty\}$.

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