

## Some Remarks on Quasi-Linear Evolution Equations in Banach Spaces

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### Introduction

This paper is concerned with the abstract Cauchy problem for a nonlinear equation of evolution of the form

$$(E) \quad (d/dt)u(t) + A(t, u(t))u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T],$$

with initial condition  $u(0) = a$ . Here  $u$  represents an unknown function taking its values  $u(t)$  in a Banach space  $X$ ;  $A(t, y)$  is a linear operator in  $X$  depending on  $t$  in  $[0, T]$  and  $y$  in a certain subset of  $X$  and  $f$  is a nonlinear operator from a certain domain of  $[0, T] \times X$  into  $X$ . The abstract equation of this form was studied by Kato [1] and [4]. According to Kato [1],  $A(t, u)u$  is called the quasi-linear part of (E) and  $f(t, u)$  the semi-linear part of (E). The operator  $f$  may be genuinely nonlinear, but is assumed to be regular and stable; hence  $f$  is regarded as a perturbation to the quasi-linear part. In this sense an equation of the form (E) is called a quasi-linear evolution equation. In [1], an existence theorem of strong solutions of (E) in a reflexive space is established by means of the method of successive approximation. This successive approximation is based on the theory of abstract linear "hyperbolic" equation advanced in [2] and [3].

The purpose of this paper is to introduce a notion of weak solution of (E) in a nonreflexive space and prove an existence and uniqueness theorem for the weak solutions of (E) by means of the method of Cauchy's polygonal approximation. This type of approximation makes it possible to prove the existence theorem under weaker conditions than those assumed in [1], [4] and [10]. In fact, it is assumed in [1], [4] and [10] that the operators  $y \rightarrow f(t, y)$  and  $y \rightarrow A(t, y)$  are  $X$ -Lipschitz continuous. But,

in this paper, these assumptions will be replaced by weaker ones. The basic idea in our method is based on the recent work of Kobayasi [8]. There are many works in which similar methods are employed, for instance, see [6], [7], [9] and the references cited there. We here show that this method can also be applied to the quasi-linear case as mentioned above.

This paper consists of three sections. In section 1, basic hypotheses are set up along with some comments. In section 2, we introduce a notion of  $\varepsilon$ -approximate solution and show that given an  $\varepsilon > 0$  an  $\varepsilon$ -approximate solution to (E) is always constructed. In section 3, a notion of weak solution is introduced and the existence and uniqueness theorem for weak solutions is established. Every strong solution is a weak solution and under additional assumptions, the converse is also true. Hence our existence theorem gives a generalization of [1; Theorem 6]. The proof is obtained by proving the convergence of  $\varepsilon$ -approximate solutions  $\{u_\varepsilon\}$  and then showing that the limit  $u(t)$  ( $= \lim u_\varepsilon(t)$ ) gives a unique weak solution.

### § 1. Basic hypotheses.

In what follows, we consider two real Banach spaces  $X$  and  $Y$  with norms  $\|\cdot\|$  and  $\|\cdot\|_Y$ , respectively. The operator norm of a bounded linear operator  $A$  on  $Y$  to  $X$  is denoted by  $\|A\|_{Y,X}$ . However, we write  $\|\cdot\|$  (resp.  $\|\cdot\|_Y$ ) for the operator norm  $\|\cdot\|_{X,X}$  (resp.  $\|\cdot\|_{Y,Y}$ ) for brevity in notation. The domain of an operator  $A$  is denoted by  $D(A)$ . Let  $X^*$  be the dual space of  $X$ . The value of  $x^* \in X^*$  at  $x \in X$  is denoted by  $(x, x^*)$ . The duality mapping of  $X$  is denoted by  $F$ , i.e.,

$$F(x) = \{x^* \in X^*; (x, x^*) = \|x\|^2 = \|x^*\|^2\} \quad \text{for } x \in X.$$

We now put basic hypotheses on the spaces  $X$ ,  $Y$  and the operators  $A(t, y)$  and  $f$ .

(X)  $Y$  is continuously and densely embedded in  $X$ . There is a linear isometry  $S$  of  $Y$  onto  $X$ .

In the following  $W$  denotes a fixed open ball in  $Y$  with center  $y_0$  and radius  $R$ , i.e.,  $W = \{y \in Y; \|y - y_0\|_Y < R\}$ . The set  $W$  is understood to be a metric space with the metric  $d(y, y') = \|y - y'\|_Y$  ( $y, y' \in W$ ).

(A) Condition for the linear operators  $A(t, y)$ :

(A1) There exist positive constants  $T_0$  and  $\alpha$  such that for each  $t \in [0, T_0]$  and  $y \in W$ ,  $-A(t, y)$  is the infinitesimal generator of a  $(C_0)$ -semigroup  $\{\exp[-sA(t, y)]\}_{s \geq 0}$  on  $X$  with  $\|\exp[-sA(t, y)]\| \leq e^{\alpha s}$  for  $s \geq 0$ ,

$t \in [0, T_0]$  and  $y \in W$ .

(A 2) For each  $t \in [0, T_0]$  and  $y \in W$ , there is a bounded linear operator  $B(t, y)$  on  $X$  into itself such that

$$SA(t, y)S^{-1} = A(t, y) + B(t, y), \quad \|B(t, y)\| \leq \lambda_1$$

for  $t \in [0, T_0]$  and  $y \in W$ , where  $\lambda_1$  is a positive constant depending only on  $T_0$  and  $W$ .

(A 3) For each  $t \in [0, T_0]$  and  $y \in W$ ,  $D(A(t, y)) \supset Y$ .

(A 4) The restriction of  $A(t, y)$  to  $Y$  (which is a bounded linear operator on  $Y$  to  $X$  by the closed graph theorem) satisfies the following: There is a constant  $K$ , depending only on  $T_0$  and  $W$ , such that  $\|A(t, y)\|_{Y, X} \leq K$  for  $t \in [0, T_0]$  and  $y \in W$ .

(A 5)  $A(t, y)y_0 \in Y$ , and  $\|A(t, y)y_0\|_Y \leq \lambda_2$  for  $t \in [0, T_0]$ ,  $y \in W$ , and some constant  $\lambda_2$  depending only on  $T_0$  and  $W$ .

(G) Condition for the nonlinear operator  $f(\cdot, \cdot) - A(\cdot, \cdot)$ :

In what follows, we write  $G(t, y, x)$  for  $f(t, y) - A(t, y)x$ .

(G 1)  $f$  is a bounded operator on  $[0, T_0] \times W$  to  $Y$  in the sense that  $\|f(t, y)\|_Y \leq \lambda_3$  for  $t \in [0, T_0]$ ,  $y \in W$ , and some positive constant  $\lambda_3$  (depending possibly upon  $T_0$  and  $W$ ).

(G 2)  $(t, x) \rightarrow G(t, x, x)$  is continuous on  $[0, T_0] \times W$  to  $X$ .

(G 3) For  $t \in [0, T_0]$ , and  $x, y \in W$ , there is an  $x^* \in F(x - y)$  such that

$$(G(t, x, x) - G(t, y, y), x^*) \leq \mu \|x - y\|^2,$$

where  $\mu$  is a positive constant independent of  $t \in [0, T_0]$ ,  $x, y \in W$  and  $x^* \in F(x - y)$ .

We here list some basic consequences that follow immediately from (X) and (A). By a well-known perturbation theorem for  $(C_0)$ -semigroups (see [5]) we infer that for each  $t \in [0, T_0]$  and  $y \in W$ ,  $A(t, y) + B(t, y)$  is the infinitesimal generator of a  $(C_0)$ -semigroup  $\{\exp[-s(A(t, y) + B(t, y))]\}_{s \geq 0}$  on  $X$  such that  $\|\exp[-s(A(t, y) + B(t, y))]\| \leq e^{(\alpha + \lambda_1)s}$  for  $s \geq 0$ . Moreover, we have  $S \exp[-sA(t, y)]S^{-1} = \exp[-s(A(t, y) + B(t, y))]$ . Therefore, we have  $\exp[-sA(t, y)](Y) \subset Y$ , and the restriction of  $\exp[-sA(t, y)]$  to  $Y$  is a  $(C_0)$ -semigroup on  $Y$  such that

$$(1.1) \quad \|\exp[-sA(t, y)]\|_Y \leq e^{(\alpha + \lambda_1)s},$$

for  $s \geq 0$ ,  $t \in [0, T_0]$  and  $y \in W$ . Detailed expositions of these facts are found in [2].

### § 2. Construction of approximate solutions.

Let  $r \in (0, R)$ , and let  $T$  be a positive number  $\leq T_0$  satisfying

$$e^{(\alpha+\lambda_1)T}[r + (\lambda_2 + \lambda_3)T] < R .$$

We fix both  $T$  and  $r$  throughout this paper. Let  $a$  be an element of  $W$  satisfying  $\|a - y_0\|_Y \leq r$ . For each partition  $P: 0 = t_0 < t_1 < \dots < t_N = T$  of the interval  $[0, T]$ , we define  $u_P: [0, T] \rightarrow W$  by the following:

$$(2.1) \quad \begin{cases} u_P(0) = a , \\ u_P(t) = \exp[-(t-t_k)A_k]u_P(t_k) + \int_{t_k}^t \exp[-(t-s)A_k]f_k ds , \\ t \in [t_k, t_{k+1}], k=0, 1, \dots, N-1 , \end{cases}$$

where  $A_k = A(t_k, u_P(t_k))$  and  $f_k = f(t_k, u_P(t_k))$ .

To justify this definition, we need the following:

**LEMMA 2.1.** *Suppose that  $u_P(t) \in W$  for  $0 \leq t \leq t_k$ , where  $0 \leq k < N$ . Then  $u_P(t) \in W$  for  $t \in [t_k, t_{k+1}]$ .*

**PROOF.** By the assumption,  $A_k, f_k$  and  $u_P(t) \in Y, t \in [t_k, t_{k+1}]$ , are well-defined. We then show that  $\|u_P(t) - y_0\|_Y < R$  for  $t \in [t_k, t_{k+1}]$ . By (2.1), we have

$$\begin{aligned} \|u_P(t) - y_0\|_Y &\leq \| \exp[-(t-t_k)A_k](u_P(t_k) - y_0) \|_Y \\ &\quad + \| \exp[-(t-t_k)A_k]y_0 - y_0 \|_Y \\ &\quad + \int_{t_k}^t \| \exp[-(t-s)A_k]f_k \|_Y ds \end{aligned}$$

for  $t_k \leq t \leq t_{k+1}$ . By (1.1), the first term on the right hand side is bounded above by  $e^{(\alpha+\lambda_1)h_k} \|u_P(t_k) - y_0\|_Y$ , where  $h_k = t_{k+1} - t_k$ . By (1.1) and (G 1), the third term is bounded above by  $\lambda_3 h_k e^{(\alpha+\lambda_1)h_k}$ . By (1.1), (A5) and the equality

$$\exp[-(t-t_k)A_k]y_0 - y_0 = \int_0^{t-t_k} \exp[-sA_k]A_k y_0 ds ,$$

the second term is bounded above by  $\lambda_2 h_k e^{(\alpha+\lambda_1)h_k}$ . Therefore, we have

$$\|u_P(t) - y_0\|_Y \leq e^{(\alpha+\lambda_1)h_k} [\|u_P(t_k) - y_0\|_Y + (\lambda_2 + \lambda_3)h_k] .$$

By induction, it follows that

$$\begin{aligned} \|u_P(t) - y_0\|_Y &\leq e^{(\alpha+\lambda_1)t_{k+1}} \|a - y_0\|_Y + (\lambda_2 + \lambda_3) \sum_{j=0}^k h_j e^{(\alpha+\lambda_1)(t_{k+1}-t_j)} \\ &\leq e^{(\alpha+\lambda_1)T} [r + (\lambda_2 + \lambda_3)T] < R . \end{aligned}$$

Q.E.D.

The function  $u_P(t)$  defined by (2.1) is continuous on  $[0, T]$  into the metric space  $W$  and is strongly continuously differentiable on each interval  $[t_k, t_{k+1}]$ ,  $k=0, 1, \dots, N-1$ , in the sense of the strong topology of  $X$ . Moreover, on each interval  $[t_k, t_{k+1}]$ ,  $k=0, 1, \dots, N-1$ ,  $u_P(t)$  is the solution of the linear equation

$$(2.2) \quad \begin{aligned} (d/dt)u_P(t) &= -A(t_k, u_P(t_k))u_P(t) + f(t_k, u_P(t_k)) \\ &= G(t_k, u_P(t_k), u_P(t)), \quad t \in [t_k, t_{k+1}]. \end{aligned}$$

**DEFINITION 2.2.** Let  $\epsilon > 0$ , and let  $a$  be an element of  $W$  satisfying  $\|a - y_0\|_Y \leq r$ . Then a function  $u_\epsilon$  on  $[0, T]$  to  $W$  is said to be an  $\epsilon$ -approximate solution to (E) with initial value  $u_\epsilon(0) = a$ , if it satisfies the following:

- (i) There exists a partition  $P_\epsilon: 0 = t_0^\epsilon < t_1^\epsilon < \dots < t_{N_\epsilon}^\epsilon = T$  of  $[0, T]$  satisfying  $t_{k+1}^\epsilon - t_k^\epsilon \leq \epsilon$ ,  $k=0, 1, \dots, N_\epsilon - 1$ ,
- (ii)  $u_\epsilon$  is defined by (2.1) with  $P = P_\epsilon$  and  $u_\epsilon = u_{P_\epsilon}$ ,
- (iii)  $\|G(t, u_\epsilon(t), u_\epsilon(t)) - G(t_k^\epsilon, u_\epsilon(t_k^\epsilon), u_\epsilon(t))\| \leq \epsilon$  for  $t \in (t_k^\epsilon, t_{k+1}^\epsilon]$  and  $0 \leq k \leq N_\epsilon - 1$ .

Now one can show that given an  $\epsilon > 0$  there always exists an  $\epsilon$ -approximate solution of (E) in the sense of Definition 2.2.

**PROPOSITION 2.3.** For each  $\epsilon > 0$  and  $a \in W$  with  $\|a - y_0\|_Y \leq r$ , there is an  $\epsilon$ -approximate solution  $u_\epsilon$  to (E) with  $u_\epsilon(0) = a$ .

To show this, we first prepare the following:

**LEMMA 2.4.** Let  $a \in W$  be an element satisfying  $\|a - y_0\|_Y \leq r$ , and let  $P: 0 = t_0 < t_1 < \dots < t_n < \dots \leq T$  be a strictly increasing sequence in  $[0, T]$ . We define  $u_P: [0, t_\infty) \rightarrow W$  by (2.1), where  $t_\infty = \lim t_n$ . Then,  $u_P(t)$  converges in  $Y$ -norm as  $t \rightarrow t_\infty$ .

**PROOF.** For each  $t \in [0, t_\infty)$  and  $s \in [0, t]$ , we define linear operator  $U_P(t, s)$  by

$$U_P(t, s) = \exp[-(t - t_k)A_k] \exp[-(t_k - t_{k-1})A_{k-1}] \cdots \exp[-(t_{j+1} - s)A_j]$$

for  $t \in [t_k, t_{k+1}]$  and  $s \in [t_j, t_{j+1}]$ , where  $A_n = A(t_n, u_P(t_n))$ ,  $n=0, 1, \dots$ . Note that  $U_P(t, s)(Y) \subset Y$  and  $U_P(t, s) = \exp[-(t - t_k)A_k]U_P(t_k, s)$  for  $t_k \leq t \leq t_{k+1}$ . Let  $t_i \in \{t_n\}$  and  $y \in Y$ . Let  $t$  and  $s$  be any numbers satisfying  $t_\infty > t > s > t_i$ ; hence  $t \in [t_k, t_{k+1}]$  and  $s \in [t_j, t_{j+1}]$  for some  $j$  and  $k$ . Then, recalling that  $S: Y \rightarrow X$  is an isometry, we have

$$\begin{aligned}
\|u_P(t) - u_P(s)\|_Y &= \|Su_P(t) - Su_P(s)\| \\
&\leq \|Su_P(t) - SU_P(t, t_i)S^{-1}y\| + \|SU_P(t, t_i)S^{-1}y - U_P(t, t_i)y\| \\
&\quad + \|U_P(t, t_i)y - U_P(s, t_i)y\| + \|U_P(s, t_i)y - SU_P(s, t_i)S^{-1}y\| \\
&\quad + \|SU_P(s, t_i)S^{-1}y - Su_P(s)\| \\
&\equiv I + II + III + IV + V.
\end{aligned}$$

Put  $h_q = t_{q+1} - t_q$  and  $a_q = \|u_P(t_q) - U_P(t_q, t_i)S^{-1}y\|_Y$ ,  $q = i, i+1, \dots$ . Then, by (1.1) and (G 1), we have

$$\begin{aligned}
I &\leq \|\exp[-(t-t_k)A_k]u_P(t_k) - \exp[-(t-t_k)A_k]U_P(t_k, t_i)S^{-1}y\|_Y \\
&\quad + \int_{t_k}^t \|\exp[-(t-s)A_k]f_k\|_Y ds \\
&\leq e^{(\alpha+\lambda_1)h_k}a_k + \lambda_3 h_k e^{(\alpha+\lambda_1)h_k}.
\end{aligned}$$

Therefore, by induction, we have

$$\begin{aligned}
I &\leq \exp\left((\alpha + \lambda_1) \sum_{q=i}^k h_q\right) a_i + \lambda_3 \sum_{p=i}^k h_p \exp\left((\alpha + \lambda_1) \sum_{q=p}^k h_q\right) \\
&\leq e^{(\alpha+\lambda_1)T} [\|u_P(t_i) - S^{-1}y\|_Y + \lambda_3(t_{k+1} - t_i)].
\end{aligned}$$

Similarly, we have

$$V \leq e^{(\alpha+\lambda_1)T} [\|u_P(t_i) - S^{-1}y\|_Y + \lambda_3(t_{j+1} - t_i)].$$

The estimates for the terms *II*, *III* and *IV* are given in [10; Lemma 2.1]:

$$\begin{aligned}
II &\leq \lambda_1 e^{(\alpha+\lambda_1)T}(t-t_i) \|y\|, \\
IV &\leq \lambda_1 e^{(\alpha+\lambda_1)T}(s-t_i) \|y\|, \\
III &\leq K e^{(\alpha+\lambda_1)T}(t-s) \|y\|_Y.
\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned}
\limsup_{t, s \rightarrow t_\infty} \|u_P(t) - u_P(s)\|_Y \\
\leq 2e^{(\alpha+\lambda_1)T} [\|u_P(t_i) - S^{-1}y\|_Y + \lambda_3(t_\infty - t_i) + \lambda_1(t_\infty - t_i) \|y\|],
\end{aligned}$$

for every  $t_i$  and  $y \in Y$ . Recalling that  $Y$  is dense in  $X$ , and that  $t_i$  and  $y$  are arbitrary, we have

$$\lim_{t, s \rightarrow t_\infty} \|u_P(t) - u_P(s)\|_Y = 0. \quad \text{Q.E.D.}$$

**PROOF OF PROPOSITION 2.3.** For each  $\varepsilon > 0$ , we put  $t_0^* = 0$  and  $u_0(0) = a$ .

We define  $t_n^\varepsilon$  and  $u_\varepsilon(t)$  on  $(t_n^\varepsilon, t_{n+1}^\varepsilon]$ ,  $n=0, 1, \dots$ , in the following manner: Suppose that  $t_j^\varepsilon$ ,  $j=0, 1, \dots, k$ , are constructed. If  $t_k^\varepsilon < T$ , let  $t_{k+1}^\varepsilon$  be the largest number satisfying  $t_{k+1}^\varepsilon \leq T$ ,  $t_{k+1}^\varepsilon - t_k^\varepsilon \leq \varepsilon$  and

$$M_\varepsilon(t) \equiv \|G(t, u_\varepsilon(t), u_\varepsilon(t)) - G(t_k^\varepsilon, u_\varepsilon(t_k^\varepsilon), u_\varepsilon(t))\| \leq \varepsilon,$$

for  $t \in (t_k^\varepsilon, t_{k+1}^\varepsilon]$ ; and define  $u_\varepsilon(t)$  on  $[0, t_{k+1}^\varepsilon]$  by (2.1) for the partition  $P = P'_\varepsilon: 0 = t_0^\varepsilon < t_1^\varepsilon < \dots < t_{k+1}^\varepsilon$  of  $[0, t_{k+1}^\varepsilon]$ . Observe that if  $t_k^\varepsilon < T$ , then one has  $t_{k+1}^\varepsilon > t_k^\varepsilon$ , since  $M_\varepsilon(t)$  is continuous in  $t \in (t_k^\varepsilon, t_{k+1}^\varepsilon]$  and  $\lim_{t \downarrow t_k^\varepsilon} M_\varepsilon(t) = 0$  by (G 2) and (A 4). We then show that there is an  $N_\varepsilon$  such that  $t_{N_\varepsilon}^\varepsilon = T$ . Assume for the contrary that  $t_n^\varepsilon < T$  for all  $n=0, 1, \dots$ . Let  $t_\infty = \lim t_n^\varepsilon$  and let  $w$  be the limit  $\lim_{t \uparrow t_\infty} u_\varepsilon(t)$  obtained by Lemma 2.4. Since  $\|u_\varepsilon(t) - y_0\|_Y \leq e^{(\alpha+\lambda_1)T}[r + (\lambda_2 + \lambda_3)T] \equiv \rho$ , we have  $\|w - y_0\|_Y \leq \rho < R$ . Therefore, by (G 2) and (A 4), we have  $\lim_{t \uparrow t_\infty} M_\varepsilon(t) = 0$ . It follows that there is an  $m$  such that  $t_{m+1}^\varepsilon - t_m^\varepsilon \leq \varepsilon/2$  and  $M_\varepsilon(t) \leq \varepsilon/2$  for  $t \in (t_m^\varepsilon, t_{m+1}^\varepsilon]$ . This is a contradiction, since  $t_{m+1}^\varepsilon$  is the largest number satisfying  $t_{m+1}^\varepsilon \leq T$ ,  $t_{m+1}^\varepsilon - t_m^\varepsilon \leq \varepsilon$  and  $M_\varepsilon(t) \leq \varepsilon$  for  $t \in (t_m^\varepsilon, t_{m+1}^\varepsilon]$ . Thus the partition  $P_\varepsilon: 0 = t_0^\varepsilon < t_1^\varepsilon < \dots < t_{N_\varepsilon}^\varepsilon = T$  constructed as above fulfills all the requirements of Definition 2.2.

§ 3. Existence and uniqueness of weak solutions.

In this section, we discuss the construction of weak solutions to (E) and establish an existence and uniqueness theorem for the weak solutions. We begin by introducing the following:

DEFINITION 3.1. Let  $a$  be an element of  $W$  satisfying  $\|a - y_0\|_Y \leq r$ . Then a function  $u: [0, T] \rightarrow X$  is said to be a *weak solution* to (E) with initial value  $u(0) = a$ , if it satisfies the following:

- (i) There exist a sequence  $\{\varepsilon_n\}$  of positive numbers converging to 0 and a sequence  $\{u_n\}$  of  $\varepsilon_n$ -approximate solutions with  $u_n(0) = a$ ,
- (ii)  $u_n(t)$  converges in  $X$  as  $n \rightarrow \infty$  to  $u(t)$ , uniformly on  $[0, T]$ .

Note that the value  $u(t)$  of a weak solution  $u$  with initial value  $a$  satisfying  $\|a - y_0\|_Y \leq r$  belongs to the  $X$ -closure  $\bar{W}_\rho$  of  $W_\rho$ , where  $W_\rho = \{y \in Y; \|y - y_0\|_Y \leq \rho\}$  and  $\rho = e^{(\alpha+\lambda_1)T}[r + (\lambda_2 + \lambda_3)T]$ . As will be shown later, the notion of weak solution is a generalization of the following notion of strong solution:

DEFINITION 3.2. A function  $u: [0, T] \rightarrow X$  is said to be a strong solution to (E) constrained in  $W$ , if it satisfies the following:

- (i)  $u(t)$  is  $X$ -strongly absolutely continuous on  $[0, T]$ , and is  $X$ -strongly differentiable at almost every  $t \in [0, T]$ ,
- (ii)  $u(t) \in W$  and  $u$  satisfies (E) at a.e.  $t \in [0, T]$ .

We now state our existence and uniqueness theorem for the weak solutions to (E).

**THEOREM 3.3.** *Given an initial value  $a$  with  $\|a - y_0\|_Y \leq r$ , there is a unique weak solution to (E). More precisely, let  $\{\varepsilon_n\}$  be any sequence of positive numbers converging to 0 and for each  $\varepsilon_n$ , let  $u_n$  be any  $\varepsilon_n$ -approximate solution to (E) with  $u_n(0) = a$ . Then  $\{u_n\}$  converges uniformly to a unique weak solution  $u$  to (E) with  $u(0) = a$ . Furthermore, if  $u$  and  $v$  are two weak solutions to (E) with  $\|u(0) - y_0\|_Y \leq r$  and  $\|v(0) - y_0\|_Y \leq r$ , then we have*

$$(3.1) \quad e^{-\mu t} \|u(t) - v(t)\| \leq e^{-\mu s} \|u(s) - v(s)\|$$

for every pair  $t, s \in [0, T]$  with  $t \geq s$ .

**PROOF.** Let  $\{\hat{\varepsilon}_n\}$  be another sequence of positive numbers converging to 0. For each  $\hat{\varepsilon}_n$ , let  $v_n$  be an  $\hat{\varepsilon}_n$ -approximate solution to (E) with  $\|v_n(0) - y_0\|_Y \leq r$ . Then, by Proposition 2.3, there are partitions  $P_{\varepsilon_n} = \{t_k^n\}$  and  $P_{\hat{\varepsilon}_n} = \{\hat{t}_j^n\}$  of  $[0, T]$  which fulfill all the requirements of Definition 2.2 for  $u_n$  and  $v_n$ , respectively.

Let  $t \in (t_k^n, t_{k+1}^n) \cap (\hat{t}_j^m, \hat{t}_{j+1}^m)$ . Then, by (2.2), we have

$$\begin{aligned} & (d/dt) \|u_n(t) - v_m(t)\|^2 \\ &= 2(G(t_k^n, u_n(t_k^n), u_n(t)) - G(t, u_n(t), u_n(t)), x^*) \\ & \quad + 2(G(t, u_n(t), u_n(t)) - G(t, v_m(t), v_m(t)), x^*) \\ & \quad + 2(G(t, v_m(t), v_m(t)) - G(\hat{t}_j^m, v_m(\hat{t}_j^m), v_m(t)), x^*) \end{aligned}$$

for every  $x^* \in F(u_n(t) - v_m(t))$ . By (G 3), there is an  $x^* \in F(u_n(t) - v_m(t))$  such that the second term on the right hand side is bounded above by  $2\mu \|u_n(t) - v_m(t)\|^2$ . In view of Definition 2.2, the first and the third terms are bounded above by  $2\varepsilon_n \|u_n(t) - v_m(t)\|$  and  $2\hat{\varepsilon}_m \|u_n(t) - v_m(t)\|$ , respectively. Therefore, we have

$$\begin{aligned} (d/dt) \|u_n(t) - v_m(t)\|^2 &\leq 2\mu \|u_n(t) - v_m(t)\|^2 \\ & \quad + 2(\varepsilon_n + \hat{\varepsilon}_m) \|u_n(t) - v_m(t)\|. \end{aligned}$$

It follows that

$$(3.2) \quad \begin{aligned} e^{-\mu t} \|u_n(t) - v_m(t)\| &= e^{-\mu s} \|u_n(s) - v_m(s)\| \\ &\leq (\varepsilon_n + \hat{\varepsilon}_m) \int_s^t e^{-\mu r} dr \end{aligned}$$

for  $0 \leq s \leq t \leq T$ . Put  $s=0$  and  $v_n = u_n$  for  $n=1, 2, \dots$ . Then, (3.2) yields



that  $\{u_n\}$  converges uniformly on  $[0, T]$  to some function  $u$  and the limit function  $u$  is by definition a weak solution to (E) with  $u(0)=a$ . (3.1) follows from (3.2). The uniqueness of the weak solution follows from (3.1). Q.E.D.

The next two theorems show the relationship between weak and strong solutions.

**THEOREM 3.4.** *Let  $u$  be a strong solution to (E) with  $u(0)=a$ ,  $\|a-y_0\|_Y \leq r$ . Then  $u$  is a weak solution to (E) with the same initial value. Therefore, there exists at most one strong solution to (E) with initial value  $a$  satisfying  $\|a-y_0\|_Y \leq r$ .*

**PROOF.** Let  $u_n$  be a  $(1/n)$ -approximate solution to (E) with  $u_n(0)=a$ , and let  $P_n = \{t_k^n\}$  be a partition of  $[0, T]$  fulfilling all of the requirements of Definition 2.2 for  $u_n$ . Let  $t \in (t_k^n, t_{k+1}^n)$ . Then, in a way similar to the proof of Theorem 3.3, we have

$$\begin{aligned} &(d/dt) \|u(t) - u_n(t)\|^2 \\ &= 2(G(t, u(t), u(t)) - G(t, u_n(t), u_n(t)), x^*) \\ &\quad + 2(G(t, u_n(t), u_n(t)) - G(t_k^n, u_n(t_k^n), u_n(t)), x^*) \\ &\leq 2\mu \|u(t) - u_n(t)\|^2 + (2/n) \|u(t) - u_n(t)\| \end{aligned}$$

for some  $x^* \in F(u(t) - u_n(t))$ . It follows that

$$(3.3) \quad e^{-\mu t} \|u(t) - u_n(t)\| \leq (1/n) \int_0^t e^{-\mu r} dr .$$

On the other hand, by Theorem 3.3,  $\{u_n\}$  converges to a weak solution  $v$  with  $v(0)=a$ . Therefore, we have  $u=v$  by (3.3). Q.E.D.

**THEOREM 3.5.** *Let  $u$  be a weak solution to (E) with  $u(0)=a$ ,  $\|a-y_0\|_Y \leq r$ . Suppose that  $u(t_0) \in W$  for some  $t_0 \in [0, T]$ . Then the strong right derivative  $(d/dt)^+ u(t)$  exists at  $t=t_0$  and is equal to  $-A(t_0, u(t_0))u(t_0) + f(t_0, u(t_0))$ .*

**PROOF.** Put  $u_0 = u(t_0)$ ,  $A_0 = A(t_0, u_0)$ ,  $f_0 = f(t_0, u_0)$  and

$$v(t) = \exp[-(t-t_0)A_0]u_0 + \int_{t_0}^t \exp[-(t-s)A_0]f_0 ds ,$$

for  $t_0 \leq t \leq T$ . Then  $v$  is continuously differentiable on  $[t_0, T]$  into  $X$  and

$$(d/dt)v(t) = -A_0v(t) + f_0 , \quad t_0 \leq t \leq T , \quad v(t_0) = u_0 .$$

Moreover, since  $v$  is continuous on  $[t_0, T]$  into  $Y$  and  $v(t_0) \in W$ ,  $v(t) \in W$  for sufficiently small  $t > t_0$ . In what follows, we will show that

$$\|u(t) - v(t)\| = o(t - t_0), \quad \text{as } t \downarrow t_0.$$

Once this is done, then we have

$$\begin{aligned} & \left\| \frac{u(t) - u_0}{t - t_0} + A_0 u_0 - f_0 \right\| \\ & \leq \left\| \frac{u(t) - v(t)}{t - t_0} \right\| + \left\| \frac{v(t) - u_0}{t - t_0} + A_0 u_0 - f_0 \right\| \\ & \longrightarrow 0, \quad \text{as } t \downarrow t_0, \end{aligned}$$

which proves the conclusion of Theorem 3.5.

Now, we assume without loss of generality that  $v(t) \in W$  for  $t \geq t_0$ . Let  $u_n$  be a  $(1/n)$ -approximate solution to (E) with  $u_n(0) = a$ . By Theorem 3.3,  $\{u_n\}$  converges to  $u$ . Let  $P_n = \{t_k^n\}$  be a partition of  $[0, T]$  which fulfills all the requirements of Definition 2.2 for  $u_n$ , and let  $t \in (t_k^n, t_{k+1}^n) \cap (t_0, T]$ . Then we have

$$\begin{aligned} & (d/dt) \|u_n(t) - v(t)\|^2 \\ & = 2(G(t_k^n, u_n(t_k^n), u_n(t)) - G(t, u_n(t), u_n(t)), x^*) \\ & \quad + 2(G(t, u_n(t), u_n(t)) - G(t, v(t), v(t)), x^*) \\ & \quad + 2(G(t, v(t), v(t)) - G(t_0, u_0, v(t)), x^*) \\ & \leq (2/n) \|u_n(t) - v(t)\| + 2\mu \|u_n(t) - v(t)\|^2 \\ & \quad + 2 \|u_n(t) - v(t)\| \cdot \|G(t, v(t), v(t)) - G(t_0, u_0, v(t))\| \end{aligned}$$

for some  $x^* \in F(u_n(t) - v(t))$ . It follows that

$$\begin{aligned} & e^{-\mu t} \|u(t) - v(t)\| \\ & \leq \int_{t_0}^t e^{-\mu s} \|G(s, v(s), v(s)) - G(t_0, u_0, v(s))\| ds \end{aligned}$$

for  $t \in [t_0, T]$ . This implies that  $\|u(t) - v(t)\| = o(t - t_0)$  as  $t \downarrow t_0$ , since the integrand is continuous in  $s$  by (G 2) and (A 4), and vanishes at  $s = t_0$ .

Q.E.D.

**COROLLARY 3.6.** *Let  $u$  be a weak solution to (E) with  $u(0) = a$  and  $\|a - y_0\|_Y \leq r$ . Suppose that  $u(t) \in W$  for all  $t \in [0, T]$ . Then  $u$  is a unique strong solution to (E) with  $u(0) = a$ . Furthermore, we have*

$$(E') \quad (d/dt)^+ u(t) + A(t, u(t))u(t) = f(t, u(t)),$$

for all  $t \in [0, T]$ .

PROOF. By Theorem 3.5,  $u$  satisfies (E'). The uniqueness of a strong solution was proved by Theorem 3.4. Let  $u_\varepsilon$  be any  $\varepsilon$ -approximate solution to (E) with  $u_\varepsilon(0) = a$ . Then, by (A 4), (G 1) and (2.2), we have  $\|(d/dt)u_\varepsilon(t)\| \leq K(R + \|y_0\|_Y) + \lambda_3$ . Therefore,  $u(t)$  is Lipschitz continuous on  $[0, T]$  to  $X$ . Since  $t \rightarrow (u(t), x^*)$  is Lipschitz continuous for every  $x^* \in X^*$ , we have

$$(u(t) - a, x^*) = \int_0^t (d/ds)(u(s), x^*) ds = \left( \int_0^t (d/ds)^+ u(s) ds, x^* \right),$$

for  $t \in [0, T]$  and  $x^* \in X^*$ . It follows that

$$u(t) - a = \int_0^t (d/ds)^+ u(s) ds, \quad t \in [0, T].$$

This implies that  $u(t)$  is differentiable at almost every  $t \in [0, T]$ . Therefore, the relation (E') yields that  $u$  satisfies (E) at almost every  $t \in [0, T]$ .  
Q.E.D.

Let  $u$  be a weak solution to (E) with  $u(0) = a$  and  $\|a - y_0\|_X \leq r$ . Let  $W_\rho = \{y \in Y; \|y - y_0\|_Y \leq \rho\}$ , where  $\rho = e^{(\alpha + \lambda_1)T} [r + (\lambda_2 + \lambda_3)T] < R$ . We consider the following condition:

(I)  $W_\rho$  is closed in  $X$ .

If (I) is satisfied, then we have  $u(t) \in W_\rho \subset W$ . In [1], it is assumed that  $X$  is reflexive. If  $X$  is reflexive, (I) is automatically satisfied. See [1; Lemma 7.3]. Therefore, we obtain the following:

COROLLARY 3.7. *In addition to the assumptions (X), (A) and (G), suppose that  $X$  is reflexive. Let  $a \in W$  be an element satisfying  $\|a - y_0\|_Y \leq r$ . Then there exists a unique strong solution  $u$  to (E) with  $u(0) = a$ .*

REMARK 3.8. The strong solution  $u$  obtained by [1; Theorem 6] is strongly continuously differentiable on  $[0, T]$ . To obtain such solutions, it is required to impose the following condition:

(G 2') For each  $x \in Y$ , the functions  $(t, y) \rightarrow A(t, y)x$  and  $(t, y) \rightarrow f(t, y)$  are continuous on  $[0, T_0] \times W_x$  to  $X$ .

Here  $W_x$  denotes a metric space with the metric  $d_x(x, y) = \|x - y\|$  on  $W$ . In what follows, we assume (G 2'), instead of (G 2). Suppose that  $X$  is reflexive. Let  $u$  be a strong solution to (E) with  $\|u(0) - y_0\|_Y \leq r$ , obtained by Corollary 3.7. Then, by [1; Lemmas 9.2 and 9.3], the function  $t \rightarrow B(t, u(t))$  is strongly measurable in  $X$  and the function  $t \rightarrow f(t, u(t))$

is strongly measurable in  $Y$ . Therefore, in view of [8], we can apply [1; Theorem 2] to the linear equation of evolution

$$(L^*) \quad (d/dt)w(t) + A^*(t)w(t) = f^*(t), \quad 0 \leq t \leq T, \quad w(0) = u(0),$$

where  $A^*(t) = A(t, u(t))$  and  $f^*(t) = f(t, u(t))$ . The strong solution  $w$  of  $(L^*)$  exists uniquely and is continuously differentiable. On the other hand, since  $u$  is also a strong solution to  $(L^*)$ , we have  $u = w$ . Therefore,  $u$  must be continuously differentiable.

The result mentioned above is an extension of [1; Theorem 6].

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