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Some Remarks on Subvarieties of Hopf Manifolds

Masahide KATO

Sophia University

Introduction

A holomorphic automorphism g of a complex space $\mathfrak{X}$ is called a *contraction* to a point $O\in\mathfrak{X}$ if g satisfies the following three conditions:

(i) $g(O)=O$,

(ii) $\lim_{\nu\to+\infty} g^{\nu}(x)=O$ *for any point* $x\in\mathfrak{X}$,

(iii) *for any small neighborhood* U *of* O *in* $\mathfrak{X}$, *there exists an integer* ν_0 *such that* $g^{\nu}(U)\subset U$ *for all* $\nu\geqq\nu_0$,

where g^{ν} is the ν-times composite of g. By [2]*, the complex space $\mathfrak{X}$ which admits a contracting automorphism is holomorphically isomorphic to an algebraic subset of C^N for some N. We identify $\mathfrak{X}$ to the algebraic subset of C^N. Then there exists a contracting automorphism $\tilde{g}$ of C^N to the origin O such that $\tilde{g}|_{\mathfrak{X}}=g$ ([2], [3]). Obviously the action of $\tilde{g}$ on $C^N-\{O\}$ is free and properly discontinuous. Hence the quotient space $H=C^N-\{O\}/\langle\tilde{g}\rangle$ is a compact complex manifold which is called a *primary Hopf manifold*. Sometimes we indicate by H^N an N-dimensional primary Hopf manifold. The compact complex space $\mathfrak{X}-\{O\}/\langle g\rangle$ is clearly an analytic subset of a primary Hopf manifold. A compact complex manifold X of dimension n $(n\geqq2)$ is called a *Hopf manifold* if its universal covering is holomorphically isomorphic to $C^n-\{O\}$ (Kodaira [4]).

The purpose of this paper is to show several properties of subvarieties of Hopf manifolds.

§1. Hopf manifolds.

The following proposition shows that it is sufficient to consider only subvarieties of primary Hopf manifolds.

PROPOSITION 1. *Any Hopf manifold is a submanifold of a (higher dimensional) primary Hopf manifold.*

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* In [2], the condition (iii) is forgotten.

PROOF. Let X be any Hopf manifold. Then, by definition, there exists a group G of holomorphic transformations of $C^n-\{O\}$ such that $X=C^n-\{O\}/G$ $(n=\dim X\geqq 2)$. It follows from a theorem of Hartogs that any element of G can be extended to a holomorphic transformation of C^n. Hence we may assume that each element of G is a holomorphic transformation of C^n which fixes the origin $O\in C^n$. By the same argument as in [4] pp. 694-695, G contains a contraction.

For each element $x\in G$, we denote by $dx(O)$ the jacobian matrix at the origin $O\in C^n$.

LEMMA 1. *An element $x\in G$ is a contraction if and only if* $|\det(dx(O))|<1$.

PROOF. If $x\in G$ is a contraction, then any eigenvalue α of $dx(O)$ satisfies $|\alpha|<1$ (see [3] for the detail). Hence $|\det(dx(O))|<1$. Conversely, let x be an element of G satisfying $|\det(dx(O))|<1$. Let g be a contraction contained in G. Since $C^n-\{O\}/\langle g\rangle$ is compact, the index of the infinite cyclic subgroup $\{g\}$ generated by g is finite in G. Now assume that x is not a contraction. Then x^n is not a contraction for any integers n. Hence $x^n\neq g^m$ for any pair of integers n and m except $n=m=0$. This implies that $\{x\}\cap\{g\}=\{1\}$. This contradicts the fact that $\{g\}$ is of the finite index in G. Q.E.D.

Let U be a subgroup of G defined by

$$U=\{x\in G:\ |\det(dx(O))|=1\}\ .$$

Obviously U is a normal subgroup of G.

LEMMA 2. *There exists an infinite cyclic subgroup Z of G such that G is the semi-direct product of Z and U; $G=Z\cdot U$.*

PROOF. Define a group homomorphism $l: G\rightarrow \boldsymbol{R}$ by

$$l(x)=-\log|\det(dx(O))|\quad (x\in G)\ .$$

Let $g_1\in G$ be a contraction. Then the index d of the infinite cyclic group $\{l(g_1)\}$ generated by $l(g_1)$ in $l(G)$ is finite. Hence $d^{-1}l(g_1)$ is a minimum positive element of $l(G)$. Let g be an element of G such that $l(g)=d^{-1}l(g_1)$. We put $Z=\{g\}$. Then it is clear that $G=Z\cdot U$. Q.E.D.

LEMMA 3. *U is a finite normal subgroup of G.*

PROOF. Clear by Lemma 2.

Now continue the proof of Proposition 1. It is easy to see that any holomorphic transformation u of C^n which fixes the origin is linear, if u is of the finite order. Hence U is a finite subgroup of $GL(n, C)$. Hence, by H. Cartan [1], $\mathfrak{X}=C^n/U$ is a complex space with unique possible singularity at $\bar{O}$, where $\bar{O}$ is the corresponding point to the origin $O \in C^n$. The generator g of Z induces a contracting automorphism $\bar{g}$ of $\mathfrak{X}$ such that $\bar{g}(\bar{O})=\bar{O}$. Hence $X=\mathfrak{X}-\{\bar{O}\}/\langle\bar{g}\rangle$ is a submanifold of a primary Hopf manifold as we have seen in the introduction. Q.E.D.

§ 2. Line bundles defined by divisors.

Let M be an arbitrary compact complex manifold and N be a divisor of M. The line bundle $[N]$ defined by N is an element of $H^1(M, O^*)$. There is a natural homomorphism $i: H^1(M, C^*) \to H^1(M, O^*)$ induced by the natural injection $C^* \to O^*$. If $[N]$ is in the image of i, then $[N]$ is called a *flat* line bundle. In other words, $[N]$ is locally flat if and only if its transition functions can be written by constant functions.

Now let $\tilde{g}$ be any contracting automorphism of C^N which fixes the origin $O \in C^N$. Then, by L. Reich ([6], [7]), we can choose a system of coordinates of C^N such that $\tilde{g}$ can be written in the following form:

$$\begin{aligned}
&z_1' = \alpha_1 z_1 \\
&z_2' = z_1 + \alpha_2 z_2 \\
&\vdots \\
&z_{r_1}' = z_{r_1-1} + \alpha_{r_1} z_{r_1} \\
&z_{r_1+1}' = \alpha_{r_1+1} z_{r_1+1} + P_{r_1+1}(z_1, \cdots, z_{r_1}) \\
(1) \qquad &\vdots \\
&z_{r_1+r_2}' = z_{r_1+r_2-1} + \alpha_{r_1+r_2} z_{r_1+r_2} + P_{r_1+r_2}(z_1, \cdots, z_{r_1}) \\
&z_{r_1+r_2+1}' = \alpha_{r_1+r_2+1} z_{r_1+r_2+1} + P_{r_1+r_2+1}(z_1, \cdots, z_{r_1+r_2}) \\
&\vdots \\
&z_N' = z_{N-1} + \alpha_N z_N + P_N(z_1, \cdots, z_{r_1+r_2+\cdots+r_{\mu-1}}) ,
\end{aligned}$$

where $1 > |\alpha_1| \geqq \cdots \geqq |\alpha_N| > 0$, μ is the number of Jordan blocks of the linear part, P_j $(r_1+\cdots+r_s < j \leqq r_1+\cdots+r_{s+1})$ are finite sums of monomials $z_1^{m_1} \cdots z_{r_s}^{m_{r_s}}$ which satisfy

$$(2) \qquad \begin{aligned}
&\alpha_j = \alpha_1^{m_1} \cdots \alpha_{r_s}^{m_{r_s}} , \\
&m_1 + \cdots + m_{r_s} \geqq 2 \quad (\text{all} \quad m_l > 0) .
\end{aligned}$$

Let $\tilde{\omega}: C^N - \{O\} \to H = C^N - \{O\}/\langle\tilde{g}\rangle$ be the covering projection. For any analytic subset X in H, the set $\tilde{\omega}^{-1}(X)$ is an analytic subset in $C^N - \{O\}$.

If $\dim X \geqq 1$, then by a theorem of Remmert-Stein, $\mathfrak{X}=\tilde{\omega}^{-1}(X)\cup\{O\}$ is an analytic subset of C^N. In what follows, we indicate by the script letters the analytic subsets in C^N corresponding in the above manner to the analytic subsets of H written by the Roman letters. An analytic subset is called a *variety* if it is irreducible.

Assume that X is an analytic subvariety in H of $\dim X \geqq 2$ and that D is an analytic subvariety of codimension 1 in X. It is clear that $\mathfrak{X}$ and $\mathscr{D}$ are both $\tilde{g}$-invariant in C^N, i.e., $g(\mathfrak{X})=\mathfrak{X}$ and $g(\mathscr{D})=\mathscr{D}$.

LEMMA 4 ([2]). *There exists a non-constant holomorphic function f on $\mathfrak{X}$ such that $g^*f=\alpha f$ for some constant α $(0<|\alpha|<1)$ and that $f|_{\mathscr{D}}=0$.*

REMARK 1. In [2], the word "variety" is used as "analytic set".

Let X be a non-singular manifold. Consider f of Lemma 4 as a multiplicative multi-valued holomorphic function on X (K. Kodaira [4] p. 701). The divisor $D_1=(f)$ is well-defined. The equation $g^*f=\alpha f$ implies that the line bundle $[D_1]$ is flat of which the transition functions are some powers of α. We summarize these facts as follows.

THEOREM 1. *Let X be a submanifold of H and D an effective divisor on X. Assume that* $\dim X \geqq 2$. *Then then exists an effective divisor E on X such that the line bundle $[D+E]$ is flat of which the transition functions are some powers of a certain constant $\alpha \in C^*$* $(0<|\alpha|<1)$.

REMARK 2. The following example shows that there are cases such that the "additional" effective divisor E of Theorem 1 is indispensable.

Let (x_0, x_1, x_2, x_3) be a standard system of coordinates of C^4. Fix a complex number α such that $0<|\alpha|<1$. Let $\tilde{g}$ be a contracting holomorphic automorphism of C^4 defined by

$$\tilde{g}: (x_0, x_1, x_2, x_3) \longmapsto (\alpha x_0, \alpha x_1, \alpha x_2, \alpha x_3) .$$

Define $\tilde{g}$-invariant subvarieties of C^4 by

$$\mathfrak{X}:\ x_0x_1=x_2x_3$$

and

$$\mathscr{A}:\ x_3=0 .$$

Denote the intersection $\mathfrak{X}\cap\mathscr{A}$ by $\mathscr{S}$. Then $\mathscr{S}=\{x_0=x_3=0\}\cup\{x_1=x_3=0\}$. We put

$$\mathscr{S}_1=\{x_0=x_3=0\}$$

and

$$\mathscr{S}_2=\{x_1=x_3=0\}\ .$$

Then $S=\mathscr{S}-\{O\}/\langle\tilde{g}\rangle$, $S_1=\mathscr{S}_1-\{O\}/\langle\tilde{g}\rangle$ and $S_2=\mathscr{S}_2-\{O\}/\langle\tilde{g}\rangle$ are subvarieties of a compact complex manifold $X=\mathfrak{X}-\{O\}/\langle\tilde{g}\rangle$. It is clear that $[S_1+S_2]=[S]$ is flat. We shall prove that either $[S_1]$ or $[S_2]$ is not flat. Assume that both $[S_1]$ and $[S_2]$ are flat. Let $\mathfrak{U}=\{U_\lambda\}$ be a sufficiently fine finite open covering of X. We represent $[S_1]$ as a 1-cocycle $\{c_{1\lambda\mu}\}\in Z^1(\mathfrak{U}, \boldsymbol{C}^*)$. Since $\dim H^0(X, O[S_1])>0$, there exists a non-zero section φ_1 which vanishes exactly on S_1. Let $\varphi_{1\lambda}=c_{1\lambda\mu}\varphi_{1\mu}$ on $U_\lambda\cap U_\mu$. As we can easily see,

$$\eta_1=\frac{d\varphi_{1\lambda}}{\varphi_{1\lambda}}=\frac{d\varphi_{1\mu}}{\varphi_{1\mu}}=\cdots$$

is a meromorphic 1-form on X. Since $\mathfrak{X}-\{O\}$ is simply connected,

$$f_1(x)=\exp\int^x\eta_1$$

is a holomorphic function on $\mathfrak{X}-\{O\}$ such that $\tilde{g}*f_1=\beta_1 f_1$ ($\beta_1\in\boldsymbol{C}^*$, $0<|\beta_1|<1$) which vanishes exactly on $\mathscr{S}_1-\{O\}$ with multiplicity 1. Since $\mathfrak{X}$ is normal at O, f_1 uniquely extends to a holomorphic function on $\mathfrak{X}$. Comparing the initial terms of $\tilde{g}*f_1$ and f_1 at O, we see that β_1 is some power of α, i.e., $\beta_1=\alpha^{m_1}$ ($m_1\geqq 1$). By the same manner, we construct f_2 for a non-zero section $\varphi_2\in H^0(X, O[S_2])$ such that $\tilde{g}*f_2=\alpha^{m_2}f_2(m_2\geqq 1)$. Let f_0 be a restriction of a holomorphic function x_3 to $\mathfrak{X}-\{O\}$. Then $\tilde{g}*f_0=\alpha f_0$. It is easy to see that $f=f_1\cdot f_2\cdot f_0^{-1}$ is a non-vanishing holomorphic function on $\mathfrak{X}-\{O\}$ such that $\tilde{g}*f=\alpha^{m_1+m_2-1}f$ ($m_1+m_2-1\geqq 1$). But this does not occur if $\dim X>1$. In fact, using the non-vanishing holomorphic function f, we get the following commutative diagram:

$$\begin{array}{ccc}\mathfrak{X}-\{O\} & \xrightarrow{\ \tilde{g}\ } & \mathfrak{X}-\{O\}\\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f}\\ \boldsymbol{C}^* & \xrightarrow{\times\alpha^{m_1+m_2-1}} & \boldsymbol{C}^*\ .\end{array}$$

Then f induces a proper surjective holomorphic mapping $\bar{f}\colon X\to\boldsymbol{C}^*/\langle\alpha^{m_1+m_2-1}\rangle$. For any point $\tau\in\boldsymbol{C}^*/\langle\alpha^{m_1+m_2-1}\rangle$, $\bar{f}^{-1}(\tau)=X_\tau$ is a compact subvariety in X. Hence $\tilde{\omega}^{-1}(X_\tau)$ is a complex analytic subset in $\boldsymbol{C}^4-\{O\}$ whose connected components are compact, where $\tilde{\omega}$ is the covering map $\boldsymbol{C}^4-\{O\}\to\boldsymbol{C}^4-\{O\}/\langle\tilde{g}\rangle$. This implies that $\tilde{\omega}^{-1}(X_\tau)$ is a countable union of points. Hence $\dim X_\tau=0$. This contradicts $\dim X>1$. This implies that either

$[S_1]$ or $[S_2]$ is not flat.

REMARK 3. If $\dim X=2$, then $[D]$ is always flat ([3]).

§ 3. Some properties of subvarieties.

By Lemma 5 in [2], we have easily

PROPOSITION 2. *Let Y_1 and Y_2 be subvarieties of a (primary) Hopf manifold H such that $Y_1 \subset Y_2$ and $0 < n_1 = \dim Y_1 < n_2 = \dim Y_2$. Then there exists a sequence of subvarieties $W_0, W_1, \cdots, W_p$ $(p=n_2-n_1)$ in H with following properties:*

(i) $W_0 = Y_1$, $W_p = Y_2$,

(ii) $W_i \subset W_{i+1}$ $(i=0, \cdots, p-1)$, $\dim W_i + 1 = \dim W_{i+1}$.

PROPOSITION 3. *Let $H^N = C^N - \{O\}/\langle \tilde{g} \rangle$ be a primary Hopf manifold. Then*

(a) *any positive dimensional subvariety in H^N contains a curve,*

(b) *any irreducible curve in H^N is non-singular elliptic,*

(c) *for any elliptic curve C in H^N, there exist an eigenvalue α of $\tilde{g}$, a constant β and certain positive integers m, n with $\alpha^m = \beta^n$ such that C is isomorphic to $C^*/\langle \beta \rangle$.*

PROOF. (a) Let Y be a n-dimensional subvariety in $H^N (n \geqq 1)$. For any integer k $(1 \leqq k \leqq N)$, the $(N-k)$-dimensional subspace C^{N-k} defined by $z_1 = \cdots = z_k = 0$ is $\tilde{g}$-invariant. There exists an integer k such that $\dim (C^{N-(k-1)} \cap \mathscr{Y}) = 1$. Then $\tilde{\omega}((C^{N-(k-1)} \cap \mathscr{Y}) - \{O\})$ is a 1-dimensional analytic subset of Y.

(b) Let C be any irreducible curve in H^N. Then $\mathscr{C}$ is a 1-dimensional analytic subset of C^N. Let $\mathscr{C}_0$ be one of the irreducible components of $\mathscr{C}$. Then, for some positive integer n_0, g^{n_0} acts on $\mathscr{C}_0$ as a contracting automorphism of $\mathscr{C}_0$. Let $\lambda: \mathscr{C}_0^* \to \mathscr{C}_0$ be the normalization of $\mathscr{C}_0$. Then g^{n_0} naturally induces a contracting automorphism of $\mathscr{C}_0^*$. By [2], $\mathscr{C}_0^* \cong C$. It is clear that $\lambda^{-1}(O)$ consists of one point O^*. Hence $\mathscr{C}_0 - \{O\} \cong \mathscr{C}_0^* - \{O^*\} \cong C^*$. Thus C^* is an infinite cyclic unramified covering of C. Therefore C is a non-singular elliptic curve.

(c) Consider the $\tilde{g}$-invariant subspaces C^{N-k} defind in (a). For $k = 0$, C^{N-k} is the total space. Fix the integer k $(0 \leqq k \leqq N-1)$ such that $\mathscr{C} \subset C^{N-k}$ and $\mathscr{C} \not\subset C^{N-k-1}$. If $\mathscr{C} \cap C^{N-k-1}$ contains a point p other than O, then $\mathscr{C} \cap C^{N-k-1}$ contains an infinite sequence of points $\tilde{g}^n(p) \to O (n=1, 2, \cdots)$. Hence one of the irreducible components of $\mathscr{C}$ is contained in C^{N-k-1}. Since $\tilde{g}$ is transitive over all the irreducible components of $\mathscr{C}$, this implies that $\mathscr{C} \subset C^{N-k-1}$, contradiction. Therefore $\mathscr{C} \cap C^{N-k-1} = \{0\}$. Hence $f =$

$z_{k+1|C^{N-k}}$, the restriction of z_{k+1} to C^{N-k}, vanishes nowhere on $\mathscr{C}-\{O\}$. Moreover f satisfies the equation $g^*f=\alpha_{k+1}f$. Hence we get the following commutative diagram:

$$\begin{array}{ccc} \mathscr{C}-\{O\} & \xrightarrow{g} & \mathscr{C}-\{O\} \\ \downarrow f & & \downarrow f \\ C^* & \xrightarrow{\alpha_{k+1}} & C^* . \end{array}$$

This induces a covering $\bar{f}: C \to C^*/\langle\alpha_{k+1}\rangle$. Since both C and $C^*/\langle\alpha_{k+1}\rangle$ are non-singular elliptic curves, $\bar{f}$ has no branch points by the Hurwitz's formula. Hence there exist $\beta \in C^*$ and positive integers m, n such that $C \cong C^*/\langle\beta\rangle$ and $\alpha_{k+1}^m=\beta^n$. Q.E.D.

REMARK 4. By Propositions 2 and 3 (a), it follows that any n-dimensional subvariety of a Hopf manifold contains subvarieties of arbitrary dimensions less than n.

§ 4. Subvarieties of algebraic dimension 0.

In general, let M be a compact complex analytic subvariety. Then the field $\mathscr{M}(M)$ of all meromorphic functions on M has the finite transendental degree $a(M)$ over C. We call $a(M)$ the *algebraic dimension of* M. It is well-known that $a(M) \leqq \dim M$. The number $\dim M - a(M)$ is called the *algebraic codimension* of M.

THEOREM 2. *Let Y be a subvariety of dimension k in N-dimensional primary Hopf manifold H^N. Assume that $a(Y)=0$. Then the number of $(k-1)$-dimensional subvarieties in Y is at most N.*

Before proving the theorem, we shall make some preparations.

Let $\alpha_1, \cdots, \alpha_N$ be the eigenvalues of $\tilde{g}$ ((1)). Put $\theta_j=\log\alpha_j$, $(0 \leqq \arg\theta_j < 2\pi$, $j=1, 2, \cdots, N)$. Let K be a vector space over the field of rational numbers Q generated by the elements $2\pi\sqrt{-1}$, $\theta_1, \cdots, \theta_N$. Choose a basis $\tau_0, \tau_1, \cdots, \tau_\lambda$ of K so that following conditions may be satisfied:

(i) $\tau_0=2\pi\sqrt{-1}$,

(ii) $\{\tau_1, \cdots, \tau_\lambda\}$ is a subset of $\{\theta_1, \cdots, \theta_N\}$,

(iii) for any $\nu \geqq 1$, τ_ν is linearly independent to $Q\tau_0+Q\tau_1+\cdots+Q\tau_{\nu-1}$,

(iv) if $\tau_\nu=\theta_j$, $\tau_\mu=\theta_k$ and $\nu<\mu$, then $j<k$.

It is easy to check that we can choose such a basis. We denote by α_{i_ν} the element of $\{\alpha_1, \cdots, \alpha_N\}$ corresponding to τ_ν. Note that $\tau_\nu=\theta_{i_\nu}=\log\alpha_{i_\nu}$ $(\nu=1, 2, \cdots, \lambda)$. If the equation

$$\alpha_{i_\nu}=\alpha_1^{a_1}\cdots\alpha_l^{a_l} \qquad (l<i_\nu)$$

holds for some integers $a_1, \cdots, a_l$, then

$$\tau_\nu=\theta_{i_\nu}=\sum_{j=1}^{l} a_j\theta_j+p\tau_0 \qquad (p\in \boldsymbol{Z}) .$$

Since $\sum_{j=1}^{l} a_j\theta_j$ is written by a linear combination of $\tau_0, \tau_1, \cdots, \tau_{\nu-1}$, this is absurd. Therefore α_{i_ν} has no such relations. Hence by (1),

$$z_i'=\alpha_{i_\nu}z_i \qquad (\nu=1, 2, \cdots, \lambda) .$$

PROOF OF THEOREM 2. We may assume that Y can not be contained any primary Hopf manifold of dimension less than N. Let D be a subvariety of codimension 1 in Y. By Lemma 4, $\mathscr{D}$ is contained in the zero locus of a non-constant holomorphic function f on $\mathscr{Y}$ such that $\tilde{g}*f=\alpha f$ $(0<|\alpha|<1)$. There exist some integers $m, m_1, \cdots, m_\lambda$ such that

$$\alpha^m=\alpha_{i_1}^{m_1}\cdots\alpha_{i_\lambda}^{m_\lambda} .$$

Put

$$h=z_{i_1}^{m_1}\cdots z_{i_\lambda}^{m_\lambda} .$$

Since Y is not contained in any lower dimensional primary Hopf manifold, h is not equal to zero on $\mathscr{Y}$. Hence both f^m and h are eigenfunctions of $\tilde{g}^*$ of which the eigenvalues are the same α^m. Then h/f^m defines a non-zero meromorphic function on Y. By the assumption $a(Y)=0$, $h/f^m=$ constant$=c\neq 0$. Hence we get

$$h=cf^m . \tag{3}$$

Let Z_{i_ν} $(\nu=1, \cdots, \lambda)$ be analytic subsets of Y corresponding to $\{z_{i_\nu}=0\}\cap\mathscr{Y}$. The equation (3) implies that D is contained in $\bigcup_{\nu=1}^{\lambda} Z_{i_\nu}$. Since $\lambda\leqq N$, this proves the theorem. Q.E.D.

§ 5. C^*-actions.

PROPOSITION 4. *There exists a holomorphic mapping*

$$\begin{array}{ccc} \tilde{\varphi}\colon \boldsymbol{C}\times \boldsymbol{C}^N & \longrightarrow & \boldsymbol{C}^N \\ \cup & & \cup \\ (t, z) & \longmapsto & \tilde{\varphi}_t(z) \end{array}$$

which satisfies the following properties:

(i) *for every $t\in \boldsymbol{C}$, $\tilde{\varphi}_t$ is a holomorphic automorphism of $\boldsymbol{C}^N$ which fixes the origin,*

(ii) $\tilde{\varphi}_{t+s}=\tilde{\varphi}_t\circ\tilde{\varphi}_s$,

(iii) *there exists an integer* n_0 *such that* $\tilde{\varphi}_1=\tilde{g}^{n_0}$,

(iv) *every* $\tilde{g}$*-invariant subvarieties in* C^N *is* $\tilde{\varphi}_t$*-invariant for all* $t\in C$.

We say that an analytic subset of C^N is $\tilde{\varphi}$*-invariant*, if it is $\tilde{\varphi}_t$-invariant for all $t\in C$.

PROOF. Let $\alpha_{i_1},\cdots,\alpha_{i_\lambda}$ be the eigenvalues of $\tilde{g}$ considered in §4. For any eigenvalue α_j of $\tilde{g}$, there exist some integers $m_j, m_{j_1},\cdots,m_{j_\lambda}$ such that

$$\alpha_j^{m_j}=\alpha_{i_1}^{m_{j_1}}\cdots\alpha_{i_\lambda}^{m_{j_\lambda}}\qquad(j=1,2,\cdots,N)\ .$$

Put $n_0=m_1\cdots m_N$ and $g_0=g^{n_0}$. We define

$$\alpha_{i_\nu}^t=\exp t\tau_\nu\qquad(t\in C,\ \nu=1,2,\cdots,\lambda)\ ,\tag{4}$$

and

$$\alpha_j^{n_0t}=\exp\Big(tn_j\sum_{\nu=1}^{\lambda}m_{j_\nu}\tau_\nu\Big)\qquad(n_j=n_0m_j^{-1},\ j=1,2,\cdots,N)\ .\tag{5}$$

Let $R(\alpha_1^{n_0},\cdots,\alpha_N^{n_0})=1$ be any relation among the eigenvalues of g_0, where $R(u_1,\cdots,u_N)$ is a product of some (possibly negative) powers of u_j $(j=1,2,\cdots,N)$, u_j being indeterminates. Now let $R(u_1,\cdots,u_N)=u_1^{a_1}\cdots u_N^{a_N}$ $(a_j\in Z)$. Then, for $t\in C$,

$$\begin{aligned}R(\alpha_1^{n_0t},\cdots,\alpha_N^{n_0t})&=\alpha_1^{a_1n_0t}\cdots\alpha_N^{a_Nn_0t}\\&=\exp\Big(t\sum_{j=1}^{N}a_jn_j\sum_{\nu=1}^{\lambda}m_{j_\nu}\tau_\nu\Big)\\&=\exp\Big(t\sum_{\nu=1}^{\lambda}\Big(\sum_{j=1}^{N}a_jn_jm_{j_\nu}\Big)\tau_\nu\Big)\ .\end{aligned}\tag{6}$$

Put $t=1$ in (6). Then we get

$$\sum_{\nu=1}^{\lambda}\Big(\sum_{j=1}^{N}a_jn_jm_{j_\nu}\Big)\tau_\nu=p\tau_0\qquad(p\in Z)\ .$$

Hence we get $p=0$ and $\sum_{j=1}^{N}a_jn_jm_{j_\nu}=0$ $(\nu=1,2,\cdots,\lambda)$. Therefore

$$R(\alpha_1^{n_0t},\cdots,\alpha_N^{n_0t})=1\tag{7}$$

for all $t\in C$. Put $\beta_j=\alpha_j^{n_0}$. By (1), the j-th coordinate of the point $g_0^n(z)$ is given by

$$(g_0^n(z))_j=\beta_j^n\{z_j+Q_j(n,z_1,\cdots,z_{j-1})\}\ ,\tag{8}$$

where Q_j is a polynomial of $n, z_1, \cdots, z_{j-1}$. Replace n and β_j^n of (8) by t and $\alpha_j^{n_0 t}=\beta_j^t$, respectively. Then we get a holomorphic automorphism $\tilde{\varphi}_t$ of C^N defined by

$$(\tilde{\varphi}_t(z))_j=\beta_j^t\{z_j+Q_j(t, z_1, \cdots, z_{j-1})\} .$$

We shall prove that $\tilde{\varphi}=\{\tilde{\varphi}_t\}_{t\in C}$ satisfies the desired conditions. The condition (i) and (iii) are clearly satisfied. To prove the condition (ii) is satisfied we put

$$z=\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \quad Q(t, z)=\begin{pmatrix} Q_1(t, z) \\ \cdots \\ Q_N(t, z) \end{pmatrix} \quad \text{and} \quad A^t=\begin{pmatrix} \beta_1^t & & 0 \\ & \ddots & \\ 0 & & \beta_N^t \end{pmatrix} .$$

We write $\tilde{\varphi}_t(z)$ as

$$\tilde{\varphi}_t(z)=A^t(z+Q(t, z)) . \tag{9}$$

Again we put

$$d(t, s, z)=\tilde{\varphi}_{t+s}(z)-\tilde{\varphi}_t\circ\tilde{\varphi}_s(z) . \tag{10}$$

It is sufficient to prove that $d(t, s, z)$ vanishes identically. By (9),

$$\begin{aligned} d(t, s, z)&=A^{t+s}(z+Q(t+s, z))-A^t(A^s(z+Q(s, z))+Q(t, A^s(z+Q(s, z)))) \\ &=A^{t+s}Q(t+s, z)-A^{t+s}Q(s, z)-A^tQ(t, A^s(z+Q(s, z))) . \end{aligned} \tag{11}$$

Let $Q_j(s, z)=\sum q_{i_1\cdots i_{j-1}}(s)z_1^{i_1}\cdots z_{j-1}^{i_{j-1}}$ be the j-th component of $Q(s, z)$, where $i_1, \cdots, i_{j-1}$ satisfy $\beta_1^{i_1}\cdots\beta_{j-1}^{i_{j-1}}=\beta_j$ and $i_l>0$. Then, by (7),

$$\begin{aligned} &Q_j(t, A^s(z+Q(s, z))) \\ &\quad=\sum q_{i_1\cdots i_{j-1}}(t)\{\beta_1^s(z_1+Q_1(s, z))\}^{i_1}\cdots\{\beta_{j-1}^s(z_{j-1}+Q_{j-1}(s, z))\}^{i_{j-1}} \\ &\quad=\beta_j^s\sum q_{i_1\cdots i_{j-1}}(t)(z_1+Q_1(s, z))^{i_1}\cdots(z_{j-1}+Q_{j-1}(s, z))^{i_{j-1}} . \end{aligned}$$

Hence we get

$$A^tQ(t, A^s(z+Q(s, z)))=A^{t+s}Q(t, z+Q(s, z)) . \tag{12}$$

Combining (11) with (12), we obtain

$$d(t, s, z)=A^{t+s}(Q(t+s, z)-Q(s, z)-Q(t, z+Q(s, z))) .$$

Hence it is sufficient to show that

$$d_1(t, s, z)=Q(t+s, z)-Q(s, z)-Q(t, z+Q(s, z))$$

vanishes identically. Note that every component of $d_1(t, s, z)$ is a polynomial of t, s, and z.

Fix any integer $t=m$. Since $d_1(m, n, z)$ vanishes identically for any $n \in \boldsymbol{Z}$, the algebraic subset in $\boldsymbol{C}^{N+1}$ defined by

$$\{(s, z) \in \boldsymbol{C}^{N+1} : d_1(m, s, z)=0\}$$

contains infinitely many N-dimensional subspaces of $\boldsymbol{C}^{N+1}$. Hence we infer that $d_1(m, s, z)$ vanishes identically for any integer m. Again, since $d_1(m, s, z)=0$ for any $m \in \boldsymbol{Z}$, the algebraic subset in $\boldsymbol{C}^{N+2}$ defined by $d_1(t, s, z)=0$ contains infinitely many $(N+1)$-dimensional subspaces of $\boldsymbol{C}^{N+2}$. Hence we conclude that d_1 vanishes identically on $\boldsymbol{C}^{N+2}$. Therefore the condition (ii) is satisfied.

Next we prove that the condition (iv) is satisfied. We need the following

LEMMA 5. *Let $\mathscr{Y}$ be a $\tilde{g}$- and φ-invariant analytic subvariety in $\boldsymbol{C}^N$. Let $\mathscr{X}$ be a pure 1-codimensional $\tilde{g}$-invariant analytic subset of $\mathscr{Y}$. Then each irreducible component of $\mathscr{X}$ is $\tilde{\varphi}$-invariant.*

PROOF. By Lemma 4, there exists a holomorphic function f on $\mathscr{Y}$ such that $\tilde{g}*f=\alpha f$ $(0<|\alpha|<1)$ and that $f|_{\mathscr{X}}=0$. Here we shall prove the following equation:

$$\tilde{\varphi}_t^* f=\alpha^t f . \tag{13}$$

Once the equation (13) is proved, the lemma is clear. In fact, each irreducible component of $\mathscr{X}$ is an irreducible component of the zero locus of f. Since everything continuously varies depending on t, (13) implies that the irreducible components of $\mathscr{X}$ is $\tilde{\varphi}$-invariant.

We put

$$M(\alpha)=\{h \in \mathscr{O}_{\mathscr{Y}} : \tilde{g}^*h=\alpha h\} .$$

Then $M(\alpha)$ is a finite dimensional vector space over $\boldsymbol{C}$ (cf. [2]). Let $\sigma_1, \cdots, \sigma_s$ be a basis of $M(\alpha)$. Put $\sigma_i^t(z)=\sigma_i(\tilde{\varphi}_t(z))$ $(i=1, 2, \cdots, s)$. Since $\mathscr{Y}$ is $\tilde{\varphi}_t$-invariant, the elements $\sigma_1^t, \cdots, \sigma_s^t$ form another basis of $M(\alpha)$. Hence there exist some constants $c_{ij}(t)$ depending on t such that

$$\sigma_i^t=\sum_{j=1}^{s} c_{ij}(t)\sigma_j .$$

We claim that $C(t)=(c_{ij}(t))$ is holomorphically dependent on t. In fact, we can choose points $z_1, \cdots, z_s \in \mathscr{Y}$ such that

$$S=\begin{pmatrix}\sigma_1(z_1) & \cdots & \sigma_1(z_s)\\ \vdots & & \vdots\\ \sigma_s(z_1) & \cdots & \sigma_s(z_s)\end{pmatrix}$$

is a non-singular matrix. Then,

$$\begin{pmatrix}\sigma_1^t(z_1) & \cdots & \sigma_1^t(z_s)\\ \vdots & & \vdots\\ \sigma_s^t(z_1) & \cdots & \sigma_s^t(z_s)\end{pmatrix}S^{-1}=C(t)\ . \tag{14}$$

Since the left hand side of (14) is holomorphically dependent on t, $C(t)$ is holomorphic.

It is easy to see that $\{C(t)\}_{t\in C}$ is a 1-parameter subgroup of GL $(s, \boldsymbol{C})$, satisfying the equality,

$$C(n)=\alpha^n I \qquad (n\in \boldsymbol{Z})\ . \tag{15}$$

Hence there exist a matrix A which has the Jordan canonical form and a non-singular matrix P such that

$$C(t)=P^{-1}\exp(tA)P\ .$$

By (15), A is a diagonal matrix. Put $P^{-1}\sigma_j=\tau_j$ $(j=1, 2, \cdots, s)$. Then,

$$\tau_j^t=(\text{ext}\ ta_j)\tau_j \qquad (j=1, 2, \cdots, s)\ , \tag{16}$$

where $a_1, \cdots, a_s$ are the diagonal components of A. Comparing the initial terms of the both sides of (16), we get

$$\exp ta_j=\exp\sum_{\nu=1}^{\lambda} tn_{j_\nu}\tau_\nu \qquad (j=1, 2, \cdots, s)\ , \tag{17}$$

for some integers n_{j_ν}. Letting $t=1$, we get

$$\alpha=\exp a_j=\exp\sum_{\nu=1}^{\lambda} n_{j_\nu}\tau_\nu \qquad (j=1, 2, \cdots, s)\ .$$

Hence for any i and j,

$$\sum_{\nu=1}^{\lambda}(n_{j_\nu}-n_{i_\nu})\tau_\nu=p_{ij}\tau_0\ ,$$

choosing some integers p_{ij}. Since $\tau_0, \tau_1, \cdots, \tau_\lambda$ are linearly independent over $\boldsymbol{Q}$, this implies that $n_{j_\nu}=n_{i_\nu}$ and $p_{ij}=0$. Hence $\exp ta_j=\exp ta_i$ for any i and j. Therefore $C(t)$ is a scalar matrix:

$$C(t)=\alpha^t I \qquad (\alpha^t=\exp ta_j)\ .$$

Since $f \in M(\alpha)$, f can be expressed as

$$f = c_1\tau_1 + \cdots + c_s\tau_s \qquad (c_j \in C) .$$

Then $\tilde{\varphi}_t^* f = \sum_j c_j \tilde{\varphi}_t^* \tau_j = \alpha^t \sum c_j \tau_j = \alpha^t f.$ Q.E.D.

Proof of (iv). By Lemma 5 [2], there exists a sequence $\{\mathscr{W}_j : j = 0, 1, \cdots, p\}$ of $\tilde{g}$-invariant subvarieties of C^N such that $\mathscr{W}_0$ = a given $\tilde{g}$-invariant subvariety $\mathscr{W}$, $\mathscr{W}_j \subset \mathscr{W}_{j+1}$, $\dim \mathscr{W}_j + 1 = \dim \mathscr{W}_{j+1}$ and $\mathscr{W}_p = C^N$ $(p = N - \dim \mathscr{W}_0)$. Since C^N is obviously $\tilde{g}$- and $\tilde{\varphi}$-invariant, we infer that $\mathscr{W}$ is $\tilde{\varphi}$-invariant by the previous lemma. Q.E.D.

As a corollary, we obtain

THEOREM 3. *For any primary Hopf manifold H^N, there exists another primary Hopf manifold H'^N with following properties:*

(i) *H'^N is a finite cyclic unramified covering of H^N,*

(ii) *H'^N has a free C^*-action $\varphi = \{\varphi_\tau\}_{\tau \in C^*}$ such that every positive dimensional subvariety in H'^N is φ-invariant.*

PROOF. Let $H' = C^N - \{O\}/\langle \tilde{g}^{n_0} \rangle$. Then everything is clear from Proposition 4.

COROLLARY. *The Euler number of a submanifold of a Hopf manifold is equal to* 0.

PROOF. By Theorem 3, every submanifold of a Hopf manifold has a finite unramified covering which admits a free S^1-action. Hence the Euler number vanishes. Q.E.D.

§ 6. Subvarieties of algebraic codimension 1.

Let Y be a n-dimensional $(n \geqq 2)$ subvariety of a primary Hopf manifold H^N. Take another primary Hopf manifold H'^N of Theorem 3. Let $\omega: H'^N \to H^N$ be the covering map. We denote by Y' a connected component of $\omega^{-1}(Y)$.

THEOREM 4. *The algebraic dimension of Y is $n-1$ if and only if the C^*-action φ on Y' reduces to a complex torus action.*

PROOF. Assume that $a(Y) = n-1$. Since $a(Y') = a(Y) = n-1$, Y' has an $(n-1)$-dimensional algebraic family of elliptic curves.

The moduli of curves depends continuously on the parameters. Hence, by Proposition 3, the moduli are constant. Since every curve in Y is φ-invariant, the C^*-action reduces to a complex torus action on the open dense subset of Y' and therefore on the whole Y'.

Conversely, assume that φ reduces to a complex torus action ψ on Y'. Then $\mathscr{Y}'$ is an affine variety in C^N with the C^*-action $\tilde{\psi}$ induced by $\tilde{\varphi}$. Moreover the action $\tilde{\psi}$ is compatible with g', where g' is a contracting automorphism to O of C^N defining H'^N. It is not difficult to check that the C^*-action $\tilde{\psi}$ on $\mathscr{Y}'$ is algebraic. (Construct a contracting automorphism on $C\times\mathscr{Y}'\times\mathscr{Y}'$ which leaves invariant the closure $\bar{\Gamma}$ of the graph Γ of $\tilde{\psi}$, where $\bar{\Gamma}$ is an analytic subset of $C\times\mathscr{Y}'\times\mathscr{Y}'$. Use the result of [2] to show that $\bar{\Gamma}$ is an algebraic subset of $C\times\mathscr{Y}'\times\mathscr{Y}'$.) Hence, by Proposition (1.1.3) in Orlik-Wagreich [5], there is an embedding $j: \mathscr{Y}'\to C^{N'}$ for some N' and a C^*-action $\tilde{\psi}'$ on $C^{N'}$ such that $j(\mathscr{Y}')$ is $\tilde{\psi}'$-invariant and that $\tilde{\psi}'$ induces $\tilde{\psi}$ on $\mathscr{Y}'$. Moreover, by a suitable choice of coordinates $(z_1, \cdots, z_N)$ on $C^{N'}$, the action $\tilde{\psi}'$ on $C^{N'}$ can be written as

$$\tilde{\psi}'(\rho, (z_1, \cdots, z_{N'}))=(\rho^{q_1}z_1, \cdots, \rho^{q_{N'}}z_{N'}) ,$$

where the q_i's are positive integers. There exists a constant α such that $\tilde{\psi}'_\alpha$ induces g' on $\mathscr{Y}'$. Then $Y'=\mathscr{Y}'-\{O\}/\langle g'\rangle$ can be considered as a submanifold of $C^{N'}-\{O\}/\langle\tilde{\psi}'_\alpha\rangle$.

The following theorem is known.

THEOREM (Ueno [8]). *Let M_1 be a subvariety of a compact complex variety M_0. Then*

$$\dim M_1-a(M_1)\leqq\dim M_0-a(M_0) . \tag{18}$$

Now it is clear that $a(C^{N'}-\{O\}/\langle\tilde{\psi}'_\alpha\rangle)=N'-1$. Hence, by (18), we get $a(Y')\geqq\dim Y'-1$. Since $a(Y')<\dim Y'$, we obtain $a(Y')=a(Y)=n-1$.

Q.E.D.

REMARK 5. Topologically, *any submanifold of a Hopf manifold is diffeomorphic to a fibre bundle over a* 1*-dimensional circle of which the transition function has a finite order as an element of the diffeomorphism group of the fibre.* This can be seen without difficulty from Theorem 3.

REMARK 6. *A compact complex surface S is a submanifold of a Hopf manifold if and only if S is a relatively minimal surface of class VI_0, VII_0-elliptic or a Hopf surface* (see [3] for the proof of the "if" part). Let S be a submanifold of a Hopf manifold. It is clear by Proposition 3 that S is relatively minimal. By Theorem 1, S is not algebraic. Hence $a(S)\leqq 1$. Assume that $a(S)=1$. Then, by Theorem 1, there exists a flat line bundle L on S such that the mapping $\Phi_L: S\to P^n$ defined by the linear system $|L|$ gives an algebraic reduction of S which is defined everywhere. Put $\Delta=\Phi_L(S)$. Let η be the line bundle on Δ associated to a hyperplane section of Δ. Then we have $\Phi_L^*\eta = L$. We note that every fibre of $\Phi_L: S \to \Delta$ is a non-singular elliptic curve (Proposition 3). We indicate by $b_i(M)$ the i-th Betti number of a manifold M. It is clear that $b_1(\Delta) \leqq b_1(S) \leqq b_1(\Delta)+2$. Assume first that $b_1(\Delta)=b_1(S)$. Since L is a flat line bundle on S, L is raised from a group representation ρ of $H_1(S, \boldsymbol{Z})$ into $\boldsymbol{C}^*$. Let m be a certain positive integer such that ρ^m is trivial on the torsion part of $H_1(S, \boldsymbol{Z})$. Then, in view of $b_1(\Delta)=b_1(S)$, there exists a flat line bundle ξ on Δ such that $\Phi_L^*\xi = L^m$. Hence we get $\Phi_L^*\zeta = \Phi_L^*\eta^m$. Since $\Phi_L^*: H^1(\Delta, O^*) \to H^1(S, O^*)$ is an injection, this implies that the ample line bundle η on Δ is flat. This is absurd. Hence we get $b_1(\Delta) < b_1(S)$. Next assume that $b_1(S)=b_1(\Delta)+2$. By Corollary to Theorem 3, we get $b_2(S)=2b_1(\Delta)+2$. This implies that the dual of the homology class represented by a general fibre is a Betti base of $H^2(S, \boldsymbol{Z})$. This contradicts Theorem 1. Hence we conclude that $b_1(S)=b_1(\Delta)+1$. Therefore $b_1(S)$ is odd. Hence S is either a surface of VI_0 or VII_0-elliptic. Consider the case $a(S)=0$. By the classification theory of surfaces [4], a relatively minimal surface with no non-constant meromorphic functions and vanishing Euler number is either a complex torus or a surface of VII_0. A complex torus has a positive algebraic dimension if it contains a divisor. Hence by Proposition 3 we infer that S is of VII_0-class. Moreover $b_1(S)=1$ and $b_2(S)=0$. Hence, by Theorem 34 [4], S is a Hopf surface.

References

[1] H. Cartan, Quotient d'un espace analytique par un groupe d'automorphismes, *Algebraic Geometry and Topology*, Princeton Univ. Press, 1964, 90-102.

[2] Ma. Kato, A generalization of Bieberbach's example, Proc. Japan Acad., **50** (1974), 329-333.

[3] Ma. Kato, Complex structures on $S^1 \times S^5$, J. Math. Soc. Japan, **28** (1976), 550-576.

[4] K. Kodaira, On the structure of compact complex analytic surfaces, II, IV, Amer. J. Math., **88** (1966), 682-721, **90** (1968), 1048-1066.

[5] P. Orlik and P. Wagreich, Isolated singularities of algebraic surfaces with $\boldsymbol{C}^*$ action, Ann. of Math., **93** (1971), 205-228.

[6] L. Reich, Das Typenproblem bei formal-biholomorphen Abbildungen mit anziehendem Fixpunkt, Math. Ann., **179** (1969), 227-250.

[7] L. Reich, Normalformen biholomorpher Abbildungen mit anziehendem Fixpunkt, Math. Ann., **180** (1969), 233-255.

[8] K. Ueno, Classification of algebraic varieties and compact complex spaces, Lecture notes in Math., **439**, Springer.

[3A] Ma. Kato, On a characterization of submanifolds of Hopf manifolds, Complex analysis and algebraic geometry, Iwanami, Tokyo, 191-206.

Present Address:
Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku, Tokyo 102