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On Linear Integro-Differential Equations in a Banach Space

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Introduction

In this paper we study a linear Volterra integro-differential equation of the form

$$\text{(E)} \qquad \frac{d}{dt}u(t)=Au(t)+\int_0^t B(t-s)u(s)ds+f(t) \qquad \text{for} \quad t>0,\ u(0)=x\ ,$$

in a Banach space X with norm $\|\cdot\|$. Here $f\colon R_+=[0, \infty)\to X$ is continuous and A is the infinitesimal generator of a semi-group of class (C_0) on X. For each $t\in R_+$ $B(t)$ is a (in general unbounded) linear operator with domain dense in X. Let $B(X)$ denote the set of all bounded linear operators from X into itself.

It is well known [2] that on a finite dimensional space $X=R^n$ (the n-dimensional space of column vectors with the usual norm $|\cdot|$),

$$\text{(0.1)} \qquad u(t)=U(t)x+\int_0^t U(t-s)f(s)ds \qquad \text{for} \quad t>0\ ,$$

is a unique solution of (E) for $x\in X$. In this case A and $B(t)$ are $n\times n$ matrices, $B(t)$ is a locally integrable function on R_+ and the $n\times n$ matrix function $U(t)$ is the solution of the equation

$$\frac{d}{dt}U(t)=AU(t)+\int_0^t B(t-s)U(s)ds=U(t)A+\int_0^t U(t-s)B(s)ds \qquad \text{for} \quad t>0\ ,$$
$$U(0)=I \text{ (the identity matrix).}$$

In a general Banach space X, it is also known [5, 12] that if $\{B(t); t\in R_+\}$ is in $B(X)$ and $B(t)x\colon R_+\to X$ is continuous for each $x\in X$, then there exists a one-parameter family $\{U(t); t\in R_+\}$ in $B(X)$ which satisfies the following two equations

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$$(\mathrm{U}_1)\quad \frac{d}{dt}U(t)x=AU(t)x+\int_0^t B(t-s)U(s)xds \qquad \text{for} \quad t>0,\ U(0)x=x\in D(A)\ ,$$

and

$$(\mathrm{U}_2)\quad \frac{d}{dt}U(t)x=U(t)Ax+\int_0^t U(t-s)B(s)xds \qquad \text{for} \quad t>0,\ U(0)x=x\in D(A)\ ,$$

and the function u given by (0.1) gives the unique solution of (E) for $x\in D(A)$ and f strongly continuously differentiable on R_+.

Our purpose of this paper is to generalize the results in [2], [5] and [12] to the case in which $\{B(t);\, t\in R_+\}$ is not necessarily in $B(X)$. The paper is organized as follows. In section 1 we construct a one-parameter family $\{U_T(t);\, t\in R_+\}$in $B(X)$ satisfying certain integral equation in X by using "successive approximations" method which has been used for perturbation theory of semi-group generators by Miyadera [8, 9] and Voigt [13], and for Volterra integro-differential equations in a Banach space by the author [12]. In section 2 we give some sufficient conditions on $B(t)$ under which $U_T(t)$ satisfies (U_2) for $x\in D$, where D is a dense linear subset of X. In section 3 an existence and uniqueness theorem is obtained for (E) under appropriate conditions which also guarantee that $U_T(t)$ satisfies (U_1) and (U_2) for $x\in D$. Further in section 3 "the variation of constants" formula of the form (0.1) is obtained. Our results in section 3 will partly correspond to Miller's theorem [6] which has been obtained by studies of well-posedness of Volterra integro-differential equations in a Banach space.

For results on linear Volterra integro-differential equations in a Hilbert space see Hannsgen [3, 4].

Let I be a subinterval of $R=(-\infty,\infty)$. As usual $L^1(I)$ will denote the Lebesgue space of all extended real-valued measurable functions $g\colon I\to\bar{R}$ such that $\int_I |g(t)|dt<\infty$, where $\bar{R}$ denotes the set of all extended real numbers. $L^1_{\mathrm{loc}}(R_+)$ will denote the set of all $\bar{R}$-valued functions g which are locally of class $L^1(R_+)$; that is $\int_0^T |g(t)|dt<\infty$ for every $T>0$. $C(I)$ will denote the set of all real-valued continuous functions $g\colon I\to R$ and $C^1(I)$ will denote the set of all real-valued continuously differentiable functions $g\colon I\to R$. $L^1(I;X)$ will denote the set of all $g\colon I\to X$ such that g is Bochner integrable on I. $L^1_{\mathrm{loc}}(R_+;X)$ will denote the set of all X-valued functions g which are locally of class $L^1(R_+;X)$. $C(I;X)$ will denote the set of all X-valued continuous functions $g\colon I\to X$ and $C^1(I;X)$ will denote the set of all X-valued strongly continuously differentiable functions $g\colon I\to X$.

§ 1. We begin with the integral equation of the form

$$U(t)x = V(t)x + \int_0^t U(t-s)F(s)x\,ds \qquad \text{for} \quad t>0\,, \tag{1.1}$$

where V and F are given operator-valued functions. We now introduce a class of linear operators.

DEFINITION 1.1. A one-parameter family of linear operators $F(t)$ defined on a dense linear subset D of X for each $t \in R_+$ is said to be *of class* $(F(\cdot))$ if

(F_1) $F(t)x \in L^1_{\mathrm{loc}}(R_+; X)$ for each $x \in D$,

(F_2) for any $t \in R_+$

$$\sup\left\{\int_0^t \|F(s)x\|\,ds;\ x \in D,\ \|x\| \leqq 1\right\} < \infty\ .$$

REMARK 1.2. Each of the following conditions is equivalent to (F_2).

(F_3) For some $\lambda \in R$ and any $t \in R_+$

$$(*) \qquad \sup\left\{\int_0^t \exp(-\lambda s)\|F(s)x\|\,ds;\ x \in D,\ \|x\| \leqq 1\right\} < \infty\ .$$

(F_4) For any $\lambda \in R$ and $t \in R_+$ (*) holds.

If $\{F(t);\ t \in R_+\}$ belongs to $(F(\cdot))$, we define

$$L_\lambda(t) = \sup\left\{\int_0^t \exp(-\lambda s)\|F(s)x\|\,ds;\ x \in D,\ \|x\| \leqq 1\right\}$$

for any $\lambda \in R$ and $t \in R_+$.

Our first result is the following proposition. The proof of the result uses the same techniques as those used in [9] and [12].

PROPOSITION 1.3. *Let* $\{F(t);\ t \in R_+\}$ *be of class* $(F(\cdot))$. *Suppose that there exist constants* $0 < t_0 \leqq \infty$ *and* $\lambda \in R$ *such that* $L_\lambda(t_0) < 1$. *Then for each strongly continuous family* $\{V(t);\ t \in R_+\}$ *in* $B(X)$ *satisfying*

$$\|V(t)\| \leqq M_0 \exp(\lambda t) \quad \textit{for} \quad t \in R^+ \quad \textit{and some} \quad M_0 > 0 \tag{1.2}$$

there exists a one-parameter family $\{U_V(t);\ t \in R_+\}$ *in* $B(X)$ *with the properties*:

(i) $U_V(t)$ *is strongly continuous on* R_+,

(ii) *for each* $x \in D$ $U_V(t)x$ *satisfies the integral equation* (1.1), *and*

(iii) *there exists a nondecreasing function* $M(t)$ *defined on* R_+ *such that*

$$||U_V(t)||\leqq M(t)\exp(\lambda t) \qquad for \quad t\in R_+ .$$

In particular if $t_0=\infty$ *we have* $||U_V(t)||\leqq M_0(1-L_\lambda(\infty))^{-1}\exp(\lambda t)$ *for* $t\in R_+$.

To prove this proposition we use the following lemma.

LEMMA 1.4. *Let* $\{U(t);\, t\in R_+\}$ *be in* $B(X)$ *and* $U(t)x\in C(R_+;\, X)$ *for each* $x\in X$. *If* $f\in L^1_{\text{loc}}(R_+;\, X)$, *then as a function of* s, $U(t-s)f(s)\in L^1([0,\, t];\, X)$ *for* $t>0$. *Further if we define* $g(t)=\int_0^t U(t-s)f(s)ds$ *for* $t>0$, *and* $g(0)=0$, *then* $g\in C(R_+;\, X)$.

The proof of Lemma 1.4 can be carried out by standard arguments, and let us note that under the condition of Proposition 1.3 the equality $U_V(0)=V(0)$ holds, if $\{U_V(t);\, t\in R_+\}$ exists, since Lemma 1.4 implies $\lim_{t\downarrow 0}\int_0^t U_V(t-s)F(s)xds=0$ and D is dense.

PROOF OF PROPOSITION 1.3. Fix $T>0$. It follows from the definition of $L_\lambda(t)$ that $L_\lambda(t)\leqq L_\lambda(t_0)<1$ whenever $0<t\leqq t_0$. Therefore we can choose some $0<t_1\leqq T$ such that $t_1\leqq t_0$ and $L_\lambda(t_1)<1$. In fact if $t_0=\infty$ or $T<t_0$ we can set $t_1=T$ and if $T\geqq t_0$ we can set $t_1=t_0$. Let $\{V(t);\, t\in R_+\}$ be a family in $B(X)$ and $V(t)x\in C(R_+;\, X)$ for each $x\in X$ with the estimate (1.2). Then

$$(1.3) \qquad ||V(t)x||\leqq M_0\exp(\lambda t)||x|| \quad \text{for every} \quad x\in X \quad \text{and} \quad t>[0,\, t_1] .$$

For each nonnegative integer n and $t\in[0,\, t_1]$ we define a bounded linear operator $U_n(t)$ on D as follows: for $x\in D$

$$(1.4) \qquad U_0(t)x=V(t)x\,, \quad U_n(t)x=\int_0^t \bar{U}_{n-1}(t-s)F(s)xds \qquad \text{for} \quad t\in(0,\, t_1]$$
$$\text{and} \quad U_n(0)x=0 \qquad \text{for} \quad n=1,2,\cdots,$$

where $\bar{U}_{n-1}(t)$ denotes the extension of $U_{n-1}(t)$ onto X. To observe that $U_n(t)$ are well defined and bounded on D, we show that for every n and $x\in D$, $U_n(t)x\in C(0,\, t_1];\, X)$ and

$$(1.5) \qquad ||U_n(t)x||\leqq M_0(L_\lambda(t_1))^n\exp(\lambda t)||x|| \qquad \text{for} \quad t\in[0,\, t_1] .$$

By (F_1), (1.4) and Lemma 1.4 it follows that for $t\in(0,\, t_1]$

$$U_1(t)x=\int_0^t \bar{U}_0(t-s)F(s)xds \quad \text{and} \quad U_1(0)x=0$$

are well defined and $U_1(t)x \in C([0, t_1]; X)$ for $x \in D$. Moreover, by (1.3) and the definition of $L_\lambda(t_1)$ one has

$$\|U_1(t)x\| \leqq M_0 \exp(\lambda t)\int_0^t \exp(-\lambda s)\|F(s)x\|ds$$
$$\leqq M_0 \exp(\lambda t)L_\lambda(t_1)\|x\| \qquad \text{for} \quad t \in (0, t_1]$$

and hence $\|U_1(t)x\| \leqq M_0 L_\lambda(t_1)\exp(\lambda t)\|x\|$ for $t \in [0, t_1]$. Since D is dense in X, $U_1(t)$ can be extended onto X. Now we see by induction that $U_n(t)$ is well defined and $U_n(t)x \in C([0, t_1]; X)$ for $x \in D$, and (1.5) holds. Consequently it follows that for every n $\bar{U}_n(t)$ is strongly continuous on $[0, t_1]$ and

$$\|\bar{U}_n(t)\| \leqq M_0(L_\lambda(t_1))^n \exp(\lambda t) \qquad \text{for} \quad t \in [0, t_1]\,. \tag{1.6}$$

Since $L_\lambda(t_1) \leqq L_\lambda(t_0) < 1$, $\sum_{n=0}^\infty \bar{U}_n(t)$ converges absolutely in the uniform operator topology and uniformly in t on $[0, t_1]$. Define $U_V(t) \in B(X)$ by

$$U_V(t) = \sum_{n=0}^\infty \bar{U}_n(t) \qquad \text{for} \quad t \in [0, t_1]\,, \tag{1.7}$$

then $U_V(t)$ is strongly continuous on $[0, t_1]$. Clearly $U_V(0) = V(0)$ and by the definition of $\bar{U}_n(t)$ and (1.7) $U_V(t)$ satisfies the integral equation (1.1) for $x \in D$ and $t \in (0, t_1]$, and from (1.6) and (1.7) one has

$$\|U_V(t)\| \leqq M_0(1 - L_\lambda(t_0))^{-1}\exp(\lambda t) \qquad \text{for} \quad t \in [0, t_1]\,. \tag{1.8}$$

Translate (1.1) by t_1 to see that $W(t) = U_V(t + t_1)$ must satisfy

$$W(t)x = V(t+t_1)x + \int_0^{t+t_1} U_V(t+t_1-s)F(s)xds$$
$$= \left[V(t+t_1)x + \int_t^{t+t_1} U_V(t+t_1-s)F(s)xds\right] + \int_0^t W(t-s)F(s)xds$$

for $t > 0$ and $x \in D$. Clearly the term in brackets, we say, the new forcing function $V_1(t)x$ is strongly continuous on R_+ by (F_1) and Lemma 1.4 and satisfies (1.3), since from (1.8) and (F_2) one has

$$\|V_1(t)x\| \leqq M_0 \exp(\lambda(t+t_1))\left[\|x\| + (1 - L_\lambda(t_0))^{-1}\int_t^{t+t_1} \exp(-\lambda s)\|F(s)x\|ds\right]$$
$$\leqq M_1 \exp(\lambda t)\|x\| \quad \text{for} \quad t \in [0, t_1] \quad \text{and} \quad x \in D\,,$$

where $M_1 = M_0[1 + L_\lambda(2t_0)(1 - L_\lambda(t_0))^{-1}]\exp(\lambda t_1)$. Therefore the same argument can be repeated to obtain a solution of (1.1) on $(t_1, 2t_1]$, $(2t_1, 3t_1], \cdots$ until $(Nt_1, T]$ where $(N+1)t_1 \geqq T$, and $\|U_V(t)\| \leqq M(t)\exp(\lambda t)$ for $t \in [0, T]$

and some nondecreasing function $M(t)$ which depends on t and λ. Since T is an arbitrary positive number, this proves the existence of a one-parameter family $U_V(t)\in B(X)$ on R_+ which is strongly continuous, and satisfies the integral equation (1.1) and the estimate (iii) for $t\in R_+$.

Q.E.D.

REMARK 1.5. The condition (L_1): there exists t_0, $0<t_0\leqq\infty$ such that $L_\lambda(t_0)<1$ for some $\lambda\in R$, is equivalent to the following condition

(L_2) $\lim_{t\downarrow 0} L_\lambda(t)<1$ for some $\lambda\in R$.

We use this fact in section 2.

We now give a simple condition which guarantees the assumptions of Proposition 1.3.

LEMMA 1.6. *Let $\{F(t);\, t\in R_+\}$ be a one-parameter family of linear operators defined on a dense linear subset D of X which satisfies* (F_1). *If there exists a non-negative function $\phi\in L^1_{\mathrm{loc}}(R_+)$ such that*

$$(F_5)\qquad \|F(t)x\|\leqq\phi(t)\|x\| \quad \textit{for} \quad t\in R_+ \quad \textit{and} \quad x\in D\,,$$

then $F(t)$ is of class $(F(\cdot))$ and there exist constants $0<t_0\leqq\infty$ and $\lambda\in R$ such that $L_\lambda(t_0)<1$.

§2. A one-parameter family $\{T(t);\, t\in R_+\}$ in $B(X)$ is called a *semi-group* of class (C_0) on X if it satisfies

$T(0)=I$ (the identity operator), $T(t+s)=T(t)T(s)(t,\, s\in R_+)$, and $T(t)x\in C(R_+;\, X)$ for each $x\in X$. The *infinitesimal generator A of $\{T(t);\, t\in R_+\}$ is defined by*

$$D(A)=\{x\in X;\ Ax=\lim_{h\downarrow 0} h^{-1}[T(h)-I]x \text{ exists}\}\,.$$

It is well known that A is a densely defined, closed linear operator in X. We denote by $\rho(A)$ and $R(\lambda;\, A)$ the resolvent set and resolvent of A, respectively: $R(\lambda;A)=(\lambda-A)^{-1}$, $\lambda\in\rho(A)$. For $t>0$ $T(t)$ and A commute on $D(A)$, and for $x\in D(A)$ $T(t)x$ is strongly continuously differentiable on R_+ and is the unique solution of the differential equation $dT(t)/dt=AT(t)x$ with the initial condition $T(0)x=x$. Moreover $\int_0^t T(s)xds\in D(A)$ and $A\int_0^t T(s)xds=T(t)x-x$ for $x\in X$ and $t>0$. It is also well known that there exist constants $M\geqq 1$ and $\omega\in R$ such that

$$(2.1)\qquad \|T(t)\|\leqq M\exp(\omega t) \qquad \text{for} \quad t\in R_+ \quad \text{and} \quad \{\lambda;\ \lambda>\omega\}\subset\rho(A)\,.$$

Moreover $R(\lambda; A)=\int_0^\infty \exp(-\lambda t)T(t)xdt$ for $x\in X$ and $\lambda>\omega$.

Concerning further results of a semi-group of operators of class (C_0) see for example Dunford-Schwartz [1] or Pazy [10].

In what follows A will be the infinitesimal generator of a (C_0) semi-group $\{T(t);\ t\in R_+\}$ and $B(t)$ be a linear operator defined on $D(B(t))$ with range in X.

DEFINITION 2.1. A one-parameter family $\{B(t);\ t\in R_+\}$ is said to be *of class* $(B(\cdot))$ if

(B_1) there exists a dense linear subset D of X such that $D\subset D(A)\cap D(B(t))$ for all $t\in R_+$, $T(t)D\subset D$ for each $t\in R_+$ and $B(t)x\colon R_+\to X$ is measurable for each $x\in D$,

(B_2) there exists a function $\beta\in L^1_{loc}(R_+)$ such that

$$\|B(t)x\|\leqq\beta(t)\|x\|_A \quad \text{for all} \quad t\in R_+ \quad \text{and} \quad x\in D,$$

where $\|x\|_A=\|x\|+\|Ax\|$, and

(B_3) for any $t\in R_+$

$$\sup\left\{\int_0^t\left\|\int_0^s B(s-r)T(r)xdr\right\|ds;\ x\in D,\ \|x\|\leqq 1\right\}<\infty .$$

In order to show that (B_3) makes sense we need the following lemma. For a proof see Lemma 1.1 in [6].

LEMMA 2.2. *Let* $\{B(t);\ t\in R_+\}$ *satisfy* (B_1) *and* (B_2). *Let* $u\colon R_+\to D$ *be any function such that* $u(t)$ *and* $Au(t)$ *are continuous on* R_+. *Then as a function of* s, $B(t-s)u(s)\in L^1([0,t]; X)$ *for* $t>0$ *and if we define* $g(t)=\int_0^t B(t-s)u(s)ds$ *for* $t>0$ *and* $g(0)=0$, *then* $g\in C(R_+; X)$.

The operator of the form $B(t)=b(t)A$ will serve as an important example which forms a family of class $(B(\cdot))$, where $b\in L^1_{loc}(R_+)$ is a given function.

PROPOSITION 2.3. *Let* $\{B(t);\ t\in R_+\}$ *satisfy*

(B_4) $D(A)\subset D(B(t))$ *for all* $t\in R_+$, $B(t)x\colon R_+\to X$ *is measurable for each* $x\in D(A)$ *and there exists a function* $\beta\in L^1_{loc}(R_+)$ *such that for any* y *in* X

$$\|B(t)R(\mu; A)y\|\leqq\beta(t)\|y\| \quad \textit{for all } t\in R_+ \textit{ and some } \mu\in\rho(A),$$

and (B_3) *with* D *replaced by* $D(A)$. *Then* $\{B(t); t\in R_+\}$ *forms a family of class* $(B(\cdot))$.

PROOF. Clearly (B_4) implies (B_1) and (B_2) with D replaced by $D(A)$. Q.E.D.

In the sequel it will be assumed that B is a linear operator defined on the domain $D(B)$ with range in X. The following result is a direct consequence of Proposition 2.3.

COROLLARY 2.4. *Let* $b\in L^1_{loc}(R_+)$ *and* $|b(t)|<\infty$ *for* $t\in R_+$. *Let* B *satisfy*

(B_5) $\qquad D(A)\subset D(B)$ *and* $BR(\mu; A)\in B(X)$ *for some* $\mu\in\rho(A)$,

and (B_3) *with* D *and* $B(t)$ *replaced by* $D(A)$ *and* $b(t)B$, *respectively. Then* $\{b(t)B; t\in R_+\}$ *forms a family of class* $(B(\cdot))$.

PROOF. *Set* $B(t)=b(t)B$. Then $B(t)$ is well defined for all $t\in R_+$ and one has $D(B)\subset D(B(t))$ for $t\in R_+$. Now (B_4) follows from (B_5). Q.E.D.

PROPOSITION 2.5. *Let* $\{B(t); t\in R_+\}$ *be of class* $(B(\cdot))$. *Set*

$$F(t)x=\int_0^t B(t-s)T(s)xds \qquad \text{for} \quad t\in R_+ \quad \text{and} \quad x\in D\,. \tag{2.2}$$

Then $\{F(t); t\in R_+\}$ *forms a family of class* $(F(\cdot))$ *and hence*

$$L_\lambda(t)=\sup\left\{\int_0^t \exp(-\lambda s)\left\|\int_0^s B(s-r)T(r)xdr\right\|ds;\ x\in D,\ \|x\|\leqq 1\right\} \tag{2.3}$$

is finite for all $\lambda\in R$ *and* $t\in R_+$.

PROOF. The conditions (F_1) and (F_2) follow from Lemma 2.2 and (B_3) respectively. Q.E.D.

We now consider an integro-differential equation of the form

$$\frac{d}{dt}U(t)x=U(t)Ax+\int_0^t U(t-s)B(s)xds \qquad \text{for} \quad t>0\,, \quad U(0)x=x\,. \tag{2.4}$$

DEFINITION 2.6. A one-parameter family $\{U_T(t); t\in R_+\}$ in $B(X)$ is said to be an *adjoint kernel* on R_+ if

(i) $U_T(t)x\in C(R_+; X)$ for each $x\in X$ and there exist $\lambda\in R$ and a nondecreasing function M such that $\|U_T(t)\|\leqq M(t)\exp(\lambda t)$ for $t\in R_+$,

(ii) there exists some dense linear subset D of X such that $D\subset D(A)\cap D(B(t))$ for all $t\in R_+$ and $U_T(t)x\in C^1(R_+; X)$ for each $x\in D$, and

(iii) for $t>0$ and $x \in D$ $U_T(t)x$ satisfies the integro-differential equation (2.4) with

(iv) $U_T(0)=I$.

Our first main result is the following

THEOREM 2.7. *Let $\{B(t); t \in R_+\}$ be of class $(B(\cdot))$. Suppose that there exist constants $0<t_0 \leqq \infty$ and $\lambda \geqq \omega$ such that $L_\lambda(t_0)<1$, where L_λ is the function defined by* (2.3). *Then there exists an adjoint kernel $\{U_T(t); t \in R_+\}$.*

The following lemma will be useful in the proof of Theorem 2.7 and in the remainder of this paper.

LEMMA 2.8. *Let $\{U(t); t \in R_+\}$ be in $B(X)$ and satisfy*

(i) *$U(t)x \in C([0, \infty); X)$ for each $x \in X$, and*

(ii) *there exists a function $\hat{\beta} \in {}^1_{\mathrm{loc}}(R_+)$ such that*

$$\|U(t)\| \leqq \hat{\beta}(t) \qquad \textit{for all} \quad t \in R_+ .$$

If the hypotheses of Lemma 2.2 are satisfied, then we have

$$\int_0^t\left(\int_0^s U(t-s)B(s-r)u(r)dr\right)ds=\int_0^t\left(\int_r^t U(t-s)B(s-r)u(r)ds\right)dr \quad \textit{for} \quad t>0 .$$

PROOF. Fix $t>0$. Define $B(r, s)x=B(s-r)x$ for $(r, s) \in \bar{\Omega} \equiv \{(r, s); 0 \leqq r \leqq s \leqq t\}$ and $x \in D$. Then $B(r, s)x$ is measurable on $\bar{\Omega}$ for each $x \in D$ and from (B_2) one has $\|B(r, s)x\| \leqq \beta(s-r)\|x\|_A$ for $x \in D$. Clearly the function $\beta(r, s)=\beta(s-r)$ is integrable on $\bar{\Omega}$. Since $u(r)$ is strongly continuous on $[0, t]$, the function $u(r, s)$ on $\bar{\Omega}$ to X defined by the formula $u(r, s)=u(r)$ is continuous. Further we define $U(r, s)=U(t-s)$ for $(r, s) \in \bar{\Omega}$. Then $U(r, s) \in B(X)$ for $(r, s) \in \bar{\Omega}$ and $U(r, s)x$ is strongly continuous on $\bar{\Omega}\backslash\{(r, s); s=t\}$ for $x \in X$. Using the similar arguments as those used in the proof of Lemma 1.1 in [6] it is seen that $v(r, s)=B(r, s)u(r, s)$ is strongly measurable on $\bar{\Omega}$ and $U(r, s)v(r, s)$ is also strongly measurable on $\bar{\Omega}(\varepsilon)=\{(r, s); 0 \leqq r \leqq s<t-\varepsilon\}$ for sufficiently small $\varepsilon>0$. Thus $U(r, s)v(r, s)$ is strongly measurable on $\bar{\Omega}$. Moreover from (B_2) one has

$$\int_0^t\left(\int_0^s\|U(r, s)v(r, s)\|dr\right)ds \leqq \sup\{\|u(r)\|_A; 0 \leqq r \leqq t\}\left(\int_0^t \hat{\beta}(s)ds\right)\left(\int_0^t \beta(s)ds\right)<\infty .$$

Thus by Corollary III.11.15 in [1] and Fubini's theorem we have

$$\int_0^t\left(\int_0^s U(r, s)v(r, s)dr\right)ds=\int_0^t\left(\int_r^t U(r, s)v(r, s)\, ds\right)dr \qquad \text{for} \quad t>0 . \qquad \text{Q.E.D.}$$

Proof of Theorem 2.7. Set $V(t)=T(t)$ in Proposition 1.3. Then Propositions 2.5 and 1.3 imply that there exists a one-parameter family $\{U_T(t);\, t\in R_+\}$ in $B(X)$ which satisfies (i) and (iv) of Definition 2.6 and the integral equation of the form

$$U_T(t)x=T(t)x+\int_0^t U_T(t-s)\Big(\int_0^s B(s-r)\,T(r)xdr\Big)ds \tag{2.5}$$

for $t>0$ and $x\in D$. From (B_1), (B_2) and Lemma 1.4 it follows that as a a function of s, $U_T(t-s)B(s)x\in L^1([0, t]; X)$ for $t>0$ and $x\in D$ and if we define $\hat{B}(t)x=\int_0^t U_T(t-s)B(s)xds$ for $t>0$ and $\hat{B}(0)x=0$ for each $x\in D$, then $D\subset D(\hat{B}(t))$ for $t\in R_+$ and $\hat{B}(t)x\in C(R_+; X)$ for $x\in D$. Furthermore by (i) of Definition 2.6 and (B_2) we have

$$\|\hat{B}(t)x\|\leqq M(t)\exp(\lambda t)\Big(\int_0^t\beta(s)ds\Big)\|x\|_A\equiv\hat{\beta}(t)\|x\|_A \qquad \text{for} \quad t>0\,. \tag{2.6}$$

Therefore if we define $\hat{\beta}(0)=0$, then it is seen that $\hat{\beta}\in L^1_{\text{loc}}(R_+)$ and $\hat{\beta}(t)<\infty$ for $t\in R_+$, and hence $\hat{\beta}(t)$ satisfies (B_1) and (B_2). Thus Lemma 2.2 implies that $\int_0^t\hat{B}(t-r)T(r)xdr$ is well defined and continuous in t for $t>0$ and $x\in D$. Since $U_T(t)$ is strongly continuous on R_+, $\|U_T(t)\|$ is bounded and measurable on each finite interval of R_+. Thus from Lemma 2.8 we have

$$\begin{aligned}\int_0^t U_T(t-s)\Big(\int_0^s B(s-r)T(r)xdr\Big)ds&=\int_0^t\Big(\int_r^t U_T(t-s)B(s-r)T(r)xds\Big)dr\\ &=\int_0^t\Big(\int_0^{t-r} U_T(t-r-s)B(s)T(r)xds\Big)dr\\ &=\int_0^t\hat{B}(t-r)T(r)xdr \quad \text{for } t>0 \text{ and } x\in D\,.\end{aligned}$$

Substituting this formula into (2.5) we have

$$U_T(t)x=T(t)x+\int_0^t\hat{B}(t-s)T(s)xds \qquad \text{for} \quad t>0 \quad \text{and} \quad x\in D\,. \tag{2.7}$$

Let $h>0$. Define $I_1(t; h)x=h^{-1}\int_0^h\hat{B}(t+r)xdr-\hat{B}(t)x$ and $I_2(t; h)x=h^{-1}\int_0^h\hat{B}(t+h-r)(T(r)x-x)dr$ for $t\in R_+$ and $x\in D$. We wish to show that

$$U_T(t+h)x=U_T(t)T(h)x+h\hat{B}(t)x+hI_1(t; h)x+hI_2(t; h)x \tag{2.8}$$

for $t\in R_+$ and $x\in D$. When $t=0$ this formula is easily obtained from (2.7). Let $t>0$. Then it follows from (2.7) and (B_1) that

$$\begin{aligned} U_T(t+h)x &= T(t+h)x+\int_0^{t+h}\hat{B}(t+h-r)T(r)xdr \\ &= T(t)T(h)x+\int_0^t \hat{B}(t-r)T(r)T(h)xdr+h\hat{B}(t)x \\ &\quad +\left(\int_0^h \hat{B}(t+r)xdr-h\hat{B}(t)x\right)+\int_0^h \hat{B}(t+h-r)(T(r)x-x)dr \\ &= U_T(t)T(h)x+h\hat{B}(t)x+hI_1(t;\,h)x+hI_2(t;\,h)x\,. \end{aligned}$$

This proves (2.8).

We now show that $U_T(t)x \in C^1(R_+;\,X)$ for each $x \in D$. Since $\hat{B}(t)x \in C(R_+;\,X)$ for $x \in D$, it follows that $\lim_{h\downarrow 0} I_1(t;\,h)x=0$ for $t \in R_+$. Let $t \in R_+$. Choose $T>0$ such that $t+h<T$. Since $M(t)$ is nondecreasing, (2.6) yields

$$\begin{aligned} \|I_2(t;\,h)x\| &\leqq h^{-1}\int_0^h \hat{\beta}(t+h-r)\|T(r)x-x\|_A dr \\ &\leqq M(T)\exp(\lambda T)\left(\int_0^T \beta(s)ds\right)\sup\{\|T(r)x-x\|_A;\ 0\leqq r\leqq h\}\,. \end{aligned}$$

Therefore this shows that

$$\begin{aligned} \frac{d^+}{dt}U_T(t)x &= \lim_{h\downarrow 0} h^{-1}[U_T(t+h)x-U_T(t)x] \\ &= U_T(t)\lim_{h\downarrow 0} h^{-1}(T(h)x-x)+\hat{B}(t)x+\lim_{h\downarrow 0}(I_1(t;\,h)x+I_2(t;\,h)x) \\ &= U_T(t)Ax+\hat{B}(t)x \qquad \text{for} \quad t\in R_+ \quad \text{and} \quad x \in D\,. \end{aligned}$$

Since the right-hand side of this equality is continuous on R_+ we have

$$\frac{d}{dt}U_T(t)x=U_T(t)Ax+\hat{B}(t)x \qquad \text{for} \quad t>0 \quad \text{and} \quad x\in D\,.$$

Moreover it is easily seen that $\lim_{t\downarrow 0}(d/dt)U_T(t)x=Ax=(d^+/dt)U_T(t)x|_{t=0}$.

Q.E.D.

As a simple consequence of Theorem 2.7 we have the following which is one of the main results in [12].

COROLLARY 2.9. *Let $\{B(t);\,t\in R_+\}$ be a strongly continuous family in $B(X)$. Then there exists an adjoint kernel $\{U_T(t);\,t\in R_+\}$.*

PROOF. Since $\|B(t)\| \in L^1_{loc}(R_+)$, it is easy to see that $\{B(t);\,t\in R_+\}$ forms a family of class $(B(\cdot))$ with D replaced by $D(A)$. Also for any $\lambda \in R$ we have $\lim_{t\downarrow 0} I_\lambda(t)=0$. Therefore we can find $t_0>0$ such that $L_\lambda(t_0)<1$. Thus the conclusion follows from Theorem 2.7. Q.E.D.

COROLLARY 2.10. *Let* $\{B(t);\ t\in R_+\}$ *satisfy*

(B_6) $D(A)\subset D(B(t))$ *for all* $t\in R_+$, *and there exists a constant* b *such that*

$$\|B(0)R(\mu;\ A)y\|\leqq b\|y\| \quad \text{for} \quad y\in X \quad \text{and some} \quad \mu\in\rho(A)\ .$$

Further there exists another family $\{B_1(t);\ t\in R_+\}$ *of linear operators in* X *which satisfies* (B_4) *with* β *replaced by some function* $\beta_1\in L^1_{\mathrm{loc}}(R_+)$ *for this* $\mu\in\rho(A)$ *and*

$$B(t)x=B(0)x+\int_0^t B_1(s)xds \qquad \text{for} \quad t>0 \quad \text{and} \quad x\in D(A)\ .$$

Then there exists an adjoint kernel $\{U_T(t);\ t\in R_+\}$.

PROOF. Observe that $B(t)x\colon R_+\to X$ is continuous for every $x\in D(A)$. From (B_6) and (B_4) it follows that

$$\text{(2.9)} \qquad \|B(t)R(\mu;\ A)y\|\leqq\|B(0)R(\mu;\ A)y\|+\left\|\int_0^t B_1(s)R(\mu;\ A)yds\right\|$$
$$\leqq\left(b+\int_0^t\beta_1(s)ds\right)\|y\|\equiv\beta_2(t)\|y\|$$

for $y\in X$ and some $\mu\in\rho(A)$. Therefore it is seen that $\{B(t);\ t\in R_+\}$ satisfies (B_1) and (B_2) with D replaced by $D(A)$ and Lemma 2.2 implies that as function of s, $\int_0^s B(s-r)T(r)xdr$ is continuous for each $s>0$ and $x\in D(A)$. Further by (B_6) and Fubini's theorem we have

$$\int_0^s B(s-r)T(r)xdr=\int_0^s B(0)T(r)xdr+\int_0^s\left(\int_r^s B_1(s-u)T(r)xdu\right)dr$$
$$=\int_0^s B(0)T(r)xdr+\int_0^s\left(\int_0^u B_1(s-u)T(r)xdr\right)du$$
$$=B(0)R(\mu;\ A)\left(\mu\int_0^s T(r)xdr-T(s)x+x\right)$$
$$+\int_0^s B_1(s-u)R(\mu;\ A)\left(\mu\int_0^u T(r)xdr-T(u)x+x\right)du\ .$$

Thus for any $\lambda\in R$ $L_\lambda(t)=\sup\left\{\int_0^t\exp(-\lambda s)\left\|\int_0^s B(s-r)T(r)xdr\right\|ds;\ x\in D(A),\ \|x\|\leqq 1\right\}$ is well defined and satisfies the inequality

$$L_\lambda(t)\leqq\int_0^t\exp(-(\lambda-\omega)s)r(s)ds\ ,$$

where r is some continuous function on R_+. This shows that $\lim_{t\downarrow 0}L_\lambda(t)=$

0. Therefore the conclusion follows from Proposition 2.3 and Theorem 2.7. Q.E.D.

COROLLARY 2.11. *Let* $b \in C^1(R_+)$ *and let* B *satisfy* (B_5). *Then there exists an adjoint kernel* $\{U_T(t);\ t \in R_+\}$ *for* $B(t) = b(t)B$.

PROOF. Clearly (B_5) and $b \in C^1(R_+)$ imply (B_6). Q.E.D.

The class $P(A)$ defined in [1] plays an important role in perturbation theory of semi-group (see also [8] and [9]). We now also define $P(A)$ as follows:

DEFINITION 2.12. A linear operator B is said to be *of class* $P(A)$ if it satisfies

(B_7) there exists a dense linear subset D of X such that $D \subset D(A) \cap D(B)$, $T(t)D \in D$ for $t \in R_+$ and $BT(t)x \in C(R_+;\ X)$ for each $x \in D$, and (B_8) with $B(t)$ replaced by $b(t)B$, where $b \in L^1_{loc}(R_+)$.

The following result can be proved by the same arguments as those in Lemma 1.1 in [6].

LEMMA 2.13. *Let* $\{U(t);\ t \in R_+\}$ *be a family in* $B(X)$ *and satisfy*
(i) $U(t)x\colon R_+ \to X$ *is measurable for* $x \in X$, *and*
(ii) *there exists a function* $\beta \in L^1_{loc}(R_+)$ *such that*

$$\|U(t)\| \leqq \beta(t) \qquad \text{for} \quad t \in R_+ .$$

If $f \in C(R_+;\ X)$, *then as a function of* s, $U(t, s)f(s) \in L^1([0, t];\ X)$ *for all* $t > 0$. *Further if we define* $g(t) = \int_0^t U(t-s)f(s)\,ds$ *for* $t > 0$ *and* $g(0) = 0$, *then* $g \in C(R_+;\ X)$.

PROPOSITION 2.14. *Let* $b \in L^1_{loc}(R_+)$ *and let* B *be of class* $P(A)$. *Then* $\{F(t);\ t \in R_+\}$ *forms a family of class* $(F(\cdot))$, *where* F *is the function defined by* (2.2) *with* $B(t)$ *replaced by* $b(t)B$.

PROOF. Set $N = \{t;\ |b(t)| = \infty\}$. Define $U(t) = b(t)I$ for $t \in R_+ \backslash N$ and $U(t) = 0$ for $t \in N$. Then $U(t)$ is well defined on R_+ and $U(t)x\colon R_+ \to X$ is measurable for $x \in X$. Since $BT(t)x \in C(R_+;\ X)$ for $x \in D$ and $b \in L^1_{loc}(R_+)$, it follows from Lemma 2.13 that $F(t)x \in C(R_+;\ X)$ for $x \in D$. Now it is easy to see that F satisfies (F_1) and (F_2). Q.E.D.

REMARK 2.15. Concerning the existence of family $\{F(t);\ t \in R_+\}$ of class $(F(\cdot))$ the condition (B_7) is rather restrictive. The following condition, for example, assures the conclusion of Proposition 2.14:

(B_7') there exists a dense linear subset D of X such that $D \subset D(B)$, $T(t)D \subset D$ for $t \in R_+$ and as a function of t, $BT(t)x \in L^1_{\mathrm{loc}}(R_+; X)$ for $x \in D$.

Corresponding to Lemma 2.8 the following result holds. We omit its proof.

LEMMA 2.16. *Let $\{B(t); t \in R_+\}$ be a family in $B(X)$ satisfying* (i) (ii) *of Lemma* 2.13. *Suppose further that $\{U(t); t \in R_+\}$ is a strongly continuous family in $B(X)$. If $u \in C(R_+; X)$, then we have*

$$\int_0^t \left(\int_0^s U(t-s)B(s-r)u(r)dr\right)ds = \int_0^t \left(\int_r^t U(t-s)B(s-r)u(r)ds\right)dr \qquad \text{for} \quad t>0\,,$$

THEOREM 2.17. *Let $b \in L^1_{\mathrm{loc}}(R_+)$ and let B be of class $P(A)$. Suppose that there exist constants $0<t_0 \leqq \infty$ and $\lambda \geqq \omega$ such that $L_\lambda(t_0)<1$, where L_λ is the function defined by* (2.3) *for $B(t)=b(t)B$. Then there exists an adjoint kernel $\{U_T(t); t \in R_+\}$.*

PROOF. Propositions 2.14 and 1.3 imply that there exists a one-parameter family $\{U_T(t); t \in R_+\}$ in $B(X)$ which satisfies (i) and (iv) of Definition 2.6 and integral equation of the form

$$U_T(t)x = T(t)x + \int_0^t U_T(t-s)\left(\int_0^s b(s-r)BT(r)xdr\right)ds$$

for $t>0$ and $x \in D$. For $B(t)=b(t)B$ define $\hat{B}(t)$ as in the proof of Theorem 2.7. Then by Lemma 1.4 $\hat{B}(t)x \in C(R_+; X)$ for $x \in D$. Moreover Lemma 2.2, (i) of Definition 2.6 and (B_2) imply that $\int_0^t \hat{B}(t-r)T(r)xdr$ is continuous in t for $t>0$ and $x \in D$ and there exists some $\hat{\beta} \in L^1_{\mathrm{loc}}(R_+)$ such that $\|\hat{B}(t)x\| \leqq \hat{\beta}(t)\|Bx\|$ for $t \in R_+$ and $x \in D$. Furthermore Lemma 2.16 implies

$$\int_0^t U_T(t-s)\left(\int_0^s b(s-r)BT(r)xdr\right)ds = \int_0^t \hat{B}(t-r)T(r)xdr$$

for $t>0$ and $x \in D$. Therefore we have

$$U_T(t)x = T(t)x + \int_0^t \hat{B}(t-r)T(r)xdr \qquad \text{for} \quad t>0 \quad \text{and} \quad x \in D\,.$$

Now by the same argument as in the proof of Theorem 2.7 one can easily complete the proof. Q.E.D.

COROLLARY 2.18. *Let $b \in L^1_{\mathrm{loc}}(R_+)$ and B satisfy* (B_7) *and*

(B_8) *for some* $t_0>0$

$$\sup\left\{\int_0^{t_0}\|BT(s)x\|ds;\ x\in D,\ \|x\|\leq 1\right\}<\infty\ .$$

Then B *is of class* $P(A)$ *and further there exists an adjoint kernel* $\{U_T(t);\ t\in R_+\}$ *for* $B(t)=b(t)B$. *If* $b\in L^1(R_+)$ *and* $t_0=\infty$, *then there exists* $\lambda>\text{Max}\{\omega, 0\}$ *such that* $\|U_T(t)\|\leq M(\lambda)\exp(\lambda t)$ *for* $t\in R_+$ *and some constant* $M(\lambda)$ *which depends on* λ *only.*

PROOF. Define $K_\lambda(t_0)=\sup\left\{\int_0^{t_0}\exp(-\lambda s)\|BT(s)x\|ds;\ x\in D,\ \|x\|\leq 1\right\}$ for $\lambda\in R$, then by the same argument as in Lemma 1 in [9] we have

$$\int_0^\infty \exp(-\lambda t)\|BT(t)x\|dt\leq L(\lambda, t_0)\|x\| \qquad \text{for}\quad x\in D \quad\text{and}\quad \lambda>\text{Max}\{\omega, 0\}\ ,$$

where $L(\lambda, t_0)=K_\lambda(t_0)[1+M\exp(-(\lambda-\omega)t_0)\{1-\exp(-(\lambda-\omega)t_0)\}]^{-1}$. Thus it follows that

$$\begin{aligned}&\int_0^t \exp(-\lambda s)\left\|\int_0^s b(s-r)BT(r)xdr\right\|ds\\ &\quad\leq\left(\int_0^t \exp(-\lambda s)|b(s)|ds\right)\left(\int_0^t \exp(-\lambda s)\|BT(s)x\|ds\right)\\ &\quad\leq L(\lambda, t_0)\left(\int_0^t \exp(-\lambda s)|b(s)|ds\right)\|x\| \qquad \text{for}\quad t>0\ .\end{aligned}$$

Therefore for $\lambda>\text{Max}\{\omega, 0\}$

$$L_\lambda(t)=\sup\left\{\int_0^t \exp(-\lambda s)\left\|\int_0^s b(s-r)BT(r)xdr\right\|ds;\ x\in D,\ \|x\|\leq 1\right\}$$

is well defined and finite for all $t>0$. Moreover the above inequality shows that $\lim_{t\downarrow 0}L_\lambda(t)=0$. Thus B is of class $P(A)$ and further the first part of the conclusion follows from Theorem 2.17.

If $b\in L^1(R_+)$ and $t_0=\infty$, then we have for $\lambda>\text{Max}\{\omega, 0\}$

$$L_\lambda(t)\leq K_\lambda(\infty)\left(\int_0^t \exp(-\lambda s)|b(s)|ds\right)\leq K_\lambda(\infty)\left(\int_0^\infty \exp(-\lambda s)|b(s)|ds\right)\ ,$$

where $K_\lambda(\infty)=\sup\left\{\int_0^\infty \exp(-\lambda s)\|BT(s)x\|ds;\ x\in D,\ \|x\|\leq 1\right\}$. Since $L_\lambda(t)$ is bounded and nondecreasing in t, $L_\lambda(\infty)=\lim_{t\to\infty}L_\lambda(t)$ exists and satisfies the inequality $L_\lambda(\infty)\leq K_\lambda(\infty)\left(\int_0^\infty \exp(-\lambda s)|b(s)|ds\right)$. $K_\lambda(\infty)$ is also nonincreasing in λ, and hence there exists $K_\infty(\infty)=\lim_{\lambda\to\infty}K_\lambda(\infty)<\infty$. Since

$b \in L^1(R_+)$, one has $\lim_{\lambda\to\infty}\left(\int_0^\infty \exp(-\lambda s)|b(s)|ds\right)=0$. Thus $\lim_{\lambda\to\infty} L_\lambda(\infty)=0$. Now by Proposition 1.3 we have $\|U_T(t)\| \leqq M(1-L_\lambda(\infty))^{-1}\exp(\lambda t)$ for $t \in R_+$ and sufficiently large $\lambda > \mathrm{Max}\{\omega, 0\}$. Q.E.D.

COROLLARY 2.19. *Let $b \in L^1_{\mathrm{loc}}(R_+)$, and let $\psi \in L^1_{\mathrm{loc}}(R_+)$ and $\psi(t)>0$ for $t \in R_+$. Assume that B is a closed linear operator in X such that*

$$(\mathrm{B}_9) \qquad T(t)X \subset D(B) \quad \text{for all} \quad t>0\,, \quad \text{and}$$

$$(\mathrm{B}_{10}) \qquad \|BT(t)x\| \leqq \psi(t)\|x\| \quad \text{for} \quad x \in X \quad \text{and} \quad t>0\,.$$

Then B is of class $P(A)$ and further there exists an adjoint kernel $\{U_T(t);\, t \in R_+\}$ for $B(t)=b(t)B$.

PROOF. Let $\delta>0$. Choose $\delta_1=\delta_1(\delta)$ such that $0<\delta_1\leqq\delta$ and $\psi(\delta_1)<\infty$. Let $t \geqq \delta$ and let $t=n\delta_1+s$, $0\leqq s<\delta_1$, $n\geqq 1$. Then

$$\begin{aligned}(2.10) \quad \|BT(t)x\| &= \|BT(\delta_1)T((n-1)\delta_1+s)x\| \\ &\leqq (\psi(\delta_1)M\exp(-\omega\delta_1))\exp(\omega t)\|x\| \equiv M(\delta)\exp(\omega t)\|x\| \\ &\qquad \text{for} \quad t\geqq\delta\,.\end{aligned}$$

Especially $\|BT(\delta)x\| \leqq M(\delta)\exp(\omega\delta)\|x\|$. Replacing δ by t, we have

$$(2.11) \qquad \|BT(t)x\| \leqq M(t)\exp(\omega t)\|x\| \quad \text{for} \quad x\in X \quad \text{and} \quad t>0\,,$$

where $M(t)<\infty$ for $t>0$. Let $t>0$. Choose $\hat{\varepsilon}$ such that $0<\hat{\varepsilon}<t$. Then the strong continuity of $BT(t)x$ in t follows from the estimates

$$\|BT(t+h)x-BT(t)x\| \leqq M(t)\exp(\omega t)\|T(h)x-x\| \quad \text{for} \quad h>0$$

and

$$\|BT(t-h)x-BT(t)x\| \leqq M(\hat{\varepsilon})\exp(\omega t)\|T(h)x-x\| \quad \text{for} \quad t-\hat{\varepsilon}>h>0\,.$$

Next we shall show that $D(A)\subset D(B)$ and $BR(\mu;A)\in B(X)$ for $\mu>\omega$. Let $\delta>0$ be fixed again. Choose $\varepsilon>0$ such that $\varepsilon<\delta$. Then it follows from (B_{10}) and (2.10) that

$$\begin{aligned}(2.12) \quad \int_\varepsilon^\infty \exp(-\mu s)\|BT(s)x\|ds &= \int_\varepsilon^\delta \exp(-\mu s)\|BT(s)x\|ds \\ &\quad + \int_\delta^\infty \exp(-\mu s)\|BT(s)x\|ds \\ &\leqq \left[\int_0^\delta \exp(-\mu s)\psi(s)ds + M(\delta)(\mu-\omega)^{-1}\right]\|x\| \\ &\equiv M_1\|x\| \qquad .\end{aligned}$$

for $x\in X$ and $\mu>\omega$. Setting $R_\varepsilon(\mu;\ A)x=\int_\varepsilon^\infty \exp(-\mu s)T(s)xds$, then $R_\varepsilon(\mu;\ A)x\in D(B)$. Moreover by (2.12) and (B_{10}) we can see that $\lim_{\varepsilon\downarrow 0} BR_\varepsilon(\mu;\ A)x$ exists. Since $\lim_{\varepsilon\downarrow 0} R_\varepsilon(\mu;\ A)x=R(\mu;\ A)x$ and B is closed, we have

$$R(\mu;\ A)x\in D(B),\ \lim_{\varepsilon\downarrow 0} BR_\varepsilon(\mu;\ A)x=BR(\mu;\ A)x \quad\text{and}\quad \|BR(\mu;\ A)x\|\leqq M_1\|x\|$$

for $x\in X$ and $\mu>\omega$. That is, $D(A)\subset D(B)$ and $BR(\mu;\ A)\in B(X)$ for $\mu>\omega$. Now it is not difficult to see that $BT(t)x\in C(R_+;\ X)$ for every $x\in D(A)$ and there exists some $t_0>0$ which satisfies (B_8) with D replaced by $D(A)$. The conclusion now follows from Corollary 2.18. Q.E.D.

§ **3.** In this section we shall deal with the inhomogeneous initial value problems

$$\text{(E)}\qquad \begin{cases} \dfrac{d}{dt}u(t)=Au(t)+\displaystyle\int_0^t B(t-s)u(s)ds+f(t) & \text{for}\quad t>0\,, \\ u(0)=x\,, \end{cases}$$

and

$$\text{(E}'\text{)}\qquad \begin{cases} \dfrac{d}{dt}u(t)=Au(t)+\displaystyle\int_0^t b(t-s)Bu(s)ds+f(t) & \text{for}\quad t>0\,, \\ u(0)=x\,. \end{cases}$$

Throughout this section we shall assume that $\{B(t);\ t\in R_+\}$ is of class $(B(\cdot))$, B is of class $P(A)$, $b\in L^1_{\text{loc}}(R_+)$ and D is the set defined in Definition 2.1 (or Definition 2.12).

DEFINITION 3.1. A function $u\colon I=[0,\ T]\to D$ is said to be a *strong solution* of (E) (or (E′)) on I if u and Au (or Bu) $\in C(I;\ X)$ and (E) (or (E′)) is satisfied at all points in $I\backslash\{0\}$, where T is some positive number.

Obvious modifications of this definition can be used when the interval I is of the form $[0,\ T)$ with $0<T\leqq\infty$.

THEOREM 3.2. *If the hypotheses of Theorem* 2.7 *are true, then the adjoint kernel* $\{U_T(t);\ t\in R_+\}$ *exists. If u is a strong solution of* (E) *on I for $f\in C(I;\ X)$ and $x\in D$, then*

$$u(t)=U_T(t)x+\int_0^t U_T(t-s)f(s)ds \qquad \textit{for}\quad t\in I\,. \tag{3.1}$$

PROOF. Theorem 2.7 and $u(t)\in D$ imply that there exists an adjoint kernel $\{U_T(t); t\in R_+\}$ and the X-valued function $g(s)=U_T(t-s)u(s)$ is strongly differentiable for $0<s<t$. Define $\hat{B}(t)$ and $\hat{\beta}(t)$ as in the proof of Theorem 2.7 and define $\hat{g}(t)=\int_0^t B(t-r)u(r)dr$ for $t\in I\backslash\{0\}$ and $\hat{g}(0)=0$, then from Lemma 2.2 we can see that $\hat{g}\in C(I; X)$. Noting that $\hat{\beta}(t)$ is finite for each $t\in R_+$ and nondecreasing in t, from (2.6) one has

$$\begin{aligned}\|\hat{B}(t-s-h)u(s+h)-\hat{B}(t-s)u(s)\|\leqq&\|\hat{B}(t-s-h)u(s)-\hat{B}(t-s)u(s)\|\\&+\hat{\beta}(T)\|u(s+h)-u(s)\|_A\end{aligned}$$

for $0\leqq s<t$ and $0<h<t-s$, and

$$\begin{aligned}\|\hat{B}(t-s+h)u(s-h)-\hat{B}(t-s)u(s)\|\leqq&\|\hat{B}(t-s+h)u(s)-\hat{B}(t-s)u(s)\|\\&+\hat{\beta}(2T)\|u(s-h)-u(s)\|_A\end{aligned}$$

for $0<s\leqq t$ and $0<h<s$. Thus it is seen that as a function of s, $\hat{B}(t-s)u(s)$ is strongly continuous on $[0, t]$. Now from (E) and (iii) of Definition 2.6 it follows that

$$\begin{aligned}g'(s)&=U_T(t-s)u'(s)-U_T'(t-s)u(s)\\&=U_T(t-s)\Big(Au(s)+\int_0^s B(s-r)u(r)dr+f(s)\Big)\\&\quad-U_T(t-s)Au(s)-\int_0^{t-s}U_T(t-s-r)B(r)u(s)dr\\&=U_T(t-s)f(s)+U_T(t-s)\hat{g}(s)-\hat{B}(t-s)u(s)\qquad\text{for}\quad 0<s<t\,.\end{aligned}$$

Here $'=d/dt$. Integrating $g'(s)$ from ε to $t-\varepsilon$ and then letting $\varepsilon\to 0$ we have

$$\begin{aligned}&u(t)-U_T(t)x-\int_0^t U_T(t-s)f(s)ds\\&\quad=\int_0^t U_T(t-s)\hat{g}(s)\,ds-\int_0^t\hat{B}(t-s)u(s)ds\\&\quad=\int_0^t U_T(t-s)\Big(\int_0^s B(s-r)u(r)\,dr\Big)ds-\int_0^t\Big(\int_0^{t-s}U_T(t-s-r)B(r)u(s)dr\Big)ds\\&\quad=\int_0^t\Big(\int_0^s U_T(t-s)B(s-r)u(r)dr\Big)ds-\int_0^t\Big(\int_s^t U_T(t-r)B(r-s)u(s)dr\Big)ds=0\,.\end{aligned}$$

The last equality follows from Lemma 2.8. Q.E.D.

THEOREM 3.2′. *The conclusion of Theorem* 3.2 *remains true with Theorem* 2.7 *and* (E) *replaced by Theorem* 2.17 *and* (E′) *respectively.*

The proof of Theorem 3.2′ can be carried out by similar arguments to those in the proof of Theorem 3.2 with using Theorem 2.17 and Lemma 2.16.

REMARK 3.3. Miller has obtained a similar result under the condition that (E) is uniformly well posed (Theorem 5.4 in [6]) or that $B(t)=b(t)A$ and $b\in L^1(R_+)\cap C^1(R_+)$ (Corollary 7.6 in [6]).

COROLLARY 3.4. *Let* $f\equiv 0$ *in* (*E*) (*or* (E′)). *If* (E) (*or* (E′)) *has a unique strong solution for every* $x\in D$, *then the adjoint kernel is uniquely determined.*

DEFINITION 3.5. The continuous function u defined by (3.1) is called the *mild solution* of (E) (or (E′)) on I.

Finally in Theorem 3.7 below we give a sufficient condition under which (E) has a unique strong solution on R_+. Our proof is similar to the proof of Lemma 7.2 in [6] and our result contains a generalization of its lemma. To prove Theorem 3.7 we need the following lemma which is proved in Chapter 4 of [10].

LEMMA 3.6. *Let* $f_1, f_2\in C(R_+; X)$ *such that*

(i) $f_1\in C^1(R_+; X)$, *and*

(ii) $f_2\in D(A)$ *for* $t\in R_+$ *and* $Af_2\in L^1_{loc}(R_+; X)$.

Then for any $x\in D(A)$,

$$\frac{d}{dt}u(t)=Au(t)+f_1(t)+f_2(t)\ , \qquad u(0)=x$$

has a unique strong solution $u\in C^1(R_+; X)$ *such that* $Au\in C(R_+; X)$. *This solution can be represented in the form*

$$u(t)=T(t)x+\int_0^t T(t-s)(f_1(s)+f_2(s))ds \qquad \text{for} \quad t>0$$

and

$$Au(t)=T(t)f_1(0)-f_1(t)+\int_0^t T(t-s)(f_1'(s)+Af_2(s))ds+AT(t)x \qquad \text{for} \quad t>0\ .$$

THEOREM 3.7. *Suppose that* $\{B(t);\ t\in R_+\}$ *satisfies* (B_0) *with* $\beta_1\in C(R_+)$. *If* $f\in C^1(R_+; X)$ *or* $f(t)\in D(A)$ *for* $t\in R_+$ *and* $Af\in L^1_{loc}(R_+; X)$, *then the mild solution of* (E) *on* R_+ *is the unique strong solution of* (E) *on* R_+.

PROOF. Fix any $T>0$ and $\mu\in\rho(A)$. Define $d(T)=\{v: v$ maps $[0, T]$ into $D(A)$ and both v and Av are continuous$\}$. Since A is closed, it is easy to see that $d(T)$ with norm $|||v|||=\sup\{\exp(-Lt)|||v(t)|||_A: 0\leqq t\leqq T$

for some $L \in R_+\}$, is a Banach space, where $|||y|||_A = ||y|| + ||(\mu - A)y||$ for every $y \in D(A)$. Given v in $d(T)$ define $S_0 v(t) = \int_0^t B(t-s)v(s)ds$ for $0 < t \leqq T$ and $S_0 v(0) = 0$. By the use of (B_6) with $\beta_1 \in C(R_+)$ and (B_4) it is seen that $S_0 v(t)$ is well defined and strongly continuously differentiable with

$$\frac{d}{dt} S_0 v(t) = B(0)v(t) + \int_0^t B_1(t-s)v(s)ds ,$$

since $S_0 v(t) = B(0)R(\mu; A)\left(\mu \int_0^t v(s)ds - \int_0^t Av(s)ds\right) + \int_0^t B_1(s)R(\mu; A)\left(\mu \int_0^{t-s} v(r)dr - \int_0^{t-s} Av(r)dr\right)ds$ and $B(0)R(\mu; A)$ and $B_1(s)R(\mu; A) \in B(X)$ for $s \in R_+$. From Lemma 3.6 it follows that any v in $d(T)$,

$$\frac{d}{dt} u(t) = Au(t) + \{S_0 v(t) + f(t)\} , \qquad u(0) = x \in D(A)$$

has a unique strong solution $u = Sv$ which is again in $d(T)$.

Since S_0 is a linear operator on $d(T)$, we can decide whether or not S is a contraction map by computing the norm when $x = 0$ and $f \equiv 0$. If $|||v||| \leqq 1$, then from (2.9)

$$||S_0 v(t)|| \leqq \int_0^t ||B(t-s)v(s)||ds \leqq \exp(Lt) \int_0^t \beta_2(s) \exp(-Ls)ds ,$$

where $\beta_2(s) = b + \int_0^s \beta_1(r)dr$. Therefore

$$\exp(-Lt)||S_0 v(t)|| \leqq \int_0^t \beta_2(s) \exp(-Ls)ds ,$$

while $x = 0$, $f \equiv 0$ and Lemma 3.6 imply that

$$\begin{aligned} ||Sv(t)|| &= \left\| \int_0^t T(t-s) S_0 v(s)ds \right\| \\ &\leqq \int_0^t M \exp(\omega(t-s)) \exp(Ls)\left(\int_0^s \beta_2(r) \exp(-Lr)\,dr\right)ds \end{aligned}$$

or

$$\exp(-Lt)||Sv(t)|| \leqq M \int_0^t \exp(-(L-\omega)(t-s))\left(\int_0^s \beta_2(r) \exp(-Lr)dr\right)ds$$

and

$$||(\mu - A)Sv(t)|| = \left\| \mu Sv(t) + S_0 v(t) - \int_0^t T(t-s)(S_0 v)'(s)ds \right\|$$

$$\leqq |\mu| \, \|Sv(t)\| + \left\|S_0 v(t) - \int_0^t T(t-s)\left(B(0)v(s) + \int_0^s B_1(s-r)v(r)dr\right)ds\right\|$$

$$\leqq |\mu| \, \|Sv(t)\| + \|S_0 v(t)\| + M \exp(Lt) \int_0^t \exp(-(L-\omega)(t-s))$$

$$\times \left(b + \int_0^s \beta_1(r) \exp(-Lr)dr\right)ds \, .$$

Thus

$$|||S||| \leqq M\Big[(1+|\mu|)\int_0^T \exp(-(L-\omega)(T-s))\left(\int_0^s \beta_2(r) \exp(-Lr)dr\right)ds$$

$$+ M^{-1}\int_0^T \beta_2(s) \exp(-Ls)\, ds + \int_0^T \exp(-(L-\omega)(T-s))$$

$$\times \left(b + \int_0^s \beta_1(r) \exp(-Lr)dr\right)ds\Big] < 1$$

for L sufficiently large, where we have used the fact that if a and b are non-negative functions, and $b(t)$ is nondecreasing in t, then

$$\int_0^t a(t-s)b(s)ds = \int_0^t a(s)b(t-s)ds$$

is also nondecreasing in t for $t>0$. Thus the contraction mapping theorem implies the existence and uniqueness of a strong solution of (E) on $[0, T]$. Since T is an arbitrary positive number, this proves the existence and uniqueness on R_+. Q.E.D.

COROLLARY 3.8. *Suppose that* $\{B(t);\, t \in R_+\}$ *satisfies* (B_6) *with* $\beta_1 \in C(R_+)$. *Then the adjoint kernel* $U_T(t)$ *maps* $D(A)$ *into* $D(A)$ *for each* $t \in R_+$ *with* $AU_T(t)x \in C(R_+;\, X)$ *and satisfies*

$$\frac{d}{dt} U_T(t)x = AU_T(t)x + \int_0^t B(t-s)U_T(s)x ds \qquad \textit{for} \quad x \in D(A) \quad \textit{and} \quad t>0 \, ,$$

and

$$\frac{d}{dt} U_T(t)x = U_T(t)Ax + \int_0^t U_T(t-s)B(s)x ds \qquad \textit{for} \quad x \in D(A) \quad \textit{and} \quad t>0 \, .$$

The following example shows that every function given by (3.1) does not satisfy equation (E).

EXAMPLE. Consider the integro-differential equation

$$\text{(E)} \qquad \begin{cases} \dfrac{d}{dt} u(t) = Au(t) + k^2 \displaystyle\int_0^t T(t-s)u(s)ds + f(t) & \text{for} \quad t>0 \, , \\ u(0) = x \, , \end{cases}$$

where k is a given constant. It is not difficult to see that the adjoint kernel $\{U_T(t); t\in R_+\}$ is given by $U_T(t)=\cosh(kt)T(t)$. Suppose there exists an $x\in X$ such that $T(t)x\notin D(A)$ for any $t>0$. Then $u(t)=U_T(t)x$ is not differentiable for all $t>0$ and thus u is not a solution of (E) with $f\equiv 0$. Moreover if $f(t)=kT(t)x$ for all $t\in R_+$, by Theorem 3.2 the solution u of (E) is, if it exists, represented by

$$\begin{aligned}u(t)&=\cosh(kt)T(t)x+k\int_0^t\cosh(k(t-s))T(t-s)T(s)xds\\&=\exp(kt)T(t)x\ .\end{aligned}$$

But this is not also differentiable for any $t>0$.

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