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The Remainder Term in the Local Limit Theorem for Independent Random Variables

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Introduction

Let $\{X_k, k=1, 2, \cdots\}$ be a sequence of independent random variables with $EX_k=0$, $EX_k^2=\sigma_k^2<\infty$ $(\sigma_k\geqq 0)$ and with distribution function $F_k(x)$, and suppose that each of X_k has a bounded density function $p_k(x)$. Furthermore, we suppose that some of σ_k^2 are not zero, in particular, we assume $\sigma_1^2>0$ without loss of generality. Write $s_n^2=\sum_{k=1}^n \sigma_k^2$, $Z_n=s_n^{-1}\sum_{k=1}^n X_k$, $f_k(t)=Ee^{itX_k}$, $\bar{f}_n(t)=Ee^{itZ_n}$, $R_k(z)=\int_{|u|>z} u^2dF_k(u)$ and $Q_k(z)=\left|\int_{|u|\leqq z} u^3dF_k(u)\right|$. Moreover, let $\bar{p}_n(x)$ be the density function of Z_n and $\phi(x)$ be the standard normal density function.

Let us denote two classes of functions $g(z)$ defined for all z as follows:

$$G=\{g(z)\,|\,g(z) \text{ is even on } (-\infty, \infty) \text{ and positive on } (0, \infty), \text{ and further } z/g(z) \text{ is non-decreasing on } (0, \infty)\}$$

and

$$G_0=\{g(z)\,|\,g(z)\in G, \text{ and in addition, } z^\alpha/g(z) \text{ is non-decreasing on } (0, \infty) \text{ for some } \alpha \text{ with } 0<\alpha<1\}\,.$$

For $g(z)\in G$, write

$$\lambda_k(g)=\sup_{z>0} g(z)R_k(z)\,, \qquad \mu_k(g)=\sup_{z>0}\frac{g(z)}{z}Q_k(z)\,,$$

$$\rho_k(g)=\lambda_k(g)+\mu_k(g)$$

and

$$T_n=\frac{\sum_{k=1}^n \rho_k(g)}{s_n^2 g(s_n)}\,.$$

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In this paper, we shall discuss rates of convergence of $\bar{p}_n(x)$ to $\phi(x)$ for independent but not necessarily identically distributed random variables, under the milder moment condition such as $\rho_k(g)<\infty$. In §1, we shall state a uniform estimate for $|\bar{p}_n(x)-\phi(x)|$, the proof of which will be given in §2. Some nonuniform bound will be stated in §3 and proved in §4.

Throughout this paper, C and c will be universal positive constants which may depend on $g(z)\in G$ and differ from one expression to another, and by the same θ we shall denote generally different real or complex numbers with $|\theta|\leqq 1$.

§1. A uniform estimate.

We first state a central limit theorem, which is a further extension of the Berry-Esseen theorem for independent random variables. It is a uniform version of Theorem 2 in [3], and also readily derived from Therem 2 in [1].

THEOREM A (*Central limit theorem*). *Let $\bar{F}_n(x)$ be the distribution function of Z_n and $\Phi(x)$ be the standard normal distribution function. Let $g(z)\in G$ and suppose $\rho_k(g)<\infty$ for $1\leqq k\leqq n$. Then we have*

$$\sup_x |\bar{F}_n(x)-\Phi(x)|\leqq CT_n .$$

The result we are going to show is the following, which is a local version of Theorem A.

THEOREM 1. *Let $g(z)\in G$. Suppose $\rho_k(g)<\infty$ for $1\leqq k\leqq n$. Furthermore, suppose that*

(a) *$s_n^2<Kn$, for some positive constant K,*
(b) *$\sup_x p_k(x)<M$, for some $M>0$ independent of k.*

Then we have

$$\sup_x |\bar{p}_n(x)-\phi(x)|\leqq CT_n .$$

It was remarked in [3] that if $g(z)\in G_0$, then $\mu_k(g)\leqq C_\alpha\lambda_k(g)$, where C_α is a positive constant depending on α, α being the number such that $z^\alpha/g(z)$ is non-decreasing on $(0, \infty)$. We thus have the following corollary.

COROLLARY 1. *Let $g(z)\in G_0$. Suppose $\lambda_k(g)<\infty$ for $1\leqq k\leqq n$. If the conditions* (a) *and* (b) *are satisfied, then*

$$\sup_x |\bar{p}_n(x)-\phi(x)| \leqq C\frac{\sum_{k=1}^n \lambda_k(g)}{s_n^2 g(s_n)} .$$

If $EX_k^2 g(X_k)<\infty$, then obviously $\rho_k(g)<\infty$. Therefore, we have the following.

COROLLARY 2. *Let $g(z)\in G$. If $EX_k^2 g(X_k)<\infty$ for $1\leqq k\leqq n$, then*

$$\sup_x |\bar{p}_n(x)-\phi(x)| \leqq C\frac{\sum_{k=1}^n EX_k^2 g(X_k)}{s_n^2 g(s_n)} ,$$

under the conditions (a) *and* (b).

The following two corollaries are also given from Theorem 1.

COROLLARY 3. *In addition to the conditions* (a) *and* (b), *suppose that* $\liminf_{n\to\infty} s_n^2/n>0$. *Let $0<\delta\leqq 1$. Then, in order that*

$$\sup_x |\bar{p}_n(x)-\phi(x)| = O(n^{-\delta/2}) ,$$

it is sufficient for $0<\delta<1$, that

$$\frac{1}{n}\sum_{k=1}^n \sup_{z>0} z^\delta R_k(z)=O(1) \quad as \quad n\to\infty , \tag{1.1}$$

and for $\delta=1$, that (1.1) *with $\delta=1$ and*

$$\frac{1}{n}\sum_{k=1}^n \sup_{z>0} Q_k(z)=O(1) \quad as \quad n\to\infty$$

hold.

This corollary is obtained from our theorem with $g(z)=|z|^\delta$, and is an extension of the local limit theorem by Ibragimov-Linnik ([2], Theorem 4.5.1) to the case of non-identically distributed random variables. Moreover we have

COROLLARY 4. *Suppose that $\{X_k\}$ is a sequence of independent, identically distributed random variables with $EX_1=0$, $EX_1^2=1$, and with bounded density $p(x)$. Let $g(z)\in G$. If*

$$\sup_{z>0} g(z)\int_{|x|>z} x^2 p(x)dx<\infty \tag{1.2}$$

and

$$\sup_{z>0}\frac{g(z)}{z}\left|\int_{|x|\leqq z} x^3 p(x)dx\right|<\infty ,$$

then

$$\sup_x |\bar{p}_n(x) - \phi(x)| \leqq \frac{C}{g(\sqrt{n})} . \tag{1.3}$$

In particular, when $g(z) \in G_0$*, only the condition* (1.2) *implies* (1.3).

This is also an extension of Ibragimov-Linnik's local limit theorem [2], and where $g(z) = |z|^\delta$, $0 < \delta < 1$.

§2. Proof of Theorem 1.

We begin with the following lemma which is a slight modification of Lemma 2 in [1].

LEMMA 1. *Let* $g(z) \in G$*, and write* $R(z) = \int_{|x|>z} x^2 dF(x)$ *for some distribution* $F(x)$*. Suppose that*

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 dF(x) < \infty \quad (\sigma \geqq 0) , \quad \lambda(g) = \sup_{z>0} g(z) R(z) < \infty .$$

Then

$$\frac{\sigma^2 g(\sigma)}{\lambda(g)} \leqq \frac{8}{3} .$$

PROOF. Since $\sigma^2 \leqq (\sigma^2/4) + R(\sigma/2)$, we have

$$\lambda(g) \geqq g\left(\frac{\sigma}{2}\right) R\left(\frac{\sigma}{2}\right) \geqq \frac{3\sigma^2}{4} g\left(\frac{\sigma}{2}\right) .$$

Noting that $g(\varepsilon z) \geqq \varepsilon g(z)$ for $0 < \varepsilon < 1$, we have

$$\lambda(g) \geqq \frac{3\sigma^2}{8} g(\sigma) ,$$

which concludes the lemma.

In what follows, we suppose that $T_n \leqq 1/27$. In the other case, the statement of the theorem trivially holds for $C = 27(M+1)$, since $|\bar{p}_n(x) - \phi(x)| \leqq M+1$ because of the condition (b). We prove the following

LEMMA 2. *For* $|t| \leqq T = (1/3) T_n^{-1/3}$*,*

$$|\bar{f}_n(t) - e^{-t^2/2}| \leqq C T_n (t^2 + |t|^3) e^{-t^2/2} .$$

This lemma plays a main role in the proof of the theorem. The technique of the proof is closely related to that of Lemma 5 in [1].

PROOF. We first note that $T\geqq 1$, since we have assumed $T_n\leqq 1/27$. Using

$$f_k(t)=1-\frac{1}{2}\sigma_k^2t^2+\int_{-\infty}^{\infty}\left(e^{itu}-1-itu+\frac{1}{2}t^2u^2\right)dF_k(u)\ ,$$

we have

$$f_k(t)=1-\frac{1}{2}\sigma_k^2t^2+\int_{|ut|<1}\left(-\frac{1}{6}it^3u^3+\frac{1}{24}\theta t^4u^4\right)dF_k(u)$$
$$+\int_{|ut|\geqq 1}\left(-\frac{1}{2}\theta t^2u^2+\frac{1}{2}t^2u^2\right)dF_k(u)\ .$$

On the other hand,

$$\int_{|ut|<1}u^4dF_k(u)=\int_0^{1/|t|}u^2d(-R_k(u))\leqq 2\int_0^{1/|t|}uR_k(u)du\ .$$

Hence

$$f_k(t)=1-\frac{1}{2}\sigma_k^2t^2+\theta\left\{\frac{1}{6}|t|^3Q_k(1/|t|)+\frac{1}{12}t^4\int_0^{1/|t|}uR_k(u)du+t^2R_k(1/|t|)\right\}$$
$$=1-\frac{1}{2}\sigma_k^2t^2+\theta\frac{t^2}{g(1/|t|)}\left\{\frac{1}{6}g(1/|t|)|t|Q_k(1/|t|)\right.$$
$$+\frac{1}{12}t^2g(1/|t|)\int_0^{1/|t|}\frac{u}{g(u)}g(u)R_k(u)du$$
$$\left.+g(1/|t|)R_k(1/|t|)\right\}\ .$$

Here

$$t^2g(1/|t|)\int_0^{1/|t|}\frac{u}{g(u)}g(u)R_k(u)du$$
$$\leqq t^2g(1/|t|)\frac{1}{|t|g(1/|t|)}\lambda_k(g)\int_0^{1/|t|}du=\lambda_k(g)\ .$$

Therefore we have

$$f_k(t)=1-\frac{1}{2}\sigma_k^2t^2-2\theta\rho_k(g)\frac{t^2}{g(1/|t|)}\ ,$$

so that

$$f_k(t/s_n)=1-u_k\ , \tag{2.1}$$

where

$$u_k=\frac{\sigma_k^2}{2s_n^2}t^2+2\theta\rho_k(g)\frac{1}{s_n^2 g(s_n/|t|)}t^2 . \tag{2.2}$$

Noting that $g(s_n)/s_n \leqq g(\sigma_k)/\sigma_k$ and using Lemma 1, we have for $|t|\leqq T$,

$$\begin{aligned}\frac{\sigma_k^2}{2s_n^2}t^2 &\leqq \frac{\sigma_k^2}{2s_n^2}T^2 \leqq \frac{\sigma_k^2(g(s_n))^{2/3}}{18 s_n^{2/3}(\sum_{k=1}^n \rho_k(g))^{2/3}} \\ &\leqq \frac{\sigma_k^{4/3}(g(\sigma_k))^{2/3}}{18(\lambda_k(g))^{2/3}} \leqq \frac{1}{18}\left(\frac{8}{3}\right)^{2/3} \leqq \frac{2}{9} .\end{aligned} \tag{2.3}$$

Moreover, using that $g(\varepsilon z)\geqq \varepsilon g(z)$ for $0<\varepsilon<1$ and $g(\varepsilon z)\geqq g(z)$ for $\varepsilon\geqq 1$, we have

$$\begin{aligned}\left|\frac{2\theta\rho_k(g)}{s_n^2 g(s_n/|t|)}t^2\right| &\leqq \frac{2\rho_k(g)}{s_n^2 g(s_n)}|t|^3 , \quad \text{if} \quad |t|\geqq 1 , \\ &\leqq \frac{2\rho_k(g)}{s_n^2 g(s_n)}t^2 , \quad \text{if} \quad |t|<1 ,\end{aligned}$$

so that we have, recalling $T\geqq 1$, for all $|t|\leqq T$,

$$\left|\frac{2\theta\rho_k(g)}{s_n^2 g(s_n/|t|)}t^2\right| \leqq \frac{2\rho_k(g)}{s_n^2 g(s_n)}T^3 \leqq \frac{2}{27} . \tag{2.4}$$

Combining (2.2)-(2.4), we have

$$|u_k|\leqq \frac{8}{27}<1 . \tag{2.5}$$

We next have

$$|u_k|^2\leqq 2\left(\frac{\sigma_k^4}{4s_n^4}t^4+\frac{4(\rho_k(g))^2}{s_n^4(g(s_n/|t|))^2}t^4\right)$$

which is

$$\leqq 2\left(\frac{\sigma_k^4}{4s_n^4}t^4+\frac{4(\rho_k(g))^2}{s_n^4(g(s_n))^2}t^6\right) , \quad \text{if} \quad |t|\geqq 1 ,$$

and is

$$\leqq 2\left(\frac{\sigma_k^4}{4s_n^4}t^4+\frac{4(\rho_k(g))^2}{s_n^4(g(s_n))^2}t^4\right) , \quad \text{if} \quad |t|<1 .$$

Hence we have

$$\sum_{k=1}^{n}|u_k|^2 \leqq \sum_{k=1}^{n} \frac{1}{s_n^2 g(s_n)}|t|^3\Big(\frac{\sigma_k^4 g(s_n)}{2s_n^2}T + \frac{8(\rho_k(g))^2}{s_n^2 g(s_n)}T^3\Big)$$
$$\leqq \frac{|t|^3}{s_n^2 g(s_n)} \sum_{k=1}^{n}\left(\frac{\sigma_k^4 g(s_n)}{6s_n^2}\Big(\frac{s_n^2 g(s_n)}{\rho_k(g)}\Big)^{1/3} + \frac{8(\rho_k(g))^2}{27 s_n^2 g(s_n)}\cdot\frac{s_n^2 g(s_n)}{\rho_k(g)}\right)$$
$$\leqq \frac{|t|^3}{s_n^2 g(s_n)} \sum_{k=1}^{n}\left(\frac{\sigma_k^4 (g(s_n))^{4/3}}{6 s_n^{4/3}(\rho_k(g))^{4/3}}\rho_k(g) + \frac{8}{27}\rho_k(g)\right).$$

Here using that $g(s_n)/s_n \leqq g(\sigma_k)/\sigma_k$ and Lemma 1 again, we have

$$\frac{\sigma_k^4 (g(s_n))^{4/3}}{s_k^{4/3}(\rho_k(g))^{4/3}} \leqq \frac{\sigma_k^4 (g(s_n))^{4/3}}{s_n^{4/3}(\lambda_k(g))^{4/3}} \leqq \frac{\sigma_k^{8/3}(g(\sigma_k))^{4/3}}{(\lambda_k(g))^{4/3}}$$
$$\leqq \Big(\frac{8}{3}\Big)^{4/3} \leqq \frac{16}{3}.$$

Therefore,

$$\sum_{k=1}^{n}|u_k|^2 \leqq \frac{|t|^3}{s_n^2 g(s_n)} \sum_{k=1}^{n}\Big(\frac{1}{6}\cdot\frac{16}{3} + \frac{8}{27}\Big)\rho_k(g) \tag{2.6}$$
$$\leqq \frac{32}{27}\frac{\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n)}|t|^3.$$

On the other hand, making use of (2.1) and (2.5), we have

$$\log f_k(t/s_n) = -u_k + \frac{1}{2}\theta\frac{|u_k|^2}{1-|u_k|} = -u_k + \frac{27}{38}\theta|u_k|^2, \tag{2.7}$$

so that from (2.2), (2.6) and (2.7),

$$\log \bar{f}_n(t) = \sum_{k=1}^{n}\log f_k(t/s_n)$$
$$= -\frac{1}{2}t^2 + 2\theta\frac{\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n/|t|)}t^2 + \frac{16}{19}\theta\frac{\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n)}|t|^3$$
$$\equiv -\frac{1}{2}t^2 + A(t),$$

say, where

$$|A(t)| \leqq \frac{54}{19}\frac{\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n)}|t|^3, \quad \text{if} \quad |t| \geqq 1, \tag{2.8}$$

$$\leqq \frac{54}{19}\frac{\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n)}t^2, \quad \text{if} \quad |t| < 1, \tag{2.9}$$

and further

$$|A(t)|\leqq\frac{54}{19}T_n T^3=\frac{2}{19} \tag{2.10}$$

for $|t|\leqq T$. Using the inequality $|e^z-1|\leqq|z|e^{|z|}$, we finally have from (2.8)-(2.10),

$$\begin{aligned}|\bar{f}_n(t)-e^{-t^2/2}|&\leqq e^{-t^2/2}|e^{A(t)}-1|\leqq e^{-t^2/2}e^{2/19}|A(t)|\\&\leqq C(t^2+|t|^3)e^{-t^2/2}T_n\end{aligned}$$

for $|t|\leqq T$, and the lemma is thus proved.

Now, let us return to the proof of the theorem. The condition (b) implies the integrability of $p_{k_1}(x)p_{k_2}(x)$ for any $1\leqq k_1\neq k_2\leqq n$, which gives us the integrability of $f_{k_1}(t)f_{k_2}(t)$ by Parseval identity. Therefore $\bar{f}_n(t)$ is integrable for $n\geqq 2$, and so we have

$$\bar{p}_n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\bar{f}_n(t)dt\ ,\quad n\geqq 2\ ,$$

and hence

$$\bar{p}_n(x)-\phi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}(\bar{f}_n(t)-e^{-t^2/2})dt\ ,\quad n\geqq 2\ .$$

We have

$$\begin{aligned}&\sup_x|\bar{p}_n(x)-\phi(x)|\\&\leqq\int_{|t|\leqq T}|\bar{f}_n(t)-e^{-t^2/2}|dt+\int_{|t|>T}|\bar{f}_n(t)|dt+\int_{|t|>T}e^{-t^2/2}dt\\&\equiv I_1+I_2+I_3\ ,\end{aligned} \tag{2.11}$$

say. It follows from Lemma 2 that

$$I_1\leqq CT_n\int_{|t|\leqq T}(t^2+|t|^3)e^{-t^2/2}dt\leqq CT_n\ . \tag{2.12}$$

As to I_3,

$$I_3\leqq\frac{1}{T^3}\int_{|t|>T}|t|^3e^{-t^2/2}dt\leqq CT_n\ , \tag{2.13}$$

since $T^3=(1/27)T_n^{-1}$. Finally, it remains to estimate I_2. In order to do it, we need the following result given by Survila [4].

LEMMA 3 ([4]). *Under the conditions* (a) *and* (b), *we have*

$$|\bar{f}_n(t)| \leqq \begin{cases} |f_{k_1}(t/s_n) f_{k_2}(t/s_n)| \exp\{-cn\}\,, & for \quad |t| \geqq \dfrac{\pi}{\sqrt{2}}\sqrt{n}\,, \\ \exp\{-ct^2\}\,, & for \quad |t| < \dfrac{\pi}{\sqrt{2}}\sqrt{n}\,. \end{cases}$$

Using this lemma, we have

$$\begin{aligned} I_2 &\leqq \int_{T<|t|<\pi\sqrt{n}/\sqrt{2}} |\bar{f}_n(t)|\,dt + \int_{|t|\geqq\pi\sqrt{n}/\sqrt{2}} |\bar{f}_n(t)|\,dt \\ &\leqq \int_{T<|t|<\pi\sqrt{n}/\sqrt{2}} e^{-ct^2}dt \\ &\quad + \int_{|t|\geqq\pi\sqrt{n}/\sqrt{2}} |f_{k_1}(t/s_n) f_{k_2}(t/s_n)|\, e^{-cn}dt \\ &\leqq \frac{C}{T^3} + C s_n e^{-cn}\,, \end{aligned}$$

because $f_{k_1}(t)f_{k_2}(t)$ is integrable. Since $T^3=(1/27)T_n^{-1}$ and $s_n^2 \leqq Kn$, we have

$$I_2 \leqq CT_n + C\sqrt{n}\, e^{-cn}\,.$$

On the other hand, since $\sigma_1^2>0$, there exists $z_0>0$ such that $\int_{|x|>z_0} x^2 dF_1(x)>0$. Therefore we have

$$\begin{aligned} T_n &= \frac{\sum_{k=1}^{n} \rho_k(g)}{s_n^2 g(s_n)} \geqq \frac{\sup_{z>0} g(z) \int_{|x|>z} x^2 dF_1(x)}{Cn^{3/2}} \\ &\geqq \frac{g(z_0) \int_{|x|>z_0} x^2 dF_1(x)}{Cn^{3/2}} = \frac{C}{n^{3/2}}\,, \end{aligned} \tag{2.14}$$

which implies $\sqrt{n}\, e^{-cn} \leqq CT_n$. Thus we have

$$I_2 \leqq CT_n\,. \tag{2.15}$$

Hence, (2.11), (2.12), (2.13) and (2.15) yield the conclusion of the theorem.

§3. A nonuniform estimate.

We state the following theorem concerning nonuniform convergence of $|\bar{p}_n(x)-\phi(x)|$.

THEOREM 2. *Let $g(z) \in G$. Under the same conditions as in Theorem 1, we have*

$$\sup_x (1+|x|)^2 |\bar{p}_n(x)-\phi(x)| \leqq CT_n .$$

Four corollaries mentioned in §1 also hold for this nonuniform estimate, that is, we can replace $\sup_x |\bar{p}_n(x)-\phi(x)|$ by $\sup_x (1+|x|)^2|\bar{p}_n(x)-\phi(x)|$ in those corollaries.

We shall prove this theorem in the next section.

§4. Proof of Theorem 2.

First of all, we consider two independent random variables X and X' with mean 0 and finite variance σ^2 ($\sigma>0$), and with the same distribution function $F(x)$. Denote by $F^s(x)$ the distribution function of $X-X'$, and define $R(z)$, $Q(z)$ and $R^s(z)$ for $F(x)$ and $F^s(x)$ by the similar way as in defining $R_k(z)$ and $Q_k(z)$ for $F_k(x)$. Let $f(t)$ be the characteristic function of X. Then we may express

$$f'(t) = -t(\sigma^2+\gamma_1(t)) , \tag{4.1}$$

$$f''(t) = -\sigma^2+\gamma_2(t) , \tag{4.2}$$

where $\lim_{t\to 0} |\gamma_i(t)|=0$, $i=1, 2$.

We show some lemmas.

Lemma 4 ([1]). *For all z, $R^s(z)\leqq 40R(z/2)$.*

Lemma 5. *Let $g(z)\in G$, and suppose that $\lambda(g)=\sup_{z>0} g(z)R(z)<\infty$ and $\mu(g)=\sup_{z>0} g(z)Q(z)/z<\infty$. Then*

$$\int_{|u|>1/|t|} u^2 dF(u) \leqq \frac{\lambda(g)}{g(1/|t|)} , \tag{4.3}$$

$$\int_{|u|\leqq 1/|t|} u^4 dF(u) \leqq \frac{2\lambda(g)}{t^2 g(1/|t|)} \tag{4.4}$$

and

$$\left|\int_{|u|\leqq 1/|t|} u^3 dF(u)\right| \leqq \frac{\mu(g)}{|t| g(1/|t|)} . \tag{4.5}$$

Proof. (4.3) and (4.5) are readily shown from the definition of $\lambda(g)$ and $\mu(g)$, and (4.4) has been shown implicitly in the beginning part of the proof of Lemma 2 in §2.

Lemma 6. *Let $g(z)\in G$ and suppose that $\rho(g)=\lambda(g)+\mu(g)<\infty$. Then*

$$|\gamma_i(t)| \leqq \frac{C\rho(g)}{g(1/|t|)} , \quad i=1, 2 ,$$

where $\gamma_i(t)$ *are the ones defined in* (4.1) *and* (4.2).

PROOF. We have

$$
\begin{aligned}
|\gamma_1(t)| &= \frac{1}{|t|}|\sigma^2 t + f'(t)| \\
&= \frac{1}{|t|}\left|\int iu(e^{itu}-1-itu)dF(u)\right| \\
&\leqq \frac{2}{|t|}\int_{|u|>1/|t|}|t|u^2 dF(u) \\
&\quad + \frac{1}{|t|}\left|\int_{|u|\leqq 1/|t|} u\left(e^{itu}-1-itu+\frac{1}{2}t^2u^2\right)dF(u)\right| \\
&\quad + \frac{1}{2|t|}\left|\int_{|u|\leqq 1/|t|} t^2u^3 dF(u)\right| \\
&\leqq 2\int_{|u|>1/|t|} u^2 dF(u) + \frac{1}{6}t^2\int_{|u|\leqq 1/|t|} u^4 dF(u) \\
&\quad + \frac{1}{2}|t|\left|\int_{|u|\leqq 1/|t|} u^3 dF(u)\right| \\
&\leqq \frac{C\rho(g)}{g(1/|t|)},
\end{aligned}
$$

because of Lemma 5. Further we have

$$
\begin{aligned}
|\gamma_2(t)| &= |\sigma^2 + f''(t)| \\
&= \left|\int u^2(e^{itu}-1)dF(u)\right| \\
&\leqq 2\int_{|u|>1/|t|} u^2 dF(u) \\
&\quad + \left|\int_{|u|\leqq 1/|t|} u^2(e^{itu}-1-itu)dF(u)\right| \\
&\quad + \left|\int_{|u|\leqq 1/|t|} itu^3 dF(u)\right| \\
&\leqq 2\int_{|u|>1/|t|} u^2 dF(u) + t^2\int_{|u|\leqq 1/|t|} u^4 dF(u) \\
&\quad + |t|\left|\int_{|u|\leqq 1/|t|} u^3 dF(u)\right| \\
&\leqq \frac{C\rho(g)}{g(1/|t|)},
\end{aligned}
$$

which completes the proof.

The following lemma is a slight modification of Lemma 3 in [1].

LEMMA 7. *Under the same assumptions as in Lemma* 1,

$$|f(t)|^2 \leqq 1-\sigma^2 t^2 + \frac{47\lambda(g)}{g(1/|t|)} t^2 .$$

PROOF. We have

$$(4.6) \qquad |f(t)|^2 = 1-\sigma^2 t^2 + \int\left(\cos tu - 1 + \frac{1}{2}t^2u^2\right) dF^s(u)$$

$$= 1-\sigma^2 t^2 + \left(\int_{|ut|<1} + \int_{|ut|\geqq 1}\right)\left(\cos tu - 1 + \frac{1}{2}t^2u^2\right) dF^s(u)$$

$$\leqq 1-\sigma^2 t^2 + \frac{1}{24}t^4 \int_{|ut|<1} u^4 dF^s(u) + \frac{1}{2}t^2 \int_{|ut|\geqq 1} u^2 dF^s(u) .$$

From Lemma 4 and (4.3),

$$(4.7) \qquad t^2 \int_{|ut|\geqq 1} u^2 dF^s(u) \leqq \frac{80\lambda(g)}{g(1/|t|)} t^2 .$$

Furthermore

$$\int_{|ut|<1} u^4 dF^s(u) = \int_0^{1/|t|} u^2 d(-R^s(u)) \leqq 2 \int_0^{1/|t|} uR^s(u) du$$

$$\leqq 80 \int_0^{1/|t|} uR\left(\frac{u}{2}\right) du$$

from Lemma 4. Since $R(z) \leqq \lambda(g)/g(z)$, we have

$$(4.8) \qquad \int_{|ut|<1} u^4 dF^s(u) \leqq 80 \int_0^{1/|t|} \frac{u\lambda(g)}{g(u/2)} du \leqq \frac{80\lambda(g)}{|t| g(1/(2|t|))} \int_0^{1/|t|} du$$

$$= \frac{160\lambda(g)}{t^2 g(1/|t|)} .$$

The estimates (4.6)-(4.8) give us the desired inequality.

Now, noting that $f_k'(t)$ and $f_k''(t)$ exist, we have

$$(4.9) \qquad \bar{f}_n''(t) = \frac{1}{s_n^2}\left\{\sum_{j=1}^n f_j''\left(\frac{t}{s_n}\right) \prod_{\substack{k=1\\k\neq j}}^n f_k\left(\frac{t}{s_n}\right)\right.$$

$$\left. + \sum_{i=1}^n f_i'\left(\frac{t}{s_n}\right) \sum_{\substack{j=1\\j\neq i}}^n f_i'\left(\frac{t}{s_n}\right) \prod_{\substack{k=1\\k\neq j\neq i}}^n f_k\left(\frac{t}{s_n}\right)\right\} .$$

Then we have the following lemma.

LEMMA 8 ([4]). *Under the conditions* (a) *and* (b), *we have for* $1 \leqq k_1 \neq k_2 \leqq n$,

$$|\bar{f}_n''(t)| \leqq \begin{cases} \left| f_{k_1}\left(\frac{t}{s_n}\right) f_{k_2}\left(\frac{t}{s_n}\right) \right| (n+1) \exp\{-cn\}, & \textit{for} \quad |t| \geqq \frac{\pi}{\sqrt{2}}\sqrt{n}, \\ (1+t^2)\exp\{-ct^2\}, & \textit{for} \quad |t| < \frac{\pi}{\sqrt{2}}\sqrt{n}. \end{cases}$$

In what follows, we suppose that $T_n \leqq 1/94$. In the other case, the statement of the theorem trivially holds for $C = 94C^*$, since $(1+|x|)^2 |\bar{p}_n(x) - \phi(x)| \leqq C^*$ for some positive constant C^* under our conditions (a) and (b), which was implicitly shown by Survila [4].

LEMMA 9. *For* $|t| \leqq T^{(1)} \equiv (1/94) T_n^{-1}$,

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} \left| f_k\left(\frac{t}{s_n}\right) \right| \leqq e^{-t^2/8}, \quad 1 \leqq j \leqq n$$

and

$$\prod_{\substack{k=1 \\ k \neq j \neq i}}^{n} \left| f_k\left(\frac{t}{s_n}\right) \right| \leqq e^{-t^2/8}, \quad 1 \leqq j \neq i \leqq n. \tag{4.10}$$

PROOF. It suffices to prove (4.10), because

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} \left| f_k\left(\frac{t}{s_n}\right) \right| \leqq \prod_{\substack{k=1 \\ k \neq j \neq i}}^{n} \left| f_k\left(\frac{t}{s_n}\right) \right|.$$

Note that $T^{(1)} \geqq 1$. From Lemma 7, we have

$$\begin{aligned} \left| f_k\left(\frac{t}{s_n}\right) \right|^2 &\leqq 1 - \frac{\sigma_k^2}{s_n^2} t^2 + \frac{47\lambda_k(g)}{s_n^2 g(s_n/|t|)} t^2 \\ &\leqq \exp\left\{ -\frac{\sigma_k^2}{s_n^2} t^2 + \frac{47\lambda_k(g)}{s_n^2 g(s_n/|t|)} t^2 \right\}. \end{aligned}$$

Since $\lambda_k(g) \leqq \rho_k(g)$,

$$\begin{aligned} \prod_{\substack{k=1 \\ k \neq j \neq i}}^{n} \left| f_k\left(\frac{t}{s_n}\right) \right| &\leqq \exp\left\{ -\frac{1}{2} t^2 \left(1 - \frac{\sigma_j^2 + \sigma_i^2}{s_n^2} - \frac{47 \sum_{k=1, k \neq j \neq i}^{n} \rho_k(g)}{s_n^2 g(s_n/|t|)} \right) \right\} \\ &\leqq \exp\left\{ -\frac{1}{2} t^2 \left(1 - \frac{\sigma_j^2 + \sigma_i^2}{s_n^2} - \frac{47 \sum_{k=1}^{n} \rho_k(g)}{s_n^2 g(s_n/|t|)} \right) \right\}. \end{aligned}$$

Here, for $1 \leqq |t| \leqq T^{(1)}$,

$$\frac{47\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n/|t|)} \leqq \frac{47\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n)}|t| \leqq \frac{1}{2},$$

and for $|t|<1$,

$$\frac{47\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n/|t|)} \leqq \frac{47\sum_{k=1}^{n}\rho_k(g)}{s_n^2 g(s_n)} = \frac{1}{2T^{(1)}} \leqq \frac{1}{2}.$$

On the other hand, since $T_n \leqq 1/94$, we have, using Lemma 1,

$$\begin{aligned}\frac{\sigma_k^2}{s_n^2} &\leqq \frac{\sigma_k^2(g(s_n))^{2/3}}{s_n^{2/3}(\lambda_k(g))^{2/3}} T_n^{2/3} \\ &\leqq \frac{\sigma_k^{4/3}(g(\sigma_k))^{2/3}}{(\lambda_k(g))^{2/3}} T_n^{2/3} \\ &\leqq \left(\frac{8}{3}\right)^{2/3}\left(\frac{1}{94}\right)^{2/3} < \frac{1}{9}.\end{aligned}$$

Therefore, these estimates imply (4.10).

LEMMA 10. *Under the conditions* (a) *and* (b), *we have for* $|t| \leqq T^* = \min(T^{(1)}, T^{(2)})$, $T^{(2)} \equiv (1/3)T_n^{-1/3}$,

$$|\bar{f}_n''(t) - (t^2-1)e^{-t^2/2}| \leqq C(1+|t|^5)e^{-ct^2}T_n.$$

PROOF. Note that $T^{(2)} \geqq 1$. From (4.9),

$$\begin{aligned}&|\bar{f}_n''(t) - (t^2-1)e^{-t^2/2}| \\ &\leqq \left|\frac{1}{s_n^2}\sum_{j=1}^{n} f_j''\left(\frac{t}{s_n}\right)\prod_{\substack{k=1\\k\neq j}}^{n} f_k\left(\frac{t}{s_n}\right) + e^{-t^2/2}\right| \\ &\quad + \left|\frac{1}{s_n^2}\sum_{i=1}^{n} f_i'\left(\frac{t}{s_n}\right)\sum_{\substack{j=1\\j\neq i}}^{n} f_j'\left(\frac{t}{s_n}\right)\prod_{\substack{k=1\\k\neq j\neq i}}^{n} f_k\left(\frac{t}{s_n}\right) - t^2e^{-t^2/2}\right| \\ &\equiv J_1 + J_2,\end{aligned}$$

say. We express $f_k'(t)$ and $f_k''(t)$ in the following form, respectively:

$$f_k'(t) = -t(\sigma_k^2 + \gamma_{k,1}(t)),$$

$$f_k''(t) = -\sigma_k^2 + \gamma_{k,2}(t).$$

Then we have

$$J_1 = \left|\frac{1}{s_n^2}\sum_{j=1}^{n}\left(-\sigma_j^2 + \gamma_{j,2}\left(\frac{t}{s_n}\right)\right)\prod_{\substack{k=1\\k\neq j}}^{n} f_k\left(\frac{t}{s_n}\right) + e^{-t^2/2}\right|$$

$$\leqq \left| -\frac{1}{s_n^2}\sum_{j=1}^{n}\sigma_j^2 \prod_{\substack{k=1\\k\neq j}}^{n} f_k\left(\frac{t}{s_n}\right) + e^{-t^2/2}\right|$$

$$+\frac{1}{s_n^2}\sum_{j=1}^{n}\left|\gamma_{j,2}\left(\frac{t}{s_n}\right)\right| \prod_{\substack{k=1\\k\neq j}}^{n}\left|f_k\left(\frac{t}{s_n}\right)\right|$$

$$\equiv J_{11}+J_{12}\ ,$$

say. On the other hand, similarly as in (2.7),

$$\text{(4.11)}\qquad \log \prod_{\substack{k=1\\k\neq j}}^{n} f_k\left(\frac{t}{s_n}\right)=\sum_{\substack{k=1\\k\neq j}}^{n}\log f_k\left(\frac{t}{s_n}\right)$$

$$=-\frac{t^2}{2s_n^2}(s_n^2-\sigma_j^2)+2\theta\frac{\sum_{\substack{k=1\\k\neq j}}^{n}\rho_k(g)}{s_n^2 g(s_n/|t|)}t^2+\frac{16}{19}\theta\frac{\sum_{\substack{k=1\\k\neq j}}^{n}\rho_k(g)}{s_n^2 g(s_n)}|t|^3$$

$$\equiv -\frac{t^2}{2}+\frac{\sigma_j^2}{2s_n^2}+B_j(t)\ ,$$

say, where

$$|B_j(t)|\leqq\frac{54}{19}T_n(t^2+|t|^3)$$

and

$$|B_j(t)|\leqq\frac{2}{19}$$

for $|t|\leqq T^{(2)}$. Therefore we have

$$J_{11}\leqq e^{-t^2/2}\left|1-\frac{1}{s_n^2}\sum_{j=1}^{n}\sigma_j^2\exp\left\{\frac{\sigma_j^2}{2s_n^2}+B_j(t)\right\}\right|$$

$$=e^{-t^2/2}\frac{1}{s_n^2}\sum_{j=1}^{n}\sigma_j^2\left|1-\exp\left\{\frac{\sigma_j^2}{2s_n^2}+B_j(t)\right\}\right|$$

$$\leqq e^{-t^2/2}\frac{1}{s_n^2}\sum_{j=1}^{n}\sigma_j^2\left(\frac{\sigma_j^2}{2s_n^2}+|B_j(t)|\right)\exp\left\{\frac{\sigma_j^2}{2s_n^2}+|B_j(t)|\right\}$$

$$\leqq Ce^{-t^2/2}\left(\frac{1}{2s_n^4}\sum_{j=1}^{n}\sigma_j^4+\frac{1}{s_n^2}\sum_{j=1}^{n}\sigma_j^2|B_j(t)|\right)$$

$$\leqq Ce^{-t^2/2}\left(\frac{1}{2s_n^4}\sum_{j=1}^{n}\sigma_j^4+(t^2+|t|^3)T_n\right).$$

Here we have, using Lemma 1 again,

$$\frac{1}{s_n^4}\sum_{j=1}^{n}\sigma_j^4=\frac{1}{s_n^2 g(s_n)}\sum_{j=1}^{n}\frac{\sigma_j^4 g(s_n)}{s_n^2}\leqq\frac{1}{s_n^2 g(s_n)}\sum_{j=1}^{n}\frac{\sigma_j^4 g(\sigma_j)\lambda_j(g)}{s_n\sigma_j\lambda_j(g)}$$

$$\leqq \frac{1}{s_n^2 g(s_n)} \sum_{j=1}^{n} \frac{\sigma_j^2 g(\sigma_j)\lambda_j(g)}{\lambda_j(g)} \leqq \frac{1}{s_n^2 g(s_n)} \sum_{j=1}^{n} \left(\frac{8}{3}\right)\lambda_j \leqq (g)\frac{8}{3} T_n .$$

Hence we have

$$J_{11} \leqq C e^{-t^2/2}(1+t^2+|t|^3) T_n . \tag{4.12}$$

Next, from Lemmas 6 and 9,

$$\begin{aligned} J_{12} &\leqq \frac{C}{s_n^2 g(s_n/|t|)} \sum_{j=1}^{n} \rho_j(g) \prod_{\substack{k=1\\k\neq j}}^{n} \left|f_k\left(\frac{t}{s_n}\right)\right| \\ &\leqq \frac{C \sum_{j=1}^{n} \rho_j(g)}{s_n^2 g(s_n/|t|)} e^{-ct^2} \\ &\leqq \frac{C \sum_{j=1}^{n} \rho_j(g)}{s_n^2 g(s_n)} |t| e^{-ct^2} , \quad \text{if} \quad |t| \geqq 1 \\ &\leqq \frac{C \sum_{j=1}^{n} \rho_j(g)}{s_n^2 g(s_n)} e^{-ct^2} , \quad \text{if} \quad |t| < 1 , \end{aligned} \tag{4.13}$$

so that

$$J_{12} \leqq C(1+|t|) e^{-ct^2} T_n . \tag{4.14}$$

As to J_2,

$$\begin{aligned} J_2 = \Bigg| &\frac{1}{s_n^2} \sum_{i=1}^{n} \left(-\frac{t}{s_n}\left(\sigma_i^2 + \gamma_{i,1}\left(\frac{t}{s_n}\right)\right)\right) \\ &\times \sum_{\substack{j=1\\j\neq i}}^{n} \left(-\frac{t}{s_n}\left(\sigma_j^2 + \gamma_{j,1}\left(\frac{t}{s_n}\right)\right)\right) \prod_{\substack{k=1\\k\neq j\neq i}}^{n} f_k\left(\frac{t}{s_n}\right) - t^2 e^{-t^2/2} \Bigg| \\ \leqq t^2 \Bigg| &\frac{1}{s_n^4} \sum_{i=1}^{n} \sum_{\substack{j=1\\j\neq i}}^{n} \left\{\sigma_i^2\sigma_j^2 + \sigma_i^2 \gamma_{j,1}\left(\frac{t}{s_n}\right) + \sigma_j^2 \gamma_{i,1}\left(\frac{t}{s_n}\right) \right. \\ &\left. + \gamma_{i,1}\left(\frac{t}{s_n}\right)\gamma_{j,1}\left(\frac{t}{s_n}\right)\right\} \prod_{\substack{k=1\\k\neq j\neq i}}^{n} f_k\left(\frac{t}{s_n}\right) - e^{-t^2/2} \Bigg| \\ \leqq t^2 \Bigg| &\frac{1}{s_n^4} \sum_{i=1}^{n} \sum_{\substack{j=1\\j\neq i}}^{n} \sigma_i^2\sigma_j^2 \prod_{\substack{k=1\\k\neq j\neq i}}^{n} f_k\left(\frac{t}{s_n}\right) - e^{-t^2/2} \Bigg| \\ &+ t^2 \frac{1}{s_n^4} \sum_{i=1}^{n} \sum_{\substack{j=1\\j\neq i}}^{n} \sigma_i^2 \left|\gamma_{j,1}\left(\frac{t}{s_n}\right)\right| \prod_{\substack{k=1\\k\neq j\neq i}}^{n} \left|f_k\left(\frac{t}{s_n}\right)\right| \\ &+ t^2 \frac{1}{s_n^4} \sum_{i=1}^{n} \sum_{\substack{j=1\\j\neq i}}^{n} \sigma_j^2 \left|\gamma_{i,1}\left(\frac{t}{s_n}\right)\right| \prod_{\substack{k=1\\k\neq j\neq i}}^{n} \left|f_k\left(\frac{t}{s_n}\right)\right| \\ &+ t^2 \frac{1}{s_n^4} \sum_{i=1}^{n} \sum_{\substack{j=1\\j\neq i}}^{n} \left|\gamma_{i,1}\left(\frac{t}{s_n}\right)\right| \left|\gamma_{j,1}\left(\frac{t}{s_n}\right)\right| \prod_{\substack{k=1\\k\neq j\neq i}}^{n} \left|f_k\left(\frac{t}{s_n}\right)\right| \end{aligned}$$

$$\equiv J_{21}+J_{22}+J_{23}+J_{24} ,$$

say. Similarly as in (4.11),

$$\log \prod_{\substack{k=1\\k\neq j\neq i}}^{n} f_k\left(\frac{t}{s_n}\right)=-\frac{t^2}{2}+\frac{\sigma_i^2+\sigma_j^2}{2s_n^2}+D_{ji}(t) ,$$

where

$$D_{ji}(t)=2\theta\frac{\sum_{k=1,\,k\neq j\neq i}^{n}\rho_k(g)}{s_n^2 g(s_n/|t|)}t^2+\frac{16}{19}\theta\frac{\sum_{k=1,\,k\neq j\neq i}^{n}\rho_k(g)}{s_n^2 g(s_n)}|t|^3 ,$$

and

$$|D_{ji}(t)|\leqq\frac{54}{17}T_n(t^2+|t|^3) , \quad |D_{ji}(t)|\leqq\frac{2}{19} .$$

Hence we have

$$\begin{aligned}(4.15)\qquad J_{21}&\leqq e^{-t^2/2}t^2\frac{1}{s_n^4}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sigma_i^2\sigma_j^2\left|1-\exp\left\{\frac{\sigma_i^2+\sigma_j^2}{2s_n^2}+D_{ji}(t)\right\}\right|+e^{-t^2/2}t^2\frac{1}{s_n^4}\sum_{i=1}^{n}\sigma_i^4\\
&\leqq Ce^{-t^2/2}t^2\left(\frac{1}{2s_n^6}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i^2\sigma_j^2(\sigma_i^2+\sigma_j^2)+\frac{1}{s_n^4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i^2\sigma_j^2|D_{ji}(t)|+\frac{1}{s_n^4}\sum_{i=1}^{n}\sigma_i^4\right)\\
&\leqq Ce^{-t^2/2}t^2\left(\frac{1}{s_n^4}\sum_{j=1}^{n}\sigma_j^4+\frac{1}{s_n^4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i^2\sigma_j^2|D_{ji}(t)|\right)\\
&\leqq Ce^{-t^2/2}t^2(1+t^2+|t|^3)T_n .\end{aligned}$$

Furthermore, by a way similar to one when we have had (4.13), we have still from Lemmas 6 and 9,

$$\begin{aligned}(4.16)\qquad J_{22}&\leqq Ct^2\frac{\sum_{j=1}^{n}\rho_j(g)}{s_n^2 g(s_n/|t|)}e^{-ct^2}\\
&\leqq C(t^2+|t|^3)\frac{\sum_{j=1}^{n}\rho_j(g)}{s_n^2 g(s_n)}e^{-ct^2}\\
&\leqq C(t^2+|t|^3)e^{-ct^2}T_n ,\end{aligned}$$

and similarly

$$(4.17)\qquad J_{23}\leqq C(t^2+|t|^3)e^{-ct^2}T_n .$$

Finally,

$$J_{24}\leqq Ct^2\frac{1}{s_n^4}\sum_{j=1}^{n}\left|\gamma_{j,1}\left(\frac{t}{s_n}\right)\right|^2 e^{-ct^2}$$

$$\leqq Ct^2\frac{1}{s_n^4}\sum_{j=1}^{n}\left(\frac{\rho_j(g)}{g(s_n/|t|)}\right)^2 e^{-ct^2}$$

$$\leqq Ce^{-ct^2}(t^2+t^4)\sum_{j=1}^{n}\frac{(\rho_j(g))^2}{s_n^4(g(s_n))^2}\,,$$

where

$$\frac{\sum_{j=1}^{n}(\rho_j(g))^2}{s_n^4(g(s_n))^2}\leqq T_n^2\leqq\frac{1}{94}T_n\,.$$

Therefore,

$$J_{24}\leqq C(t^2+t^4)e^{-ct^2}T_n\,. \tag{4.18}$$

Thus, (4.12), (4.14), (4.15), (4.16), (4.17) and (4.18) imply the conclusion of the lemma.

Now we proceed to the proof of Theorem 2. We have shown in Theorem 1 that $\sup_x|\bar{p}_n(x)-\phi(x)|\leqq CT_n$, so that it suffices to prove

$$\sup_x x^2|\bar{p}_n(x)-\phi(x)|\leqq CT_n\,. \tag{4.19}$$

Recalling the form (4.9), we see from the condition (b) that $\bar{f}_n''(t)$ is integrable for $n\geqq 4$, and that

$$\sup_x x^2|\bar{p}_n(x)-\phi(x)|\leqq\int|\bar{f}_n''(t)-(t^2-1)e^{-t^2/2}|\,dt\,. \tag{4.20}$$

(For $n\leqq 3$, the statement of the theorem holds trivially, because $(1+|x|)^2|\bar{p}_n(x)-\phi(x)|$ is bounded.)

From (4.20), we have

$$\begin{aligned}&\sup_x x^2|\bar{p}_n(x)-\phi(x)| \\ &\leqq\int_{|t|\leqq T^*}|\bar{f}_n''(t)-(t^2-1)e^{-t^2/2}|\,dt+\int_{|t|>T^*}|\bar{f}_n''(t)|\,dt \\ &\quad+\int_{|t|>T^*}(1+t^2)e^{-t^2/2}dt \\ &\equiv K_1+K_2+K_3\,,\end{aligned} \tag{4.21}$$

say. It follows from Lemma 10 that

$$K_1\leqq CT_n\,. \tag{4.22}$$

As to K_2, using Lemma 8, we have

$$
\begin{aligned}
K_2 &= \int_{T^*<|t|<\pi\sqrt{n}/\sqrt{2}} |\bar{f}_n''(t)|\,dt + \int_{|t|\geqq \pi\sqrt{n}/\sqrt{2}} |\bar{f}_n''(t)|\,dt \\
&\leqq \int_{T^{(1)}<|t|<\pi\sqrt{n}/\sqrt{2}} |\bar{f}_n''(t)|\,dt + \int_{T^{(2)}<|t|<\pi\sqrt{n}/\sqrt{2}} |\bar{f}_n''(t)|\,dt \\
&\quad + \int_{|t|\geqq \pi\sqrt{n}/\sqrt{2}} |\bar{f}_n''(t)|\,dt \\
&\leqq \int_{T^{(1)}<|t|<\pi\sqrt{n}/\sqrt{2}} (1+t^2)e^{-ct^2}dt \\
&\quad + \int_{T^{(2)}<|t|<\pi\sqrt{n}/\sqrt{2}} (1+t^2)e^{-ct^2}dt \\
&\quad + \int_{|t|\geqq \pi\sqrt{n}/\sqrt{2}} \left|f_{k_1}\!\left(\frac{t}{s_n}\right)f_{k_2}\!\left(\frac{t}{s_n}\right)\right|(n+1)e^{-cn}dt \\
&\leqq \frac{C}{T^{(1)}} + \frac{C}{(T^{(2)})^3} + Cn^{3/2}e^{-cn}\,,
\end{aligned}
$$

since $f_{k_1}(t)f_{k_2}(t)$ is integrable by the condition (b). Noting that $T^{(1)}=(1/94)T_n^{-1}$ and $(T^{(2)})^3=(1/27)T_n^{-1}$, and using $T_n \geqq Cn^{-3/2}$ which has been shown in (2.14), we have

$$K_2 \leqq CT_n\,. \tag{4.23}$$

Similarly, we have

$$
\begin{aligned}
K_3 &\leqq \int_{|t|>T^{(1)}} (1+t^2)e^{-t^2/2}dt + \int_{|t|>T^{(2)}} (1+t^2)e^{-t^2/2}dt \\
&\leqq \frac{C}{T^{(1)}} + \frac{C}{(T^{(2)})^3} \leqq CT_n\,,
\end{aligned}
\tag{4.24}
$$

so that (4.21)-(4.24) conclude (4.19) and therefore the theorem.

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