TOKYO J. MATH.
VOL. 3, NO. 2, 1980

Weak-* Maximality of Certain Hardy Algebras $H^{\infty}(m)$

SHÛICHI OHNO

Waseda University

(Communicated by J. Wada)

The purpose of this paper is to discuss the weak-*maximality of certain Hardy algebras $H^{\infty}(m)$. Merrill [7] obtained conditions for the maximality of Hardy algebras for logmodular algebras. In this paper we study this problem for hypo-Dirichlet algebras and obtain a similar result as one of Merrill. We also discuss as an application the(uniform) maximality of certain classes of hypo-Dirichlet algebras.

§1. Preliminaries.

Let A be a *uniform algebra* on a compact Hausdorff space X, i.e., let A be a closed subalgebra in $C(X)$ separating points in X and containing constant functions on X, where $C(X)$ denotes the Banach algebra of complex-valued continuous functions on X with the supremum norm. A is called a *hypo-Dirichlet algebra* on X if there exist finite elements $Z_1, Z_2, \cdots, Z_\sigma$ in the family A^{-1} of invertible elements of A such that the real linear space of functions of the form of

$$\mathrm{Re}\,(f)+\sum_{i=1}^{\sigma} c_i \log|Z_i| \quad (f\in A,\ c_i\in \boldsymbol{R})$$

is dense in the space $C_R(X)$ of real continuous functions on X.

Now let A be a hypo-Dirichlet algebra and M_A be the maximal ideal space of A. Then each element ϕ of M_A has a finite dimensional set M_ϕ of representing measures on X for ϕ. And every $\phi\in M_A$ has a unique Arens-Singer measure m on X. A positive measure m on X is called an *Arens-Singer measure* for ϕ if $\log|\phi(f)|=\int\log|f|\,dm$ for all $f\in A^{-1}$ ([1]; [4], p. 116).

The abstract Hardy spaces $H^p(m)$, $1\leqq p\leqq\infty$, associated with A are defined as follows; for $1\leqq p<\infty$, $H^p(m)$ is the $L^p(m)$-closure of A and $H^{\infty}(m)$ is the weak-*closure of A in $L^{\infty}(m)$. We see that $H^{\infty}(m)$ is an

Received February 7, 1980

algebra. For $1 \leqq p \leqq \infty$, $H_0^p(m)=\left\{f \in H^p(m): \int f dm=0\right\}$. Let N^p be the real annihilator of A in $L_R^p(m)$ $(1 \leqq p \leqq \infty)$ and N_c^p be the complexification of N^p. Then we have the following ([4], p. 109).

$$N^1=N^p=N^\infty ,$$
$$H^\infty(m)=H^p(m) \cap L^\infty(m) \qquad (1 \leqq p<\infty) ,$$

and

$$L^p(m)=H^p(m) \oplus \overline{H_0^p(m)} \oplus N_c^\infty \qquad (1<p<\infty) .$$

Let P be a Gleason part of M_A containing ϕ. When ϕ has a unique Arens-Singer measure m (where ϕ has not a unique representing measure), it is known that P is non-trivial, i.e., P is not a singleton ([1], Theorem 12.2). Though ϕ can be extended to $H^\infty(m)$, we shall denote the extended one by ϕ again whenever no confusion arises. Let $\tilde{P}$ be the Gleason part of ϕ in $M_{H^\infty(m)}$, the maximal ideal space of $H^\infty(m)$. Then $\tilde{P}=\left\{\tilde{\psi}: f(\tilde{\psi})=\int f d\psi,\ d\psi \text{ is a representing measure for } \psi \in P \text{ and } f \in H^\infty(m)\right\}$. The space $\tilde{P}$, endowed with the induced topology of $M_{H^\infty(m)}$, can be compactified by adding a boundary Γ so that $\tilde{P} \cup \Gamma$ can be given the structure of a finite compact bordered Riemann surface and the functions in $H^\infty(m)$ are analytic on $\tilde{P}$. There is a natural isometric embedding of the algebra $H^\infty(\tilde{P})$ of bounded analytic functions on $\tilde{P}$ into $H^\infty(m)$ so that $H^\infty(m)$ is the direct sum of $H^\infty(\tilde{P})$ and the ideal I of functions in $H^\infty(m)$ which vanish identically on $\tilde{P}$ ([4], p. 161; [6]).

A closed (weak-* closed for $p=\infty$) subspace M of $L^p(m)$ $(1 \leqq p \leqq \infty)$ is called *invariant* if $f \in A$ and $g \in M$ imply that $fg \in M$. Ahern and Sarason [1] said that an invariant subspace M of $L^p(m)$ is *of type B* if $A_0 M$ is not dense in M (for $p=\infty$, not weak-*dense), where A_0 is the kernel of the functional ϕ. And they offered the conjecture whether every invariant subspace of $L^p(m)$ of type B is of the form $wH^p(m)$, where w is a function in $L^\infty(m)$ that agrees in modulus almost everywhere with $|Z_1|^{\alpha_1} \cdots |Z_\sigma|^{\alpha_\sigma}$ for some real numbers $\alpha_1, \cdots, \alpha_\sigma$. They called such a function w a *rigid function* and such a subspace $wH^p(m)$ a *Beurling subspace*. For example, if the invariant subspace M of $L^p(m)$ is generated by f such that $\log|f|$ is summable, it is known that M is of type B and so a Beurling subspace ([1], Lemma 11.1). In general, they proved that if the subspace $H_\psi^p(m)$ is a Beurling subspace for every ψ in P, then every invariant subspace of $L^p(m)$ of type B is a Beurling subspace ([1], Theorem 13.1) and Gamelin answered that $H_\psi^p(m)$ is a Beurling subspace for every ψ in P ([6], Theorem 8.6).

§2. Weak-*maximality of $H^\infty(m)$.

We need the following theorem, essentially due to Gamelin (cf. [4], p. 177, Lemma 8.1; [5]), in order to prove our main theorem.

THEOREM 2.1. *Let A be a hypo-Dirichlet algebra on a compact Hausdorff space X and m be a unique Arens-Singer measure on X for $\phi \in P$, a non-trivial Gleason part of M_A. Then the following properties are equivalent*:

(i) *$H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$;*

(ii) *If $f \in L^1(m)$, $f \neq 0$, $h \in L^\infty(m)$ and $fh^n \in H^1(m)$ for $n=0, 1, 2, \cdots$, then $h \in H^\infty(m)$;*

(iii) *If $f \in L^1(m)$, $f \neq 0$, $h \in L^\infty(m)$ and $fh^n \in H^1(m) + N_c^\infty$ for $n=0, 1, 2, \cdots$, then $h \in H^\infty(m)$;*

(iv) *If M is a non-zero closed invariant subspace of $L^1(m)$ which can not be reduced to the form $\chi_E L^1(m)$, χ_E the characteristic function of a set E, and if $h \in L^\infty(m)$ satisfies $hM \subset M$, then $h \in H^\infty(m)$.*

PROOF. (i) $\Rightarrow$ (iv). Let M be a non-zero closed invariant subspace in $L^1(m)$ and B be the family of $f \in L^\infty(m)$ with $fM \subset M$. Then B is a weak-* closed subalgebra of $L^\infty(m)$ containing $H^\infty(m)$. By (i), $B=H^\infty(m)$ or $B=L^\infty(m)$. If $B=L^\infty(m)$, M must be the form $\chi_E L^1(m)$. This contradicts the assumption of (iv). Hence $B=H^\infty(m)$. From this, if $h \in L^\infty(m)$ satisfies $hM \subset M$, then $h \in H^\infty(m)$.

(iv) $\Rightarrow$ (iii). Assume (iv) and if M is the closed invariant subspace in $L^1(m)$ generated by fh^n, $n=0, 1, 2, \cdots$, then M satisfies the assumption of (iv). Indeed, if M is of the form $\chi_E L^1(m)$, then $\chi_E L^1(m) = M \subset H^1(m) + N_c^\infty$. Since $H^1(m) + N_c^\infty$ is an invariant subspace of type B, $H^1(m) + N_c^\infty = wH^1(m)$ for a rigid function w. So $w^{-1}\chi_E L^1(m) \subset H^1(m)$, and hence $\chi_E \in H^1(m)$ since $w \in L^1(m)$. This contradicts the antisymmetric property of $H^1(m)$. So $h \in H^\infty(m)$ by (iv).

(iii) $\Rightarrow$ (ii). It is clear because $H^1(m) \subset H^1(m) + N_c^\infty$.

(ii) $\Rightarrow$ (i). Let $h \in L^\infty(m)$ and $h \notin H^\infty(m)$. Let B denote the weak-* closed subalgebra generated by $H^\infty(m)$ and h. Then B is a weak-*closed subalgebra of $L^\infty(m)$ and contains $H^\infty(m)$ properly. We prove only that $B=L^\infty(m)$. If $f \in L^1(m)$ is orthogonal to B, fh^n is orthogonal to A for $n=0, 1, 2, \cdots$. In particular, $fh^n \perp A_0$. So $fh^n \in H^1(m) + N_c^\infty$ ($n=0, 1, 2, \cdots$) ([1], Theorem 11.1). Since $H^1(m) + N_c^\infty$ is of the form $wH^1(m)$ for a rigid function w, $w^{-1}fh^n \in H^1(m)$ ($n=0, 1, 2, \cdots$). By (ii) and the fact that $h \notin H^\infty(m)$, $w^{-1}f=0$, and hence $f=0$. It follows that $B=L^\infty(m)$.

The implication (ii) $\Rightarrow$ (i) of the theorem above is due to Dr. T.

Nakazi. We are now in a position to give our main theorem. This is an analogue of results of Merrill ([7], Theorems 1 and 2) in the case when A is a hypo-Dirichlet algebra.

THEOREM 2.2. *Let A be a hypo-Dirichlet algebra on a compact Hausdorff space X and m be a unique Arens-Singer measure on X for $\phi\in M_A$. Suppose that $\tilde{P}$ is the (non-trivial) Gleason part of ϕ in $M_{H^\infty(m)}$. Then the following properties are equivalent*:

(i) *$H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$;*

(ii) *If $f\in H^\infty(m)$ vanishes on $\tilde{P}$, then $f=0$;*

(iii) *Each non-zero invariant subspace M in $H^2(m)$ is of the form*

$$M=wH^2(m)\ ,$$

where w is a rigid function in $H^\infty(m)$.

PROOF. (i)$\Rightarrow$(ii). Let Γ be the ideal boundary of $\tilde{P}$. Then we can regard $C_R(\Gamma)$ as a subspace of $L_R^\infty(m)$. Let I be the ideal of functions in $H^\infty(m)$ which vanish on $\tilde{P}$. Then since $C_R(\Gamma)I\subset I$, we have $L_R^\infty(\Gamma)I\subset I$, where $L_R^\infty(\Gamma)$ denotes the weak-*closure of $C_R(\Gamma)$ in $L_R^\infty(m)$ ([6]). Suppose now that (ii) is not true, then $I\neq\{0\}$. If M is the $L^1(m)$-closure of I, then M is a non-zero invariant subspace of $L^1(m)$. And M is not reduced to the form $\chi_E L^1(m)$. This is because of the antisymmetric property of $H^1(m)$. Now we have a $\chi_E\in L_R^\infty(\Gamma)$ such that $\chi_E I\subset I$ and $0<\chi_E<1$. Hence $\chi_E M\subset M$. But $\chi_E\in L^\infty(m)$ and $\chi_E\notin H^\infty(m)$. Thus, by Theorem 2.1 (i)$\Rightarrow$(iv), $H^\infty(m)$ is not maximal.

(ii)$\Rightarrow$(iii). Let M be a non-zero invariant subspace in $H^2(m)$ and let $M'=M\cap L^\infty(m)$. In order to show that M has the form $wH^2(m)$ for a rigid function w, we show only that M' is of the form $wH^\infty(m)$ since the closure of $M\cap L^\infty(m)$ in $L^2(m)$ is M ([4], p. 131, Theorem 6.1). By (ii) we have $H^\infty(m)=H^\infty(\tilde{P})$ and so $M'\subset H^\infty(\tilde{P})$.

Case I. If M' contains a function which does not vanish at ϕ, then M' is of type B and so is of the form $wH^\infty(m)$.

Case II. Suppose that all functions in M' vanish at ϕ. They have a zero of a finite order at ϕ. Let k be the smallest positive integer among their orders at ϕ. Let g be a function in $H^\infty(\tilde{P})$ which has a simple zero at ϕ and vanishes nowhere else on $\tilde{P}\cup\Gamma$ and $\log|g|$ be summable (for example, [6], p. 139). Then $g^{-k}M'$ is a non-zero invariant subspace containing a function which does not vanish at ϕ. Hence $g^{-k}M'$ is of the form $wH^\infty(m)$, and so $M'=wg^kH^\infty(m)$. Consequently, as M' is generated by a function wg^k in $H^\infty(m)$ such that $\log|wg^k|$ is summable, M' is of type B and M' itself is of the form $w_0H^\infty(m)$ for a rigid function w_0.

(iii)$\Rightarrow$(i). If (iii) is true, it is easy to prove that any non-zero invariant subspace M in $H^1(m)$ has the form $wH^1(m)$. Suppose that B is a properly weak-*closed subalgebra of $L^\infty(m)$ containing $H^\infty(m)$. We must prove $B=H^\infty(m)$. Let M be an annihilator of B in $L^1(m)$. Then M is a non-zero invariant subspace of $L^1(m)$ which is contained in $H^1(m)+N_c^\infty$. Since the invariant subspace $H^1(m)+N_c^\infty$ has the form $w_0H^1(m)$ for a rigid function w_0, $w_0^{-1}M$ is a non-zero invariant subspace contained in $H^1(m)$ and hence $w_0^{-1}M=w'H^1(m)$ for a rigid function w'. It follows that $M=w'w_0H^1(m)=w'(H^1(m)+N_c^\infty)$, and so $B=(w')^{-1}H_0^\infty(m)$. As $H_0^\infty(m)$ is a Beurling subspace, we can denote that $B=wH^\infty(m)$ for a rigid function w. Now B is an algebra, so it contains w^2 and there is a function h in $H^\infty(m)$ with $w^2=wh$. Hence $w=h\in H^\infty(m)$. Consequently $B=H^\infty(m)$.

In the proof of (i)$\Rightarrow$(ii) above of Merrill in the case A is a logmoduarl algebra, the Wermer's embedding function Z plays an important rôle. Our proof of (i)$\Rightarrow$(ii) is based on results of Gamelin ([6], p. 139-140).

REMARK. It is known that the corona conjecture is true for a finite open Riemann surface; if R is a finite open Riemann surface, then R is dense in $M_{H^\infty(R)}$ (Alling [2]). So we can easily obtain the following; under the hypothesis of Theorem 2.2, the property of that theorem is equivalent to

(iv) $\tilde{P}$ is dense in $M_{H^\infty(m)}$.

In fact, if (ii) of the theorem is true, $H^\infty(m)$ is isometrically isomorphic to $H^\infty(\tilde{P})$. So $M_{H^\infty(m)}$ is homeomorphic to $M_{H^\infty(\tilde{P})}$. As $\tilde{P}$ is dense in $M_{H^\infty(\tilde{P})}$, $\tilde{P}$ is dense in $M_{H^\infty(m)}$. Conversely if f in $H^\infty(m)$ vanishes on $\tilde{P}$, $f=0$ by (iv).

EXAMPLES. Now we present the examples of hypo-Dirichlet algebras such that $H^\infty(m)$ is maximal.

(1) Let R be a finite open Riemann surface and X be its boundary. Let A be the algebra of all functions on X that are restrictions of functions continuous on $R\cup X$ and analytic in R. Then A is a hypo-Dirichlet algebra. Fix m a harmonic measure for some point in R. Constructing the abstract Hardy algebra $H^\infty(m)$, $H^\infty(m)$ is a maximal weak-* closed subalgebra of $L^\infty(m)$ ([1]).

(2) Let K be a compact subset of the complex plane with a non-empty interior whose complement has finitely many components and X be its boundary. Let A be the algebra of all functions on X that can be uniformly approximated by rational functions whose poles lie off K.

Then A is a hypo-Dirichlet algebra. Fix m a unique Arens-Singer measure on X for some point in K and construct the abstract Hardy algebra $H^\infty(m)$. Then $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$.

§3. The uniform maximality of hypo-Dirichlet algebras.

Let K be a compact finitely connected subset in C, the interior K^0 be connected and $A=R(K)|_{bK}$, where bK denotes the topological boundary of K. Then A is a hypo-Dirichlet algebra on bK (see examples in §2). Here we see that only non-trivial Gleason part of M_A is precisely the interior K^0 of K ([4], p. 149). As an application of Theorem 2.2, we can show the following, using a similar method as in a theorem of Merrill ([7], Theorem 5).

THEOREM 3.1. *A uniform algebra $A=R(K)|_{bK}$ as above is maximal as a uniformly closed subalgebra of $C(bK)$.*

PROOF. Suppose that B is a uniform algebra containing A and m is a unique Arens-Singer measure for a point of K^0. Then $H^\infty(m)\subset B^\infty\subset L^\infty(m)$, where B^∞ is the weak-*closure of B in $L^\infty(m)$. B^∞ is a weak-* closed subalgebra of $L^\infty(m)$. That B^∞ is an algebra is proved as follows; it is clear that $BB^\infty\subset B^\infty$. Taking the weak-*closure, we have $B^\infty B^\infty\subset B^\infty$. Now, since $H^\infty(m)$ is weak-*closed and maximal, $B^\infty=H^\infty(m)$ or $B^\infty=L^\infty(m)$. Let $B^\infty=H^\infty(m)$ and let μ be a measure on bK which is orthogonal to A. By the theorem of Wilken ([4], p. 47) μ is absolutely continuous with respect to a unique Arens-Singer measure for any z in the interior K^0 of K. This is because of that K^0 is only non-trivial Gleason part of the maximal ideal space of A. In particular, μ is absolutely continuous with respect to m, and so $\mu=hdm$ for a function $h\in L^1(m)$. Since A and B have the same weak-*closure, $\mu\perp H^\infty(m)=B^\infty$, and so $\mu\perp B$. It follows that $A=B$. When $B=L^\infty(m)$, suppose that μ is a measure on bK which is orthogonal to B. Since $\mu\perp A$, using the theorem of Wilken again, $\mu=hdm$ for some $h\in L^1(m)$. Hence $\mu\perp B^\infty=L^\infty(m)$ since $\mu\perp B$, and so $\mu=0$. Consequently $B=C(bK)$. It proves the theorem.

As a special case of the theorem above, we have the following.

COROLLARY 3.2 (*Björk and de Paepe* [3]). *Let $K=\{z\in C: r\leqq|z|\leqq 1$, where $0<r<1\}$, and $A=R(K)|_{bK}$. Then A is a maximal uniformly closed subalgebra of $C(bK)$.*

ACKNOWLEDGMENTS. The author wishes to express their hearty thanks to Dr. T. Nakazi and Dr. J. Tanaka for their valuable advices and constant encouragments.

References

[1] P. R. AHERN and D. SARASON, The H^p spaces of a class of function algebras, Acta Math., **117** (1967), 123-163.

[2] N. L. ALLING, A proof of the corona conjecture for finite open Riemann surfaces, Bull. Amer. Math. Soc., **70** (1964), 110-112.

[3] J.-E. BJÖRK and P. J. DE PAEPE, Maximal subalgebras, Papers from the summer gathering on function algebras at Aarhus, July 1969, 36-39, Various publications series no. **9**, Matematisk Institut, Aarhus, 1969.

[4] T. W. GAMELIN, Uniform Algebras, Prentice-Hall, Englewood Cliffs, N. J., 1969.

[5] T. W. GAMELIN, Remarks on compact groups with ordered duals, Rev. Un. Mat. Argentina, **23** (1967), 97-108.

[6] T. W. GAMELIN, Embedding Riemann surfaces in maximal ideal spaces, J. Functional Analysis, **2** (1968), 123-146.

[7] S. MERRILL, Maximality of certain algebras $H^\infty(dm)$, Math. Z., **106** (1968), 261-266.

Present Address:
DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES AND ENGINEERINGS
WASEDA UNIVERSITY
NISHIOKUBO, SHINJUKU-KU, TOKYO 160