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The Riemann-Hilbert Problem and its Application to Analytic Functions of Several Complex Variables

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Introduction

In this paper we shall prove the local existence of holomorphic functions in an analytic cover (a ramified Riemann domain) $\pi: Y \to X$ by using a solution of the Riemann-Hilbert problem (see §6). The existence of such functions was earlier proved in 1958 by H. Grauert and R. Remmert [10] and in 1960 by R. Kawai [11] by different methods. We can consider the functions on Y as many-valued functions on X which may have the branch points along the critical locus D of the analytic cover $\pi: Y \to X$. We shall construct such many-valued functions on X from the solutions of the total differential equation (1.1) whose monodromy representation is the one associated with the analytic cover $\pi: Y \to X$ (see §5). For this purpose, in §3, using the results of P. Deligne [6], we solve the Riemann-Hilbert problem in the following situation; let X be a connected Stein manifold and let D be a divisor of X (*not necessarily normal crossing*). Suppose that a representation ρ of $\pi_1(X-D, x_0)$ in $\mathrm{GL}_q(\boldsymbol{C})$ is given. We shall construct a total differential equation (1.1) whose monodromy is the given ρ. We can study in detail the case of $\dim X=2$ than that of $\dim X \geqq 3$, more precisely, when $\dim X=2$, if $H^2(X, \boldsymbol{Z})=0$, we can solve the Riemann-Hilbert problem *without apparent singularities* (Theorem 3). As an application of Proposition 2 of §3, we shall give a remark to the Riemann-Hilbert problem *in the restricted sense*, when X is a two-dimensional connected complex manifold. This problem was treated by K. Aomoto [1] by different method when X is an n-dimensional complex projective space (see §4). In solving the Riemann-Hilbert problem, we do not use the existence of resolution of X satisfying the condition that the inverse image of D is normal crossing, but we use essentially the extension theorems of coherent analytic sheaves of J.-P. Serre [15] and Y.-T. Siu

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[16] (see §2).

§1. Preliminaries.

1.1. In what follows we assume that all manifolds under consideration are *paracompact*. Let X be a connected complex manifold and we fix a base point $x_0 \in X$. Suppose that γ_1 and γ_2 be closed curves in X starting from x_0. Then we denote by $\gamma_1 \cdot \gamma_2$ the closed curve defined by

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_2(2t) & \text{for } 0 \leqq t \leqq 1/2 \\ \gamma_1(2t-1) & \text{for } 1/2 \leqq t \leqq 1 . \end{cases}$$

The constant sheaf with coefficients in C^q is denoted by $\underline{C}^q$. In this paper, a locally constant sheaf $\underline{V}$ on X of rank q means always the sheaf which is locally isomorphic to the constant sheaf $\underline{C}^q$. Let γ be a closed curve starting from x_0; i.e., let $\gamma: [0, 1] \to X$ be a continuous map with $\gamma(0) = \gamma(1) = x_0$. Then $\gamma^*(\underline{V})$ is a locally constant sheaf on $[0, 1]$; hence it is a constant sheaf. Thus there is a unique isomorphism between $\gamma^*(\underline{V})$ and the constant sheaf on $[0, 1]$ with coefficients in V_{x_0}. It follows that γ determines an isomorphism $\gamma_* \in \mathrm{GL}(\underline{V}_{x_0})$ and γ_* depends only on the homotopy class of γ. It is evident that $(\gamma_1 \cdot \gamma_2)_* = (\gamma_1)_* \cdot (\gamma_2)_*$. Hence one can determine a homomorphism $\rho: \pi_1(X, x_0) \to \mathrm{GL}(\underline{V}_{x_0})$ by $\rho(\gamma) = \gamma_*$.

Let $\underline{V}$ be as above. There exists a sufficiently fine open covering $X = \bigcup_{j \in J} U_j$ such that $\underline{V}|_{U_j}$ is constant; hence there is an isomorphism $\varphi_j: \underline{C}^q \to \underline{V}|_{U_j}$. Since $\varphi_i^{-1} \cdot \varphi_j$ is an isomorphism of constant sheaf $\underline{C}^q$ on $U_i \cap U_j$, there exists a matrix $g_{ij} \in \mathrm{GL}_q(C)$ for any $U_i \cap U_j \neq \emptyset$ such that

$$\varphi_i(\xi_i) = \varphi_j(\xi_j) , \quad \text{where} \quad \xi_i, \xi_j \in \underline{C}^q$$

if and only if $\xi_i = g_{ij} \cdot \xi_j$. It is obvious that g_{ij} satisfy the cocycle conditions:

$$g_{ij} \cdot g_{jk} = g_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset ;$$

hence there is determined a flat vector bundle E of rank q with the transition functions g_{ij}. There is a simple relation between $\underline{V}$ and E, i.e., $\underline{V}$ is isomorphic to $C(E)$, where $C(E)$ is the sheaf of germs of locally constant sections of E. Thus we have seen that a flat vector bundle determines a representation ρ of $\pi_1(X, x_0)$ in $\mathrm{GL}(\underline{V}_{x_0})$. Let us consider the converse. Suppose that a representation ρ of $\pi_1(X, x_0)$ in $\mathrm{GL}_q(C)$ be given. There is an open covering $X = \bigcup_{j \in J} U_j$ such that each U_j and $U_j \cap U_k$ are simply connected. We suppose $x_0 \in U_0$, and choose a point

$x_j \in U_j$. Since X is connected, there is a path ℓ_j in X from x_0 to x_j. For any $x \in U_i \cap U_j$, let $d_{ij}(x)$ be a path in U_i from x_i to x. If γ is a closed curve starting from x_0, we denote by $[\gamma]$ the homotopy class of γ. Write

$$g_{ij}: =\rho([\ell_i^{-1} \cdot d_{ij}^{-1}(x) \cdot d_{ji}(x) \cdot \ell_j]) \quad \text{for} \quad x \in U_i \cap U_j .$$

Since each U_j and $U_i \cap U_j$ are simply connected, g_{ij} is constant on $U_i \cap U_j$. It follows that

$$g_{ij} \cdot g_{jk} = g_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset .$$

Hence $\{g_{ij}\}$ satisfies the cocycle conditions, and one can determine a flat vector bundle E with the transition functions g_{ij}. Let $C(E)=\underline{V}$, and let $\gamma: [0, 1] \to X$ be a closed curve starting from x_0, then there is an open covering $\gamma([0, 1]) \subset \bigcup_{j \in J} U_j$ (if necessary, change the indices of $\{U_i\}$) such that $U_i \cap U_{i+1} \neq \emptyset$ for $i=0, \cdots, m$, where $U_{m+1}=U_0$. By the definition of E, there is a frame $e^{(i)}=(e_1^{(i)}, \cdots, e_q^{(1)})$ of E on U_i such that, any section ξ of E is identified with the collection of vectors $\{\xi_i\}$ such that $\xi_i = g_{ij} \cdot \xi_j$, where $\xi_i = {}^t(\xi_i^1, \cdots, \xi_i^q)$ and $\xi = \sum_{\alpha=1}^q \xi_i^\alpha e_\alpha^{(i)}$. Let ξ_0 be a local section of $C(E)$ on a neighborhood of x_0. Using the frame $e^{(0)}$, we can identify the vector space $\underline{V}_{x_0}$ with the complex number space C^q; hence we can consider $\gamma_* \in \mathrm{GL}(\underline{V}_{x_0})$ as a matrix $A_{\gamma_*} \in \mathrm{GL}_q(C)$. Then, by the definition of γ_*, it follows that

$$\begin{aligned} A_{\gamma_*} &= g_{0m} \cdot g_{m,m-1} \cdot \cdots \cdot g_{10} \cdot \xi_0 \\ &= \rho([\ell_0^{-1} d_{0m}^{-1} d_{m0} \ell_m]) \cdot \cdots \cdot \rho([\ell_1^{-1} d_{10}^{-1} d_{01} \ell_0]) \xi_0 \\ &= \rho([\gamma]) \xi_0 \end{aligned}$$

because the closed curve $(\ell_0^{-1} d_{0m}^{-1} d_{m0} \ell_m) \cdots (\ell_1^{-1} d_{10}^{-1} d_{01} \ell_0)$ is homotopic to γ. Hence we have that

$$\gamma_*(e_1^{(0)}, \cdots, e_q^{(0)}) = (e_1^{(0)}, \cdots, e_q^{(0)}) \rho([\gamma]) ,$$

where $\gamma_*(e_1^{(0)}, \cdots, e_q^{(0)})$ is a $1 \times q$ matrix $(\gamma_* e_1^{(0)}, \cdots, \gamma_* e_q^{(0)})$ of q sections of $C(E)$ on U_0. Thus we have that, *given a representation ρ of $\pi_1(X, x_0)$ in $\mathrm{GL}_q(C)$, there exists a flat vector bundle E on X satisfying the conditions that the action of $\pi_1(X, x_0)$ to $C(E)_{x_0}$ is identified with the given ρ provided that we choose properly the basis of $C(E)_{x_0}$.*

1.2. Let E be a holomorphic vector bundle of rank q on X, and let $\mathcal{O}(E)$ be the sheaf of germs of holomorphic sections of E. We denote by Ω_X^p the sheaf of germs of holomorphic p-forms on X. A holomorphic connection ∇ on E is a C-linear homomorphism

$$\nabla\colon \mathcal{O}(E)\longrightarrow \Omega^1_X \bigotimes_{\mathcal{O}_X} \mathcal{O}(E)$$

which satisfies the Leibniz formula

$$\nabla(fs)=df\otimes s+f\nabla s$$

for any local sections f of $\mathcal{O}_X$ and s of $\mathcal{O}(E)$. Given ∇, there is one and only one C-linear homomorphism

$$\hat{\nabla}\colon \Omega^1_X \bigotimes_{\mathcal{O}_X} \mathcal{O}(E)\longrightarrow \Omega^2_X \bigotimes_{\mathcal{O}_X} \mathcal{O}(E)$$

which satisfies the Leibniz formula

$$\hat{\nabla}(\theta\otimes s)=d\theta\otimes s-\theta\wedge\nabla s$$

for any local sections θ of Ω^1_X and s of $\mathcal{O}(E)$. Now let us consider the composition

$$K=\hat{\nabla}\circ\nabla\colon \mathcal{O}(E)\longrightarrow \Omega^2_X \bigotimes_{\mathcal{O}_X} \mathcal{O}(E) .$$

By simple computation, it follows that the correspondence $s(x)\rightarrow K(s)(x)$ defines a holomorphic section of holomorphic vector bundle $\mathrm{Hom}\,(E, \wedge^2 T^*_X\otimes E)\cong \wedge^2 T^*_X\otimes \mathrm{End}\,(E)$, where T^*_X is the cotangent bundle of X and $\mathrm{End}\,(E)=\mathrm{Hom}\,(E, E)$, so we have $K\in\Gamma(X, \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{O}(\mathrm{End}\,(E)))$. This section $K=K_\nabla$ is called the curvature tensor of the connection ∇. A connection ∇ is called *integrable* if its curvature tensor K_∇ is zero. Let $e=(e_1, \cdots, e_q)$ be a holomorphic frame of E on a neighborhood U in X. Then we define the connection matrix $\omega=(\omega_{ij})$ associated with the frame e by setting

$$\nabla e_i=\sum_{j=1}^{q}\omega_{ji}e_j \quad \text{for} \quad i=1, \cdots, q ,$$

where $\omega_{ji}\in\Gamma(U, \Omega^1_X)$. Note that

$$K(e_i)=\hat{\nabla}\Big(\sum_{j=1}^{q}\omega_{ji}e_i\Big)$$
$$=\sum_{j=1}^{q}K_{ji}\otimes e_j ,$$

where we have set

$$K_{ij}=d\omega_{ij}+\sum_{k=1}^{q}\omega_{ik}\wedge\omega_{kj}\in\Gamma(U, \Omega^2_X) ,$$

i.e., in matrix notation $K=d\omega+\omega\wedge\omega$, $K=(K_{ij})$. Hence ∇ is integrable if and only if the connection matrix ω satisfies the differential equation $d\omega+\omega\wedge\omega=0$. Using the frame e, we write a local section s in the form $s=\sum_{i=1}^{q} u_i e_i$; then the relation $\nabla s=0$ is equivalent to the total differential equation

$$d\begin{pmatrix} u_1 \\ \vdots \\ u_q \end{pmatrix}+\begin{pmatrix} \omega_{11} & \cdots & \omega_{1q} \\ & \cdots & \\ \omega_{q1} & \cdots & \omega_{qq} \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_q \end{pmatrix}=0 .$$

Hence, by the classical existence theorem of differential equations, if ∇ is integrable, then the subsheaf $\operatorname{Ker}\nabla$ (of $\mathcal{O}(E)$) of local solutions of $\nabla s=0$ is a locally constant sheaf of rank q. Conversely, let E be a flat vector bundle of rank q. Since $\mathcal{O}(E)=C(E)\otimes_{C}\mathcal{O}_X$, we can define a C-linear homomorphism ∇: $\mathcal{O}(E)\to\Omega_X^1\otimes_{\mathcal{O}_X}\mathcal{O}(E)$ as follows: $\nabla(s\otimes f):=df\otimes s$ for any local sections s of $C(E)$ and f of $\mathcal{O}_X$. It is easy to check that ∇ is an integrable connection on E such that $\operatorname{Ker}\nabla=C(E)$.

1.3. Let D be a normal crossing divisor of X, i.e., D is locally defined by the equation $\{z_1\cdots z_k=0\}$, where $(z_1,\cdots,z_n)$ is a local coordinate system. Write $X^*=X-D$. Suppose that E is a holomorphic vector bundle on X and ∇ is an integrable connection on $E|_{X^*}$. Suppose that there exists a local coordinate system $(z_1,\cdots,z_n)$ in a neighborhood U of a point $x\in D$ such that $U\cap D=\{z_1\cdots z_k=0\}$. Then ∇ is said to have at most *logarithmic pole* along D, if the connection matrix $(\omega_{ij})=\omega$ associated with any frame has at most logarithmic pole along $U\cap D$, i.e., each ω_{ij} is written in the form

$$\omega_{ij}=\sum_{\nu=1}^{k}\alpha_\nu(dz_\nu/z_\nu)+\eta ,$$

where α_ν is holomorphic on U and η is a holomorphic 1-form on U. Write $U\cap D=:\bigcup_{i=1}^{k}C_i$ where $C_i=\{z_i=0\}$, then we write $\operatorname{res}_{C_\nu}\omega_{ij}:=\alpha_\nu|_{C_\nu}$ and call $\operatorname{res}_{C_\nu}\omega_{ij}$ the residue of ω_{ij} along C_ν. We set $\operatorname{res}_{C_\nu}\omega:=(\operatorname{res}_{C_\nu}\omega_{ij})$ and call it the residue of the connection ∇ along C_ν. Let $D=\bigcup D_j$ be the decomposition into irreducible components of D. It is shown that

$$\operatorname{res}_{D_i}\omega\in\Gamma(D_i,\mathcal{O}(\operatorname{End}(E)|_{D_i})\otimes_{\mathcal{O}_{D_i}}\tilde{\mathcal{O}}_{D_i})$$

where $\tilde{\mathcal{O}}_{D_i}$ is the sheaf of germs of weakly holomorphic functions on D_i (see [5], p. 78).

1.4. Let D, E, ∇ be as above. Let $\Delta=\{z\in C\,|\,|z|<1\}$. Let ϕ: $\Delta\to X$ be an arbitrary holomorphic map such that $\phi^{-1}(D)=\{0\}$, and let $\phi^*\nabla$ and ϕ^*E

be the inverse of ∇ and E by ϕ respectively. We say that ∇ is *regular singular* along D if the connection $\phi^*\nabla$ on ϕ^*E is regular singular at $z=0$ in the usual sense of ordinary differential equation (see [6], p. 85). Let

$$d\begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} + \begin{pmatrix} \Omega_{11} & \cdots & \Omega_{1q} \\ & \cdots & \\ \Omega_{q1} & \cdots & \Omega_{qq} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} = 0 \tag{1.1}$$

be a total differential equation, where every Ω_{ij} has at most pole along D. Suppose that (1.1) is completely integrable on $X-D$. D is said to be the *apparent singularity* of (1.1) if, for every $x \in D$, any solution of (1.1) in a small simply-connected neighborhood of x is single-valued and meromorphic there.

§2. Extension of flat vector bundles.

2.1. Let X be a connected complex manifold and let D be a divisor of X. Let $X^*: =X-D$ and $x_0 \in X^*$. Suppose that a representation ρ of $\pi_1(X^*, x_0) \to \mathrm{GL}_q(C)$ is given. We shall attempt to construct a completely integrable total differential equation of the form (1.1) which satisfies the following two conditions:

1) the equation (1.1) is regular singular along D, moreover there exists a divisor A in X *along which* (1.1) *may have apparent singularities.*

We choose q linearly independent solutions $f_1, \cdots, f_q$ of (1.1) at $x_0 \notin A$ properly, and let γ be any closed curve in X^* starting from x_0. We denote by $\gamma_*[f_1, \cdots, f_q]$ the result of analytic continuation of the function element $[f_1, \cdots, f_q]$ along the curve γ.

We require that

2) $\quad \gamma_*[f_1, \cdots, f_q]=[f_1, \cdots, f_q]\rho([\gamma])$ for any $[\gamma] \in \pi_1(X^*, x_0)$.

For a given representation ρ of $\pi_1(X^*, x_0)$ in $\mathrm{GL}_q(C)$, we shall call *the Riemann-Hilbert problem the problem of constructing the equation* (1.1) *which satisfies the above two conditions.*

As is constructed in $n^\circ 1$ and $n^\circ 2$ of §1, there exist a flat vector bundle E on X^* associated with ρ, and a unique integrable holomorphic connection ∇ on E such that the sheaf of germs of local solutions of $\nabla s=0$ coinsides with $C(E)$. For the pair (∇, E), Y. Manin showed ([6], p. 94) that E can be extended uniquely to a holomorphic vector bundle E_1 on $X-\mathrm{Sing}(D)$, where $\mathrm{Sing}(D)$ means the singular locus of D, satisfying the following two conditions:

(M.1) For any point $x \in D - \mathrm{Sing}(D)$, there exists an open neighborhood U of x in $X - \mathrm{Sing}(D)$ such that, for any holomorphic frame $e = (e_1, \cdots, e_q)$ of E_1 on U, if we write

$$\nabla e_i = \sum_{j=1}^{q} \omega_{ji} e_j \quad \text{for} \quad i = 1, \cdots, q ,$$

then any ω_{ij} has at most logarithmic poles along $D \cap U$.

(M.2) Let $\omega = (\omega_{ij})$ be a connection matrix. By $n°3$ of §1, we have $\mathrm{res}\,\omega \in \Gamma(D \cap U, \mathcal{O}(\mathrm{End}(E_1)|_D) \otimes_{\mathcal{O}_D} \tilde{\mathcal{O}}_D)$. Suppose that $D \cap U = \bigcup_{i=1}^{m} C_i$ be the decomposition into irreducible components of $D \cap U$. Then, by the simple computation (See [5], p. 79.), the eigenvalues $\alpha_1, \cdots, \alpha_q$ of the matrix $\mathrm{res}_{C_i}\,\omega$ are constant on C_i. Then the following inequality must be satisfied

$$0 \leqq \mathrm{Re}\,\alpha_i < 1 \quad \text{for} \quad i = 1, \cdots, q .$$

2.2. First we consider *two-dimensional case*. Write $S = \mathrm{Sing}(D)$. In this case, S is at most countable discrete point set in X; hence for any $s_0 \in S$, there exists an open neighborhood U of s_0 in X such that $S \cap U = \{s_0\}$. By iteration of σ-process centered at s_0, we see that the inverse image of $D \cap U$ is normal crossing. Doing this procedure at every point of S, we have the proper modification $\tau: \tilde{X} \to X$ as follows:

1) $\tilde{X}$ is a complex manifold,
2) $\tau^{-1}(D)$ is a normal crossing divisor in $\tilde{X}$,
3) $\tau: \tilde{X} - \tau^{-1}(S) \to X - S$ is biholomorphic.

Since by 3), $\tilde{X} - \tau^{-1}(D)$ is biholomorphic to $X - D$, there exists a flat vector bundle F on $\tilde{X} - \tau^{-1}(D)$ such that $\tau_*(\mathcal{O}(F)) = \mathcal{O}(E)$. By 2) and a result of Y. Manin cited above, F can be extended uniquely to a holomorphic vector bundle F_1 on $\tilde{X}$ which satisfies (M.1) and (M.2). On the other hand, E can be also extended uniquely to a holomorphic vector bundle E_1 on $X - S$ satisfying the conditions (M.1) and (M.2). Considering that $\tilde{X} - \tau^{-1}(S)$ is biholomorphic to $X - S$ and that the extension is uniquely determined by the above two conditions, it follows easily that

$$\tau_*(\mathcal{O}(F_1|_{\tilde{X} - \tau^{-1}(S)})) = \mathcal{O}(E_1) .$$

By H. Grauert and R. Remmert ([9], p. 424) the direct image $\tau_*(\mathcal{O}(F_1))$ of $\mathcal{O}(F_1)$ is a coherent analytic sheaf on X; hence *$\mathcal{O}(E_1)$ can be extended to a coherent analytic sheaf $\tau_*(\mathcal{O}(F_1))$ on X*. Let $j: X - S \to X$ be a canonical injection. Since S is a two-codimensional analytic subset of X, by a theorem of J.-P. Serre ([15], Th. 1), we have the direct image $j_*(\mathcal{O}(E_1))$ is a coherent analytic sheaf on X. Since the locally free sheaf

$\mathcal{O}(E_1)$ is reflexive, we see that $j_*(\mathcal{O}(E_1))$ is *reflexive* ([15], Prop. 7). On the other hand, Serre ([15], Remarques 2) stated, without proof, the following:

PROPOSITION 1. *Let $A=C\{z_1, z_2\}$ be a two-dimensional regular analytic local C-algebra and let M be a finitely generated A-module. If M is reflexive, M is a free A-module.*

Since Theorem 3 depends essentially on this fact, we shall give the proof below;

PROOF. Let $A=C\{z_1, z_2\}$ be the ring of convergent power series of two variables z_1 and z_2, and let $P(A)$ be the set of all prime ideals *of height equal to one*, and for an A-module M we put

$$Z(M)=\{f \in A \mid \exists x \in M,\ x\neq 0 \text{ with } fx=0\}\ .$$

We denote by $\mathrm{prof}_A M$ the homological codimension of M. Since M is reflexive, we can consider M as a lattice of some finite dimensional K-vector space with respect to A, where K is the quotient field of A, (see [4], p. 50). So, there exist free A-submodule L_1 and L_2 of V such that $L_1 \subset M \subset L_2$ and $\mathrm{rg}_A L_1=\dim_K V$. It follows that $Z(M)=\{0\}$, and especially $z_1 \notin Z(M)$; hence $\mathrm{prof}_A M\geqq 1$. If $\mathrm{prof}_A M=1$, we have $\mathrm{prof}_A(M/z_1M)=\mathrm{prof}_A M-1=0$. By the definition of homological codimension, we see that the maximal ideal $\mathfrak{m}$ of A is contained in $Z(M/z_1M)$, especially $z_2 \in Z(M/z_1M)$. So, there exists $m_1 \notin z_1M$ such that $z_2m_1=z_1m_2$ where $m_2 \in M$ and $m_2\neq 0$. Let $\mathfrak{p}_1:=Az_1 \in P(A)$ and $\mathfrak{p}_2:=Az_2 \in P(A)$. If we write $n_1:=m_1/z_1$ and $n_2:=m_2/z_2$, then we have $n_1 \in M_{\mathfrak{p}_2}$ and $n_2 \in M_{\mathfrak{p}_1}$ where $M_{\mathfrak{p}_i}$ is the localization of M with respect to the prime ideal $\mathfrak{p}_i$. We can consider M as the subset of V, and so $M_{\mathfrak{p}} \subset V$ for any $\mathfrak{p} \in P(A)$. Therefore we have that $n_1=n_2=:\alpha \in V$. If $\mathfrak{p} \in P(A)$ is an ideal containing z_1, then we have $\mathfrak{p}=Az_1$, because $\mathfrak{p}$ is minimal and Az_1 is prime. The same situation holds for z_2. So it follows that if $\mathfrak{p} \in P(A)$ is not equal to $\mathfrak{p}_1$ and $\mathfrak{p}_2$, then we have $z_1 \notin \mathfrak{p}$ and $z_2 \notin \mathfrak{p}$. Hence α is contained in $M_{\mathfrak{p}}$ for any $\mathfrak{p} \in P(A)$. Since M is reflexive, we have $M=\bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$ by ([4], p. 50), and so $\alpha \in M$. Thus we have $m_1=z_1\alpha \in z_1M$, which is a contradiction. Therefore we have $\mathrm{prof}_A M\geqq 2$. Since $\dim_A M\leqq \dim A=2$ and since $2\leqq \mathrm{prof}_A M\leqq \dim_A M$, we see that $\mathrm{prof}_A M=\dim_A M=2$; hence M is a Cohen-Macaulay module of $\dim_A M=2$. A being regular, we conclude that M is a free A-module (see for example [8], p. 142). Q.E.D.

From Proposition 1, it follows that $j_*(\mathcal{O}(E_1))$ is a *locally free sheaf* on X. Hence we have the following:

PROPOSITION 2. *Let X be a connected two-dimensional complex manifold and let D, X^* and $x_0 \in X^*$ be as in n° 2.1. We assume that a representation ρ of $\pi_1(X^*, x_0)$ in $\mathrm{GL}_q(\boldsymbol{C})$ is given. If E is a flat vector bundle on $X-D=X^*$ associated with ρ, then E can be uniquely extended to a holomorphic vector bundle E_1 on $X-\mathrm{Sing}(D)$ satisfying* (M.1) *and* (M.2); *moreover the direct image $j_*(\mathcal{O}(E_1))$ is a locally free sheaf on X.*

2.3. We consider the general case of $\dim X \geqq 3$. Let us recall the definition of absolute gap-sheaves. Suppose that $\mathcal{S}$ is a coherent analytic sheaf on a complex manifold X. We define the sheaf $\mathcal{S}^{[d]}$ on X by the following presheaf:

$$U \longrightarrow \varinjlim_{A \in \mathfrak{A}_d(U)} \Gamma(U-A, \mathcal{S}) ,$$

where $\mathfrak{A}_d(U)$ is the directed set of all analytic subset of U of $\dim A \leqq d$. We call $\mathcal{S}^{[d]}$ the *d-th absolute gap-sheaf* of $\mathcal{S}$. Let $D=D_1 \times D_2 \subset \boldsymbol{C}^{n-2} \times \boldsymbol{C}^2 = \boldsymbol{C}^n(z_1, \cdots, z_n)$ be a polydisc centered at the origin, where $(z_1, \cdots, z_n)$ is the coordinate system of $\boldsymbol{C}^n$. Put $V=\{z \in D \mid z_{n-1}=z_n=0\}$, and let $\mathcal{F}$ be a coherent analytic sheaf on $D-V$. For any $t \in D_1$, we denote the analytic restriction of $\mathcal{F}$ to the linear subspace $\{z \in \boldsymbol{C}^n \mid z_1=t_1, \cdots, z_{n-2}=t_{n-2}\}$ by

$$\mathcal{F}(t) := \mathcal{F} \otimes_{\mathcal{O}_{D-V}} (\mathcal{O}_{D-V}/(z_1-t_1, \cdots, z_{n-2}-t_{n-2})\mathcal{O}_{D-V}) .$$

We use the following:

LEMMA 1 (Y.-T. Siu [16], p. 243). *Let $\mathcal{F}$ be a coherent analytic sheaf on $D-V$ such that $\mathcal{F}^{[n-2]}=\mathcal{F}$. Suppose that $\mathcal{F}(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times D_2$ for any $t \in D_1$. Then $\mathcal{F}$ can be extended uniquely to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on $D=D_1 \times D_2$ satisfying the condition $\tilde{\mathcal{F}}^{[n-2]}=\tilde{\mathcal{F}}$.*

Using this lemma, we shall prove the following theorem.

THEOREM 1. *Let X be a connected complex manifold and let D be a divisor of X. We assume that a representation ρ of $\pi_1(X-D, x_0)$ in $\mathrm{GL}_q(\boldsymbol{C})$ is given. Let E be the flat vector bundle associated with ρ. Then E can be extended to the unique vector bundle E_1 on $X-\mathrm{Sing}(D)$ satisfying the conditions* (M.1) *and* (M.2) *in* n°2.1. *Moreover $\mathcal{O}(E_1)$ can be extended to a coherent analytic sheaf on X, in particular $j_*(\mathcal{O}(E_1))$ is coherent.*

PROOF. Let $S_1=\mathrm{Sing}(D)$, $S_2=\mathrm{Sing}(S_1)$, $\cdots$, $S_k=\mathrm{Sing}(S_{k-1})$ be a decreasing sequence of analytic subset of X where $\dim S_i = n_i$ for $i=1, \cdots, k$ and S_k is smooth. Write $\mathscr{F}_1 := \mathcal{O}(E_1)$. First we show the following:

LEMMA 2. *The locally free sheaf $\mathscr{F}_1$ on $X-S_1$ can be extended uniquely to a coherent analytic sheaf $\mathscr{F}_2$ on $X-S_2$ satisfying $\mathscr{F}_2^{[n-2]} = \mathscr{F}_2$.*

PROOF OF LEMMA 2. Let $x_0 \in S_1 - S_2$, then x_0 is a smooth point of S_1. There exists a local coordinate system $(z_1, \cdots, z_n)$ is a small neighborhood U of x_0 such that $U \cap S_2 = \varnothing$, $\{z_1 = \cdots = z_{n-1} = 0\} \cap D \cap U = \{x_0\}$ and $U \cap S_1 = \{z_{n_1+1} = \cdots = z_n\} = 0$, where $x_0 = (0, \cdots, 0)$. Hence there exists a small polydisc

$$\Delta = \{z \in U \mid |z_i| < \varepsilon_i,\ i=1, \cdots, n\}$$

as follows:

1) Put $\Delta' = \{(z_1, \cdots, z_{n-1}) \mid |z_i| < \varepsilon_i,\ i=1, \cdots, n-1\}$ and $\Delta'' = \{z_n \in \boldsymbol{C} \mid |z_n| < \varepsilon_n\}$ and let $\pi: \Delta \cap D \to \Delta'$ be a holomorphic map induced by the natural projection: $\Delta \to \Delta'$. Then π is proper.

2) Write $\Delta_1 = \{(z_1, \cdots, z_{n-2}) \mid |z_i| < \varepsilon_i,\ i=1, \cdots, n-2\}$ $\Delta_2 = \{(z_{n-1}, z_n) \mid |z_i| < \varepsilon_i,\ i=n-1, n\}$ and $V = \{z \in \Delta \mid z_{n-1} = z_n = 0\}$. Then $\Delta \cap S_1 \subset V$. Since $\mathscr{F}_1$ is *locally free on $\Delta - V$*, we have $\mathscr{F}_1^{[n-2]} = \mathscr{F}_1$ on $\Delta - V$ by the definition of absolute $(n-2)$-th gap-sheaves and Hartogs' continuation theorem. Let $t \in \Delta_1$ and put $D(t) := (\{t\} \times \Delta_2) \cap D$. Since π is proper, we have $D(t) \subsetneqq \Delta_2$, i.e., $D(t)$ is a divisor of Δ_2. Suppose that $f(x) = 0$ is a defining equation of D in Δ. Then, after some linear change of coordinate of $(z_1, \cdots, z_n)$ if necessary, (Write $f(x)$ in the form of Weierstrass polynomial and consider the discriminant of $f(x)$.) it follows that

1) $f(t, z_{n-1}, z_n) = 0$ is a defining equation of $D(t)$,

2) either $\partial f(t, z_{n-1}, z_n)/\partial z_{n-1} \neq 0$ or $\partial f(t, z_{n-1}, z_n)/\partial z_n \neq 0$ at a smooth point u of $D(t)$. Thus *(t, u) is a smooth point of D if u is a smooth point of $D(t)$*. Put $(\{t\} \times \Delta_2)^* := \{t\} \times \Delta_2 - \operatorname{Sing}(D(t))$. Then the sheaf $\mathscr{F}_1(t)$ is isomorphic to $\mathcal{O}(E_1|_{(\{t\} \times \Delta_2)^*})$ where $E_1|_{(\{t\} \times \Delta_2)^*}$ is the restriction of the vector bundle E_1 to $(\{t\} \times \Delta_2)^*$. Since E_1 is a flat vector bundle on $X-D$, $E_1|_{\{t\} \times \Delta_2 - D(t)}$ is also a flat vector bundle. On the other hand, there is a unique connection ∇ on E_1 satisfying (M.1), (M.2) and the condition "$\operatorname{Ker} \nabla = \boldsymbol{C}(E_1)$ on $X-D$". So the integrable meromorphic connection ∇' is induced on $E_1|_{(\{t\} \times \Delta_2)^*}$ for which (M.1), (M.2) and the condition "$\operatorname{Ker} \nabla' = \boldsymbol{C}(E_1|_{\{t\} \times \Delta_2 - D(t)})$ on $\{t\} \times \Delta_2 - D(t)$" are satisfied. In fact, suppose that $u \in D(t)$ is a smooth point of $D(t)$. Then (t, u) is a smooth point of D; hence there is a small neighborhood N of (t, u) in Δ such that $N \cap S_1 = \varnothing$ and $N \cap (\{t\} \times \Delta_2) \cap \operatorname{Sing}(D(t)) = \varnothing$. For an arbitrary holomorphic frame $e = (e_1, \cdots, e_q)$ of E_1 on N, we can write $\nabla e_i = \sum_{j=1}^{q} \omega_{ji} e_j$. Let $N' =$

$N\cap(\{t\}\times\Delta_2)$ and let $e'=e|_{N'}$ be the restriction of the frame e to N', which is the frame of $E_1|_{N'}$ on N'. By the definition of ∇', we see that $\nabla' e'_i=\sum_{j=1}^{q}(\omega_{ji}|_{N'})e'_j$. Thus $\omega_{ij}|_{N'}$ has at most logarithmic pole along $N'\cap D(t)$, and the eigenvalues $\alpha_1,\cdots,\alpha_q$ of $(\mathrm{res}(\omega_{ij}|_{N'}))$ satisfy the inequality $0\leqq\mathrm{Re}\,\alpha_i<1$ for $i=1,\cdots,q$. Hence the pair $(E_1|_{(\{t\}\times\Delta_2)^*},\nabla')$ satisfies the conditions (M.1) and (M.2). Applying the Proposition 2 to $E_1|_{(\{t\}\times\Delta_2-D(t))}$ we see that $\mathscr{F}_1(t)$ can be extended to a coherent analytic sheaf on $\{t\}\times\Delta_2$. Thus all the conditions of Lemma 1 are satisfied. So $\mathscr{F}_1$ can be extended to a coherent analytic sheaf $\tilde{\mathscr{F}}_1$ on Δ satisfying $\tilde{\mathscr{F}}_1^{[n-2]}=\tilde{\mathscr{F}}_1$. On the other hand, since this extension is unique by Lemma 1, we can glue $\tilde{\mathscr{F}}_1$ to get the coherent analytic sheaf $\mathscr{F}_2$ on $X-S_2$. Thus Lemma 2 is proved. Q.E.D.

LEMMA 3. *Let $\mathscr{F}_i$ be a coherent analytic sheaf on $X-S_i$ constructed inductively from $\mathscr{F}_1$ satisfying $\mathscr{F}_i^{[n-2]}=\mathscr{F}_i$. Then $\mathscr{F}_i$ can be extended uniquely to a coherent analytic sheaf $\mathscr{F}_{i+1}$ on $X-S_{i+1}$ which satisfies $\mathscr{F}_{i+1}^{[n-2]}=\mathscr{F}_{i+1}$.*

PROOF OF LEMMA 3. Let $x_0\in S_i-S_{i+1}$. As in Lemma 2, there exists a local coordinate system $(z_1,\cdots,z_n)$ in a small neighborhood U of x_0 in X such that $U\cap S_{i+1}=\emptyset$, $\{z_1=\cdots=z_{n-1}=0\}\cap U\cap D=\{x_0\}$ and $S_i\cap U=\{z_{n_i+1}=\cdots=z_n=0\}$. Hence there exists a polydisc Δ in U centered at x_0 such that $\pi\colon\Delta\cap D\to\Delta'$ is proper, where π, Δ', and Δ are as in Lemma 2. Since $\dim S_i\leqq n-2$, we have that $S_i\cap\Delta\subset\{z_{n-1}=z_n=0\}$. Let $t\in\Delta_1$, then $(\{t\}\times\Delta_2)\cap D=D(t)$ is a divisor of $\{t\}\times\Delta_2$. In the same way as in Lemma 2, we have that $\mathscr{F}_i(t)$ is isomorphic to $\mathscr{O}(E_1|_{(\{t\}\times\Delta_2-D(t))})$ on $\{t\}\times\Delta_2-D(t)$ and that $\mathscr{F}_i(t)$ can be extended to a coherent analytic sheaf $\tilde{\mathscr{F}}_i$ satisfying $\tilde{\mathscr{F}}_i^{[n-2]}=\tilde{\mathscr{F}}_i$ on Δ. Gluing $\tilde{\mathscr{F}}_i$ at every point of S_i-S_{i+1}, $\mathscr{F}_i$ can be extended to a coherent analytic sheaf $\mathscr{F}_{i+1}$ on $X-S_{i+1}$ satisfying $\mathscr{F}_{i+1}^{[n-2]}=\mathscr{F}_{i+1}$. Q.E.D.

The proof of Theorem 1 is actually done by using Lemma 2 and Lemma 3 inductively. *This completes the proof of Theorem* 1.

§3. The Riemann-Hilbert problem on Stein manifolds.

3.1. Let X be a connected Stein manifold and let D be a divisor of X. Suppose that a representation ρ of $\pi_1(X-D,x_0)$ in $\mathrm{GL}_q(C)$ is given where x_0 is a base point of $X-D$. Let E be the flat vector bundle associated with ρ, and let E_1 be the unique vector bundle on $X-\mathrm{Sing}\,(D)$ satisfying the conditions (M.1) and (M.2). By Theorem 1, $\mathscr{O}(E_1)$ can be extended as a coherent analytic sheaf $\mathscr{F}$ on X. Let $D=\bigcup_{i\in I}D_i$ be the decomposition of D into its irreducible components and let $x_i\in D_i-\mathrm{Sing}\,(D)$. Then $V=\{x_i\in X\mid i\in I\}$ is a discrete point set of X, and consequently a zero-dimensional analytic subset of X. Let us take an element $\varphi\in\Gamma(X,\mathscr{F})$. We denote by $\mathfrak{m}_{X,x_i}$ the maximal ideal of the local ring $\mathscr{O}_{X,x_i}$ at x_i, and let φ_{x_i} be the germ at x_i defined by φ. Noting that $\mathscr{F}_{x_i}=\mathscr{O}(E_1)_{x_i}$, the quotient $\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i}$ is isomorphic to C^q. We will denote by $\varphi(x_i)$ the residue class of $\varphi_{x_i} \bmod \mathfrak{m}_{X,x_i}\mathscr{F}_{x_i}$ in C^q and $\varphi(x_i)$ is said to be the value of φ at x_i.

LEMMA 4. *There exists a global section $\varphi\in\Gamma(X,\mathscr{F})$ which has the prescribed value in $\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i}\cong C^q$ at every point $x_i\in V$.*

PROOF. Let $\mathscr{I}$ be the coherent analytic sheaf of ideals defined by V, then we have the exact sequence of sheaves

$$0\longrightarrow\mathscr{I}\longrightarrow\mathscr{O}_X\xrightarrow{\ p\ }\mathscr{O}_X/\mathscr{I}\longrightarrow 0$$

where p is the natural projection. Making tensor product with $\mathscr{F}$, we have the exact sequence

$$\mathscr{I}\otimes_{\mathscr{O}_X}\mathscr{F}\longrightarrow\mathscr{O}_X\otimes_{\mathscr{O}_X}\mathscr{F}\xrightarrow{p\otimes 1}(\mathscr{O}_X/\mathscr{I})\otimes_{\mathscr{O}_X}\mathscr{F}\longrightarrow 0\ .$$

Since $\mathrm{Ker}\,(p\otimes 1)=:\mathscr{K}$ is coherent, and $(\mathscr{O}_X/\mathscr{I})\otimes_{\mathscr{O}_X}\mathscr{F}$ is isomorphic to $\amalg_{i\in I}(\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i})$, where $\amalg$ means disjoint union, we have the exact sequence

$$0\longrightarrow\mathscr{K}\longrightarrow\mathscr{F}\longrightarrow\coprod_{i\in I}(\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i})\longrightarrow 0\ .$$

By Theorem B of Oka-Cartan-Serre on Stein manifolds, we have $H^1(X,\mathscr{K})=0$; hence $\Gamma(X,\mathscr{F})\to\amalg_{i\in I}(\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i})$ is surjective. This is to be proved. Q.E.D.

Choose q linearly independent vectors in $C^q(\cong\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i})$ and apply Lemma 4. Then there exist global sections $\varphi_1,\cdots,\varphi_q\in\Gamma(X,\mathscr{F})$ such that the value $\varphi_1(x_i),\cdots,\varphi_q(x_i)$ are linearly independent in C^q at every point $x_i\in V$. Put $X':=X-\mathrm{Sing}\,(D)$. Since $\mathscr{F}|_{X'}=\mathscr{O}(E_1)$, φ_α can be considered as a global section of $\mathscr{O}(E_1)$. Let $\mathfrak{U}=\{U_j\}$ be a sufficiently fine open covering of X' and let $\{g_{jk}\}$ be the transition functions of E_1 with respect to $\mathfrak{U}$, where g_{jk} is $\mathrm{GL}_q(C)$-valued holomorphic function on $U_j\cap U_k$. Then a global section φ_α of E_1 is identified with collection $\{\varphi_{\alpha,j}\}$ where $\varphi_{\alpha,j}={}^t(\varphi^1_{\alpha,j},\cdots,\varphi^q_{\alpha,j})$ is C^q-valued holomorphic function on U_j such that $\varphi_{\alpha,j}=g_{jk}\varphi_{\alpha,k}$ on $U_j\cap U_k$, and the values $\varphi_\alpha(x_i)\in\mathscr{F}_{x_i}/\mathfrak{m}_{X,x_i}\mathscr{F}_{x_i}$ $(x_i\in U_j)$ is identified with the value $\varphi_{\alpha,j}(x_i)$ of the holomorphic function $\varphi_{\alpha,j}$ on U_j. The set $\Psi_j=(\varphi_{1,j}, \cdots, \varphi_{q,j})$ can be considered as a (q, q)-matrix-valued holomorphic function on U_j. On the other hand, we have $\Psi_j=g_{jk}\Psi_k$ on $U_j\cap U_k$. So, putting $\psi_j:=\det\Psi_j$, we see that $\psi_j=(\det g_{jk})\psi_k$ in $U_j\cap U_k$. Let G be the line bundle defined by the transition functions $\{\det g_{jk}\}$, i.e., $G=\{\det g_{jk}\}\in Z^1(\mathfrak{U}, \mathcal{O}^*_{X'})$. Then we have $\psi:=\{\psi_j\}\in\Gamma(X', \mathcal{O}(G))$. Since the values $\varphi_1(x_i), \cdots, \varphi_q(x_i)$ are linearly independent in $\boldsymbol{C}^q$, it follows that $\psi(x_i)\neq 0$ at every point $x_i\in V$; hence $A':=\{x\in X'\mid\psi(x)=0\}$ defines either a divisor or an empty set. Since $X-X'=\operatorname{Sing}(D)$ is an analytic subset of X *of codimension at least two* at every point of Sing (D), the closure $\bar{A}'$ of A' in X is a divisor of X *by the continuation theorem of Thullen* [17]. Thus we have the following:

LEMMA 5. *There exist a divisor A of X and q global sections $s_1, \cdots, s_q\in\Gamma(X', \mathcal{O}(E_1))$ of E_1 such that $(s_1, \cdots, s_q)$ is a frame of E_1 on $X'-A$ and such that $D_i\not\subset A$ for any irreducible component of D.*

3.2. Let ∇ be the unique connection on E_1 satisfying (M.1) and (M.2) such that $\operatorname{Ker}\nabla=\boldsymbol{C}(E)$ on $X-D$. Let $s_1, \cdots, s_q\in\Gamma(X', \mathcal{O}(E_1))$ be as above. We write ∇s_i on $X'-A$ in the form:

$$\nabla s_i=\sum_{j=1}^{q}\Omega_{ji}s_j \quad \text{for} \quad i=1, \cdots, q\,.$$

By (M.1) Ω_{ij} has at most logarithmic pole along $(X'-A)\cap D$.

LEMMA 6. *Ω_{ij} is a meromorphic form on X for $i, j=1, \cdots, q$.*

PROOF. Let $x\in(A-D)\cap X'$; then one can find a small open neighborhood U of X such that there is a holomorphic frame $e=(e_1, \cdots, e_q)$ of E_1 on U and that $U\cap D=\emptyset$. We can write $s_i=\sum_{j=1}^q h_{ij}e_j$ where $h_{ij}\in\Gamma(U, \mathcal{O}_U)$. Then the matrix $h:=(h_{ij})$ is non-singular at every point of $U-(A\cap U)$. We write $\nabla e_i=\sum_{j=1}^q\omega_{ji}e_j$ for $i=1, \cdots, q$, where ω_{ji} is a holomorphic one-form on U. Then we have

$$\begin{aligned}\nabla s_i&=\nabla\Big(\sum_{j=1}^{q}h_{ij}e_j\Big)\\&=\sum_{j=1}^{q}dh_{ij}e_j+\sum_{j=1}^{q}h_{ij}\nabla e_j\\&=\sum_{j=1}^{q}\Big(dh_{ij}+\sum_{k=1}^{q}h_{ik}\omega_{jk}\Big)e_j\,.\end{aligned}$$

On the other hand, on $U-(U\cap A)$, we have

$$\nabla s_i=\sum_{j=1}^{q}\Omega_{ji}s_j=\sum_{j=1}^{q}\Big(\sum_{k=1}^{q}\Omega_{ki}h_{kj}\Big)e_j\ ;$$

hence, on $U-(U\cap A)$, we obtain

$$dh_{ij}+\sum_{k=1}^{q}h_{ik}\omega_{jk}=\sum_{k=1}^{q}\Omega_{ki}h_{kj}\quad\text{for}\quad i,j=1,\cdots,q\ .$$

The above equation can be written in the matrix notation,

$$dh+h\cdot{}^t\omega={}^t\Omega\cdot h\ ,$$

or

$$(3.1)\qquad {}^t\Omega=(dh)\cdot h^{-1}+h\cdot{}^t\omega\cdot h^{-1}\quad\text{on}\quad U-(U\cap A)\ .$$

Since h^{-1} has at most pole along $U\cap A$, so has ${}^t\Omega$, i.e., Ω_{ij} is a meromorphic one-form on $X-\mathrm{Sing}(D)\cup(A\cap D)$. We know by Lemma 5 codim $(A\cap D)\geqq 2$ and codim $(\mathrm{Sing}(D))\geqq 2$, so Ω_{ij} *is extended to a meromorphic one-form on* X *by the continuation theorem of Levi.* Q.E.D.

Let Ω_{ij} and $s_1,\cdots,s_q\in\Gamma(X',\mathcal{O}(E_1))$ be as above and let $u=\sum_{i=1}^{q}y_is_i$ be a local section of $\mathcal{O}(E_1)$ around $x\in X-(A\cup D)$. From the relation

$$\nabla u=\sum_{i=1}^{q}\Big(dy_i+\sum_{j=1}^{q}\Omega_{ij}y_j\Big)s_i\ ,$$

it follows that u *is a horizontal section of* ∇ *if and only if* $u=\sum_{i=1}^{q}y_is_i$ *satisfies the total differential equation*

$$(3.2)\qquad d\begin{pmatrix}y_1\\ \vdots\\ y_q\end{pmatrix}+\begin{pmatrix}\Omega_{11}&\cdots&\Omega_{1q}\\ &\cdots&\\ \Omega_{q1}&\cdots&\Omega_{qq}\end{pmatrix}\begin{pmatrix}y_1\\ \vdots\\ y_q\end{pmatrix}=0\ .$$

Since $\mathrm{Ker}\,\nabla=C(E)$ on $X-D$ and $(s_1,\cdots,s_q)$ is a frame of E_1 on $X-(A\cup D)$, we see that the equation (3.2) is completely integrable on $X-(A\cup D)$. Let $\mathcal{S}$ be the sheaf of germs of local solutions of (3.2); then it follows that $\mathcal{S}$ is locally constant sheaf on $X-(A\cup D)$ and that $\mathcal{S}$ is isomorphic to $C(E)$ on $X-(A\cup D)$ by the map

$$(y_i)\in\mathcal{S}\longrightarrow\sum_{i=1}^{q}y_is_i\in C(E)\ .$$

LEMMA 7. *The total differential equation* (3.2) *has a regular singularity along* $A\cup D$; *moreover* A *is the apparent singularity of* (3.2).

PROOF. Let $x\in A-D$; then we can find a small neighborhood U of

x in X such that there is a holomorphic frame $e=(e_1, \cdots, e_q)$ of E_1 on U and that $U\cap D=\emptyset$. If we write $\nabla e_i=\sum_{j=1}^q \omega_{ji}e_j$ and take a horizontal section $u=\sum_{i=1}^q u_i e_i$ of ∇ on U, then we have

$$0=\nabla u=\sum_{i=1}^q \Big(du_i+\sum_{j=1}^q \omega_{ij}u_j\Big)e_i ,$$

that is,

$$du_i+\sum_{j=1}^q \omega_{ij}u_j=0 \quad \text{for} \quad i=1, \cdots, q . \tag{3.3}$$

If we write $u=\sum_{i=1}^q y_i s_i$ and $s_i=\sum_{j=1}^q h_{ij}e_j$, then we have $u_i=\sum_{j=1}^q h_{ij}y_j$. This can be written as

$$u={}^th\cdot y \quad \text{or} \quad y={}^th^{-1}\cdot u ,$$

where $u={}^t(u_1, \cdots, u_q)$, $y={}^t(y_1, \cdots, y_q)$ and $h=(h_{ij})$. Thus we have, in matrix notation,

$$\begin{aligned} dy+\Omega y &= d({}^th^{-1})+{}^th^{-1}du+\Omega\,{}^th^{-1}u \\ &= {}^th^{-1}\{du+({}^th\cdot\Omega\cdot{}^th^{-1}-(d\,{}^th)\,{}^th^{-1})u\} \\ &= {}^th^{-1}(du+\omega u) \qquad \text{(by (3.1))} \\ &= 0 . \end{aligned}$$

It follows that *if* u *is a local solution of* (3.3) *on* U, *then* $y={}^th^{-1}u$ *is a solution of* (3.2) *on* $U-(A\cap U)$. Since (3.3) is completely integrable on U, this means that A is the apparent singularity of equation (3.2). It follows from the condition (M.1) that Ω has at most logarithmic pole along $Z:=(D-\mathrm{Sing}\,(D))-A$; hence the equation (3.2) has a regular singularity along Z. From Lemma 5, we see that A does not contain any irreducible component of D. So by a result of P. Deligne ([6], p. 85), D is the regular singularity of the equation (3.2). Q.E.D.

Considering the proof of Lemma 5, we suppose that A does not contain the base point $x_0\in X-D$. Take q linearly independent solutions $f_1(x), \cdots, f_q(x)$ of (3.2) at x_0. For a closed curve γ in $X-(A\cup D)$ starting from x_0, we have (See §2.1.)

$$\gamma_*[f_1, \cdots, f_q]=[f_1, \cdots, f_q]\mu([\gamma]) ,$$

where $[\gamma]\in\pi_1(X-(A\cup D), x_0)$ and $\mu([\gamma])\in \mathrm{GL}_q(C)$. μ is called *the monodromy representation of the equation* (3.2). Let $j: X-(A\cup D)\to X-D$ be the canonical injection and let $j_*: \pi_1(X-(A\cup D), x_0)\to\pi_1(X-D, x_0)$ be the induced surjective homomorphism. Since A is the apparent singularity of (3.2), μ is naturally extended to a homomorphism

$$\hat{\mu}: \pi_1(X-D, x_0) \longrightarrow \mathrm{GL}_q(\boldsymbol{C})$$

such that $\hat{\mu}\circ j_* = \mu$ and that

$$\gamma_*[f_1, \cdots, f_q] = [f_1, \cdots, f_q]\hat{\mu}([\gamma]) \quad \text{for} \quad [\gamma] \in \pi_1(X-D, x_0)\ .$$

Since the monodromy representation of (3.2) is, by the definition, the same as that of the locally constant sheaf $\mathscr{S}$ and since $\mathscr{S}$ is isomorphic to $\boldsymbol{C}(E)$ on $X-(A\cup D)$, we see that $\hat{\mu}=\rho$, choosing the independent solutions of (3.2) properly. Thus we have the following:

THEOREM 2. *Let X be a Stein manifold and let D be a divisor of X. Suppose that a representation ρ of $\pi_1(X-D, x_0)$ in* $\mathrm{GL}_q(\boldsymbol{C})$ *is given. Then we can construct a total differential equation* (3.2) *as follows*:

1) *there exists a divisor A of X such that A does not contain any irreducible component of D.*

2) *the equation* (3.2) *is completely integrable on $X-(A\cup D)$; moreover A is the apparent singularity of* (3.2).

3) *the monodromy representation of* (3.2) *coinsides with the given representation ρ.*

3.3. On *two-dimensional Stein manifold X*, we could solve, by Proposition 2, the Riemann-Hilbert problem *without apparent singularity under some topological condition on X*. Let E, E_1, and ρ be as above and let $j: X-\mathrm{Sing}(D)\rightarrow X$ be the canonical injection. Then by Proposition 2, we have that $j_*(\mathscr{O}(E_1))$ is a *locally free sheaf on X*; so one has $j_*(\mathscr{O}(E_1))=\mathscr{O}(G)$ for a certain holomorphic vector bundle G on X. By a result of A. Andreotti and T. Frankel [2], X is *of the same homotopy type as a two-dimensional CW-complex*. So from a theorem of F. Peterson [13], it follows that a continuous complex vector bundle F on X of rank q is *trivial if and only if the first Chern class $c_1(F)$ of F is equal to zero*. Thus, by the *Oka principle* (H. Grauert [7]), $j_*(\mathscr{O}(E_1))$ *is a free sheaf if and only if $c_1(G)=0$*. So, we can find a global frame $s=(s_1, \cdots, s_q)$ of G on X. Hence, if we write $\nabla s_i=\sum_{j=1}^{q}\Omega_{ji}s_j$, the equation (3.2) has the regular singularity only along D and does not have the apparent singularity. Thus we obtain the following:

PROPOSITION 3. *Let X be a connected two-dimensional Stein manifold and let D, E, E_1, and ρ be as above. Then we obtain $j_*(\mathscr{O}(E_1))=\mathscr{O}(G)$ for a certain holomorphic vector bundle G on X. If $c_1(G)=0$, then we can construct a completely integrable total differential equation* (3.2) *which is regular singular along D and does not have the apparent singularity, and furthermore whose monodromy representation coincides with the given ρ.*

By Proposition 3, it follows easily the following theorem.

THEOREM 3. *Let X be a connected two-dimensional Stein manifold. If $H^2(X, \boldsymbol{Z})=0$, then for any divisor and representation ρ of $\pi_1(X-D, x_0)$ in $\mathrm{GL}_q(\boldsymbol{C})$, we can always find a solution without apparent singularity to the Riemann-Hilbert problem.*

REMARK. In the case of Theorem 3, let $\Omega=(\Omega_{ij})$ be the connection matrix of the equation (3.2). From the construction of the equation (3.2), we see that each Ω_{ij} is a meromorphic form with *generically logarithmic poles* along D. This notion was introduced by K. Saito [14].

§4. **A remark to a work of K. Aomoto [1]**——The Riemann-Hilbert problem in the restricted sense on two-dimensional manifolds.

4.1. Let X be a connected *two-dimensional* complex manifold and let D be a divisor of X. Let ρ be a representation of the group $\pi_1(X-D, x_0)$ in $\mathrm{GL}_q(\boldsymbol{C})$. *Suppose that $\rho(\pi_1(X-D, x_0))$ is contained in a maximal unipotent subgroup $\mathfrak{U}(q)$ of* $\mathrm{GL}_q(\boldsymbol{C})$; that is, $\mathfrak{U}(q)$ is a subgroup conjugate to the closed subgroup $\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}_q(\boldsymbol{C})\right\}$ in $\mathrm{GL}_q(\boldsymbol{C})$. Let ρ be a representation of $\pi_1(X-D, x_0)$ in $\mathfrak{U}(q)$. After K. Aomoto [1], we shall call the *Riemann-Hilbert problem in the restricted sense* the problem of constructing the total differential equation (3.2) which is regular singular along D and has the above given monodromy ρ.

Let E be the flat vector bundle associated with ρ where ρ is a representation of $\pi_1(X-D, x_0)$ in $\mathfrak{U}(q)$. By a result to P. Deligne ([6], p. 91), E can be extended to a holomorphic vector bundle E_1 on $X-\mathrm{Sing}(D)$ such that, choosing a sufficiently fine open covering $\mathfrak{B}=\{V_j\}_{j\in J}$ of $X-\mathrm{Sing}(D)$, the transition functions f_{jk} of E_1 are $\mathfrak{U}(q)$-valued holomorphic functions on $V_j\cap V_k$ for any $j, k\in J$. From Proposition 2 of §2, it follows that $j_*(\mathcal{O}(E_1))$ is a locally free sheaf on X where $j: X-\mathrm{Sing}(D)\to X$ is the canonical injection. Let $\tilde{E}$ be the holomorphic vector bundle on X corresponding to $j_*(\mathcal{O}(E_1))$. Then by the same argument as above (See [6], p. 91.), choosing a sufficiently fine suitable open covering $\mathfrak{W}=\{W_j\}$ of X, we have that the transition functions g_{jk} are $\mathfrak{U}(q)$-valued holomorphic functions on each $W_j\cap W_k$.

4.2. Now we shall prepare the following elementary

LEMMA 8. *Let X be as above and let V be a holomorphic vector bundle with the structure group* $\mathfrak{U}(q)$. *If* $H^1(X, \mathcal{O}_X)=0$, *then the vector bundle V is holomorphically trivial.*

PROOF. Without loss of generality, we can suppose that $\mathfrak{U}(q)$ is the following subgroup $\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}_q(C)\right\}$ of $\mathrm{GL}_q(C)$. We proceed by the induction on the rank of vector bundles. When $q=1$, there is nothing to prove. We suppose that Lemma 8 is true for all holomorphic vector bundle with structure group $\mathfrak{U}(m)$ of rank less than q. Choosing a sufficiently fine *Stein* covering $\mathfrak{W}=\{W_j\}_{j\in J}$, we may suppose that the transition functions $\{f_{jk}\}$ of V are $\mathfrak{U}(q)$-valued holomorphic functions on $W_j\cap W_k$ and they satisfy the cocycle conditions

$$f_{ij}\cdot f_{jk}=f_{ik} \quad \text{on} \quad W_i\cap W_j\cap W_k\ ;$$

that is, $\{f_{jk}\}\in Z^1(\mathfrak{W}, \mathfrak{U}(q))$. If we write each f_{jk} in the following form

$$f_{jk}=\left(\begin{array}{c|c} 1 & a_{jk} \\ \hline 0 & g_{jk} \end{array}\right)$$

where $\{a_{jk}\}\in C^1(\mathfrak{W}, \mathcal{O}_X^{q-1})$ and $\{g_{jk}\}\in C^1(\mathfrak{W}, \mathfrak{U}(q-1))$, the above cocycle conditions can be rewritten in the following form:

$$\begin{cases} a_{ij}g_{jk}+a_{jk}=a_{ik} \\ \quad g_{ij}g_{jk}=g_{ik} \end{cases} .$$

By the hypothesis of induction, there exists a zero cochain $\{g_j\}\in C^0(\mathfrak{W}, \mathfrak{U}(q-1))$ such that $g_{jk}=g_jg_k^{-1}$ on $W_j\cap W_k$. On the other hand, putting $\hat{a}_{ij}=a_{ij}g_j$, we have that

$$\hat{a}_{ij}g_j^{-1}g_{jk}+\hat{a}_{jk}g_k^{-1}=\hat{a}_{ik}g_k^{-1}\ ;$$

hence, using the equation $g_j^{-1}g_{jk}=g_k^{-1}$, we conclude that $\{\hat{a}_{jk}\}$ satisfies the cocycle conditions

$$\hat{a}_{ij}+\hat{a}_{jk}=\hat{a}_{ik} \quad \text{on} \quad W_i\cap W_j\cap W_k\ .$$

Since $\{\hat{a}_{jk}\}\in Z^1(\mathfrak{W}, \mathcal{O}_X^{q-1})$ and $H^1(\mathfrak{W}, \mathcal{O}_X)=0$, there is a 0-cochain $\{a_i\}\in C^0(\mathfrak{W}, \mathcal{O}_X^{q-1})$ which satisfies the equation

$$\hat{a}_{jk}=a_j-a_k\ .$$

When we put

$$f_j=\left(\begin{array}{c|c} 1 & a_j \\ \hline 0 & g_j \end{array}\right) \quad \text{on} \quad W_j\,,$$

by a simple computation, we see that

$$\begin{aligned} f_j\cdot f_k^{-1}&=\left(\begin{array}{c|c} 1 & a_j \\ \hline 0 & g_j \end{array}\right)\left(\begin{array}{c|c} 1 & -a_k^{-1}g_k^{-1} \\ \hline 0 & g_k^{-1} \end{array}\right) \\ &=\left(\begin{array}{c|c} 1 & \hat{a}_{jk}g_k^{-1} \\ \hline 0 & g_{jk} \end{array}\right)=\left(\begin{array}{c|c} 1 & a_{jk} \\ \hline 0 & g_{jk} \end{array}\right) \\ &=f_{jk}\,. \end{aligned}$$

Hence it follows that the vector bundle V with transition functions $\{f_{jk}\}$ is holomorphically trivial on X. Q.E.D.

Now let us return to the situation of n° 4.1, and let the notations be as above. Since $\tilde{E}$ is the holomorphic vector bundle on X with structure group $\mathfrak{U}(q)$, by Lemma 8 we conclude that $\tilde{E}$ is holomorphically trivial provided that $H^1(X, \mathcal{O}_X)=0$. Thus we obtain the following:

THEOREM 4 (K. Aomoto, [1]). *Let X be a connected two-dimensional complex manifold. If $H^1(X, \mathcal{O}_X)=0$, then for any divisor D and any representation ρ of $\pi_1(X-D, x_0)$ in a maximal unipotent subgroup $\mathfrak{U}(q)$ of $\mathrm{GL}_q(\boldsymbol{C})$, we can always find a solution to the Riemann-Hilbert problem in the restricted sense without apparent singularity.*

COROLLARY. *If X is a compact two-dimensional Kähler manifold such that the first Betti number is zero*; i.e., $H^1(X, \boldsymbol{C})=0$, *then we can always find a solution to the Riemann-Hilbert problem in the restricted sense without apparent singularity for any divisor and any representation ρ of $\pi_1(X-D, x_0)$ in $\mathfrak{U}(q)$.*

REMARK. In the case of Theorem 4 and its Corollary, let $\Omega=(\Omega_{ij})$ be the connection matrix of the equation (3.2). By the same reason as the Remark to Theorem 3, we see that Ω_{ij} is a meromorphic form with *generically logarithmic poles* along D.

§5. Analytic covers and the associated monodromy.

5.1. Let us recall the definition of *analytic covers* and holomorphic functions on them. Let Y be a locally compact Hausdorff space and let

X be a complex manifold. *An analytic cover* is a triple (Y, π, X) (later on we will write this in the form $\pi: Y\to X$) such that

1) π is a proper continuous map of Y *onto* X with discrete fibers.

2) There are a divisor D of X and a positive integer q such that π is a q-sheeted topological covering map from $Y-\pi^{-1}(D)$ *onto* $X-D$.

3) $Y-\pi^{-1}(D)$ is dense in Y.

4) For any point $y\in\pi^{-1}(D)$ and any connected open neighborhood U of y, there exists an open neighborhood $U'\subset U$ such that $U'-\pi^{-1}(D)\cap U'$ is connected.

D is called the *critical locus* of analytic cover $\pi: Y\to X$ and q is called the *sheet number* of it. There is a unique complex structure on $Y-\pi^{-1}(D)$ such that $\pi: Y-\pi^{-1}(D)\to X-D$ is a locally biholomorphic map; hence, $Y-\pi^{-1}(D)$ will be regarded as the complex manifold with this structure. We recall the definition of complex analytic space in the sense of Behnke-Stein [3]. Let $\pi: Y\to X$ be an analytic cover, and let U be an open set in Y. A continuous complex-valued function $f(y)$ on U is, by definition, holomorphic on U if the restriction of $f(y)$ to $U-U\cap\pi^{-1}(D)$ is holomorphic in the usual sense on the open subset $U-U\cap\pi^{-1}(D)$ of the complex manifold $Y-\pi^{-1}(D)$. Let $\mathcal{O}_Y$ be the sheaf of germs of holomorphic functions on Y; then it follows that $(Y, \mathcal{O}_Y)$ is a $\boldsymbol{C}$-local ringed space. Let W be a Hausdorff space. A $\boldsymbol{C}$-local ringed space $(W, \mathcal{O}_W)$ is, by definition, a complex analytic space in the sense of Behnke-Stein (komplexe α-Raum in [10]) if there exists an open covering $W=\bigcup U_i$ such that $(U_i, \mathcal{O}_W|_{U_i})$ is isomorphic to a ringed space $(Y, \mathcal{O}_Y)$ as above, where Y is an analytic cover. As is noted in Introduction, H. Grauert and R. Remmert [10] and R. Kawai [11] proved that $(W, \mathcal{O}_W)$ is a normal complex analytic space in the sense of Cartan-Serre [5]. Our aim is to prove this theorem by using the Riemann-Hilbert problem. For this purpose, we shall study the relation between holomorphic functions on Y and representation of $\pi_1(X-D, x_0)$ where $\pi: Y\to X$ and D are as above and x_0 is a base point of $X-D$.

For later applications, we list the following standard result about holomorphic functions on analytic covers. Let $(Y, \mathcal{O}_Y)$ be as above, where $\pi: Y\to X$ is an analytic cover and we denote by $\mathcal{O}_{Y,y}$ the stalk of $\mathcal{O}_Y$ at $y\in Y$.

LEMMA 9 ([10], p. 264). *Suppose that $x\in D$ is a smooth point of D, and let $\pi^{-1}(x)=\{y_1, \cdots, y_t\}$. Then $\mathcal{O}_{Y,y_i}$ is a regular $\boldsymbol{C}$-local algebra for $i=1, \cdots, t$. Let $Y':=Y-\pi^{-1}(\mathrm{Sing}\,(D))$. Then $(Y', \mathcal{O}_Y|_{Y'})$ is a complex manifold which contains $Y-\pi^{-1}(D)$ as the open submanifold.*

LEMMA 10 ([10], p. 266). *Let π: $Y\to X$ be as above and let q be the sheet number of Y. Let $f(y)$ be a continuous functions on Y. $f(y)$ is holomorphic on Y if and only if there is a monic polynomial*

$$\omega(Z; x)=Z^q+a_1(x)Z^{q-1}+\cdots+a_q(x) \tag{5.1}$$

such that $\omega(f(x); x)=0$ on Y, where $a_i(x)$ is holomorphic on X.

LEMMA 11 ([10], p. 267). *Let A be an analytic subset of Y, and $f(y)$ be a holomorphic function on $Y-A$. Suppose that, for every point $y\in A$, there exists an open neighborhood U of x such that $f(y)$ is bounded on $U-(U\cap A)$. Then $f(y)$ can be extended uniquely to a holomorphic function on Y.*

5.2. Let π: $Y\to X$ be an analytic cover with critical locus D whose sheet number is q. By the definition of complex analytic spaces in the sense of Behnke-Stein, the problem is local, i.e., we can assume X to be a polydisc in $\boldsymbol{C}^n$, and it is sufficient to show the existence of a holomorphic function $f(y)$ separating arbitrary two points in $\pi^{-1}(x_0)$, $x_0\in X-D$.

In fact, let φ: $Y\to X\times \boldsymbol{C}$ be a holomorphic map defined by $\varphi(y)=(\pi(y), f(y))$. Since $f(y)$ is holomorphic on Y, there is, by Lemma 9, a monic polynomial (5.1) such that $\omega(f(y); x)=0$ on Y. Putting $S:=\varphi(Y)$, it follows that S is a hypersurface in $X\times \boldsymbol{C}$ defined by $S=\{(x, z)\in X\times \boldsymbol{C} \mid \omega(z, x)=0\}$. Let $\tilde{\mathscr{O}}_S$ be the sheaf of germs of weakly holomorphic functions on S and $\Delta(x)$ be the discriminant of the polynomial $\omega(Z; x)$. It is obvious that $D\subset\{x\in X \mid \Delta(x)=0\}$. Let p: $S\to X$ be the projection induced by the one to the first component $X\times\boldsymbol{C}\to X$. Since $f(y)$ separates the values of $\pi^{-1}(x_0)$, we see that $A:=\{x\in X\mid \Delta(x)=0\}\subsetneqq X$; hence A is a divisor of X. It is evident that

$$\varphi\colon Y-\pi^{-1}(A)\longrightarrow S-p^{-1}(A)$$

is biholomorphic map. Take a point $s_0\in p^{-1}(A)$ and let N be a small neighborhood of s_0. If $g(s)$ is a holomorphic function in $N-(N\cap p^{-1}(A))$ on which $g(s)$ is bounded, then by Lemma 10 $\varphi^*(g)$ is holomorphic on some components of $\pi^{-1}(p(N))$. Applying the argument to the inverse map φ^{-1}, we conclude that the direct image $\varphi_*(\mathscr{O}_Y)$ is isomorphic to $\tilde{\mathscr{O}}_S$. By the normalization theorem of Oka [12], there exists a normal complex analytic space $\tilde{S}$ and a proper holomorphic map τ: $\tilde{S}\to S$ such that $\tau_*(\mathscr{O}_{\tilde{S}})=\tilde{\mathscr{O}}_S$. By the above facts and (4) of the definition of analytic covers, we have $(Y, \mathscr{O}_Y)=(\tilde{S}, \mathscr{O}_{\tilde{S}})$; this was to be proved.

Later on, we suppose that X is a *polydisc* in C^n. We write $Y^*:=Y-\pi^{-1}(D)$ and $X^*:=X-D$. By the definition of the complex structure of Y^*, we can consider a holomorphic function $g(y)$ on Y^* as a many-valued holomorphic function on X^*. Using this fact, we obtain the relation between holomorphic functions on Y^* and representations of $\pi_1(X^*, x_0)$. We state this in detail. Let $\pi^{-1}(x_0)=\{y_1, \cdots, y_q\}$ and fix this numbering. Since $\pi: Y^*\to X^*$ is a finite unramified covering and since X^* is a Stein manifold, it follows that Y^* is a Stein manifold. Hence there exists a holomorphic function $g(y)$ on Y^* such that $g(y_i)=i$ for $i=1, \cdots, q$. If we choose a sufficiently small polydisc $U\subset X^*$ centered at x_0, we can speak of the branches of $g(y)$ on U. Thus let $g_i(x)$ be the branch of $g(y)$ on U such that $g_i(x_0)=i$. It follows that $g_i(x_0)$ can be continued analytically on X^*, but, in general, it is not single-valued. Consider the vector-valued function $\vec{g}(x)=(g_1(x), \cdots, g_q(x))$ on U. $\vec{g}(x)$ can be continued analytically on X^*; hence it is a many-valued function on X^*. We shall show that $\vec{g}(x)$ gives a representation of $\pi_1(X^*, x_0)$; let γ be a closed curve in X^* starting from x_0. Since $\pi: Y^*\to X^*$ is a topological covering, there are the paths γ_i starting from y_i such that $\pi(\gamma_i)=\gamma$. Let us denote by $x_{\gamma_*(i)}$ the end point of γ_i; then $\begin{pmatrix} 1, & \cdots, & q \\ \gamma_*(1), & \cdots, & \gamma_*(q) \end{pmatrix}$ is a permutation of q letters $\{1, \cdots, q\}$. It follows that the result of analytic continuation of $g_i(x)$ along γ is identified with that of $g(y)$ along γ_i if we consider $g_i(x)$ as the function element of $g(y)$ at y_i; hence we have the function element of $g(y)$ at $x_{\gamma_*(i)}$. Thus we obtain that the result of analytic continuation of $g_i(x)$ along γ is the element $g_{\gamma_*(i)}(x)$. Let S_q be the symmetric group of q letters $\{1, \cdots, q\}$ and let $e_i=(0, \cdots, \overset{i\text{-th}}{1}, \cdots, 0)$ $i=1, \cdots, q$ be the standard basis of C^q. We denote by $j: S_q\to \mathrm{GL}_q(C)$ the following standard faithful representation; for $\sigma\in S_q$,

$$j(\sigma)\Big(\sum_{i=1}^{q} u_i e_i\Big)=\sum_{i=1}^{q} u_i e_{\sigma(i)}\ ;$$

thus we have

$$j(\sigma)=(a_{kl}) \quad \text{where} \quad a_{kl}=\begin{cases} 1 & \text{if} \quad k=\sigma(l) \\ 0 & \text{otherwise}\ . \end{cases}$$

Let γ be a closed curve in X^* starting from x_0, and as above we denote by $\gamma_*(\vec{g})=(g_{\gamma_*(1)}, \cdots, g_{\gamma_*(q)})$ the result of analytic continuation of $\vec{g}=(g_1, \cdots, g_q)$ along γ. It follows that

$$(g_{\gamma_*(1)}, \cdots, g_{\gamma_*(q)})=(g_1, \cdots, g_q)\rho([\gamma])$$

if we write $\rho([\gamma])=j\left(\begin{pmatrix} 1, & \cdots, & q \\ \gamma_*(1), & \cdots, & \gamma_*(q)\end{pmatrix}\right)$.

LEMMA 12. *Let* $\rho\colon \pi_1(X^*, x_0)\to \mathrm{GL}_q(C)$ *be as above. Then* ρ *is a finite representation of* $\pi_1(X^*, x_0)$.

PROOF. Let γ_1 and γ_2 be closed curves in X^* starting from x_0. We have

$$\begin{aligned}(g_1, \cdots, g_q)\rho([\gamma_1]\cdot[\gamma_2]) &= (\gamma_1\cdot\gamma_2)_*(g_1, \cdots, g_q) \\ &= (\gamma_1)_*((g_1, \cdots, g_q)\rho([\gamma_2])) \\ &= (g_1, \cdots, g_q)\rho([\gamma_1])\rho([\gamma_2])\ ;\end{aligned}$$

hence we obtain

$$\rho([\gamma_1][\gamma_2])=\rho([\gamma_1])\rho([\gamma_2])\ . \qquad \text{Q.E.D.}$$

We call ρ the *monodromy representation associated with the analytic cover* $\pi\colon Y\to X$. Note that, by the definition of the permutation $\begin{pmatrix} 1, & \cdots, & q \\ \gamma_*(1), & \cdots, & \gamma_*(q)\end{pmatrix}$, ρ is determined by the topological property of the analytic cover.

REMARK. Let ρ be as above, and let E be the flat vector bundle associated with ρ. We can show that $\pi_*(\boldsymbol{C}_{Y^*})=\boldsymbol{C}(E)$, where $\boldsymbol{C}_{Y^*}$ is a $\boldsymbol{C}$-valued constant sheaf on Y^*.

5.3. Conversely, we consider a many-valued holomorphic function $\vec{h}(x)=(h_1(x), \cdots, h_q(x))$ on X^* satisfying $\gamma_*(\vec{h}(x))=\vec{h}(x)\rho([\gamma])$ for any closed curve γ in X^* starting from x_0.

LEMMA 13. *Let* $\vec{h}(x)$ *be as above and suppose that* Y^* *is connected. Write* $h(y)\colon =h_1(\pi(y))$ *in a small polydisc in* Y^* *centered at* y_1. *Then* $h(y)$ *can be continued analytically along any path in* Y^* *starting from* y_1; *moreover it determines a single-valued holomorphic function* $\tilde{h}(y)$ *on* Y^* *whose function element at* y_i *coincides with* $h_i(\pi(y))$ *for* $i=1, \cdots, q$.

PROOF. Let ℓ be any path in Y^* starting from y_1, and let $\ell'=\pi(\ell)$. Since $h_1(x)$ can be continued analytically along the curve ℓ', it is evident that so is $h(y)$; hence $h(y)$ determines a many-valued holomorphic function $\tilde{h}(y)$ on Y^*. Suppose that $\tilde{h}(y)$ is not single-valued. Then there exists a closed curve γ in Y^* such that the result of analytic continuation of $h(y)$ along γ is not equal to the element $h(y)$. Let $\pi(\gamma)=\gamma'$, and let γ_i be a path in Y^* starting from y_i and satisfying $\pi(\gamma_i)=\gamma'$. Note that γ_i is not always closed and that $\gamma_1=\gamma$. As in n° 4.2, let $g(y)$ be a holomorphic function on Y^* satisfying $g(y_i)=i$ for $i=1, \cdots, q$. Since $g(y)$ is single-valued on Y^*, we have that $\gamma'_*(g_1, \cdots, g_q)=(g_1, *, \cdots, *)$. Hence, by $\gamma'_*(\vec{g})=\vec{g}\rho([\gamma'])$, we can write $\rho([\gamma'])$ in the form

$$\begin{pmatrix} 1, 0, \cdots, 0 \\ 0 \quad * \\ 0 \end{pmatrix}.$$

Thus we have that

$$\gamma'_*(h_1, \cdots, h_q)=(h_1, \cdots, h_q)\begin{pmatrix} 1, 0, \cdots, 0 \\ 0 \quad * \\ 0 \end{pmatrix}$$

i.e., $\gamma'_*(h_1)=h_1$. This means that the result of analytic continuation of $h(y)$ along γ is equal to $h(y)$. This is a contradiction. Since Y^* is connected, there exists a path from y_1 to y_i. Let γ_i be the path in Y^* starting from y_i such that $\pi(\gamma_i)=\pi(\gamma)=\gamma'$. Note that $\gamma=\gamma_1$ and $\gamma'_*(g_1)=g_i$. Hence, in the same way as above, we see

$$\rho([\gamma'])=\begin{pmatrix} 0 & * \cdots * \\ 0 & \\ 1 & * \\ 0 & \end{pmatrix} \quad \text{(1 is the } (i, 1)\text{-element)}.$$

By $\gamma'_*(\vec{h})=\vec{h}\rho([\gamma'])$, we obtain $\gamma'_*(h_1)=h_i$; this means that the result of analytic continuation of $h(y)$ along γ is equal to the element $h_i(\pi(y))$.

Q.E.D.

Let $\tilde{h}(y)$ be a single-valued holomorphic function on Y^* as in Lemma 12. Suppose that $\tilde{h}(y)$ is locally bounded at every point of $\pi^{-1}(D')$ where $D':=D-\operatorname{Sing}(D)$. Let $Y':=Y-\pi^{-1}(\operatorname{Sing}(D))$ and $X':=X-\operatorname{Sing}(D)$; then $\pi: Y'\rightarrow X'$ is an analytic cover. From Lemma 10, it follows that $\tilde{h}(y)$ can be extended to the unique holomorphic function on Y', which is denoted by the same letter $\tilde{h}$. By Lemma 9, we obtain the monic polynomial

$$\omega(Z; x)=Z^q+a_1(x)Z^{q-1}+ \cdots +a_q(x) ,$$

where $a_i(x)$ is holomorphic on $X-\operatorname{Sing}(D)$ and $\omega(h(y); x)=0$ on Y'. Since $\operatorname{codim}(\operatorname{Sing}(D))\geqq 2$, by Hartogs' continuation theorem, $a_i(x)$ can be extended to the unique holomorphic function on X, which is denoted by $\hat{a}_i(x)$. From the equality $\hat{\omega}(\tilde{h}(y); x)=0$ on Y' (where $\hat{\omega}=\sum_{i=0}^{q} \hat{a}_i(x)Z^{q-i}$), it follows that $\tilde{h}(y)$ is locally bounded at any point of $\pi^{-1}(\mathrm{Sing}(D))$; hence by Lemma 10, $\tilde{h}(y)$ can be extended to the unique holomorphic function on Y. Thus we obtain the following:

PROPOSITION 4. *Let $\pi: Y \to X$ be an analytic cover and let $\rho: \pi_1(X-D, x_0) \to \mathrm{GL}_q(\boldsymbol{C})$ be the monodromy representation associated with the analytic cover. Suppose that there exists a many-valued holomorphic function $\vec{h}(x)=(h_1(x), \cdots, h_q(x))$ on X^* such that*

1) $\gamma_*(\vec{h})=\vec{h}\rho([\gamma])$ *for any* $[\gamma]\in\pi_1(X-D, x_0)$

and that

2) $h_i(x_0)\neq h_j(x_0)$ *for any* $i\neq j$.

Let $\tilde{h}(y)$ be the single-valued function on $Y-\pi^{-1}(D)$ defined in Lemma 13. If $\tilde{h}(y)$ is locally bounded at every point of $\pi^{-1}(D-\mathrm{Sing}(D))$, then $\tilde{h}(y)$ can be extended to the unique holomorphic function on Y. Hence we can construct the holomorphic function on Y which is desired at the beginning of n° 4.2.

§6. Existence of holomorphic functions on analytic covers and the Riemann-Hilbert problem.

Let $\pi: Y \to X$ be an analytic cover where X is a polydisc in $\boldsymbol{C}^n$, and let q be the sheet number of Y. Let X^*, X' etc. be as before. We shall solve the problem proposed at n° 5.1. Since the problem is local, we can suppose that the critical locus D of the analytic cover Y has finite irreducible components: $D=\bigcup_{i=1}^{m} D_i$ and that $Y-\pi^{-1}(D)$ is connected by (4) of the definition of analytic cover (see n° 5.1). Let $\rho: \pi_1(X-D, x_0)\to \mathrm{GL}_q(\boldsymbol{C})$ be the monodromy representation associated with Y. Since X is a Stein manifold, there exists, by Theorem 2, a total differential equation (3.2) as follows:

1) there is a divisor A of X such that $x_0 \notin A$, $D_i \not\subset A$ and (3.2) is regular singular along $A\cup D$; moreover A is the apparent singularity of (3.2).

2) If we choose properly, q linearly independent solutions $f_1, \cdots, f_q$ of (3.2) at x_0 we have that

$$\gamma_*[f_1, \cdots, f_q]=[f_1, \cdots, f_q]\rho([\gamma])$$

for any closed curve γ in $X-D$ starting from x_0.

Put $f_i(x)={}^t(f_{1i}(x), \cdots, f_{qi}(x))$, and we define $g_j(x):=(f_{j1}(x), \cdots, f_{jq}(x))$; thus we have

$$\gamma_*(g_j)=g_j\rho([\gamma]) \quad \text{for any} \quad [\gamma]\in\pi_1(X-D, x_0) .$$

Since $f_1, \cdots, f_q$ are linearly independent, so are $g_1, \cdots, g_q$; hence there are constants $c_i\in C(i=1, \cdots, q)$ such that, putting $\vec{h}:=\sum_{i=1}^{q} c_i g_i$, we have $\vec{h}(x_0)=(1, 2, \cdots, q)$ and $\gamma_*(\vec{h})=\vec{h}\rho([\gamma])$ for any $[\gamma]\in\pi_1(X-D, x_0)$. By Lemma 12, there exists a holomorphic function $\tilde{h}(y)$ on Y^* such that $\tilde{h}(y_i)=i$ for $i=1, \cdots, q$. Since the equation (3.2) is regular singular along $A\cup D$ and since π: $Y'\to X'$ is a finite covering by a result of P. Deligne ([6], p. 64-65 and p. 85), $\tilde{h}(y)$ has at most pole along $Y'\cap\pi^{-1}(A\cup D)$. By shrinking X slightly, if necessary, we can suppose that the number of irreducible components of A is finite; $A=\bigcup_{i=1}^{l} A_i$. Since the Cousin's second problem has always a solution on X, we can write A_i and D_j in the form $A_i=\{a_i(x)=0\}$ and $D_j=\{d_j(x)=0\}$ for $i=1, \cdots, l$ and $j=1, \cdots, m$, where a_i and d_j are holomorphic on X. Since $\tilde{h}(y)$ has at most pole along $Y'\cap\pi^{-1}(A\cup D)$, there are positive integers μ_i and ν_j such that $c(\pi(y))\tilde{h}(y)$ is holomorphic on Y' when we write $c(x)=\prod_{i=1}^{l}(a_i(x))^{\mu_i}\prod_{j=1}^{m}(d_j(x))^{\nu_j}$; hence by Proposition 4, $c(\pi(y))\tilde{h}(y)$ can be extended to the unique holomorphic function $H(y)$ on Y. Since $c(x_0)\neq 0$, we have $H(y_i)\neq H(y_j)$ for any $i\neq j$. Hence we have the following:

THEOREM 5. *Let π: $Y\to X$ be an analytic cover whose critical locus is D, where X is a polydisc in C^n. Let $x_0\in X-D$. Suppose that ρ: $\pi_1(X-D, x_0)\to \mathrm{GL}_q(C)$ is the monodromy representation associated with the analytic cover Y. Then, using a solution of the Riemann-Hilbert problem for the representation ρ, by shrinking Y slightly if necessary, we can construct a holomorphic function $g(y)$ on Y which separates arbitrary two points in $\pi^{-1}(x_0)$.*

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