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# On Regular Fréchet-Lie Groups, I Some Differential Geometrical Expressions of Fourier-Integral Operators on a Riemannian Manifold

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## Preface

In this series of papers, we are going to construct a theory of infinite dimensional Lie groups, which will be called "regular Fréchet-Lie groups" throughout this series. Roughly speaking a regular Fréchet-Lie group is a Fréchet-Lie group (i.e., a Lie group modeled on a locally convex Fréchet space) on which product integrals can be defined.

For instance, consider a continuous curve $A(t)$ in the space $M(n)$ of $n \times n$ real matrices, which is naturally regarded as the Lie algebra of the general linear group $\mathrm{GL}(n)$. It is well-known that the solution $Z(t)$ of

$$\frac{d}{dt}Z(t)=A(t)Z(t)\ , \quad Z(0)=I \text{ (the identity)}$$

is given as follows: Let $\Delta=\{t_0, t_1, \cdots, t_n\}$ be a division of the interval $[0, t]$ and let $|\Delta|=\max|t_j-t_{j-1}|$. Then,

$$\lim_{|\Delta|\to 0} \prod_{j=0}^{n-1} \exp\,(t_{j+1}-t_j)A(t_j)$$

converges, and the limit which is written as $\prod_0^t (1+A(s))ds$ gives the solution $Z(t)$. $\prod_0^t (1+A(s))ds$ is called a *product integral* of $A(t)$. Remark that for each fixed $t$, $h(s, t)=\exp sA(t)$ is a smooth curve in $\mathrm{GL}(n)$ satisfying

$$(*)\qquad \begin{cases} h(s, t)\equiv I\ , \quad \left.\dfrac{\partial}{\partial s}\right|_{s=0} h(*, t)=A(t)\ , \\ h(s, t) \text{ is } C^1 \text{ in } s\ , \\ \dfrac{\partial h}{\partial s}(s, t) \text{ is continuous in } (s, t)\ . \end{cases}$$

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It is not hard to see that $\lim_{|\Delta|\to 0}\prod_{j=1}^{n-1} h(t_{j+1}-t_j, t_j)$ also converges to the same $Z(t)$, even if we replace $\exp sA(t)$ by any other $h(s, t)$ satisfying above $(*)$.

Putting the above remark in mind, one can extend easily the concept of product integrals to all finite dimensional Lie groups. Moreover, it can be extended to all Banach-Lie groups, and moreover to all strong ILB-Lie groups (cf. [9]). These facts will be proved in forthcoming papers in this series.

Anyway, a regular Fréchet-Lie group is a Fréchet-Lie group on which product integrals can be defined, and satisfies some nice properties which are verified in finite dimensional Lie groups, Banach-Lie groups and strong ILB-Lie groups. Therefore, in a regular Fréchet-Lie group $G$, one can solve the equation

$$\frac{d}{dt}g(t)=u(t)g(t)\ , \quad g(0)=g_0\in G$$

by using product integrals, where $g(t)$ is a $C^1$ curve in $G$ and $u(t)$ is a prescrived continuous curve in the Lie algebra $\mathfrak{g}$ of $G$. Using this fact, one can construct an analogous theory to that of finite dimensional Lie groups. For instance, one can prove the followings:

(a) *The structure of the Lie algebra determines the local structure of a regular Fréchet-Lie group. Moreover, a continuous homomorphism between two Lie algebras generates a local homomorphism between two regular Fréchet-Lie groups.*

(b) *Let $g(t)$ be a $C^1$ curve in that $g(0)=e$ (the identity) and $(d/dt)|_{t=0}g=u\in\mathfrak{g}$. Then, $\lim_{n\to\infty} g(t/n)^n$ converges uniformly on each compact interval to the one parameter subgroup* $\exp tu$ *generated by $u$.* (The proof of (a) and (b) will be given in forthcoming papers in this series.)

However, since general theories are sometimes very tiresome, we would like to start this series with several examples. The most important and interesting examples of regular Fréchet-Lie groups is the group of all invertible Fourier integral operators of order 0 on compact manifold $N$. This group is closely related to the group of the canonical transformations in classical mechanics, and in a sense it is understood as a quantized group of it (cf. the concept of quantized contact transformations in [11]).

Now, let $G\mathscr{F}^0(N)$ be the group of all invertible Fourier integral operators of order 0 over a compact manifold $N$. As a matter of fact, one of main purpose of this series is to show that $G\mathscr{F}^0(N)$ is a regular Fréchet-Lie group. The Lie algebra of $G\mathscr{F}^0(N)$ is the space of all pseudo-differential operators of order 1 with real principal symbols. There exists naturally a homomorphism $\Phi$ of $G\mathscr{F}^0(N)$ into the group $\mathscr{D}_\omega(S_N^*)$ of all contact transformations on the unit cosphere bundle $S_N^*$. Since $S_N^*$ is compact, $\mathscr{D}_\omega(S_N^*)$ is a strong ILB-Lie group in the sense of [9] or [10]. In fact, $\mathscr{D}_\omega(S_N^*)$ is a regular Frécht-Lie group, and $\Phi$: $G\mathscr{F}^0(N)\to\mathscr{D}_\omega(S_N^*)$ is a smooth homomorphism. The kernel of $\Phi$ is also a regular Fréchet-Lie group, and in fact it is the group of all invertible pseudo-differential operators of order 0.

There is a $C^\infty$-local section $\lambda$: $\mathscr{D}_\omega(S_N^*)\to G\mathscr{F}^0(N)$ defined on a neighborhood of the identity, which is sometimes called a *quantization*. Obviously $\lambda$ is not unique, but the authors conjecture that $\lambda$ can not be a local homomorphism in general. Now, let $\mathfrak{p}^1(N)$, $\Gamma_\omega(S_N^*)$ be the Lie algebras of $G\mathscr{F}^0(N)$ and $\mathscr{D}_\omega(S_N^*)$ respectively. Since $\Phi$ is smooth, the differential $(d\Phi)_e$ at the identity $e$ is a Lie algebra homomorphism of $\mathfrak{p}^1(N)$ onto $\Gamma_\omega(S_N^*)$. The kernel of $(d\Phi)_e$ is the space $\mathfrak{p}^0(N)$ of all pseudo-differential operators of order 0. Namely, we have an exact sequence

$$0\longrightarrow\mathfrak{p}^0(N)\longrightarrow\mathfrak{p}^1(N)\longrightarrow\Gamma_\omega(S_N^*)\longrightarrow 0\ .$$

By (a) above, $\lambda$ can be a local homomorphism, if and only if the above exact sequence splits. This problem is closely related to the second cohomology group of $\Gamma_\omega(S_N^*)$.

Next, we would like to mention about (b). Let $P$ be a pseudo-differential operator of order 1 with real principal symbol, i.e., $P\in\mathfrak{p}^1(N)$. Let $F(t)$ be a smooth curve in $G\mathscr{F}^0(N)$ such that $F(0)=0$ and $(d/dt)|_{t=0}F(t)=\sqrt{-1}\,P$. Then by (b), $\lim F(t/n)^n$ converges to the fundamental solution of the equation

$$\frac{d}{dt}\psi_t=\sqrt{-1}\ P\psi_t\ .$$

There are many methods to make a curve $F(t)$, but in many cases $F(t)$ is made from the one parameter group generated by the Hamiltonian vector field on $T_N^*$ given by the symbol of $P$. Remark that $F(t/n)^n$ is in fact a multiple integral operator. So, the fact that $\lim_{n\to\infty}F(t/n)^n$ converges to the fundamental solution has a deep relevance to Fynmann's idea of path integrals.

In this series of papers, we are going to discuss these facts by using group theoretical tools of regular Fréchet-Lie groups.

**Introduction**

A Fourier integral operator on $\boldsymbol{R}^n$ is an operator written in the form

$$(Fu)(x)=\int_{R^M}\int_{R^n} a(x, \theta, y)e^{i\phi(x,\theta,y)}u(y)dyd\theta$$

by using an oscillatory integral. $\phi(x, \theta, y)$ is a non-degenerate phase function defined on $R^n \times R^M \times R^n$, and $a(x, \theta, y)$ is called an amplitude function. It is well-known that the above expression is not unique. We may replace $(a, \phi)$ by some other $(a', \phi')$ to get the same operator. We can also change the number of $\theta$-variables. Even if we fix the phase function $\phi$, the above expression does contain an ambiguity for the amplitude.

To extend the above definition to a closed manifold $N$, we use local coordinate systems on $N$, and we define a Fourier integral operator on $N$ as an operator which can be expressed micro-locally in the above shape (cf. [1], [3], [6]). Therefore, the expression of a Fourier integral operator on $N$ contains usually a huge ambiguity. It depends on the choice of local coordinate systems on $N$, on the choice of phase functions and amplitudes. To understand the covariant meanings of the various quantities attached to the Fourier integral operators, one must understand first of all the rule of "coordinate transformations". However, it seems quite difficult to write down the general rule. It is also quite complicated in the case of pseudo-differential operators. It is known that principal symbols and subprincipal symbols have some covariant meanings.

However, what we have actually in mind is an operator $F$ acting on the space of the smooth functions on $N$, and $a$ and $\phi$ for instance are supplementary expressions when we want to compute concretely. This situation is just like a *point* in a manifold. A point in a manifold has various expressions as $n$-tuples of real numbers depending on the choice of local coordinate systems.

Obviously, expressions of a Fourier integral operators contain two sorts of ambiguities. One comes from the choice of local coordinate patches on $N$, and another comes from the choice of phase function and amplitude. Therefore, it seems to be useful to eliminate the former ambiguities in the expression. Indeed this is a main purpose of this paper. However, for that purpose we have to fix a Riemannian metric on $N$, and we have to use normal coordinate systems.

Let $\mathrm{Exp}_x$ be the exponential mapping defined by the Riemannian structure on $N$, and let $K$ be a smoothing operator. Using these notations, a classical pseudo-differential operator on $N$ can be written in the form

$$(Pu)(x)=\int_{T_x^*}\int_{T_x} a(x;\xi)e^{-i\langle\xi|X\rangle}\nu(x,\,\mathrm{Exp}_x X)u(\mathrm{Exp}_x X)dXd\xi+(K\circ u)(x)$$

where $\nu$ is a cut off function, and $a(x;\xi)$ is a $C^\infty$ function on the cotangent bundle $T_N^*$ having the following asymptotic expansion:

$$(*)\qquad a(x;\xi)\sim a_\beta(x;\hat{\xi})r^\beta+a_{\beta-1}(x;\hat{\xi})r^{\beta-1}+a_{\beta-2}(x;\hat{\xi})r^{\beta-2}+\cdots,$$

where $r=|\xi|$ and $a_\gamma(x;\hat{\xi})$'s are $C^\infty$ functions on the unit cosphere bundle $S_N^*$. The real number $\beta$ is called the *order of P*, and $a_\beta(x;\hat{\xi})r^\beta$ is called *the principal symbol*. The feature of our pseudo-differential operator is that all $a_\gamma(x;\hat{\xi})r^\gamma$ have meanings as functions on $T_N^*-\{0\}$.

To write down Fourier integral operators in similar way, we need more notations. Let $\varphi$ be a symplectic transformation of positively homogeneous of order one on $T_N^*-\{0\}$. $\varphi$ can be witten as $\varphi(x;\xi)=(\varphi_1(x;\xi);\varphi_2(x;\xi))$, $\varphi_2(x;\xi)\in T^*_{\varphi_1(x;\xi)}$. Let $\{\lambda_\alpha(x;\xi)\}$ be a suitable partition of unity on $T_N^*-\{0\}$ by $C^\infty$ functions $\lambda_\alpha$ of positively homogeneous of degree 0. Our Fourier integral operator $F$ can be written in the form

$$(F_\varphi u)(x)=\sum_\alpha\iint\lambda_\alpha a(x;\xi;X)e^{-i\langle\varphi_2(x;\xi)|X\rangle-i|\xi|A_\alpha(X)}(\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi$$
$$+(K\circ u)(x),$$

where $A_\alpha$ is a quadratic form depending on $\varphi$, and

$$(\nu u)^\cdot(y;Y)=\nu(y,\,\mathrm{Exp}_y Y)u(\mathrm{Exp}_y Y).$$

$a(x;\xi;X)$ is an amplitude function with an asymptotic expansion similar to the above $(*)$ (cf. (11) and (13)). The feature of our expression is in the explicit appearance of symplectic transformation $\varphi$, which plays of course the same role as a phase function. The wave front set of the distribution $u\rightsquigarrow(F_\varphi u)(x)$ is contained in $\varphi(T_x^*-\{0\})$ for every $x\in N$ (cf. Lemma 3.2). In this sense, our operators form much narrower class than that of [1], [5] and [6]. In their sense, if two symplectic transformations $\varphi$, $\psi$ satisfy $\varphi(T_x^*-\{0\})\cap\psi(T_x^*-\{0\})=\emptyset$ for every $x\in N$, then $F_\varphi+F_\psi$ is a Fourier integral operator, while we exclude such cases (cf. Remark just after (38)). However, our class is still enough to consider the fundamental solution of a equation $du/dt=\sqrt{-1}\,Pu$, if $P$ is a pseudo-differential operator of order 1. (This will be proved from the group theoretical view point in a forthcoming paper.)

If $\varphi$ is sufficiently close to the identity, then we may put $A_\alpha=0$ for all $\alpha$, hence $F_\varphi$ can be written as

$$(F_\varphi u)(x)=\iint a(x;\xi;X)e^{-i\langle\varphi_2(x;\xi)|X\rangle}(\nu u)^\cdot(\varphi_1(x;\xi),\,X)dXd\xi+(K\circ u)(x).$$

Moreover, one can replace $a(x; \xi; X)$ by some other $b(x; \xi)$ which does not contain the variable $X$. Therefore $F_\varphi$ can be written as

$$(F_\varphi u)(x) = \int b(x; \xi)\widetilde{\nu u}(\varphi(x; \xi))d\xi + (K \circ u)(x) ,$$

where $\widetilde{\nu u}$ is a sort of Fourier transformation of $(\nu u)^{\cdot}$, defined by

$$\widetilde{\nu u}(y; \eta) = \int_N e^{-i\langle\eta|Y\rangle}\nu(y, z)u(z)dz , \quad z = \mathrm{Exp}_y Y .$$

Now, consider Fourier integral operator of order 0. Then, the last expression can be slightly modified so that it may contain no ambiguity in expression. Let

$$b(x; \hat{\xi}) \sim b_0(x; \hat{\xi}) + b_{-1}(x; \hat{\xi})r^{-1} + b_{-2}(x: \hat{\xi})r^{-2} + \cdots$$

be the asymptotic expansion of $b$. Then, one may regard

$$\{\varphi, b_0, b_{-1}, b_{-2}, \cdots, K\}$$

as a local coordinate of $F_\varphi$, if $F_\varphi$ is sufficiently close to the identity in some sence (cf. §5).

Therefore, *one can give a coordinate system on a vicinity of the identity in the space of Fourier integral operators of order* 0.

## §1. Some geometrical tools and notations.

Throughout this paper, we denote by $N$ a compact, $n$-dimensional $C^\infty$ Riemannian manifold without boundary. By $T_N$, $T_N^*$ we denote the tangent and the cotangent bundle respectively. Fibers at $x \in N$ will be denoted by $T_x$, $T_x^*$ respectively.

Let $\mathrm{Exp}_x$: $T_x \to N$ be the exponential mapping defined by the Riemannian connection on $N$. For each $x \in N$, there is $r > 0$ such that $\mathrm{Exp}_x$: $D_x(r) \to N$ is an into-diffeomorphism, where $D_x(r)$ is an open $r$-neighborhood of 0 in $T_x$. The supremum of such $r$ will be denoted by $r_0(x)$. Since $N$ is compact

$$r_0 = \inf_{x \in N} r_0(x)$$

is still positive. We call $r_0$ the *injectivity radius* of $N$.

Let $\{e_1, \cdots, e_n\}$ be a basis of $T_x$. Then, every $X \in T_x$ can be written by a linear combination $X = X^i e_i$. The $n$-tuple of numbers $(X^1, \cdots, X^n)$ will be called a *linear chart* on $T_x$. Similarly, let $e^1, \cdots, e^n$ be the dual basis of $e_1, \cdots, e_n$. Then, every $\xi \in T_x^*$ can be written in the form $\xi = \xi_i e^i$.

$(\xi_1, \cdots, \xi_n)$ will be called the *dual linear chart* on $T_x^*$. The natural pairing $\langle\xi|X\rangle$ of $\xi$ and $X$ is given by $\langle\xi|X\rangle=\xi_i X^i$, using a linear chart and its dual linear chart.

If we fix a linear chart on $T_x$, then $\mathrm{Exp}_x: D_x(r)\to N$ gives a local chart around $x$. So, a point $X\in D_x(r)$ indicates also a point in $N$ through the exponential mapping. This point in $N$ will be denoted by $\cdot_x X$. Thus, $y=\cdot_x X$ implies $y=\mathrm{Exp}_x X$. The above local coordinate system will be called a *normal chart around* $x$.

We denote a point of $T_N$ by $(x; X)$ where $x$ is the base point of $(x; X)$ and $X\in T_x$. For a point $(y; Y)\in T_N$ such that $y\in \mathrm{Exp}_x D_x(r)$, there are $X\in D_x(r)$, $X_1\in T_x$ such that $y=\cdot_x X$, $Y=(d\,\mathrm{Exp}_x)_X X_1$. We denote the above situation by

$$(y;\, Y)=\cdot_x(X,\, X_1)\ .$$

$(X, X_1)$ will be called a *normal coordinate expression* of $(y; Y)$ at $x$. The notation $\cdot_x$ can be regarded as a mapping of $D_x(r)\times T_x$ into $T_N$. This mapping will be called a *normal, local trivialization around* $x$. If we give a linear chart on $T_x$, then $\cdot_x(X, X_1)$ can be regarded as a local coordinate system on $T_N$. The above local coordinate system will be called a *normal chart around* $T_x$.

Similarly, a point in $T_N^*$ will be denoted by $(x;\xi)$, where $x$ is the base point and $\xi\in T_x^*$. For a point $(y;\eta)\in T_N^*$ such that $y\in\mathrm{Exp}_x D_x(r)$, there are $X\in D_x(r)$, $\xi\in T_x^*$ such that $y=\cdot_x X$, $\eta=(d\,\mathrm{Exp}_x^{-1})_X^*\xi$. We denote the above situation by

$$(y;\eta)=\cdot_x(X,\,\xi)\ ,$$

and call it a *normal coordinate expression of* $(y;\eta)$. The notation $\cdot_x$ can be regarded as a mapping of $D_x(r)\times T_x^*$ into $T_N^*$. This mapping will be called a *normal*, (dual) *local trivialization around* $x$. The inverse mapping of $\cdot_x$ will be denoted by ${}^{x}\cdot$. Therefore one may denote $(X,\xi)={}^{x}\cdot(y;\eta)$, $(X, X_1)={}^{x}\cdot(y; Y)$ or $X={}^{x}\cdot y$, etc.

If we fix a linear chart $X^1, \cdots, X^n$ on $T_x$ and its dual linear chart $\xi_1, \cdots, \xi_n$ on $T_x^*$, then $\cdot_x(X,\xi)=\cdot_x(X^1, \cdots, X^n, \xi_1, \cdots, \xi_n)$ can be regarded as a local coordinate system on $T_N^*$. This local coordinate system will be called a *normal canonical chart around* $T_x^*$. Now, suppose $\cdot_x(X, X_1)=(y; Y)$, and $\cdot_x(X,\xi)=(y;\eta)$. Then we have

$$(1)\qquad \langle\eta|Y\rangle=\langle\eta|(d\,\mathrm{Exp}_x)_X X_1\rangle=\langle\eta(d\,\mathrm{Exp}_x)_X|X_1\rangle=\langle\xi|X_1\rangle\ ,$$

where $\eta(d\,\mathrm{Exp}_x)_X=(d\,\mathrm{Exp}_x)_X^*\eta$. We use this notation from time to time. By $dX$, $d\xi$ we denote the volume forms on each $T_x$, $T_x^*$ respectively, i.e., letting $g_{ij}$ be the Riemannian metric tensor with respect to a normal chart $(X^1, \cdots, X^n)$ around $x$, and $g(x)=\det(g_{ij}(0))$,

$$dX=\sqrt{g(x)}\, dX^1\wedge dX^2\wedge\cdots\wedge dX^n\,,\quad d\xi=\frac{1}{\sqrt{g(x)}}d\xi_1\wedge d\xi_2\wedge\cdots\wedge d\xi_n\,,$$

and we set

$$đX=\frac{1}{\sqrt{2\pi}^n}dX\,,\quad đ\xi=\frac{1}{\sqrt{2\pi}^n}d\xi\,.$$

Let $\pi: T_N^*\to N$ be the projection. For a tangent vector $\mathfrak{y}\in T_{(y;\eta)}T_N^*$, we set

$$\theta_{(y;\eta)}(\mathfrak{y})=\langle\eta\,|\,(d\pi)_{(y;\eta)}\mathfrak{y}\rangle\,. \tag{2}$$

Then, $\theta$ is a well-defined $C^\infty$ 1-form on $T_N^*$, called the *canonical* 1-*form*. By a normal canonical chart $(X^1, \cdots, X^n, \xi_1, \cdots, \xi_n)$ around $T_x^*$, the above $\theta$ is expressed as $\xi_i dX^i$ using the equality (1). Therefore, $\Omega=-d\theta$ is a symplectic 2-form on $T_N^*$, which is called the *canonical* 2-*form* on $T_N^*$.

Now, we consider the coordinate transformation between two different normal charts. Suppose $y=._x\bar{X}$ and suppose $|Y|+|\bar{X}|<r_0$. Then the distance between $x$ and $._yY$ is less than the injectivity radius $r_0$, hence $._yY$ can be written as $._xX$, and $Y$ depends smoothly on $X$, $\bar{X}\in T_x$. We denote this function by

$$Y=S(x;X,\bar{X})\,. \tag{3}$$

For a fixed $\bar{X}$, $S(x;*,\bar{X})$ is a $C^\infty$ mapping of $D_x(r_0-|\bar{X}|)$ into $T_{._x\bar{X}}=T_y$. Obviously, if $X=\bar{X}$, then $S=0$. Therefore $S$ can be written as follows:

$$S(x;X,\bar{X})=S_1(x;X,\bar{X})(X-\bar{X})\,, \tag{4}$$

where $S_1(x;X,\bar{X})$ is a linear mapping of $T_x$ into $T_y$. Remark that

$$S_1(x;\bar{X},\bar{X})=\frac{\partial S}{\partial X}(x;X,\bar{X})|_{X=\bar{X}}=(d\,\mathrm{Exp}_x)_{\bar{X}}\,. \tag{5}$$

Hence, $S_1(x;\bar{X},\bar{X})$ is invertible whenever $|\bar{X}|<r_0$. By Taylor's formula, we get

$$Y=S_1(x;\bar{X},\bar{X})(X-\bar{X})+Q(x;X,\bar{X})(X-\bar{X})^2\,, \tag{6}$$

where $Q(x;X,\bar{X})$ is a bilinear mapping of $T_x\oplus T_x$ into $T_y$. Note that $S_1(x;X,\bar{X})$ and $Q(x;X,\bar{X})$ are of $C^\infty$ with respect to $x$, $X$, $\bar{X}$. Since $N$ is compact, we see easily the following:

LEMMA 1.1. *$\|S_1(x; X, \bar{X})\|$, $\|Q(x; X, \bar{X})\|$ are bounded whenever $|X|$, $|\bar{X}|\leqq r_0/2$.*

Now, remark that $Y=S(x; X, \bar{X})$ is the coordinate transformation from a normal chart around $x$ to a normal chart around $y=\cdot_x\bar{X}$. Then, we get the following:

LEMMA 1.2. (1) *Suppose $y=\cdot_x\bar{X}$ and $\cdot_y(Y, Y_1)=\cdot_x(X, X_1)$. Then $Y=S(x; X, \bar{X})$ and $Y_1=(\partial S/\partial X)(x; X, \bar{X})X_1$.*

(2) *Suppose $y=\cdot_x\bar{X}$ and $\cdot_y(Y, \eta)=\cdot_x(X, \xi)$. Then $Y=S(x: X, \bar{X})$ and $\eta(\partial S/\partial X)(x; X, \bar{X})=\xi$, where $\eta(\partial S/\partial X)(x; X, \bar{X})=((\partial S/\partial X)(x; X, \bar{X}))^*\eta$.*

Let $(X^1, \cdots, X^n, \xi_1, \cdots, \xi_n)$, $(Y^1, \cdots, Y^n, \eta_1, \cdots, \eta_n)$ be normal canonical charts around $T_x^*$ and $T_y^*$ respectively. We assume $y=\cdot_x\bar{X}$ and $|\bar{X}|<r_0/2$. Suppose

$$\cdot_x(X^1, \cdots, X^n, \xi_1, \cdots, \xi_n)=\cdot_y(Y^1, \cdots, Y^n, \eta_1, \cdots, \eta_n) .$$

Then, by the above lemma we see

$$\begin{cases} Y^i=S^i(x; X^1, \cdots, X^n, \bar{X}^1, \cdots, \bar{X}^n) \\ \xi_i=\eta_j\dfrac{\partial S^j}{\partial X^i}(x; X^1, \cdots, X^n, \bar{X}^1, \cdots, \bar{X}^n) . \end{cases} \qquad (i=1, \cdots, n)$$

Hence, $\Omega=dY^i\wedge d\eta_i=dX^i\wedge d\xi_i$. Therefore

$$\Omega^n=dY^1\wedge\cdots\wedge dY^n\wedge d\eta_1\wedge\cdots\wedge d\eta_n=dX^1\wedge\cdots\wedge dX^n\wedge d\xi_1\wedge\cdots\wedge d\xi_n .$$

Remark that if $dX^1\wedge\cdots\wedge dX^n=C đX$, then $d\xi_1\wedge\cdots\wedge d\xi_n=C^{-1}đ\xi$. Thus, we get the following:

LEMMA 1.3. *Suppose $y=\cdot_x\bar{X}$, and $\cdot_y(Y, \eta)=\cdot_x(X, \xi)$. Then, $(1/(2\pi)^n)\Omega^n=đYđ\eta=đXđ\xi$. Moreover, if $đY=J(x; X, \bar{X})đX$, then $đ\eta=J(x; X, \bar{X})^{-1}đ\xi$.*

Recall that $S(x; X, \bar{X})\in T_{\cdot_x\bar{X}}$. Hence if we fix $X$ and vary $\bar{X}$, then we get a vector field. For an arbitrarily fixed $\bar{X}$, we define $\tilde{S}(x; X, \hat{X})$ by

$$(\cdot_x\hat{X}; S(x; X, \hat{X}))=\cdot_{\cdot_x\bar{X}}(Y, \tilde{S}(x; X, \hat{X})) . \tag{7}$$

$\tilde{S}(x; X, \hat{X})$ is a normal coordinate expression of $S(x; X, \hat{X})$ around $\cdot_x\bar{X}$. We define

$$\frac{\nabla S}{\partial\bar{X}}(x; X, \bar{X})=\frac{\partial}{\partial\hat{X}}\tilde{S}(x; X, \hat{X})|_{\hat{X}=\bar{X}} . \tag{8}$$

The above $\nabla S/\partial\bar{X}$ is in fact the covariant derivative of the vector field $S(x; X, *)$, and $(\nabla S/\partial\bar{X})(x; X, \bar{X})$ is a linear mapping of $T_x$ into $T_{\cdot_x\bar{X}}$.

LEMMA 1.4. *Notations being as above, we see*

$$\frac{\partial S}{\partial X}(x; \bar{X}, \bar{X})+\frac{\nabla S}{\partial\bar{X}}(x; \bar{X}, \bar{X})=0 .$$

PROOF. We use a normal coordinate expression $\tilde{S}$ of $S$ around $\cdot_x\bar{X}$. Since $\tilde{S}(x; \bar{X}, \bar{X})\equiv 0$, we get easily the desired one.

LEMMA 1.5.

$$\frac{\partial}{\partial X}\frac{\nabla}{\partial\bar{X}}S(x; X, \bar{X})\Big|_{\substack{X=0\\ \bar{X}=0}}=0$$

PROOF. By definition of $(\nabla S/\partial\bar{X})|_{\bar{X}=0}$, we have to take a normal chart around $x$. So, let $(y^1, \cdots, y^n)$ be a normal chart around $x$, and let $(X^1, \cdots, X^n)$, $(\bar{X}^1, \cdots, \bar{X}^n)$ be the coordinates of $X$, $\bar{X}$ respectively. Let $y(t)=(y^1(t), \cdots, y^n(t))$ be the length-minimizing geodesic joining $X$ and $\bar{X}$. Then, we see

$$\tilde{S}(x; X, \bar{X})=\frac{d}{dt}\Big|_{t=0} y(t) ,$$

where $\tilde{S}$ is the normal coordinate expression of $S$ around $T_x$, that is,

$$\cdot_x(\bar{X}, \tilde{S}(x; X, \bar{X}))=(\cdot_x\bar{X}; S(x; X, \bar{X})) .$$

Take the Taylor's expansion of $y(t)$ with respect to $t$ and substitute the equation of geodesics. Then, we get

$$y^i(t)=\bar{X}^i+t\tilde{S}^i-\frac{t^2}{2!}\left\{\begin{matrix} i \\ jk \end{matrix}\right\}_{\bar{X}}\tilde{S}^j\tilde{S}^k-\cdots ,$$

(see [2]). Put $t=1$, and compute the inverted series for $\tilde{S}^i$. Then, we get

$$\tilde{S}^i=X^i-\bar{X}^i+\frac{1}{2!}\left\{\begin{matrix} i \\ jk \end{matrix}\right\}_{\bar{X}}(X^j-\bar{X}^j)(X^k-\bar{X}^k)+\cdots .$$

Since $\left\{\begin{matrix} i \\ jk \end{matrix}\right\}_0=0$, we see easily $(\partial^2 S/\partial X\partial\bar{X})|_{\substack{X=0\\ \bar{X}=0}}=0$. This implies that

$$\frac{\partial}{\partial X}\frac{\nabla}{\partial\bar{X}}S(x; X, \bar{X})\Big|_{\substack{X=0\\ \bar{X}=0}}=0 .$$

Let $\theta$ be the canonical 1-form on $T_N^*$, and let $S_N^*$ be the unit cosphere bundle over $N$. The pull-back $\omega=i^*\theta$ by the natural inclusion $i: S_N^* \to T_N^*$ is then a $C^\infty$ contact form on $S_N^*$. We denote by $\mathscr{D}_\omega(S_N^*)$ the group of all $C^\infty$ contact transformations on $S_N^*$. Namely, $\hat{\varphi} \in \mathscr{D}_\omega(S_N^*)$ if and only if $\hat{\varphi}^*\omega=f\omega$, where $f$ is a non-vanishing $C^\infty$ function depending on $\hat{\varphi}$. For every $\hat{\varphi} \in \mathscr{D}_\omega(S_N^*)$, one can extend $\hat{\varphi}$ to a symplectic diffeomorphism $\varphi: T_N^*-\{0\} \to T_N^*-\{0\}$ by the following manner:

Let $(x; \hat{\xi})$ be a point in $S_N^*$ such that $x \in N$ and $\hat{\xi} \in S_x^*N$. Then, $(x; r\hat{\xi}) \in T_N^*-\{0\}$ for every $r>0$. Suppose $\hat{\varphi}(x; \hat{\xi})=(\hat{\varphi}_1(x; \hat{\xi}); \hat{\varphi}_2(x; \hat{\xi}))$. We define $\varphi(x; r\hat{\xi})$ by

$$\varphi(x; r\hat{\xi})=\left(\hat{\varphi}_1(x; \hat{\xi}); \frac{r}{f}\hat{\varphi}_2(x; \hat{\xi})\right) .$$

Since $\theta=r\omega$ on $T_N^*-\{0\}$, we get

$$\varphi^*\theta=\frac{r}{f}\hat{\varphi}^*\omega=r\omega=\theta .$$

By the above definition, $\varphi(x; \xi)=(\varphi_1(x; \xi); \varphi_2(x; \xi))$ satisfies

$$(9) \qquad \varphi_1(x; r\xi)=\varphi_1(x; \xi) , \quad \varphi_2(x; r\xi)=r\varphi_2(x; \xi)$$

for every $r>0$. We call a diffeomorphism $\varphi$ on $T_N^*-\{0\}$ to be *positively homogeneous of order* 1, if $\varphi$ satisfies (9). We denote by $\mathscr{D}_\Omega^{(1)}$ the group of all symplectic diffeomorphisms on $T_N^*-\{0\}$ of positively homogeneous of order 1.

LEMMA 1.6. *$\mathscr{D}_\Omega^{(1)}$ is isomorphic to $\mathscr{D}_\omega(S_N^*)$. Moreover, every $\varphi \in \mathscr{D}_\Omega^{(1)}$ satisfies $\varphi^*\theta=\theta$.*

PROOF. By the above argument, we have a monomorphism $\vee$ of $\mathscr{D}_\omega(S_N^*)$ into $\mathscr{D}_\Omega^{(1)}$ which is defined by $\vee(\hat{\varphi})=\varphi$ for any $\hat{\varphi} \in \mathscr{D}_\omega(S_N^*)$. Thus, we have only to show the surjectivity of $\vee$. Let $\varphi(x; \xi)=(\varphi_1(x; \xi); \varphi_2(x; \xi))$ be an element of $\mathscr{D}_\Omega^{(1)}$. Since $\|\varphi_2(x; \xi)\| \neq 0$, we define a $C^\infty$ diffeomorphism $\hat{\varphi}: S_N^* \to S_N^*$ by

$$\hat{\varphi}(x; \hat{\xi})=(\hat{\varphi}_1(x; \hat{\xi}); \hat{\varphi}_2(x; \hat{\xi}))=(\varphi_1(x; \hat{\xi}); f(x; \hat{\xi})\varphi_2(x; \hat{\xi})) ,$$

where $f(x; \hat{\xi})=\|\varphi_2(x; \hat{\xi})\|^{-1}$. Since $\varphi$ is positively homogeneous of order 1, we see that

$$(\varphi_1(x; r\hat{\xi}); \varphi_2(x; r\hat{\xi}))=\left(\hat{\varphi}_1(x; \hat{\xi}); \frac{r}{f(x; \hat{\xi})}\hat{\varphi}_2(x; \hat{\xi})\right) .$$

As $\varphi^*\Omega=\Omega$ and $\Omega=-d(r\omega)$ on $T_N^*-\{0\}$, we see

$$\varphi^* d(r\omega)=d\Big(\frac{r}{f}\hat{\varphi}^*\omega\Big)=d(r\omega)\ .$$

Hence

$$dr\wedge\frac{1}{f}\hat{\varphi}^*\omega+rd\Big(\frac{1}{f}\hat{\varphi}^*\omega\Big)=dr\wedge\omega+rd\omega\ .$$

Therefore $\hat{\varphi}^*\omega/f=\omega$, hence $\hat{\varphi}\in\mathscr{D}_\omega(S_N^*)$. It follows the desired surjectivity. Moreover, we see that $\varphi^*\theta=\theta$ by the previous argument.

Since $\mathscr{D}_\omega(S_N^*)$ is a topological group under the $C^\infty$-topology, so is $\mathscr{D}_\Omega^{(1)}$ by the above identification. As a matter of fact, $\mathscr{D}_\omega(S_N^*)$ is a strong ILH-Lie group (cf. [9]), (and hence it is a regular Fréchet-Lie group). However, this fact has no direct relevance to the present paper.

Let $\varphi\in\mathscr{D}_\Omega^{(1)}$, $\varphi(x;\xi)=(\varphi_1(x;\xi);\varphi_2(x;\xi))$. Then, $\varphi_1$ is a $C^\infty$ mapping of $T_N^*$ onto $N$. Therefore the derivative $(\partial\varphi_1/\partial\xi)(x;\xi)$ makes sense as a linear mapping of $T_x^*$ into $T_{\varphi_1(x;\xi)}$. However, since the base point varies, the derivative $(\partial\varphi_2/\partial\xi)(x;\xi)$ has no invariant meaning. We define the covariant derivative $(\nabla\varphi_2/\partial\xi)(x;\xi)$ as follows: Let $\tilde{\varphi}_2(x;\xi')$ be the normal coordinate expression of $\varphi_2(x;\xi')$ at the point $\varphi_1(x;\xi)$. Namely, we define $\tilde{\varphi}_2$ by

$$\cdot_{\varphi_1(x;\xi)}(X',\tilde{\varphi}_2(x;\xi'))=(\varphi_1(x;\xi');\varphi_2(x;\xi'))\ .$$

Then, $\nabla\varphi_2/\partial\xi$ is defined by $(\nabla\varphi_2/\partial\xi)(x;\xi)=\partial/\partial\xi'\tilde{\varphi}_2(x;\xi')|_{\xi'=\xi}$.

Now, remark that $T_N^*$ is diffeomorphic to the unit disk bundle $D_N^*$ in $T_N^*$. The diffeomorphism $\tau\colon D_N^*\to T_N^*$ may be given by

$$\tau(x;\theta\hat{\xi})=\Big(x;\Big(\tan\frac{\pi\theta}{2}\Big)\hat{\xi}\Big)\ , \tag{10}$$

where $(x;\hat{\xi})\in S_N^*$, $\theta\in[0,1)$. Let $\bar{D}_N^*$ be the closure of $D_N^*$ in $T_N^*$, and $\bar{D}_N(r_0)$ the closed disk bundle in $T_N$ of the radius $r_0$. Let $\bar{D}_N^*\oplus\bar{D}_N(r_0)$ be the Whitney sum (fiber product). We denote by $C^\infty(\bar{D}_N^*\oplus\bar{D}_N(r_0))$ the space of all $C$-valued $C^\infty$ functions on $\bar{D}_N^*\oplus\bar{D}_N(r_0)$, and by $C^\infty(\bar{D}_N^*)$ the space of all $C$-valued $C^\infty$ functions on $\bar{D}_N^*$, where a function $f$ defined on a manifold with boundary is called to be smooth if $f$ can be extended smoothly to a neighborhood of this manifold.

Under the $C^\infty$-topology, $C^\infty(\bar{D}_N^*\oplus\bar{D}_N(r_0))$ is a locally convex Fréchet space, and $C^\infty(\bar{D}_N^*)$ can be naturally imbedded in $C^\infty(\bar{D}_N^*\oplus\bar{D}_N(r_0))$ as a closed subspace. Define a diffeomorphism

$$\tilde{\tau}: \bar{D}_N^* \oplus \bar{D}_N(r_0) \longrightarrow T_N^* \oplus \bar{D}_N(r_0)$$

by

$$\tilde{\tau}(x; \theta\hat{\xi}, X) = \left(x; \left(\tan \frac{\pi\theta}{2}\right)\hat{\xi}, X\right).$$

For every $f \in C^\infty(\bar{D}_N^* \oplus \bar{D}_N(r_0))$, $a(x; \xi, X) = \tilde{\tau}^{-1*}f$ is a $C^\infty$ function on $T_N^* \oplus \bar{D}_N(r_0)$ having the following asymptotic expansion corresponding to the Taylor's expansion of $f$ at $\theta = 1$:

$$\begin{aligned} (11) \qquad & a(x; r\hat{\xi}, X) \\ & \sim a_0(x; \hat{\xi}, X) + a_{-1}(x; \hat{\xi}, X)r^{-1} + \cdots + a_{-l}(x; \hat{\xi}, X)r^{-l} + \cdots, \end{aligned}$$

where $r > 0$, and $a_{-j}(x; \hat{\xi}, X)$'s are $C^\infty$ functions on $S_N^* \oplus \bar{D}_N(r_0)$.

We define the space $\tilde{\sum}_C^0$ by

$$\tilde{\sum}_C^0 = \{a(x; \xi, X); a(x; \xi, X) = \tilde{\tau}^{-1*}f, \quad f \in C^\infty(\bar{D}_N^* \oplus \bar{D}_N(r_0))\}.$$

The space $\tilde{\sum}_C^0$ will be called *the space of symbols of order* 0 or *the space of amplitudes of order* 0. $\tilde{\sum}_C^0$ is a locally convex Fréchet space through the identification $\tilde{\tau}^*$. Similarly, we set

$$\sum_C^0 = \{a(x; \xi); a(x; \xi) = \tau^{-1*}f; f \in C^\infty(\bar{D}_N^*)\}.$$

$\sum_C^0$ is a closed linear subspace of $\tilde{\sum}_C^0$.

Let $\mu(r)$ be a $C^\infty$ non-decreasing function on $[0, \infty)$ such that $\mu(r) \equiv 1$ on $[0, 1]$ and $\mu(r) = r$ on $[2, \infty)$. For any real number $\beta$, we denote $\tilde{\sum}_C^\beta = \mu(r)^\beta \tilde{\sum}_C^0$, $\sum_C^\beta = \mu(r)^\beta \sum_C^0$.

Let $C^\infty(N \times N)$ be the space of all $C$-valued $C^\infty$ functions on $N \times N$ with the $C^\infty$-topology. Since $N$ is compact, $C^\infty(N \times N)$ is a locally convex Fréchet space. For every $K(x, y) \in C^\infty(N \times N)$, it defines a linear operator

$$(12) \qquad (K \circ u)(x) = \int_N K(x, y)u(y)dy, \quad u \in C^\infty(N),$$

where $C^\infty(N)$ is a Fréchet space of all $C^\infty$ functions on $N$, and $dy$ is the volume element on $N$. An operator such as (12) is called a *smoothing operator*, and a function $K$ is called the *smooth kernel* of $K$.

A $C^\infty$ function $\nu(x, y)$ on $N \times N$ will be called a *cut off* function, if $\nu(x, y)$ satisfies the following;

(a) $0 \leqq \nu(x, y) \leqq 1$, $\nu(x, y) = \nu(y, x)$.

(b) There is $\varepsilon$ such that $0 \leqq \varepsilon \leqq r_0$ and $\nu(x, y) \equiv 1$ if $\rho(x, y) \leqq \varepsilon/3$, where $\rho$ is the distance function on $N$.

(c) $\nu(x, y)\equiv 0$ if $\rho(x, y)\geqq 2\varepsilon/3$.
The number $\varepsilon$ will be called the *breadth* of $\nu$.

## §2. Non-degenerate phase functions.

Let $\varphi=(\varphi_1; \varphi_2)\in\mathscr{D}_\Omega^{(1)}$. For a function $b(y; \eta, Y)\in\tilde{\sum}_C^\beta$, we set

$$a(x; \xi; X)=b(\varphi_1(x; \xi); \varphi_2(x; \xi), X) . \tag{13}$$

$a(x; \xi; X)$ is a $C^\infty$ function on the pull-back bundle of $T_N$ by the mapping $\varphi_1$: $T_N^*-\{0\}\rightarrow N$. We denote by $\tilde{\sum}_\varphi^\beta$ the space of all such functions. For a cut off function $\nu$ and $u\in C^\infty(N)$, we set

$$(\nu u)^\cdot(y; Y)=\nu(y, \cdot_y Y)u(\cdot_y Y) . \tag{14}$$

Fourier integral operators that we are going to define in this paper are written in the form

$$\begin{aligned}(15)\quad &(F(\varphi, a, \nu)u)(x)\\ &=\sum_\alpha\int_{T_x^*}\int_{T\varphi_1(x;\xi)}\lambda_\alpha(x; \xi)a(x; \xi; X)e^{-i\langle\varphi_2(x;\xi)|X\rangle-i|\xi|A_\alpha(X)}(\nu u)^\cdot(\varphi_1(x; \xi); X)dXd\xi\end{aligned}$$

where $\varphi\in\mathscr{D}_\Omega^{(1)}$, $a\in\tilde{\sum}_\varphi^\beta$, $\{\lambda_\alpha\}$ is a suitable partition of unity on $T_N^*$, and $A_\alpha$'s are quadratic forms in $X$.

In this section, we are mainly concerned with the phase function

$$\langle\varphi_2(x; \xi)|X\rangle+|\xi|A_\alpha(X) .$$

Especially, we discuss here how we should fix $A_\alpha(X)$ to make the phase function non-degenerate.

For a fixed $y\in N$, let $\cdot_y(Y^1, \cdots, Y^n, \eta_1, \cdots, \eta_n)$ be a normal canonical chart around $T_y^*$. We fix a point $\bar{\eta}\in T_y^*$. Let $(0, \cdots, 0, \bar{\eta}_1, \cdots, \bar{\eta}_n)$ be the coordinate of $(y; \bar{\eta})$ by the above coordinate system, i.e., $(y; \bar{\eta})=\cdot_y(0, \cdots, 0, \bar{\eta}_1, \cdots, \bar{\eta}_n)$. Set $\eta_i=\bar{\eta}_i+\zeta_i$, and $(Z^1, \cdots, Z^n, \zeta_1, \cdots, \zeta_n)$ can be naturally regarded as a linear chart of $T_{(y;\bar{\eta})}T_N^*$. The canonical 2-form $\Omega$ on $T_N^*$ defines naturally a symplectic bi-linear form $\sigma$ on $T_{(y;\bar{\eta})}T_N^*$. If $\mathfrak{z}_1=(Z_1, \zeta^1)$, $\mathfrak{z}_2=(Z_2, \zeta^2)$ in $T_{(y;\bar{\eta})}T_N^*$, then $\sigma$ is given by

$$\sigma(\mathfrak{z}_1, \mathfrak{z}_2)=\langle\zeta^2|Z_1\rangle-\langle\zeta^1|Z_2\rangle . \tag{16}$$

If we change a linear chart on $T_y$ by $'Y^i=A_k^iY^k$, then the dual linear chart is changed by $'\eta_i=B_i^k\eta_k$ such that $B_i^kA_k^j=\delta_i^j$. Therefore, the splitting of $T_{(y;\bar{\eta})}T_N^*$ into the $Z$-space and the $\zeta$-space does not depend on the choice of normal canonical charts. We can naturally identify the $Z$-space by $T_y$ and the $\zeta$-space by $T_y^*$. Moreover, the above argument shows that the expression of $\sigma$ does not depend on the choice of normal canonical charts. Let $\pi_Z$ (resp. $\pi_\zeta$) be the projection of $T_{(y;\bar{\eta})}T_N^*$ onto the $Z$-(resp. $\zeta$-)space.

Now, suppose we have a Lagrangean subspace $E$ in $T_{(y;\bar{\eta})}T_N^*$.

LEMMA 2.1. *$\langle\pi_\zeta(\mathfrak{x})|\pi_Z(\mathfrak{y})\rangle=\langle\pi_\zeta(\mathfrak{y})|\pi_Z(\mathfrak{x})\rangle$ for every $\mathfrak{x}, \mathfrak{y}\in E$.*

Proof is trivial, because $E$ is a Lagrangean subspace, hence $\sigma(\mathfrak{x}, \mathfrak{y})=0$ for any $\mathfrak{x}, \mathfrak{y}\in E$. (cf. (16).)

The above lemma shows that $\langle\pi_\zeta(\mathfrak{x})|\pi_Z(\mathfrak{y})\rangle$ is a symmetric bi-linear form on $E$, and if $\pi_\zeta(\mathfrak{x})=0$, then $\langle\pi_\zeta(E)|\pi_Z(\mathfrak{x})\rangle=0$. Let $K_\zeta$ be the kernel of $\pi_\zeta\colon E\to T_y^*$. Then, it is evident that $\pi_Z\colon K_\zeta\to T_y$ is an injection. Since $N$ is a Riemannian manifold, one can take the orthogonal compliment $F$ of $\pi_Z(K_\zeta)$ in $T_y$. Hence $T_y=F\oplus\pi_Z(K_\zeta)$ and every $Y\in T_y$ can be written as $Y=Y_1+Y_2$, $Y_1\in F$, $Y_2\in\pi_Z(K_\zeta)$. We define a quadratic form

$$a_E(Y)=|Y_2|^2 \tag{17}$$

on $T_y$. As a matter of course, if $K_\zeta=\{0\}$, then we set $a_E(Y)=0$.

Now, we shall give coordinate expressions of $\pi_\zeta$, $\pi_Z$ and $a_E$. Let $Y^1, \cdots, Y^n$ be an orthonormal chart on $T_y$ such that $\pi_Z(K_\zeta)$ is given by $Y^1=\cdots=Y^l=0$. Let $\eta_1, \cdots, \eta_n$ be the dual linear chart of $Y^1, \cdots, Y^n$. Since $\langle\pi_\zeta(E)|\pi_Z(K_\zeta)\rangle=0$ by the above lemma, and since $\langle\eta|Y\rangle=\eta_iY^i$, we see that $\pi_\zeta(E)$ is given by $\eta_{l+1}=\cdots=\eta_n=0$. Obviously, $a_E(Y)=\sum_{j=l+1}^n(Y^j)^2$.

Let $\varphi=(\varphi_1;\varphi_2)$ be an element of $\mathscr{D}_\Omega^{(1)}$. For a fixed $(\bar{x};\bar{\xi})\in T_N^*-\{0\}$, we set $(y;\bar{\eta})=\varphi(\bar{x};\bar{\xi})=(\varphi_1(\bar{x};\bar{\xi});\varphi_2(\bar{x};\bar{\xi}))$. Since $\varphi$ is a symplectic diffeomorphism, $(d\varphi)_{(\bar{x};\bar{\xi})}T_{\bar{x}}^*=E$ is a Lagrangean subspace of $T_{(y;\bar{\eta})}T_N^*$. Thus, one can define a quadratic form on $T_y$ by the above manner. We denote this quadratic form by $\tilde{A}_{(y;\bar{\eta})}(Y)$. As a matter of course, if $K_\zeta=\{0\}$, then $\tilde{A}_{(y;\bar{\eta})}=0$. Such a point $(y;\bar{\eta})=\varphi(\bar{x};\bar{\xi})$ will be called a *non-degenerate point* of $\varphi$. We extend this quadratic form $\tilde{A}_{(y;\bar{\eta})}$ on $T_y$ to a neighborhood of $T_y$ by the following:

$$A_{(y;\bar{\eta})}(y';Y')=\tilde{A}_{(y;\bar{\eta})}((d\,\mathrm{Exp}_y^{-1})_{y'}Y')\ . \tag{18}$$

Now, we recall the defining equality (15) for a while. We define $\bar{Y}_0$, $Y_1$ by

$$\begin{cases}\varphi_1(x;\xi)=\cdot_y\bar{Y}_0\\ \cdot_{\varphi_1(x;\xi)}X=\cdot_yY_1\ .\end{cases} \tag{19}$$

Since $\varphi$ is positively homogeneous of order 1 (cf. (9)), $\bar{Y}_0=\bar{Y}_0(x;\xi)$ is well-defined on a conic neighborhood $U'$ of $(\bar{x};\bar{\xi})$, and $X$ is given by

(20) $$X=S(y;\, Y_1,\, \bar{Y}_0(x;\xi))\ .$$

Using these notations, (15) can be rewritten as follows:

$$F(a,\varphi,\nu)u=\sum_\alpha F(\lambda_\alpha a,\varphi,\nu)u\ ,$$

$$(F(\lambda_\alpha a,\varphi,\nu)u)(x)$$
$$=\int\int a_\alpha^* e^{-i\phi_\alpha(x;\xi|Y_1)}\nu(\cdot_y\bar{Y}_0,\cdot_y Y_1)u(\cdot_y Y_1)dY_1d\xi\ ,$$

where

(21) $$\begin{cases} a_\alpha^*(x;\xi\,|\,Y_1)=\lambda_\alpha(x;\xi)a(x;\xi;X)dX/dY_1 \\ \phi_\alpha(x;\xi\,|\,Y_1)=\langle\varphi_2(x;\xi)\,|\,X\rangle+|\xi|A_\alpha(X)\ , \end{cases}$$

and

$$X=S(y;\, Y_1,\, \bar{Y}_0(x;\xi))\ .$$

There are a conic neighborhood $U$ of $(\bar{x};\bar{\xi})$ and a neighborhood $V$ of 0 in $T_y$ such that $a_\alpha^*$ and $\phi_\alpha$ are well-defined $C^\infty$ functions on $U\times V$.

DEFINITION 2.2 ([6]). A function $\phi(x;\xi\,|\,Y_1)$ defined on an open conic subset $\Gamma$ in $U\times V$ such that $\phi(x;r\xi\,|\,Y_1)=r\phi(x;\xi\,|\,Y_1)$, $r>0$, is called a *non-degenerate phase function*, if for each fixed $x$, $\phi$ has no critical point in $\Gamma$ and at any point in the set $C_x$ defined by

$$C_x=\left\{(Y_1,\xi);\ \frac{\partial\phi}{\partial\xi}(x;\xi\,|\,Y_1)=0\right\}$$

the differentials

$$d\left(\frac{\partial\phi}{\partial\xi_1}\right),\ \cdots,\ d\left(\frac{\partial\phi}{\partial\xi_n}\right)\quad\text{(where } x \text{ is fixed)}$$

are linearly independent.

Now, we go back to our situation. For a Lagrangean subspace $E=(d\varphi)_{(\bar{x};\bar{\xi})}T_{\bar{x}}^*$, we have defined a quadratic form $A_{(y;\bar{\eta})}(y';\,Y')$ around $T_y$. We consider the following function

$$\langle\varphi_2(x;\xi)\,|\,X\rangle+|\xi|A_{(y;\bar{\eta})}(\varphi_1(x;\xi);\,X)\ .$$

If we take (21) in mind, we can understand the meaning of the following:

**PROPOSITION 2.3.** *Notations being as above, the function $\phi$ defined by*

$$\begin{cases}\phi(x;\xi\,|\,Y_1)=\langle\varphi_2(x;\xi)\,|\,X\rangle+|\xi|\,A_{(y;\bar{\eta})}(\varphi_1(x;\xi);X)\ ,\\ X=S(y;\ Y_1,\ \bar{Y}_0(x;\xi))\ ,\quad \cdot_y\bar{Y}_0(x;\xi)=\varphi_1(x;\xi)\ ,\end{cases}$$

*is non-degenerate on $U'\times V'$, where $U'$ is a conic neighborhood of $(\bar{x};\bar{\xi})\in T_N^*-\{0\}$ and $V'$ is a neighborhood of 0 in $T_y$.*

Although the proof will be given in several lemmas below, we should remark at first that we may prove the above lemma by using normal coordinate expressions.

To get a normal coordinate expression around $y$, we set

$$(22)\qquad \begin{cases}(\varphi_1(x;\xi);X)=\cdot_y(\bar{Y}_0(x;\xi),\tilde{X})\\ (\varphi_1(x;\xi);\varphi_2(x;\xi))=\cdot_y(\tilde{\varphi}_1(x;\xi),\tilde{\varphi}_2(x;\xi))\ .\end{cases}$$

Obviously, $\bar{Y}_0=\tilde{\varphi}_1$. Since $\langle\varphi_2(x;\xi)\,|\,X\rangle=\langle\tilde{\varphi}_2(x;\xi)\,|\,\tilde{X}\rangle$ (cf. (1)), the normal coordinate expression of $\phi(y;\xi\,|\,Y_1)$ is given by

$$(23)\qquad \langle\tilde{\varphi}_2(x;\xi)\,|\,\tilde{X}\rangle+|\xi|\,\tilde{A}_{(y;\bar{\eta})}(\tilde{X})\ .$$

Let $\tilde{S}(y;\ Y_1,\ \bar{Y}_0(x;\xi))$ be the normal coordinate expression of $S$ around $T_y$ i.e.,

$$(\cdot_y\bar{Y}_0(x;\xi);S(y;\ Y_1,\ \bar{Y}_0(x;\xi)))=\cdot_y(\bar{Y}_0(x;\xi),\tilde{S}(y;\ Y_1,\ \bar{Y}_0(x;\xi)))\ .$$

Therefore, we see

$$(24)\qquad \tilde{X}=\tilde{S}(y;\ Y_1,\ \bar{Y}_0(x;\xi))\ ,\quad y=\varphi_1(\bar{x};\bar{\xi})\ .$$

**LEMMA 2.4.**

$$\frac{\partial}{\partial\xi}\phi\,|_{Y_1=\bar{Y}_0(x;\xi)}\equiv 0\ .$$

**PROOF.** Since $Y_1=\bar{Y}_0(x;\xi)$ implies $X=0$, we have only to show that

$$\frac{\partial}{\partial\xi}\langle\varphi_2(x;\xi)\,|\,S(y;\ Y_1,\ \bar{Y}_0(x;\xi))\rangle\,|_{Y_1=\bar{Y}_0(x;\xi)}\equiv 0\ .$$

The left hand side is equal to $\langle\varphi_2(x;\xi)\,|\,(\nabla S/\partial\bar{Y}_0)(y;\ \bar{Y}_0,\ \bar{Y}_0)(\partial\bar{Y}_0/\partial\xi)\rangle$. Since

$$S(y;\ \bar{Y}_0(x;\xi),\ \bar{Y}_0(x;\xi))\equiv 0\ ,$$

we see (cf. Lemma 1.4)

$$\frac{\partial S}{\partial Y_1}(y;\ \bar{Y}_0,\ \bar{Y}_0)\frac{\partial\bar{Y}_0}{\partial\xi}+\frac{\nabla S}{\partial\bar{Y}_0}(y;\ \bar{Y}_0,\ \bar{Y}_0)\frac{\partial\bar{Y}_0}{\partial\xi}\equiv 0\ .$$

Thus, we have only to show that

$$\left\langle \varphi_2(x;\xi) \middle| \frac{\partial S}{\partial Y_1}(y;\ \bar{Y}_0(x;\xi),\ \bar{Y}_0(x;\xi))\frac{\partial\tilde{\varphi}_1(x;\xi)}{\partial\xi}\right\rangle \equiv 0 .$$

On the other hand, if we put $\varphi_1(x;\xi')=\cdot_{\varphi_1(x;\xi)}\hat{\varphi}_1(x;\xi')$ then

$$\frac{\partial\varphi_1(x;\xi)}{\partial\xi}=\cdot_y\Big(\tilde{\varphi}_1(x;\xi),\ \frac{\partial\tilde{\varphi}_1(x;\xi')}{\partial\xi'}\Big|_{\xi'=\xi}\Big)=\cdot_{\varphi_1(x;\xi)}\Big(0,\ \frac{\partial\hat{\varphi}_1(x;\xi')}{\partial\xi'}\Big|_{\xi'=\xi}\Big) .$$

Thus by Lemma 1.2 (1), we have

$$\frac{\partial\hat{\varphi}_1(x;\xi')}{\partial\xi'}\Big|_{\xi'=\xi}=\frac{\partial S}{\partial Y_1}(y;\ \bar{Y}_0,\ \bar{Y}_0)\frac{\partial\tilde{\varphi}_1(x;\xi')}{\partial\xi'}\Big|_{\xi'=\xi} .$$

Therefore, we have only to show $\langle\varphi_2(x;\xi)|(\partial\hat{\varphi}_1/\partial\xi)(x;\xi)\rangle=0$. Since $(\partial\hat{\varphi}_1/\partial\xi)(x;\xi)$ can be naturally identified with $\partial\varphi_1/\partial\xi(x;\xi)$, we may write the left hand side as $\langle\varphi_2(x;\xi)|(\partial\varphi_1/\partial\xi)(x;\xi)\rangle$.

Now, let $\cdot_x(X^1,\cdots,X^n,\xi_1,\cdots,\xi_n)$, $\cdot_{\varphi_1(x;\xi)}(\acute{X}^1,\cdots,\acute{X}^n,\acute{\xi}_1,\cdots,\acute{\xi}_n)$ be normal canonical charts around $T^*_x$, $T^*_{\varphi_1(x;\xi)}$ respectively. Since $\varphi\in\mathscr{D}^{(1)}_\Omega$ and hence $\varphi^*\theta=\theta$ (cf. Lemma 1.6), we see that if

$$\begin{cases}\acute{X}^i=\varphi^i_1(\bar{X}^1,\cdots,\bar{X}^n,\xi_1,\cdots,\xi_n) ,\\ \acute{\xi}_i=\varphi_{2,i}(\bar{X}^1,\cdots,\bar{X}^n,\xi_1,\cdots,\xi_n) ,\end{cases}$$

then

$$\acute{\xi}_i d\acute{X}^i=\acute{\xi}_i\frac{\partial\acute{X}^i}{\partial\bar{X}^j}d\bar{X}^j+\acute{\xi}_i\frac{\partial\acute{X}^i}{\partial\xi_j}d\xi_j=\xi_j d\bar{X}^j .$$

Therefore,

$$\text{(25)}\qquad\begin{cases}\acute{\xi}_i\dfrac{\partial\acute{X}^i}{\partial\bar{X}^j}=\xi_j\\[2ex] \acute{\xi}_i\dfrac{\partial\acute{X}^i}{\partial\xi_j}=0 .\end{cases}$$

The second equality implies $\langle\varphi_2(x;\xi)|(\partial\varphi_1/\partial\xi)(x;\xi)\rangle=0$.

Now, we consider at the fixed point $(\bar{x};\bar{\xi})$. Since $y=\varphi_1(\bar{x};\bar{\xi})$, and $\cdot_y\bar{Y}_0(x;\xi)=\varphi_1(x;\xi)$, it is clear that $\bar{Y}_0(\bar{x};\bar{\xi})=0$.

**LEMMA 2.5.** *Set $(y;\bar{\eta})=(\varphi_1(\bar{x};\bar{\xi});\varphi_2(\bar{x};\bar{\xi}))$, and $\partial^2\phi/\partial Y_1\partial\xi\big|_{x=\bar{x},\ \xi=\bar{\xi},\ Y_1=0}$ is a non-singular matrix.*

PROOF. Let $._{\bar{x}}(\bar{X}^1, \cdots, \bar{X}^n, \xi_1, \cdots, \xi_n)$ and $._y(Y^1, \cdots, Y^n, \eta_1, \cdots, \eta_n)$ be normal canonical chart around $T^*_{\bar{x}}$ and $T^*_y$ respectively. By the same argument between (17) and (18), one may assume that $Y^1, \cdots, Y^n$, and hence $\eta_1, \cdots, \eta_n$ are orthonormal charts and $\pi_\zeta(d\varphi)_{(\bar{x};\bar{\xi})}T^*_{\bar{x}} = \mathbb{T}_\zeta(E)$ is given by $\eta_{l+1}=0, \cdots, \eta_n=0$. If we put

$$(\varphi_1; \varphi_2) = ._y(\tilde{\varphi}_1^1, \cdots, \tilde{\varphi}_1^n, \tilde{\varphi}_{2,1}, \cdots, \tilde{\varphi}_{2,n})$$

$$(._y\bar{Y}_0; S) = ._y(\bar{Y}_0^1, \cdots, \bar{Y}_0^n, \tilde{S}^1, \cdots, \tilde{S}^n)$$

by the above normal chart, then

$$\phi(\bar{x}; \xi \mid Y_1) = \tilde{\varphi}_{2,i}(\bar{x}; \xi)\tilde{S}^i(y; Y_1, \bar{Y}_0(\bar{x}; \xi)) + |\xi| \textstyle\sum_{j=l+1}^n \tilde{S}^j(y; Y_1, \bar{Y}_0(\bar{x}; \xi))^2 .$$

Remark that $\bar{Y}_0(\bar{x}; \bar{\xi})=0$ and $S(y; Y_1, \bar{Y}_0)=0$ if $Y_1=\bar{Y}_0$. Hence

$$\left.\frac{\partial^2\phi}{\partial Y_1^k \partial\xi_j}\right|_{\substack{\xi=\bar{\xi}\\ Y_1=0}} = \frac{\partial\tilde{\varphi}_{2,i}}{\partial\xi_j}(\bar{x}; \bar{\xi})\frac{\partial\tilde{S}^i}{\partial Y_1^k}(y; 0, 0) + \tilde{\varphi}_{2,i}(\bar{x}; \bar{\xi})\left.\frac{\partial^2\tilde{S}^i}{\partial\xi_j\partial Y_1^k}\right|_{\substack{\xi=\bar{\xi}\\ Y_1=0}}$$
$$+2|\bar{\xi}| \textstyle\sum_{i=l+1}^n \frac{\partial\tilde{S}^i}{\partial Y_1^k}(y; 0, 0)\left.\frac{\partial S^i}{\partial\xi_j}\right|_{\substack{\xi=\bar{\xi}\\ Y_1=0}} .$$

Since $S(y; Y_1, 0)=Y_1$, we see $\partial\tilde{S}^i/\partial Y_1^k(y; 0, 0)=\delta_k^{\ i}$. Moreover,

$$\left.\frac{\partial^2\tilde{S}^i}{\partial\xi_j\partial Y_1^k}\right|_{\substack{\xi=\bar{\xi}\\ Y_1=0}} = \left.\frac{\partial^2\tilde{S}^i}{\partial\bar{Y}_0^l\partial Y_1^k}\right|_{\substack{Y_1=0\\ \bar{Y}_0=0}} \frac{\partial\bar{Y}_0^l}{\partial\xi_j}(\bar{x}; \bar{\xi})=0$$

because of Lemma 1.5. Therefore, we obtain

$$\left.\frac{\partial^2\phi}{\partial Y_1^k\partial\xi_j}\right|_{\substack{\xi=\bar{\xi}\\ Y_1=0}} = \frac{\partial\tilde{\varphi}_{2,k}}{\partial\xi_j}(\bar{x}; \bar{\xi}) + \begin{cases} 2|\bar{\xi}|\left.\dfrac{\partial\tilde{S}^k}{\partial\xi_j}\right|_{\substack{\xi=\bar{\xi}\\ Y_1=0}} & (k\geqq l+1) \\ 0 & (k\leqq l) . \end{cases}$$

Note that

$$\frac{\partial\tilde{S}^k}{\partial\xi_j}(\bar{x}; 0, \bar{Y}_0(\bar{x}; \bar{\xi})) = -\left.\frac{\partial\tilde{S}^k}{\partial Y_1^l}\right|_{\substack{\bar{Y}_0=0\\ Y_1=0}} \frac{\partial\bar{Y}_0^l}{\partial\xi_j}(\bar{x}; \bar{\xi}) = -\frac{\partial\bar{Y}_0^k}{\partial\xi_j}(\bar{x}; \bar{\xi}) = -\frac{\partial\tilde{\varphi}_1^k}{\partial\xi_j}(\bar{x}; \bar{\xi}) .$$

The last equality is simply because of $\tilde{\varphi}_1=\bar{Y}_0$

Now, remark that $(\partial\tilde{\varphi}_{2,k}/\partial\xi_j)(\bar{x}; \bar{\xi})=0$ for $k\geqq l+1$ because that $\pi_\zeta(d\varphi)_{(\bar{x};\bar{\xi})}T^*_{\bar{x}}$ is given by $\eta_{l+1}=\cdots=\eta_n=0$. By a suitable linear change of normal chart $._{\bar{x}}(\bar{X}^1, \cdots, \bar{X}^n, \bar{\xi}_1, \cdots, \bar{\xi}_n)$, we may assume that

$$\left(\frac{\partial\tilde{\varphi}_{2,k}}{\partial\xi_j}(\bar{x}; \bar{\xi})\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)\begin{matrix}\}l \\ \}n-l\end{matrix} .$$

Hence the kernel $K_\zeta$ of $\pi_\zeta: E \to T_y^*$ $(y=\varphi_1(\bar{x}; \bar{\xi}))$ is given by $(d\varphi)_{(\bar{x};\bar{\xi})}E'$, where $E'$ is the space spanned by $\xi_{l+1}, \cdots, \xi_n$. Since $\pi_Z: K_\zeta \to T_y$ is injective, we see $\pi_Z(d\varphi)_{(\bar{x};\bar{\xi})}: E' \to T_y$ is injective. Recall that the image $\pi_Z(d\varphi)_{(\bar{x};\bar{\xi})}E'$ is given by $\bar{X}^1=\cdots=\bar{X}^l=0$. Therefore, we get that

$$\left(\frac{\partial\tilde{\varphi}_1^k}{\partial\xi_j}(\bar{x}; \bar{\xi})\right)_{l+1\leq k, j\leq n}$$

is non-singular. It follows the non-singularity of $(\partial^2\phi/\partial Y_1^k\partial\xi_j)(\bar{x}; \bar{\xi}|0)$.

By the above lemma, there are a conic neighborhood $U$ of $(\bar{x}; \bar{\xi})$ in $T_N^*-\{0\}$ and a neighborhood $V$ of 0 in $T_y$ such that $\phi(x; \xi|Y_1)$ (cf. the statement of Proposition 2.3) satisfies that $\partial^2\phi/\partial Y_1\partial\xi$ is non-singular on $U\times V$. Therefore, the equation

$$\frac{\partial\phi}{\partial\xi}(z; \zeta|Y_1)=0\ , \quad (z; \zeta|Y_1)\in U\times V$$

defines for each fixed $z$ an $n$-dimensional submanifold $L_z$, which contains the submanifold $\{Y_1=\bar{Y}_0(z; \zeta)\}$. Because of the non-singularity of $\phi$, one may assume the following without loss of generality:

(A) For a point $(z; \zeta|Y_1)\in U\times V$, $(\partial\phi/\partial\xi)(z; \zeta|Y_1)=0$ if and only if $Y_1=\bar{Y}_0(z; \zeta)$.

The following lemma completes the proof of Proposition 2.3:

LEMMA 2.6. *On $U\times V$, the function $\phi(z; \zeta|Y_1)$ is a non-degenerate phase function.*

PROOF. It is clear that $\phi(z; r\zeta|Y_1)=r\phi(z; \zeta|Y_1)$ for every $r>0$. Therefore, we have only to show that $\phi$ has no critical point. So, suppose $(z; \zeta|Y_1)$ is a critical point of $\phi$ with respect to $\zeta$, $Y_1$. By the above property (A), we get $Y_1=\bar{Y}_0(z; \zeta)$. Hence $X=S(z; Y_1, \bar{Y}_0(z; \zeta))=0$. Therefore,

$$\frac{\partial\phi}{\partial Y_1}(z; \zeta|Y_1)=0$$

if and only if $\langle\varphi_2(z; \zeta)|(\partial S/\partial Y_1)(z; \bar{Y}_0(z; \zeta), \bar{Y}_0(z; \zeta))\rangle=0$. Note that

$$\frac{\partial S}{\partial Y_1}(z; \bar{Y}_0, \bar{Y}_0)=(d\,\mathrm{Exp}_z)_{\bar{Y}_0}\ ,$$

and we may assume without loss of generality that $(d\,\mathrm{Exp}_z)_{\bar{Y}_0}$ is non-singular, if $U$ and $V$ are sufficiently small. Therefore, $\phi(z; \zeta|Y_1)$ is critical if and only if $\varphi_2(z;\zeta)=0$, but this is impossible because $\varphi$ is a diffeomorphism of $T_N^*-\{0\}$ onto itself.

As an immediate conclusion from Proposition 2.3 and the above argument we get the following:

COROLLARY 2.7. *Let* $\varphi\in\mathscr{D}_\Omega^{(1)}$. *For a fixed point* $(\bar{x};\bar{\xi})\in T_N^*-\{0\}$, *we set* $(y;\eta)=\varphi(\bar{x};\bar{\xi})$. *Then, there are a conic neighborhood* $U$ *of* $(\bar{x};\bar{\xi})$ *in* $T_N^*-\{0\}$ *and a neighborhood* $V$ *of* $0$ *in* $T_y$ *such that the function* $\phi$ *defined in Proposition* 2.3 *satisfies the followings*:

(a) $\phi(x;\xi\,|\,Y_1)$ *is a non-degenerate phase function on* $U\times V$.

(b) *For each fixed* $x$, $(\partial\phi/\partial\xi)(x;\xi\,|\,Y_1)=0$ *if and only if* $Y_1=\bar{Y}_0(x;\xi)$, *where* $\cdot_y\bar{Y}_0(x;\xi)=\varphi_1(x;\xi)$.

## §3. Fourier integral operators on a compact manifold.

Suppose we have a symplectic diffeomorphism $\varphi\in\mathscr{D}_\Omega^{(1)}$ and an amplitude $a(x;\xi;X)\in\widetilde{\sum}{}_\varphi^\beta$. By Corollary 2.7, for each point $(\bar{x};\bar{\xi})\in S_N^*$, there are a conic neighborhood $U_{(\bar{x};\bar{\xi})}$ of $(\bar{x};\bar{\xi})$ in $T_N^*-\{0\}$ and a neighborhood $V_{\varphi_1(\bar{x};\bar{\xi})}$ of $0$ in $T_{\varphi_1(\bar{x};\bar{\xi})}$ such that the function $\phi$ defined by

$$(26)\qquad \begin{cases}\phi(x;\xi\,|\,Y_1)=\langle\varphi_2(x;\xi)\,|\,X\rangle+|\xi\,|\,A_{(y;\bar{\eta})}(\varphi_1(x;\xi);X)\ ,\\ X=S(y;\,Y_1,\;\bar{Y}_0(x;\xi))\ ,\quad \cdot_y\bar{Y}_0(x;\xi)=\varphi_1(x;\xi)\ ,\\ y=\varphi_1(\bar{x};\bar{\xi})\end{cases}$$

satisfies (a) and (b) in Corollary 2.7 on $U_{(\bar{x};\bar{\xi})}\times V_{\varphi_1(\bar{x};\bar{\xi})}$.

Now, we set

$$W_{(\bar{x};\bar{\xi})}^{\delta}=\{(x;\xi)\in T_N^*-\{0\};\,\rho(\varphi_1(\bar{x};\bar{\xi}),\,\varphi_1(x;\xi))<\delta\}\ .$$

Since $\varphi$ is of order 1 (cf. (9)), $W_{(\bar{x};\bar{\xi})}^{\delta}$ is a conic neighborhood of $(\bar{x};\bar{\xi})$ in $T_N^*-\{0\}$. We can choose $\delta=\delta(\bar{x};\bar{\xi})$ so that it may satisfy that $W_{(\bar{x};\bar{\xi})}^{\delta}\subset U_{(\bar{x};\bar{\xi})}$ and that the $2\delta$-neighboorhood of $0$ in $T_{\varphi_1(\bar{x};\bar{\xi})}$ is contained in $V_{\varphi_1(\bar{x};\bar{\xi})}$.

As $S_N^*$ is compact, there are finite number of points $\{(\bar{x}_\alpha;\bar{\xi}_\alpha)\}$ $1\leqq\alpha\leqq k$ in $S_N^*$ such that $T_N^*-\{0\}=\bigcup_\alpha W_\alpha$, where $W_\alpha=W_{(\bar{x}_\alpha;\bar{\xi}_\alpha)}^{\delta(\bar{x}_\alpha;\bar{\xi}_\alpha)}$. Set

$$(27)\qquad \delta_0=\min_\alpha\delta(\bar{x}_\alpha;\bar{\xi}_\alpha)\ ,$$

and let $\{\lambda_\alpha(x;\xi)\}_{1\leq\alpha\leq k}$ be a partition of unity subordinate to the above covering $\{W_\alpha\}_{1\leq\alpha\leq k}$ such that $\lambda_\alpha(x;r\xi)=\lambda_\alpha(x;\xi)$ for every $r>0$. Let $\nu$ be a cut off function of the breadth less than $\delta_0$. Putting $(y_\alpha:\bar{\eta}_\alpha)=\varphi(\bar{x}_\alpha:\bar{\xi}_\alpha)$, we set

$$\begin{cases}\phi_\alpha(x;\xi;X)=\langle\varphi_2(x;\xi)\,|\,X\rangle+|\xi|\,A_\alpha(\varphi_1(x;\xi);X)\\ A_\alpha(\varphi_1(x;\xi);X)=A_{(y_\alpha;\bar\eta_\alpha)}(\varphi_1(x;\xi);X)\,,\end{cases}$$

where

$$\begin{cases}X=S(y_\alpha\colon Y_1,\ \bar{Y}_0(x;\xi))\,, \quad \cdot_{y_\alpha}\bar{Y}_0(x;\xi)=\varphi_1(x;\xi)\,,\\ y_\alpha=\varphi_1(\bar{x}_\alpha;\bar{\xi}_\alpha)\,.\end{cases}$$

Under these notations, we consider the operator written in the form (15). To prove $F(a,\varphi,\nu)u$ is well-defined is to prove that each

(28) $$(F(\lambda_\alpha a,\varphi,\nu)u)(x) =\int_{T_x^*}\int_{T_{\varphi_1(x;\xi)}}\lambda_\alpha(x;\xi)a(x;\xi;X)e^{-i\phi_\alpha(x;\xi;X)}(\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi$$

is well-defined. To do so, we use the normal coordinate expressions around $T_{\bar{x}_\alpha}^*$ and $T_{y_\alpha}$, where $y_\alpha=\varphi_1(\bar{x}_\alpha;\bar{\xi}_\alpha)$. Thus, set

$$\varphi_1(x;\xi)=\cdot_{y_\alpha}\bar{Y}_0(x;\xi)\,, \quad \cdot_{\varphi_1(x;\xi)}X=\cdot_{y_\alpha}Y_1\,.$$

Then, (28) can be rewritten as

$$\int_{T_x^*}\int_{T_y}a_\alpha(x;\xi\,|\,Y_1)e^{-i\phi_\alpha(x;\xi|Y_1)}\nu(\cdot_{y_\alpha}\bar{Y}_0,\ \cdot_{y_\alpha}Y_1)u(\cdot_{y_\alpha}Y_1)dY_1d\xi\,,$$

where $\phi_\alpha$ is defined by (26) replacing $A_{(y;\bar\eta)}$ by $A_\alpha$, and

(29) $$\begin{cases}a_\alpha(x;\xi\,|\,Y_1)=\lambda_\alpha(x;\xi)a(x;\xi;X)dX/dY_1\\ X=S(y_\alpha;\,Y_1,\ \bar{Y}_0(x;\xi))\end{cases}.$$

Since we may assume $(x;\xi)\in W_\alpha$, $\varphi_1(x;\xi)$ is always contained in a $\delta(\bar{x}_\alpha;\bar{\xi}_\alpha)$-neighborhood of $\varphi_1(\bar{x}_\alpha;\bar{\xi}_\alpha)=y_\alpha$ and hence $|\bar{Y}_0(x;\xi)|<\delta(\bar{x}_\alpha;\bar{\xi}_\alpha)$. Since $\nu=0$ if $|X|\geqq\delta_0=\min_\alpha\delta(\bar{x}_\alpha;\bar{\xi}_\alpha)$, we have only to consider the points $Y_1\in T_{y_\alpha}$ such that $|Y_1|<2\delta(\bar{x}_\alpha;\bar{\xi}_\alpha)$. Remark that such a point is contained in $V_{\varphi_1(\bar{x}_\alpha;\bar{\xi}_\alpha)}$. Since $W_\alpha\subset U_{(\bar{x}_\alpha;\bar{\xi}_\alpha)}$, the phase function $\phi_\alpha(x;\xi|\,Y_1)$ is non-degenerate on $W_\alpha\times V_{\varphi_1(\bar{x}_\alpha;\bar{\xi}_\alpha)}$. Thus, by Theorem 1.4.1 [6], we obtain the following:

THEOREM 3.1. *$F(\lambda_\alpha a,\varphi,\nu)$ is well-defined by an oscillatory integral as a continuous linear mapping of $C^\infty(N)$ into itself. Therefore,*

$$F(a,\varphi,\nu)=\sum\nolimits_\alpha F(\lambda_\alpha a,\varphi,\nu)$$

*is an well-defined operator of $C^\infty(N)$ into itself.*

For an arbitrarily fixed $x\in N$, $u\longmapsto(F(a,\varphi,\nu)u)(x)$ can be regarded as a distribution.

LEMMA 3.2. *The wave front set* (cf. 2.5 [6]) *of* $u \rightsquigarrow (F(a, \varphi, \nu)u)(x)$ *is contained in* $\varphi(T_x^* - \{0\})$.

PROOF. We have only to show that the wave front set of $u \mapsto (F(\lambda_\alpha a, \varphi, \nu)u)(x)$ is contained in $\varphi(W_\alpha) \cap \varphi(T_x^* - \{0\})$. In accordance with 2.5.7 in [6], we consider the subset

$$\left\{\left(Y_1, \frac{\partial \phi_\alpha}{\partial Y_1}(x; \xi \mid Y_1)\right); (x; \xi \mid Y_1) \in W_\alpha \times V_{y_\alpha}, \frac{\partial \phi_\alpha}{\partial \xi}(x; \xi \mid Y_1) = 0\right\} . \tag{30}$$

In a normal coordinate expression around $y_\alpha$, the wave front set is contained in the above set. By the property (6) in Corollary 2.7, we see that $(\partial\phi_\alpha/\partial\xi)(x; \xi \mid Y_1) = 0$ if and only if $Y_1 = \bar{Y}_0(x; \xi)$. Remark that $Y_1 = \bar{Y}_0(x; \xi)$ implies $X = 0$. Hence

$$\frac{\partial \phi_\alpha}{\partial Y_1}(x; \xi \mid Y_1)\Big|_{Y_1 = \bar{Y}_0(x;\xi)} = \left\langle \varphi_2(x; \xi) \,\Big|\, \frac{\partial S}{\partial Y_1}(y_\alpha; \bar{Y}_0(x; \xi), \bar{Y}_0(x; \xi))\right\rangle . \tag{31}$$

Remark that $(\partial S/\partial Y_1)(y_\alpha; \bar{Y}_0(x; \xi), \bar{Y}_0(x; \xi)) = (d\,\mathrm{Exp}_{y_\alpha})_{\bar{Y}_0}$ and hence

$$\left\langle \varphi_2(x; \xi) \,\Big|\, \frac{\partial S}{\partial Y_1}(y_\alpha; \bar{Y}_0, \bar{Y}_0)\right\rangle = (d\,\mathrm{Exp}_{y_\alpha})^*_{\bar{Y}_0} \varphi_2(x; \xi) , \tag{32}$$

hence the right hand side of (31) is the normal coordinate expression of $\varphi_2(x; \xi)$. Namely, if we set

$$\cdot_y(\tilde{\varphi}_1(x; \xi), \tilde{\varphi}_2(x; \xi)) = (\varphi_1(x; \xi); \varphi_2(x; \xi)) ,$$

then

$$\begin{cases} \tilde{\varphi}_1(x; \xi) = \bar{Y}_0(x; \xi) \\ \tilde{\varphi}_2(x; \xi) = \dfrac{\partial \phi_\alpha}{\partial Y_1}(x; \xi \mid \bar{Y}_0(x; \xi)) . \end{cases}$$

This implies the wave front set is contained in $\varphi(W_\alpha) \cap \varphi(T_x^* - \{0\})$.

REMARK. If $F(a, \varphi, \nu)$ in the above lemma is the identity operator, then the wave front set of $u \mapsto (F(a, \varphi, \nu)u)(x)$ is $T_x^* - \{0\}$ for every $x \in N$. Therefore, we see that $\varphi(T_x^* - \{0\}) \supset T_x^* - \{0\}$ and hence $\varphi(T_x^* - \{0\}) = T_x^* - \{0\}$. Therefore $\varphi =$ identity, by virtue of Lemma 5.6.

LEMMA 3.3. *Let* $\nu'$ *be another cut off function of the breadth less than* $\delta_0$ (cf. (27)). *Then*

$$F(a, \varphi, \nu) - F(a, \varphi, \nu')$$

*is a smoothing operator.*

PROOF. Notations being as above, we have only to show that

$$F(\lambda_\alpha a, \varphi, \nu)-F(\lambda_\alpha a, \varphi, \nu')$$

is a smoothing operator. Now, remark

$$\begin{aligned}((F(\lambda_\alpha a, \varphi, \nu)-F(\lambda_\alpha a, \varphi, \nu'))u)(x) \\ =\int\int \lambda_\alpha a(x; \xi; X)e^{-i\phi_\alpha(x;\xi;X)}((\nu-\nu')u)^\cdot(\varphi_1(x; \xi); X)dXd\xi .\end{aligned}$$

Since $(\nu-\nu')(\varphi_1(x; \xi), \cdot_{\varphi_1(x;\xi)}X)\equiv 0$ if $X$ is sufficiently close to 0,

$$\begin{cases} a''_\alpha(x; \xi\,|\,Y_1)=\lambda_\alpha a(x; \xi; X)(\nu-\nu')(\varphi_1(x; \xi), \cdot_{\varphi_1(x;\xi)}X) , \\ X=S(\varphi_1(\bar{x}_\alpha; \bar{\xi}_\alpha); Y_1, \bar{Y}_0(x; \xi)) \end{cases}$$

vanishes on a conic neighborhood of

$$C_\alpha=\left\{(x; \xi\,|\,Y_1) \in W_\alpha\times V_{\nu_\alpha}; \frac{\partial\phi_\alpha}{\partial\xi}(x; \xi\,|\,Y_1)=0\right\} ,$$

because $C_\alpha$ is given by

$$\{(x; \xi\,|\,\bar{Y}_0(x; \xi)); \xi \in T^*_x-\{0\}\}\cap W_\alpha\times V_{\nu_\alpha} .$$

Hence by Proposition 1.2.4 or 5 in [6], we get the desired results.

Now, suppose we have a $C^\infty$ diffeomorphism $\psi$ of $N$ onto itself. Then, $\psi$ defines a symplectic diffeomorphism $\tilde{\psi} \in \mathscr{D}^{(1)}_\Omega$ as follows:

$$\tilde{\psi}(x; \xi)=(\psi(x); (\psi^{-1})^*_x\xi) . \tag{33}$$

Therefore, the group $\mathscr{D}(N)$ of all $C^\infty$ diffeomorphism on $N$ is naturally imbedded in the group $\mathscr{D}^{(1)}_\Omega$. Note that each point $(y: \bar{\eta}) \in T^*_N-\{0\}$ is a non-degenerate point of $\tilde{\psi}$ (cf. between (17) and (18)). Thus, if $\varphi \in \mathscr{D}^{(1)}_\Omega$ is sufficiently close to the group $\mathscr{D}(N)$, then $\varphi$ has no degenerate point, and we have no need to consider the quadratic forms $A_\alpha(X)$.

In such a situation, our Fourier integral operator (15) can be written in the form

$$\begin{aligned}(F(a, \varphi, \nu)u)(x) \\ =\int_{T^*_x}\int_{T_{\varphi_1(x;\xi)}} a(x; \xi; X)e^{-i\langle\varphi_2(x;\xi)|X\rangle}(\nu u)^\cdot(\varphi_1(x; \xi); X)dXd\xi .\end{aligned} \tag{34}$$

If we set $z=\cdot_{\varphi_1(x;\xi)}X$, $dz=J(\varphi_1(x; \xi); X)dX$, where $dz=(2\pi)^{-n/2}\times$(volume element), and $a'(x; \xi; X)=a(x; \xi; X)J(\varphi(x; \xi); X)^{-1}$, then (34) is changed into

(35) $$(\tilde{F}(a', \varphi, \nu)u)(x) = \int_{T_x^*}\int_N a'(x;\xi;X)e^{-i\langle\varphi_2(x;\xi)|X\rangle}\nu(\varphi_1(x;\xi), z)u(z)dzd\xi\ ,$$

$$X = {}^{\varphi_1(x;\xi)^*}z\ .$$

Suppose furthermore that the amplitude $a'$ does not contain the variable $X$. (See also the next section, the variable $X$ can be always eliminated if $\varphi$ is sufficiently close to the identity.) If this is the case the above operator (35) can be written in the following simple form:

(36) $$(\tilde{F}(a', \varphi, \nu)u)(x) = \int_{T_x^*} a'(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi\ ,$$

where $\widetilde{\nu u}$ is a sort of Fourier transformation of $(\nu u)^{\cdot}$, defined by

(37) $$\widetilde{\nu u}(y;\eta) = \int_N e^{-i\langle\eta|y\cdot z\rangle}\nu(y, z)u(z)dz\ .$$

The right hand side of (36) can be defined without using oscillatory integrals, because $(\nu u)^{\cdot}(y;Y) = \nu(y, \cdot_y Y)u(\cdot_y Y)$ has a compact support in $Y$, and hence the Fourier transform of $J(y;Y)\nu(y, \cdot_y Y)u(\cdot_y Y)$ is rapidly decreasing in $\eta$.

If $\varphi$=identity in (34), such an operator is called a *pseudo-differential operator*. In this case, the variable $X$ in the amplitude can be eliminated by the same method as in [8]. Hence a pseudo-differential operator on $N$ can be written in the form

(38) $$(\tilde{F}(a, 1, \nu)u)(x) = \int_{T_x^*} a(x;\xi)\widetilde{\nu u}(x;\xi)d\xi\ .$$

REMARK. It is not hard to see that the symbol $a(x;\xi)$ may be recaptured modulo rapidly decreasing functions from the action of the operator (cf. [8]). Moreover, every pseudo-differential operator in the sense of [8] with an amplitude function $a \in \tilde{\sum}_C^\beta$ can be written in the above shape (38) modulo smoothing operators.

However, there is an essential difference between our operators (15) and Fourier integral operators in the sense of [6] or [4]. Consider two elements $\varphi, \psi \in \mathscr{D}_\Omega^{(1)}$ such that $\varphi(T_x^* - \{0\}) \cap \psi(T_x^* - \{0\}) = \varnothing$ for every $x \in N$. Then, $F(a_1, \varphi, \nu) + F(a_2, \psi, \nu)$ is a Fourier integral operator in the sense of [5] and [6]. Our operators are contained in much narrower class than that of [6]. However, we restrict to this narrower class from the group theoretical view point.

EXAMPLE. For a positive number $\varepsilon$ such that $0<\varepsilon<r_0$, we denote by $S_x(\varepsilon)$ the $\varepsilon$-sphere whose center is the origin in the tangent space $T_x$ at a point $x$ of $N$. Let $d\sigma$ the volume element on $S_x(\varepsilon)$. The operator $(\mathscr{M}_\varepsilon u)(x)=\int_{S_x(\varepsilon)} u(\mathrm{Exp}_x X)d\sigma$ is called the *$\varepsilon$-spherical mean* on $N$. It is known in [12] that $\mathscr{M}_\varepsilon$ is a Fourier integral operator of order $-(n-1)/2$ in the sense of [6]. However, $\mathscr{M}_\varepsilon$ is in fact the sum of two operators of the type (15).

For simplicity we assume that $\varepsilon<r_0/3$ and let $\nu$ be cut off function of the breadth $r_0$. Define symplectic diffeomorphisms

$$\varphi_+(x;\xi)=\cdot_x\Big(\frac{\varepsilon\xi}{|\xi|},\xi\Big),\quad \varphi_-(x;\xi)=\cdot_x\Big(\frac{-\varepsilon\xi}{|\xi|},\xi\Big)$$

by identifying $T_x$ and $T_x^*$ through the Riemannian metric. Then, in fact

$$\mathscr{M}_\varepsilon u=F(a_+,\varphi_+,\nu)u+F(a_-,\varphi_-,\nu)u\ ,$$

where $a_\pm\in\Sigma_C^{-(n-1)/2}$. More precisely, $a_\pm$ is given as follows: Set $z=\cdot_x X$ and $dz=J(x;X)\bar{d}X$. Then

$$a_\pm(x;\xi)=e^{\mp i\varepsilon|\xi|}\int_{S^{n-1}} e^{i\varepsilon|\xi|\langle\xi/|\xi|,\hat{X}\rangle}J(x;\varepsilon\hat{X})d\sigma(\hat{X})\ .$$

Using the stationary phase method, we get the asymptotic expansion

$$\begin{aligned}a_\pm(x;\xi)\sim(2\pi)^{(n-1)/2}e^{\pm i(n-1)/4}J(x;\varepsilon\xi/|\xi|)(\varepsilon|\xi|)^{-(n-1)/2}+A_1(\varepsilon|\xi|)^{-(n+1)/2}\\ +A_2(\varepsilon|\xi|)^{-(n+3)/2}+\cdots\ .\end{aligned}$$

Now, assume that $\varphi\in\mathscr{D}_\Omega^{(1)}$ is sufficiently close to the identity in the $C^1$-topology. Then, one may assume that $\varphi$ has no degenerate point. Using a normal chart at each $x\in N$, we set

$$\begin{cases}(\varphi_1(x;\xi);\varphi_2(x;\xi))=\cdot_x(\bar{X}_0(x;\xi),\tilde{\xi}(x;\xi))\ ,\\ (\varphi_1(x;\xi);X)=\cdot_x(\bar{X}_0(x;\xi),\tilde{X})\ .\end{cases}$$

Since $\langle\varphi_2(x;\xi)|X\rangle=\langle\tilde{\xi}(x;\xi)|\tilde{X}\rangle$, our operator (35) can be rewritten as

(39)
$$\begin{aligned}&(\tilde{F}(a,\varphi,\nu)u)(x)\\ &\quad=\int_{T_x^*}\int_N a'(x;\xi;X)e^{-i\langle\tilde{\xi}|\tilde{X}\rangle}\nu(\varphi_1(x;\xi),z)u(z)\bar{d}z\bar{d}\xi\ ,\\ &\qquad z=\cdot_{\varphi_1(x;\xi)}X\ .\end{aligned}$$

Remark that $\tilde{\xi}=\tilde{\xi}(x;\xi)$ can be solved with respect to $\xi$, hence we set $\xi=\xi(x;\tilde{\xi})$.

Set

$$a'(x, \tilde{\xi}\,|\,X)=a(x; \xi(x, \tilde{\xi}); X)\frac{d\xi}{d\tilde{\xi}}\,,$$

and replace the variable $\xi$ by $\tilde{\xi}$. If we set $\cdot_{\varphi_1(x;\xi)}X=\cdot_x X_1$, then

$$\tilde{X}=\tilde{S}(x; X_1, \bar{X}_0(x; \xi)) \quad \text{(cf. (7))}\,.$$

Replacing $X$ or $\tilde{X}$ by $\tilde{S}(x; X_1, \bar{X}_0(x; \xi))$, (39) is changed into the form

$$\begin{aligned}(40) \qquad &(\tilde{F}(a, \varphi, \nu)u)(x)\\ &=\int_{T_x^*}\int_N a''(x; \tilde{\xi}, X_1)e^{-i\langle\tilde{\xi}|\tilde{S}\rangle}\nu(\varphi_1(x; \xi(x; \tilde{\xi})), z)u(z)dzd\tilde{\xi}\,.\end{aligned}$$

Using the normal coordinate expression of (5) and (6), we see that

$$\begin{aligned}(41) \qquad \tilde{X}&=\tilde{S}(x; X_1, \bar{X}_0(x; \xi(x; \tilde{\xi})))\\ &=X_1-\bar{X}_0(x; \xi(x; \tilde{\xi}))+\tilde{Q}(x; X_1, \bar{X}_0)(X_1-\bar{X}_0)^2\,.\end{aligned}$$

Recall that $\varphi$ is assumed to be sufficiently close to the identity and hence by Lemma 1.1, we may assume that $\|\tilde{Q}(x; X, \bar{X}_0)\|$ is bounded.

LEMMA 3.4. *Notations and assumptions being as above,*

$$\begin{cases}\phi_t(x; \tilde{\xi}, X_1)=\langle\tilde{\xi}\,|\,X_1-\bar{X}_0\rangle+t\langle\tilde{\xi}\,|\,\tilde{Q}(x; X_1, \bar{X}_0)(X_1-\bar{X}_0)^2\rangle\\ \bar{X}_0=\bar{X}_0(x; \xi(x; \tilde{\xi}))\end{cases}$$

*is a non-degenerate phase function on* $(T_N^*-\{0\})\oplus D_N(\delta_0)$ *for every* $t$, $0\leqq t\leqq 1$.

PROOF. It is obvious that $\xi(x; r\tilde{\xi})=r\xi(x; \tilde{\xi})$ and $\bar{X}_0(x; r\xi)=\bar{X}_0(x; \xi)$. Therefore, $\phi_t(x; \tilde{\xi}, X_1)$ is positively homogeneous of degree 1.

Suppose $\varphi=\mathrm{id}$. Then, $\bar{X}_0=0$ and hence $\phi_t(x; \tilde{\xi}, X_1)=\langle\tilde{\xi}\,|\,X_1\rangle$ because $S(x; X_1, 0)=X_1$. Therefore, $\phi_t(x; \tilde{\xi}, X_1)$ has no critical point on $(T_x^*-\{0\})\oplus D_x(\delta_0)$ for every $x\in N$. Since $\varphi$ is sufficiently close to the identity in the $C^1$-topology, we may assume that $\phi_t(x; \tilde{\xi}, X_1)$ associated with $\varphi$ has no critical point on $\bar{V}_x\oplus D_x(\delta_0)$ for every $x$ and $t$, $0\leqq t\leqq 1$, where $\bar{V}_x$ is a compact neighborhood of $S_x^*N$. However, since $\phi_t$ is positively homogeneous of degree 1, the above fact shows that $\phi_t(x; \tilde{\xi}, X_1)$ is a phase function for every $t$, $0\leqq t\leqq 1$.

It is obvious that $\partial\phi_t/\partial\tilde{\xi}=0$ on a submanifold $X_1=\bar{X}_0(x, \tilde{\xi})$ for every $x\in N$. Compute $\partial^2\phi_t/\partial\tilde{\xi}\partial X_1|_{X_1=\bar{X}_0}$. Then, we get

$$\left.\frac{\partial^2\phi_t}{\partial\tilde{\xi}\partial X_1}\right|_{X_1=\bar{X}_0}=I-2t\left\langle\tilde{\xi}\,\middle|\,\tilde{Q}\frac{\partial X_1}{\partial X_1}\frac{\partial\bar{X}_0}{\partial\tilde{\xi}}\right\rangle\,.$$

Since $\varphi$ is sufficiently close to the identity, we may assume that $|\tilde{\xi}|\partial\bar{X}_0/\partial\tilde{\xi}$ is close to 0 uniformly. Hence $\partial^2\phi_t/\partial\tilde{\xi}\partial X_1|_{X_1=\bar{X}_0}$ is non-singular for every $t$, $0\leqq t\leqq 1$. It follows that $\phi_t$ $(0\leqq t\leqq 1)$ is a non-degenerate phase function.

LEMMA 3.5. *There is a $C^\infty$ fiber preserving diffeomorphism $\Phi$ of order 1 on $T_N^*-\{0\}$ onto itself such that putting $\Phi(x;\tilde{\xi})=(x;\Phi_2(x;\tilde{\xi}))$*

$$\begin{aligned}&\langle\Phi_2(x;\tilde{\xi})\,|\,X_1-\bar{X}_0(x;\xi(x;\Phi_2(x;\tilde{\xi})))\rangle\\&\quad=\langle\tilde{\xi}\,|\,X_1-\bar{X}_0(x;\xi(x;\tilde{\xi}))\rangle-\langle\tilde{\xi}\,|\,Q(x;X_1,\bar{X}_0)(X_1-\bar{X}_0)^2\rangle\,.\end{aligned}$$

Proof is seen in [6] pp. 139-140. Although his proof is given on a conic neighborhood of a point, the same proof can be applied in our case by virtue of the previous lemma.

Now, for simplicity we set

$$\begin{cases}\zeta=\Phi_2(x;\tilde{\xi})\\X_0'=\bar{X}_0(x;\xi(x;\Phi_2(x;\tilde{\xi})))=X_0'(x;\zeta)\,.\end{cases}\tag{42}$$

Remark that $X_0'(x;\zeta)$ is an expression of $\varphi(T_x^*-\{0\})$ as a graph, i.e.,

$$\varphi(T_x^*-\{0\})=\{\cdot_x(X_0'(x;\zeta);\zeta);\ \zeta\in T_x^*-\{0\}\}\,.$$

Then, operator (40) can be rewritten in the form

$$\begin{aligned}&(\tilde{F}(a,\varphi,\nu)u)(x)\\&\quad=\int_{T_x^*}\int_N\bar{a}(x;\zeta,X_1)e^{-i\langle\zeta|X_1-X_0'(x;\zeta)\rangle}\nu(\varphi_1(x;\xi(\Phi^{-1}(x;\zeta)),z))u(z)dzd\zeta\,,\end{aligned}\tag{43}$$

where

$$\bar{a}(x;\zeta,X_1)=a''(\Phi^{-1}(x;\zeta),X_1)\frac{d\tilde{\xi}}{d\zeta}\,.$$

Now, recall that $z=\cdot_{\varphi_1(x;\xi)}X=\cdot_xX_1$. Then, $dz=J(x;X_1)dX_1$. Hence we may rewrite (43) as follows:

$$\begin{aligned}&(\tilde{F}(a,\varphi,\nu)u)(x)\\&\quad=\int_{T_x^*}\int_{T_x}\hat{a}(x;\zeta,X_1)e^{-i\langle\zeta|X_1\rangle+i\hat{\phi}(x;\zeta)}u(\cdot_xX_1)J(x;X_1)dX_1d\zeta\,,\end{aligned}\tag{44}$$

where

$$\begin{cases}\hat{\phi}(x;\zeta)=\langle\zeta\,|\,X_0'(x;\zeta)\rangle\,,\\\hat{a}(x;\zeta,X_1)=\bar{a}(x;\zeta,X_1)\nu(\varphi_1(x;\xi(\Phi_2^{-1}(x;\zeta)),\cdot_xX_1))\,.\end{cases}\tag{45}$$

The above expression is the most familiar one of the expressions of Fourier integral operators.

## §4. Elimination of $X_1$ in the amplitude $a(x; \xi; X_1)$.

In this section, we consider a Fourier integral operator $\tilde{F}(a, \varphi, \nu)$ written in the form (35), where $a \in \tilde{\sum}_\varphi^\beta$ and $\varphi$ is still assumed to be sufficiently close to the identity in the $C^1$-topology. Then, by the argument between (39)~(40), $F(a, \varphi, \nu)$ can be written in the form (43). The purpose of this section is to show the following:

PROPOSITION 4.1. *There is $b(x; \xi) \in \sum_C^\beta$ such that*

$$(\tilde{F}(a, \varphi, \nu)u)(x)=\int_{T_x^*} b(x; \xi)\widetilde{\nu u}(\varphi(x; \xi))d\xi+K\circ u(x) \quad \text{(cf. (36))},$$

*where $K$ is a smoothing operator.*

The proof will be given in several lemmas below.

Let $\delta$ be the breadth of the cut off function $\nu$ in (35). Since the computation in the proof of the above proposition is modulo smoothing operators, one may replace the cut off function $\nu$ by another one with a smaller breadth by virtue of Lemma 3.3. Hence one may assume that $\delta$ is sufficiently small.

Now, recall the definition of $\delta_0$ (cf. (27)). Since $\varphi$ is sufficiently close to the identity, one may assume that $\delta_0 \geqq r_0/4$. (To ensure this, consider the case $\varphi=$id., in Lemma 2.5~Corollary 2.7.) Moreover, one may assume $\max_{(x;\xi)\in T_N^*-\{0\}} \rho(x, \varphi_1(x; \xi))<\delta$, and $6\delta<\delta_0$. Let $\nu_1(x, z)$ be a cut off function with the breadth $6\delta$. Then it is clear that

$$\nu(\varphi_1(x; \xi), z)\nu_1(x; z)\equiv\nu(\varphi_1(x; \xi), z) \tag{46}$$

for every $\xi$. Therefore the operator (43) can be written in the form

$$\int_{T_x^*}\int_N \hat{a}(x; \zeta, X_1)e^{-i\langle\zeta|X_1-X_0'(x;\zeta)\rangle}\nu_1(x, z)u(z)dzd\zeta \quad \text{(cf. (45))}. \tag{47}$$

It is not hard to see $\hat{a}(x; \zeta, X_1) \in \tilde{\sum}_C^\beta$ if $a(x; \xi; X) \in \tilde{\sum}_\varphi^\beta$. Since

$$\begin{cases} \zeta \longrightarrow \zeta \\ X_1 \longrightarrow X_1-X_0'(x; \zeta) \end{cases}$$

is a $C^\infty$ diffeomorphism on $(T_x^*-\{0\})\oplus T_x$ for every $x\in N$, there is $a'(x; \zeta, X_1)\in\tilde{\sum}_C^\beta$ such that $a'(x; \zeta, X_1-X_0'(x; \zeta))=\hat{a}(x; \zeta, X_1)$.

Let $\tilde{a}'(x; \zeta, \xi^1)$ be the Fourier transform of $a'(x; \zeta, X_1)$ with respect to $X_1$. Evidently, $\tilde{a}'$ is rapidly decreasing in $\xi^1$, and

$$\hat{a}(x;\zeta, X_1)=a'(x;\zeta, X_1-X_0'(x;\zeta))=\int_{T_x^*}\tilde{a}'(x;\zeta,\xi^1)e^{i\langle\xi^1|X_1-X_0'(x;\zeta)\rangle}d\xi^1 .$$

Substituting this into (47) and replacing $\zeta-\xi^1$ by $\zeta'$, we get

$$(48)\qquad (\tilde{F}(a,\varphi,\nu)u)(x) =\int_{T_x^*}\int_N b_1(x;\zeta')e^{-i\langle\zeta'|X_1-X_0'(x;\zeta')\rangle}\nu_1(x,z)u(z)dz d\zeta' ,$$

$z=\cdot_x X_1$, where

$$(49)\qquad b_1(x;\zeta')=\int_{T_x^*}\tilde{a}'(x;\zeta'+\xi^1,\xi^1)e^{i\langle\zeta'|X_0'(x;\zeta'+\xi^1)-X_0'(x;\zeta')\rangle}d\xi^1 .$$

Remark that if $\xi^1=0$, then $\langle\zeta'|X_0'(x;\zeta'+\xi^1)-X_0'(x;\zeta')\rangle=0$. Hence, we can write

$$(50)\qquad \langle\zeta'|X_0'(x;\zeta'+\xi^1)-X_0'(x;\zeta')\rangle=\langle\xi^1|A(x;\xi^1,\zeta')\rangle .$$

Since

$$\tilde{a}'(x;\zeta'+\xi^1,\xi^1)=\int_{T_x}a'(x;\zeta'+\xi^1, X_2)e^{-i\langle\xi^1|X_2\rangle}dX_2 ,$$

we have

$$(51)\qquad b_1(x;\zeta')=\int_{T_x^*}\int_{T_x}a'(x;\zeta'+\xi^1, X_3+A(x;\xi^1,\zeta'))e^{-i\langle\xi^1|X_3\rangle}dX_3 d\xi^1 ,$$

by replacing $X_2-A(x;\xi^1,\zeta')$ by $X_3$.

LEMMA 4.2. *Notations and assumptions being as above,* $b_1(x;\zeta')$ *defined by* (51) *is contained in* $\sum_C^\beta$.

PROOF. By Taylor's formula,

$$a'(x;\zeta'+\xi^1, X_3+A(x;\xi^1,\zeta')) =\sum_{|\alpha|<K}\frac{1}{\alpha!}a_\alpha(x;\zeta'+\xi^1, A(x;\xi^1,\zeta'))X_3^\alpha+R_K .$$

Hence, we get

$$b_1(x;\zeta')=\sum_{|\alpha|<K}\frac{1}{\alpha!}(D_{\xi^1}^\alpha a_\alpha)(x;\zeta', A(x;0,\zeta'))+\iint R_k e^{-i\langle\xi^1|X_3\rangle}dX_3 d\xi^1 ,$$

where $D_\xi^\alpha=D_{\xi_1}^{\alpha_1}\cdots D_{\xi_n}^{\alpha_n}$, $D_{\xi_j}=-\sqrt{-1}\,\partial/\partial\xi_j$.

Recall the definition of $A$ (cf. (50)). Then we see $A(x; 0, \zeta')=\langle\zeta'|(\partial X_0'/\partial\zeta')(x; \zeta')\rangle=0$ because $X_0'(x; \zeta')$ gives a Lagrangean submanifold and $X_0'(x; \zeta')$ is positively homogeneous of degree 0. Therefore, it is easy to see that

$$D_\xi^\alpha a_\alpha(x; \zeta', 0)\in\sum_C^{\beta-|\alpha|}$$

and hence $b_1(x; \zeta')\in\sum_C^\beta$.

By the above lemma, we have only to consider the operator written in the form (48) with amplitude $b_1(x; \xi)\in\sum_C^\beta$. Recall the fiber preserving diffeomorphism $\Phi$: $T_N^*-\{0\}\to T_N^*-\{0\}$ of order 1 defined in Lemma 3.5. Set $\Phi(x; \tilde{\xi})=(x; \Phi_2(x; \tilde{\xi}))$ and $b_2(x; \tilde{\xi})=b_1(\Phi(x; \tilde{\xi}))d\zeta'/d\tilde{\xi}$. Then (48) is changed into

$$(52)\qquad (\tilde{F}(a, \varphi, \nu)u)(x)=\iint b_2(x; \tilde{\xi})e^{-i\langle\Phi_2(x;\tilde{\xi})|X_1-X_0'(\Phi(x;\tilde{\xi}))\rangle}\nu_1(x, z)u(z)dzd\tilde{\xi}\ ,$$

by putting $\zeta'=\Phi_2(x; \tilde{\xi})$.

Recall (42) and (41). Then, we obtain

$$(53)\qquad (\tilde{F}(a, \varphi, \nu)u)(x)=\iint b_2(x; \tilde{\xi})e^{-i\langle\tilde{\xi}|\tilde{S}(x;X_1,\bar{X}_0(x;\xi(x;\tilde{\xi})))\rangle}\nu_1(x, z)u(z)dzd\tilde{\xi}\ ,\quad z=\cdot_x X_1\ .$$

Note that the wave front set of $u\rightsquigarrow(\tilde{F}(a, \varphi, \nu)u)(x)$ is contained in $\varphi(T_x^*-\{0\})$ (cf. Lemma 3.2), and in fact this is given by

$$\{\cdot_x(\bar{X}_0(x; \xi(x; \tilde{\xi}), \tilde{\xi})); \tilde{\xi}\in T_x^*-\{0\}\}\ .$$

Since $\varphi$ is sufficiently close to the identity, $\varphi(x; \xi)$ has a normal coordinate expression around $x\in N$, that is,

$$\varphi(x; \xi)=(\varphi_1(x; \xi); \varphi_2(x; \xi))=\cdot_x(\tilde{\varphi}_1(x; \xi), \tilde{\varphi}_2(x; \xi))\ .$$

Hence, putting $\tilde{\xi}=\tilde{\varphi}_2(x; \xi)$, we get $\tilde{\varphi}_1(x; \xi)=\bar{X}_0(x; \xi)$. Set

$$\tilde{X}=\tilde{S}(x; X_1, \bar{X}_0(x; \xi))\ ,$$

and $\cdot_x(\bar{X}_0(x; \xi), \tilde{X})=(\cdot_x\bar{X}_0(x; \xi); X)$. Then $z=\cdot_{\varphi_1(x;\xi)}X$, and (53) can be rewritten as

$$\iint b(x; \xi)e^{-i\langle\varphi_2(x;\xi)|X\rangle}\nu_1(x, z)u(z)dzd\xi\ ,\quad z=\cdot_{\varphi_1(x;\xi)}X\ ,$$

where

$$b(x;\xi)=b_2(x;\tilde{\xi}(x;\xi))\frac{d\tilde{\xi}}{d\xi}.$$

Recall (46). If we replace $\nu_1(x,z)$ by $\nu(\varphi_1(x;\xi),z)$, then the difference of operators is a smoothing operator by virtue of Lemma 3.3. Hence, we obtain

$$\begin{aligned}(\tilde{F}(a,\varphi,\nu)u)(x)\\ &=\iint b(x;\xi)e^{-i\langle\varphi_2(x;\xi)|X\rangle}\nu(\varphi_1(x;\xi),z)u(z)dzd\xi+(K\circ u)(x)\\ &=\int_{T_x^*} b(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi+(K\circ u)(x)\,,\end{aligned}$$

where $K$ is a smoothing operator.

## §5. Elimination of the ambiguities in expressions.

In this section, we consider operators $F(a,\varphi,\nu)$ written in the form (15) with amplitude functions of order 0, i.e., $a\in\tilde{\sum}_\varphi^0$.

DEFINITION 5.1. An operator $F: C^\infty(N)\to C^\infty(N)$ will be called a *Fourier integral operator of order* 0, if there are $\varphi\in\mathscr{D}_\Omega^{(1)}$, $a\in\tilde{\sum}_\varphi^0$ and a smoothing operator $K_1$ such that

$$Fu=F(a,\varphi,\nu)u+K_1\circ u\,,\quad u\in C^\infty(N)\quad (\text{cf. }(15))\,.$$

By arguments in the previous section, there is a neighborhood $U_e$ of identity $e$ in $\mathscr{D}_\Omega^{(1)}$ in the $C^1$-topology (cf. Lemma 1.6) such that if $\varphi\in U_e$, then $F(a,\varphi,\nu)$ defined by (15) can be written in the form

$$\begin{aligned}(F(a,\varphi,\nu)u)(x)\\ &=\int_{T_x^*}\int_N b_1(x;\xi)e^{-i\langle\xi|X_1-X_0'(x;\xi)\rangle}\nu_1(x,z)u(z)dzd\xi\,,\quad z={\cdot}_xX_1\,,\\ &\qquad\qquad(\text{cf. }(48))\,,\end{aligned}$$

or

$$=\int_{T_x^*} b(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi+K_2\circ u\quad(\text{Proposition 4.1})\,.$$

Therefore, if $\varphi\in U_e$, a Fourier integral operator of order 0 can be written in the following form:

$$(Fu)(x)=\int_{T_x^*} b(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi+(K\circ u)(x)\,,\tag{54}$$

where $b(x;\xi)\in\sum_c^0$ and $K$ is a smoothing operator.

DEFINITION 5.2. A Fourier integral operator $F$ is called to be *in a vicinity of the identity*, if $\varphi$ in the above definition is contained in $U_\epsilon$ and $b(x;\xi)$ in the expression (54) never vanishes.

PROPOSITION 5.3. *Let $F$ be a Fourier integral operator of order* 0 *contained in a vicinity of the identity. Suppose there are two expressions of $F$;*

$$(Fu)(x)=\int_{T_N^*}b(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi+(K\circ u)(x)$$
$$=\int_{T_N^*}c(x;\xi)\widetilde{\nu u}(\psi(x;\xi))d\xi+(K'\circ u)(x)\ ,$$

*where $b, c\in\sum_c^0$, $\varphi, \psi\in U_\epsilon$ and $K$, $K'$ are smoothing operators. Then, $\varphi=\psi$ and there is a rapidly decreasing function $h(x;\xi)$ such that $b(x;\xi)=c(x;\xi)+h(x;\xi)$.*

The above proposition will be proved in several lemmas below.

By the same argument between (39) and (43), the former expression of $F$ can be changed into the form

(55) $$(Fu)(x)$$
$$=\int_{T_x^*}\int_N b_1(x;\xi)e^{-i\langle\xi|X_1-X_0'(x;\xi)\rangle}\nu_1(x,z)u(z)dzd\xi+(K\circ u)(x)\ ,$$

by using (46) and (47). Here, we should remark that if $b(x;\xi)$ does not vanish then so does $b_1(x;\xi)$. Similarly the second expression is changed into

(56) $$(Fu)(x)$$
$$=\int_{T_x^*}\int_N c_1(x;\xi)e^{-i\langle\xi|Y_1-Y_0'(x;\xi)\rangle}\nu_1(x,z)u(z)dzd\xi+(K_3\circ u)(x)\ ,$$

where $Y_1$, $Y_0'$ are corresponding quantities to $X_1$, $X_0'$ defined by (48) replacing $\varphi$ by $\psi$. Similarly, $c_1$ is defined by the same manner as $b_1$ by (55).

Let $L(x;X')$ (resp. $M(x;X')$) be the Fourier transform of $b_1(x;\xi)e^{i\langle\xi|X_0'(x;\xi)\rangle}$ (resp. $c_1(x;\xi)e^{i\langle\xi|Y_0'(x;\xi)\rangle}$) as a distribution. By (55) and (56), we see

(57) $$\int_N L(x;X_1)\nu_1(x,z)u(z)dz+\int_N K(x,z)u(z)dz$$
$$=\int_N M(x;Y_1)\nu_1(x,z)u(z)dz+\int_N K'(x,z)u(z)dz\ ,$$

$$z = \cdot_x X_1 = \cdot_x Y_1 \ .$$

Therefore, we get

$$L(x; {}^x\cdot z)\nu_1(x, z) + K(x, z) = M(x; {}^x.z)\nu_1(x, z) + K'(x, z) \ . \tag{58}$$

Note that the singular support (cf. [6]) of the distribution $u \rightsquigarrow (Fu)(x)$ is contained in $\pi\varphi(T_x^* - \{0\}) \cap \pi\psi(T_x^* - \{0\})$, where $\pi$: $T_N^* \to N$ is the natural projection. Remark that $\nu_1(x, z)$ is so chosen that it may identically equal 1 on a neighborhood of the above singular support for each $x \in N$. Therefore, we obtain the following:

LEMMA 5.4. *Notations and assumptions being as above,* $L(x; X') - M(x; X')$ $(X' = {}^x\cdot z)$ *is a smooth function on* $T_N$. *Hence,*

$$b_1(x; \xi)e^{i\langle\xi|X_0'(x;\xi)\rangle} - c_1(x; \xi)e^{i\langle\xi|Y_0'(x;\xi)\rangle}$$

*is rapidly decreasing function in* $\xi$.

PROOF. The first part is obvious, for $L(x; X') - M(x; X')$ is a smooth function on a neighborhood of the singular support of the operator. To prove the second part is to prove that $L(x; X') - M(x; X')$ is rapidly decreasing in $X'$. It is enough to show that

$$\lim_{|X'|\to\infty} X'^\alpha \frac{\partial^\beta}{\partial X'^\beta} L(x; X') = 0 \ ,$$

for every multi-indices $\alpha$, $\beta$, because once we get above, then we get the same result for $M$ by the same computation.

Since $|X'^\alpha| \leqq |X'|^{|\alpha|} \leqq |X'|^\lambda$ if $\lambda \geqq |\alpha|$, we have only to prove

$$\lim_{|X'|\to\infty} |X'|^\lambda \int (i\xi)^\beta b(x; \xi)e^{-i\langle\xi|X' - X_0(x;\xi)\rangle} d\xi = 0 \ ,$$

for $\lambda > |\beta| + n + 1$. Set $t = |\xi|$, $\hat{\xi} = \xi/|\xi|$, $s = |X'|$, $Z = X'/|X'|$. What we have to show is

$$\lim_{s\to\infty} s^\lambda \int_0^\infty \int_{S_x^* N} (i\hat{\xi})^\beta b(x; t\hat{\xi}) t^{|\beta|+n-1} e^{-its\langle\hat{\xi}|Z-(1/s)X_0'(x;\hat{\xi})\rangle} dt d\hat{\xi} = 0 \ .$$

For each fixed $Z \in S_x N$, we devide $S_x^* N$ into domains

$$E_x'(Z) = \{\hat{\eta} \in S_x^* N : |\langle\hat{\eta}|Z\rangle| \leqq 2/3\} \ ,$$

$$E_x(Z) = \{\hat{\eta} \in S_x^* N ; |\langle\hat{\eta}|Z\rangle| > 1/3\} \ .$$

For a sufficiently large $s$, one may assume that $\langle\hat{\xi}|Z - (1/s)X_0'(x; \hat{\xi})\rangle$ has no critical point in $E'_x(Z)$. Hence by the stationary phase method, we have

$$\lim_{s\to\infty} s^{\lambda}\int_0^{\infty}\int_{E'_x(Z)} \kappa(\hat{\xi})(i\hat{\xi})^{\beta}b(x;\, t\hat{\xi})t^{|\beta|+n-1}e^{-its\langle\hat{\xi}|Z-(1/s)X'_0(x;\hat{\xi})\rangle}d\hat{\xi}dt=0\ ,$$

where $\kappa(\hat{\xi})$ is a cut off function such that $\operatorname{supp}\kappa\subset E'_x(Z)$ and $\kappa\equiv 1$ on $S^*_xN-E_x(Z)$. Thus, we have only to consider on $E_x(Z)$. Remark that for a sufficiently large $s$, we may assume that $|\langle\hat{\xi}|Z-(1/s)X'_0(x;\hat{\xi})\rangle|>1/6$. Hence, we have only to show

$$\begin{aligned} I_{\lambda}=&\lim_{s\to\infty}\int_{-\infty}^{\infty}\int_{S^*_xN} s^{\lambda}\Big\langle\hat{\xi}\,|\,Z-\frac{1}{s}X'_0(x;\,\hat{\xi})\Big\rangle^{\lambda}t^{|\beta|+n-1}(1-\kappa)(i\hat{\xi})^{\beta}b(x;\,t\hat{\xi})\\ &\times e^{-its\langle\hat{\xi}|Z-(1/s)X'_0(x;\xi)\rangle}d\hat{\xi}dt\\ =&\lim_{s\to\infty}\int_{-\infty}^{\infty}\int_{S^*_xN} t^{|\beta|+n-1}(i\hat{\xi})^{\beta}(1-\kappa)\Big(\frac{1}{i}\frac{d}{dt}\Big)^{\lambda}b(x;\,t\hat{\xi})e^{-its\langle\hat{\xi}|Z-(1/s)X'_0(x;\hat{\xi})\rangle}d\hat{\xi}dt\\ =&0\ . \end{aligned}$$

Remark that $(-id/dt)^{\lambda}b(x;\,t\hat{\xi})\in\sum_c^{-\lambda}$. Hence, the above integral exists for every $\lambda$, $\beta$ such that $\lambda>|\beta|+n+1$. Replace $\lambda$ by $\lambda'$ such that $\lambda'>\lambda$ and we see $I_{\lambda}$ must be 0.

Since $F$ is in a vicinity of the identity, we may assume that $b(x;\,\xi)$ and hence $b_1(x;\,\xi)$ does not vanish. Note that if $h(x;\,\xi)$ is rapidly decreasing in $\xi$, then so is $h(x;\,\xi)e^{-i\langle\xi|X'_0(x;\xi)\rangle}$. Hence,

$$e^{i\langle\xi|X'_0(x;\xi)\rangle-i\langle\xi|Y'_0(x;\xi)\rangle}=\frac{c_1(x;\,\xi)}{b_1(x;\,\xi)}+f(x;\,\xi)\ , \tag{59}$$

where $f$ is a rapidly decreasing function in $\xi$. Remark that $c_1/b_1\in\sum_c^0$. Differentiate both sides by $\xi$ and take the limit $|\xi|\to\infty$. Then, we obtain

$$\langle\xi\,|\,X'_0(x;\,\xi)\rangle=\langle\xi\,|\,Y'_0(x;\,\xi)\rangle\ . \tag{60}$$

Hence, by (42) and (54), we get the following:

LEMMA 5.5. *$\varphi(T^*_x-\{0\})=\psi(T^*_x-\{0\})$ for every $x\in N$. Moreover, by (60), we get that $b_1-c_1$ is rapidly decreasing in $\xi$.*

PROOF. It is trivial because

$$\varphi(T^*_x-\{0\})=\{\cdot_x(\bar{X}_0(x;\,\tilde{\xi}),\,\tilde{\xi});\,\tilde{\xi}\in T^*_x-\{0\}\}\ .$$

LEMMA 5.6. *Suppose $\varphi$, $\psi\in\mathscr{D}^{(1)}_{\Omega}$ satisfy $\varphi(T^*_x-\{0\})=\psi(T^*_x-\{0\})$ for* *every* $x \in N$. *Then* $\varphi=\psi$.

PROOF. Set $\Psi=\psi\circ\varphi^{-1}$. Then, $\Psi$: $T_N^*-\{0\}\to T_N^*-\{0\}$ is a fibre preserving symplectic transformation of order 1. Apply the normal chart expression of $\Psi$ of (25). Since $\Psi(x;\xi)=(x;\Psi_2(x;\xi))$, we see $\partial\bar{X}_1^i/\partial X_0^j=\delta_j^i$ in (25). Therefore, $\xi_i'=\xi_i$. This implies $\Psi_2(x;\xi)=\xi$.

Recall the argument from (52) to the last line of the previous section. We obtain that $b-c$ is a rapidly decreaisng function in $\xi$. This implies the proof of Proposition 5.3.

Let $F$ be a Fourier integral operator of order 0 contained in a vicinity of identity. By Proposition 5.3, we can find a symplectic transformation $\varphi\in\mathscr{D}_\Omega^{(1)}$. $\varphi$ will be called a *phase transformation* of $F$. Moreover, the asymptotic expansion of $b(x;\xi)$;

$$b(x;\xi)\sim b_0(x;\hat{\xi})+b_{-1}(x;\hat{\xi})r^{-1}+b_{-2}(x;\hat{\xi})r^{-2}+\cdots$$

is uniquely determined by $F$.

Now, let $C^\infty(S_N^*)^\infty$ be the space of all series

$$B=(b_0, b_{-1}, b_{-2}, \cdots, b_{-k}, \cdots)$$

of $C^\infty$-functions on $S_N^*$. Asymptotic expansions give a linear mapping $\alpha$ of $\sum_C^0$ into $C^\infty(S_N^*)^\infty$. However, it is already known the following.

LEMMA 5.7. *There is a mapping (not necessarily linear) $\beta$ of $C^\infty(S_N^*)^\infty$ into $\sum_C^0$ such that $\alpha\beta=$identity.*

PROOF. Remark that $\sum_C^0$ is canonically isomorphic to the space $C^\infty(\bar{D}_N^*)$ and the asymptotic expansions correspond to Taylor's expansion at $\theta=1$ (cf. (11)). Therefore, this lemma can be proved by the same technique as [7] p. 38. See, also [8] p. 153.

Now, suppose $F$ is written as

$$(Fu)(x)=\int_{T_x^*} b(x;\xi)\nu\tilde{u}(\varphi(x;\xi))d\xi+(K'\circ u)(x)\,.$$

Note that the asymptotic expansion of $b$ is uniquely determined by $F$. Hence, $\alpha(b)$ is determined by $F$. Thus, remarking that $\beta\alpha(b)-b$ is rapidly decreasing in $\xi$, we get a unique expression of $F$.

$$(Fu)(x)=\int_{T_x^*}\beta\alpha(b)\nu\tilde{u}(\varphi(x;\xi))d\xi+(K\circ u)(x)\,. \tag{61}$$

Thus, we have the following theorem:

THEOREM 5.8. *A vicinity of the identity in the space of Fourier integral operators of order* 0 *is coordinatized by* $(\varphi, \alpha(b), K)$.

REMARK. The condition that $b(x;\xi)\neq 0$ can be easily replaced by $b_0(x;\xi)\neq 0$, because this is the zeroth order term in the corresponding Taylor expansion at $\theta=1$ (cf. (11)).

Let $U_\varepsilon$ be the neighborhood defined in the first part of this section, and let $\Lambda$ be the subset in $C^\infty(S_N^*)^\infty$ such that $b_0(x;\xi)$ does not vanish. For every point $(\varphi, B, K)$ in $U_\varepsilon\times\Lambda\times C^\infty(N\times N)$, we can define a Fourier integral operator $F$ of order 0 by

$$(Fu)(x)=\int_{T_x^*}\beta(B)\widetilde{\nu u}(\varphi(x;\xi))d\xi+(K\circ u)(x)\ .$$

By the above argument, we see that $\varphi$ and $B$ are uniquely determined by $F$. Therefore, so does $K$. We denote the above $F$ by $\Psi(\varphi, B, K)$. $\Psi$ is naturally regarded as a mapping of $U_\varepsilon\times\Lambda\times C^\infty(N\times N)$ onto a vicinity of the identity in the space of Fourier integral operators of order 0. Evidently, $\Psi$ is one-to-one.

The above mapping $\Psi$ will give in near future a local coordinate system at the identity of the group of all invertible Fourier integral operators of order 0.

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