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Analytic Functionals on a Countably Infinite Dimensional Topological Vector Space

Yoshihisa FUJIMOTO

University of Tokyo
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Introduction

Recently, the concept of holomorphic functions was extended to that of holomorphic functions on infinite dimensional topological vector spaces, and their properties have been discussed by many authors (see for example the references in [16] and the articles in Séminaire Pierre Lelong [17]). Among others, A. Martineau [14] investigated holomorphic functions on the space $\mathcal{O}(C^n)$ of holomorphic functions on C^n.

In this paper we will investigate holomorphic functions on the countably infinite dimensional topological vector space $\sum C$ of polynomials of one complex variable. It is remarkable that the theory of holomorphic functions on $\sum C$ is very similar to that on the finite dimensional space C^n. We also treat infinitely differentiable functions on $\sum R$. We will give a definition of hyperfunctions on $\sum R$, which may contribute to further discussion of the theory of hyperfunctions on topological vector spaces.

In §1, we will introduce the space $\sum C$ as the inductive limit of C^n and will show that any open subset of $\sum C$ is paracompact (Proposition 1.2). We will recall the properties of pseudo-convex open subsets of $\sum C$.

In §2, we investigate the space of holomorphic functions on $\sum C$ and its dual space, which is the space of analytic functionals on $\sum C$. The problem of supports of analytic functionals on $\sum C$ can be reduced to that on the finite dimensional space C^n. We can define the Fourier-Borel transformation for analytic functionals on $\sum C$ as in the finite dimensional case and prove that the Fourier-Borel transformation induces a topological isomorphism of the space of analytic functionals on $\sum C$ onto the space of entire functions of exponential type on the dual space $\prod C$ of $\sum C$ (Corollary 2.17). At the end of §2, we will study the space

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of infinitely differentiable functions on $\sum R$ and its dual space, which is the space of distributions with compact supports on $\sum R$. We can prove a theorem of Paley-Wiener type (Theorem 2.25).

In §3, we will investigate the p-th cohomology group $H^p(U, \mathcal{O})$ of an open set U of $\sum C$ with coefficients in the sheaf $\mathcal{O}$ of germs of holomorphic functions on $\sum C$. We will construct a fine resolution of $\mathcal{O}$ and will prove the vanishing of $H^p(U, \mathcal{O})$ for $p \geqq 1$, when U is pseudo-convex (Theorem 3.6).

In the final section, we will give definitions of hyperfunctions and distributions on $\sum R$.

§1. Properties of $\sum C$.

We begin with the definition of $\sum C$.

DEFINITION 1.1. We denote by $\sum C$ (resp. $\sum C^2$) the direct sum of complex planes C (resp. C^2) endowed with the inductive limit topology of the sequence of the spaces $\{C^n; u^n_{n+1}\}$ (resp. $\{C^{2n}; v^n_{n+1}\}$), where $u^n_{n+1}: C^n \to C^{n+1}$ (resp. $v^n_{n+1}: C^{2n} \to C^{2(n+1)}$) is defined by $u^n_{n+1}((z_1, \cdots, z_n)) = (z_1, \cdots, z_n, 0)$ (resp. $v^n_{n+1}((z_1, \cdots, z_{2n})) = (z_1, \cdots, z_{2n}, 0, 0)$).

We denote by u_n (resp. v_n) the canonical injection from C^n (resp. C^{2n}) into $\sum C$ (resp. $\sum C^2$) and by p_n (resp. p'_n) the projection from $\prod C$ (resp. $\prod C^2$) onto C^n (resp. C^{2n}), $\prod C$ (resp. $\prod C^2$) being the topological dual of $\sum C$ (resp. $\sum C^2$).

REMARK 1. We can easily see that $\sum C$ and $\sum C^2$ are dual Fréchet-Schwartz nuclear spaces.

REMARK 2. One can define $\sum R$, $\sum R^2$, $\prod R$ and $\prod R^2$ quite analogously.

The following proposition plays a fundamental role in this paper.

PROPOSITION 1.2. *Any open set U in $\sum C$ is paracompact.*

PROOF. Let $\{U_\alpha\}$ be an arbitrary open covering of U. We shall construct a locally finite refinement of $\{U_\alpha\}$. We identify $u_n(C^n)$ with C^n. We can assume without loss of generality that U contains the origin and that each $U_n = U \cap C^n$ is connected in C^n.

1. First, we choose an exhausting sequence $\{X_k\}$ of compact subsets of U as follows. Let X_1 be an arbitrary closed disc with radius r_1 (<1) contained in $U \cap C$. If X_{k-1} has been already chosen, put

$$X_k=\left\{z\in U\cap C^k;\ |z|_k\leqq k,\ \inf_{y\in\complement(U\cap C^k)}|z-y|_k\geqq\frac{1}{r_k}\right\},$$

where r_k satisfies $r_k>k$ and $\inf_{x\in X_{k-1},y\in\complement(U\cap C^k)}|x-y|_k>1/r_k$ $(k\geqq 2)$ and $|\cdot|_k$ denotes the Euclidean norm on C^k. X_n is a compact subset of U and we have $U=\bigcup_{n=1}^{\infty}X_n$. Thus, X_i is covered by a finite subfamily $\{U_{ij}\}_{1\leqq j\leqq n_i}$ of $\{U_\alpha\}$. Therefore $\{U_{ij}\}_{1\leqq j\leqq n_i,1\leqq i}$ is a refinement of $\{U_\alpha\}$.

2. We will construct inductively a locally finite refinement $\{U'_{ik}\}$ of $\{U_{ij}\}$. Put

$$X_{n,\varepsilon}=\{z\in C^{n+1};\ \inf_{y\in X_n}|z-y|_n\leqq\varepsilon\}\qquad\text{for}\quad\varepsilon>0\,.$$

Since $\{U_{1j}\cap C^2\}_{1\leqq j\leqq n_1}$ is an open covering of X_1 in $C^2\cap U$, there exists a refinement $\{V_{1j}\}_{1\leqq j\leqq n'_1}$ of $\{U_{1j}\cap C^2\}_{1\leqq j\leqq n_1}$ such that $V_{1j}\subset X_2$ and $\bigcup_{j=1}^{n'_1}V_{1j}\supset X_1$. Put $U'_{1j}=p_2^{-1}(V_{1j})\cap U_{1q_1(j)}$, where $q_1(j)$ denotes an integer q such that $V_{1j}\subset U_{1q}\cap C^2$. Thus, $\{U'_{1j}\}_{1\leqq j\leqq n'_1}$ is a refinement of $\{U_{1j}\}$ in $\sum C$ and $U'_{1j}\cap C^2\subset X_2$. Suppose that for any $i\leqq k-1$, we have constructed $\{U'_{ij}\}_{1\leqq j\leqq n'_i}$ which satisfy the following conditions:

a) $U'_{ij}\cap C^{i+1}\subset X_{i+1}$,

b) there exists $\varepsilon_{i-1}>0$ such that $U'_{ij}\cap X_{i-1,\varepsilon_{i-1}}=\varnothing$,

c) $\{U'_{ij}\}_{1\leqq j\leqq n'_i}$ is a refinement of $\{U_{ij}\}_{1\leqq j\leqq n_i}$.

Then, since $\{U_{kj}\cap C^{k+1}\}_{1\leqq j\leqq n_k}$ is an open covering of $X_k-\bigcup_{i=1}^{k-1}\bigcup_{j=1}^{n'_i}(U'_{ij}\cap C^k)$ in $C^{k+1}\cap U$, there exists a refinement $\{V_{kj}\}_{1\leqq j\leqq n'_k}$ of $\{U_{kj}\cap C^{k+1}\}_{1\leqq j\leqq n_k}$ in $C^{k+1}\cap U$, which satisfy the following condition:

(+) There exists $\varepsilon_{k-1}>0$ such that $V_{kj}\cap X_{k-1,\varepsilon_{k-1}}=\varnothing$ and $V_{kj}\subset X_{k+1}$.

Put $U'_{kj}=p_{k+1}^{-1}(V_{kj})\cap U_{kq_k(j)}$, where $q_k(j)$ denotes an integer q such that $V_{kj}\subset U_{kq}\cap C^{k+1}$. Hence, $\{U'_{kj}\}_{1\leqq j\leqq n'_k}$ is a refinement of $\{U_{kj}\}_{1\leqq j\leqq n_k}$ such that $U'_{kj}\cap C^{k+1}\subset X_{k+1}$ and $U'_{kj}\cap X_{k-1,\varepsilon_{k-1}}=\varnothing$. Therefore, by induction, we have constructed an open covering $\{U'_{ij}\}_{1\leqq j\leqq n'_i,1\leqq i}$ which satisfies the above conditions a), b) and c). Thus, $\{U'_{ij}\}_{1\leqq j\leqq n'_i,1\leqq i}$ is a refinement of $\{U_{ij}\}_{1\leqq j\leqq n_i,1\leqq i}$.

3. It only remains to show that the above covering is locally finite. Let x be an arbitrary point of U. The symbol m_0 denotes the minimum integer in $\{m\in N;\ x\in X_m\}$. Then, there exist $U'_{i_0j_0}$ such that $x\in U'_{i_0j_0}$ $(i_0\leqq m_0,\ 1\leqq j_0\leqq n'_{i_0})$ and an open neighborhood W_{m_0} of x in C^{m_0+1} contained in $U'_{i_0j_0}\cap C^{m_0+1}\cap X_{m_0,\varepsilon_{m_0}}$. We can choose an open neighborhood W_{m_0+1} of x in C^{m_0+2} contained in $p_{m_0+1}^{-1}(W_{m_0})\cap(U'_{i_0j_0}\cap C^{m_0+2})\cap X_{m_0+1,\varepsilon_{m_0+1}}$ such that $W_{m_0+1}\cap C^{m_0+1}=W_{m_0}$. Suppose that we have constructed $\{W_{m_0+q}\}_{q\leqq k-1}$ such that W_{m_0+q} is an open neighborhood of x in C^{m_0+q+1} and that $W_{m_0+q}\cap C^{m_0+q}=W_{m_0+q-1}$. Then, there exists an open set W_{m_0+k} in C^{m_0+k+1}

contained in $p_{m_0+k}^{-1}(W_{m_0+k-1})\cap(U'_{i_0j_0}\cap C^{m_0+k+1})\cap X_{m_0+k,\varepsilon_{m_0+k}}$ such that $W_{m_0+k}\cap C^{m_0+k}=W_{m_0+k-1}$. Therefore, by induction, we have constructed $\{W_m\}_{m\geq m_0}$. Put $W=\bigcup_{m=m_0}^{\infty}W_m$, then W is an open set of U. The construction of $\{W_m\}$ shows that W does not intersect with $\{U'_{ij}\}_{m_0+1\leq i,1\leq j\leq n'_i}$. Thus, we have proved that the covering $\{U'_{ij}\}$ is locally finite. Q.E.D.

REMARK. Changing $\sum C$ by $\sum C^2$, $\sum R$ or $\sum R^2$, this proposition still holds.

Now, we shall discuss other properties of subsets of $\sum C$. For the definitions of a holomorphically convex open set, a pseudo-convex open set, a polynomially convex open (compact) set, etc., see for example [16]. For an open set U in $\sum C$, the following properties are equivalent (see for example [5], [6], [16]):

$$\text{(1.1)}\quad \begin{cases} 1)\ \ U \text{ is a domain of existence of a holomorphic function,}\\ 2)\ \ U \text{ is a domain of holomorphy,}\\ 3)\ \ U \text{ is a pseudo-convex open set,}\\ 4)\ \ U \text{ is a holomorphically convex open set.}\end{cases}$$

Similar to the case of finite dimensions, we obtain the following proposition in the case of $\sum C$.

PROPOSITION 1.3. *Any polynomially convex compact subset of $\sum C$ has a fundamental system of neighborhoods consisting of polynomially convex open subsets.*

PROOF. Let K be an arbitrary polynomially convex compact set and let U be an arbitrary open neighborhood of K in $\sum C$.

1. Since K is compact, there exists a positive integer n such that $K\subset C^n$ and that K is a polynomially convex compact set in C^n. Thus, there exists a polynomially convex open set V in C^n such that $K\subset V\subset U\cap C^n$ by the well-known fact.

2. Let us put $I_r=\{z\in C;\ |z|<r\}$. Since $V\subset C^{n+1}\cap U$, we have $V\times I_r\subset C^{n+1}\cap U$ for sufficiently small $r>0$. Picking up one of such r, say, r_1, we write $I_1=I_{r_1}$. It is easy to see that $V\times I_1$ is a polynomially convex open set in C^{n+1}.

3. Suppose that a polynomially convex open set $V\times I_1\times\cdots\times I_m\subset C^{n+m}\cap U$ has been constructed for $m=k-1$, where $I_j=I_{r_j}$ for some $r_j>0$. Then, for sufficiently small $r>0$ we have $V\times I_1\times\cdots\times I_{k-1}\times I_r\subset C^{n+k}\cap U$. Picking up one of such r, say, r_k, $V\times I_1\times\cdots\times I_k$ is a polynomially convex open set in C^{n+k}, where $I_k=I_{r_k}$. By induction we have constructed $\{V\times I_1\times\cdots\times I_k\}$. Put $W=\bigcup_{k=1}^{\infty}V\times I_1\times\cdots\times I_k$. Since W is a finitely

polynomially convex open subset of $\sum C$, W is a polynomially convex open set (see for the details [16]). The construction shows that $K\subset W\subset U$. Q.E.D.

By the same method we can easily prove the following

PROPOSITION 1.4. *Every compact set in $\sum R$ is a polynomially convex compact subset of $\sum C$.*

§ 2. Properties of $\mathscr{O}(U)$, $\mathscr{E}(U)$ and their dual spaces.

In this section we consider the spaces of holomorphic functions and infinitely differentiable functions on $\sum C$ or $\prod C$. For the definition of a holomorphic function on a topological vector space, see for example [2] and [3].

DEFINITION 2.1. Let U be an arbitrary open set in $\sum C$ or $\prod C$. We denote by $\mathscr{O}(U)$ the topological vector space of all holomorphic functions on U endowed with the topology of the uniform convergence on each compact set in U.

Hereafter, we denote by $\mathscr{O}_n$ the sheaf of germs of holomorphic functions on C^n and we write $U_n=U\cap C^n$ for an open set U in $\sum C$.

PROPOSITION 2.2. *Let U be an arbitrary open set in $\sum C$. Then, we have the isomorphism $\mathscr{O}(U)\simeq\varprojlim_n \mathscr{O}_n(U_n)$ as topological vector spaces.*

PROOF. We can easily check the conditions of Lemma 1 of 5.5 in Chapter XI in Kantrovich-Akilov [10], which ensure that the canonical map $\mathscr{O}(U)\to\varprojlim \mathscr{O}_n(U_n)$ is an algebraic isomorphism. Obviosuly, the family of seminorms on $\mathscr{O}(U)$ defining the topology of $\mathscr{O}(U)$ coincides with that of $\varprojlim_n \mathscr{O}_n(U_n)$. Q.E.D.

COROLLARY 2.3. *$\mathscr{O}(U)$ is a Fréchet-Schwartz nuclear space.*

PROOF. Since $\mathscr{O}(U)$ is the projective limit of $\{\mathscr{O}_n(U_n)\}$ and since each $\mathscr{O}_n(U_n)$ is a Fréchet-Schwartz nuclear space, the general theory shows that $\mathscr{O}(U)$ is also a Fréchet-Schwartz nuclear space. Q.E.D.

REMARK. It was shown in Noverraz [15] that $\mathscr{O}(U)$ is a Fréchet space.

The presheaf $\{\mathscr{O}(U)\}$ defines a sheaf over $\sum C$, which is denoted by $\mathscr{O}$.

The maximum principle and the principle of analytic continuation hold not only for holomorphic functions on $\sum C$, but also for those on more general topological vector spaces; see for example [3] and [16].

Since a Gâteaux-analytic function is an analytic function (see [4]), the proposition similar to Hartogs' theorem of holomorphy holds obviously in the case of $\sum C$. We have the following proposition in the case of $\sum C$.

PROPOSITION 2.4. *Let U be a connected open subset of $\sum C$ and let K be a compact subset of U. Then, for any $f \in \mathcal{O}(U \backslash K)$, there exists a unique function $g \in \mathcal{O}(U)$ such that $f=g$ on $U \backslash K$.*

PROOF. For a sufficiently large positive integer n, K is a compact subset of $U \cap C^n$ and $(U \backslash K) \cap C^n$ is connected in $U \cap C^n$. By Hartogs' theorem, there exists a unique $g_n \in \mathcal{O}(U \cap C^n)$ such that $g_n = f|_{C^n}$ on $(U \backslash K) \cap C^n$, where $f|_{C^n} \in \mathcal{O}((U \backslash K) \cap C^n)$. On the other hand, $g_{n+1}|_{C^n} - g_n = 0$ on $(U \backslash K) \cap C^n$. Thus, by the principle of analytic continuation, $g_{n+1}|_{C^n} = g_n$ on $U \cap C^n$. In view of Proposition 2.2, there exists a unique function $g \in \mathcal{O}(U)$ such that $g|_{C^n} = g_n$ and $g=f$ on $U \backslash K$. Q.E.D.

We consider holomorphic functions on $\prod C$.

PROPOSITION 2.5. *Let U be an arbitrary open set in $\prod C$. Then, we have the isomorphism $\varinjlim_n \mathcal{O}_n(U^n) \simeq \mathcal{O}(U)$ as topological vector spaces, where $U^n = p_n(U)$ for $n>0$.*

PROOF. First, we shall check the conditions of Lemma 2 of 5.5 in Chapter XI in Kantrovich-Akilov [10] to show the canonical map $\varinjlim \mathcal{O}_n(U^n) \to \mathcal{O}(U)$ is an algebraic isomorphism. We define the map Ω_n from $\mathcal{O}_n(U^n)$ to $\mathcal{O}(U)$ by $\Omega_n(f) = f \circ p_n$ for $f \in \mathcal{O}_n(U^n)$. The other conditions being obviously satisfied, it only remains to show that $\mathcal{O}(U) \subset \bigcup_{n=1}^{\infty} \Omega_n(\mathcal{O}_n(U^n))$. Since $f \in \mathcal{O}(U)$ is continuous on U, for any $x \in U$ and any $\varepsilon > 0$ there exists an open neighborhood $V_x = D_x \times \prod_{i=q+1}^{\infty} C_i$ of x such that $|f(z) - f(x)| < \varepsilon$ holds for $z = (z_i) \in V_x$, where D_x is an open neighborhood of $p_q(x)$ in C^q for some positive integer q and C_i denotes the i-th coordinate axis of $\prod C$. In fact, every open set W in $\prod C$ has a form $W = W' \times \prod_{i=r+1}^{\infty} C_i$, where W' is an open set in C^r for some positive integer r. Thus, if we regard f as a function of only one variable z_m for sufficiently large positive integer m (the other variables being fixed), then f is bounded. Since f is an entire function of one complex variable z_m, f does not depend on z_m. Hence, $\mathcal{O}(U) \subset \bigcup_{n=1}^{\infty} \Omega_n(\mathcal{O}_n(U^n))$. We have obtained the algebraic isomorphism $\varinjlim \mathcal{O}_n(U^n) \simeq \mathcal{O}(U)$. The topology of $\varinjlim_n \mathcal{O}_n(U^n)$ is defined by all the seminorms $\{p\}$ for which $p \circ \Omega_n$ is continuous on $\mathcal{O}_n(U^n)$ for any $n>0$. On the other hand, let $U = W_m \times \prod_{i=m+1}^{\infty} C_i$. Then the family of compact sets $\{K \times \prod_{i=m+1}^{\infty} K_i\}$ is a basis of compact subsets of U, where K runs over compact sets in U and K_i runs over compact sets in C_i. In view of the above facts, it is easy to see that the topologies of $\mathcal{O}(U)$ and $\varinjlim_n \mathcal{O}_n(U^n)$ are equivalent. Q.E.D.

COROLLARY 2.6. *Let U be an open set in $\prod C$. Then, $\mathcal{O}(U)$ is a strict inductive limit of Fréchet-Schwartz spaces.*

PROOF. We define $\omega_n: \mathcal{O}_n(U^n) \to \mathcal{O}_{n+1}(U^{n+1})$ by $\omega_n(f) = f \circ p_n^{n+1}$ for $f \in \mathcal{O}_n(U^n)$, where p_n^{n+1} is the projection from C^{n+1} to C^n. Since ω_n is an isomorphism from $\mathcal{O}_n(U^n)$ onto $\omega_n(\mathcal{O}_n(U^n))$, the result follows from Proposition 2.5. Q.E.D.

Now, we introduce a subspace of $\mathcal{O}(\prod C)$.

DEFINITION 2.7. We denote by $\mathrm{Exp}(\prod C)$ the space of all functions of $\mathcal{O}(\prod C)$ that satisfy the following inequality:

$$|F(\zeta)| \leqq C \exp(r_1|\zeta_1| + \cdots + r_n|\zeta_n|) \qquad (\zeta = (\zeta_i) \in \prod C)$$

for some constant $C>0$, some positive integer n and some $(r_1, \cdots, r_n)$ $(r_i>0,\ i=1, \cdots, n)$. We call an element of $\mathrm{Exp}(\prod C)$ an entire function of exponential type on $\prod C$.

We also denote by $\mathrm{Exp}(C^n)$ the space of all entire functions of exponential type on C^n.

Put $H_K(\zeta) = \sup\{\mathrm{Re}\langle z, \zeta\rangle; z \in K\}$ for any convex compact set K in $\sum C$. Set $\mathrm{Exp}^b(\prod C, K) = \{F \in \mathrm{Exp}(\prod C); \|F\|_K < \infty\}$, where $\|F\|_K = \sup\{|F(\zeta)| \exp(-H_K(\zeta)); \zeta \in \prod C\}$. Obviously, $\mathrm{Exp}^b(\prod C, K)$ is a Banach space with the norm $\|\cdot\|_K$. We define the topology of $\mathrm{Exp}(\prod C)$ by $\varinjlim_K \mathrm{Exp}^b(\prod C, K)$, where K runs over all compact subsets of $\sum C$. This topology is a dual Fréchet-Schwartz topology. In view of the above discussion, we have the following

PROPOSITION 2.8. *We have the isomorphism $\varinjlim_n \mathrm{Exp}(C^n) \simeq \mathrm{Exp}(\prod C)$ as topological vector spaces.*

Next, we go on to the dual space of $\mathcal{O}(U)$.

DEFINITION 2.9. Let U be an open set in $\sum C$. $\mathcal{O}'(U)$ denotes the topological dual space of $\mathcal{O}(U)$. An element of $\mathcal{O}'(U)$ is called an analytic functional on U.

$\mathcal{O}'(U)$ is a dual Fréchet-Schwartz nuclear space.

LEMMA 2.10. *Let U be a pseudo-convex open set in C^{n+1}. Then, the restriction map from $\mathcal{O}_{n+1}(U)$ to $\mathcal{O}_n(U')$ is surjective, where $U'=U\cap C^n$.*

PROOF. Consider the following exact sequence of sheaves on C^{n+1},

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{n+1} \longrightarrow u^n_{n+1*}\mathcal{O}_n \longrightarrow 0 ,$$

where $\mathcal{I}$ is the sheaf of ideals of C^n and $u^n_{n+1*}\mathcal{O}_n$ denotes the direct image of $\mathcal{O}_n$. Because the sheaf $u^n_{n+1*}\mathcal{O}_n$ is a coherent sheaf of $\mathcal{O}_{n+1}$-modules by Theorem 3 of §3 in Chapter I in Grauert and Remmert [8], $\mathcal{I}$ is a coherent sheaf of $\mathcal{O}_{n+1}$-modules. Therefore, $H^1(U, \mathcal{I})=0$ holds. Thus, we obtain the result. Q.E.D.

PROPOSITION 2.11. *Let U be a pseudo-convex open set in $\sum C$. Then, we have the isomorphism $\varinjlim_n \mathcal{O}'_n(U_n)\simeq\mathcal{O}'(U)$ as topological vector spaces.*

PROOF. By Proposition 2.2 and Lemma 2.10, the restriction map from $\mathcal{O}(U)$ to $\mathcal{O}_n(U_n)$ is surjective. The restriction map from $\mathcal{O}_{n+1}(U_{n+1})$ to $\mathcal{O}_n(U_n)$ maps a bounded set to a relatively compact set. Therefore, the result follows from Theorem 5.13 in Komatsu [12]. Q.E.D.

We treat the problem of supports of analytic functionals on $\sum C$.

DEFINITION 2.12. Let U be an arbitrary open set in $\sum C$.

i) We say that $T\in\mathcal{O}'(U)$ is carried by an open set $W\subset U$ if $T\in {}^t\iota(\mathcal{O}'(W))$, where ι is the restriction map from $\mathcal{O}(U)$ to $\mathcal{O}(W)$ and ${}^t\iota$ is the transpose of ι.

ii) We say that $T\in\mathcal{O}'(U)$ is carried by a compact set K if T is carried by any open set W such that $U\supset W\supset K$.

Put $\delta(x)=\bigotimes_{j=1}^n\delta(x_j)$ $(n\leqq\infty)$. Let U be a domain of holomorphy in $\sum C$ and let T be an analytic functional other than $\delta(x)$ on U. Then, owing to Proposition 2.18, there exists the smallest positive integer n such that $T=T'\bigotimes_{j=n+1}^\infty\delta(x_j)$, where $T'\in\mathcal{O}'_n(U_n)$ is not the δ-function on U_n. We have the following

PROPOSITION 2.13. *Assume $T=T'\bigotimes_{j=n+1}^\infty\delta(x_j)$ as above. Let K be a compact subset of U such that $K\subset C^m$. In the case $m\leqq n$, T is carried by K if and only if T' is carried by K. In the case $m>n$, T is carried by K if and only if $T'\bigotimes_{j=n+1}^m\delta(x_j)$ is carried by K.*

LEMMA 2.14. *$T\in\mathcal{O}'(U)$ is carried by a compact set K ($\subset C^m$) if and only if for any compact set L such that $K\subset L\subset U_m$, there exists a constant $C_L>0$ such that*

$$|T(f)|\leqq C_L\sup_{z\in L}|f(z)| \qquad \textit{for any} \quad f\in\mathcal{O}(U)\,.$$

PROOF. Suppose that there exists a compact set L such that for any constant $C>0$ there exists a function $f_C\in\mathcal{O}(U)$ for which

$$|T(f_C)|>C\sup_{z\in L}|f_C(z)|\,.$$

Since $f_C(z)$ is continuous on U, there exists an open neighborhood ω of L in U such that

$$|T(f_C)|>C\sup_{z\in\omega}|f_C(z)|\,.$$

This inequality implies that

$$T\notin {}^t r(\mathcal{O}'(\omega))$$

where r is the restriction map from $\mathcal{O}(U)$ to $\mathcal{O}(\omega)$. It contradicts the assumption that T is carried by K.

Conversely, let W be an open neighborhood of K. For a compact set L such that $W\supset L\supset K$, there exists a constant $C_L>0$ such that

$$|T(f)|\leqq C_L\sup_{z\in L}|f(z)| \qquad \text{for any} \quad f\in\mathcal{O}(U)\,.$$

This implies that ${}^t\iota(\mathcal{O}'(W))\ni T$, where ι is the restriction map from $\mathcal{O}(U)$ to $\mathcal{O}(W)$. Q.E.D.

Using this lemma we will prove Proposition 2.13.

PROOF OF PROPOSITION 2.13. Since the proof in the case of $m\leqq n$ is almost same as that in the case of $m>n$, we will prove only the case of $m>n$. Suppose that T is carried by K. Owing to Lemma 2.14, if $K\subset C^m$, for any compact set L such that $K\subset L\subset U_m$, there exists a constant $C_L>0$ such that

$$|T(f)|\leqq C_L\sup_{z\in L}|f(z)| \qquad \text{for any} \quad f\in\mathcal{O}_n(U_n)\,.$$

Since $T=T'\bigotimes_{j=n+1}^{\infty}\delta(x_j)$,

$$|(T'\bigotimes_{j=n+1}^{m}\delta(x_j))(f)|\leqq C_L\sup_{z\in L}|f(z)| \qquad \text{for any} \quad f\in\mathcal{O}(U)|_{C^m}=\mathcal{O}_m(U_m)\,.$$

This implies that $T'\bigotimes_{j=n+1}^{m}\delta(x_j)$ is carried by K.

Conversely, suppose that $T'\bigotimes_{j=n+1}^{m}\delta(x_j)$ is carried by K. Then, for any compact set L in C^m such that $K\subset L\subset C^m$, there exists a constant $C_L>0$ such that

$$|(T'\bigotimes_{j=n+1}^{m}\delta(x_j))(f)|\leqq C_L\sup_{z\in L}|f(z)| \qquad \text{for any} \quad f\in\mathcal{O}_m(U_m)\ .$$

Since $T=T'\bigotimes_{j=n+1}^{\infty}\delta(x_j)$, we get

$$|T(f)|\leqq C_L\sup_{z\in L}|f(z)| \qquad \text{for any} \quad f\in\mathcal{O}(U)\ .$$

For any compact subset M of C^m such that $L\subset M$ and $K\subset M\subset U_m$, we get obviously,

$$|T(f)|\leqq C_L\sup_{z\in M}|f(z)| \qquad \text{for any} \quad f\in\mathcal{O}(U)\ .$$

In view of Lemma 2.14, this implies that T is carried by K. Q.E.D.

Summing up, the problem of supports of $T\in\mathcal{O}'(U)$ is reduced to that of $T'\in\mathcal{O}'_n(U_n)$.

Now, we consider the Fourier-Borel transformation for analytic functionals on $\sum C$. See [9] and [13] for the Fourier-Borel transformation in finite dimensional spaces.

Definition 2.15. We call the function $\tilde{T}(\zeta)=T_z(e^{\langle z,\zeta\rangle})$ ($z\in\sum C$, $\zeta\in\prod C$) on $\prod C$ the Fourier-Borel transform of $T\in\mathcal{O}'(\sum C)$.

Theorem 2.16. *If $T\in\mathcal{O}'(\sum C)$ is carried by a compact set K ($\subset C^n$), then $M(\zeta)=\tilde{T}(\zeta)$ is an entire function on $\prod C$ satisfying the following inequality: For any $\delta>0$, there exists a constant $C_\delta>0$ such that*

$$|M(\zeta)|\leqq C_\delta\exp(H_K(\zeta)+\delta||\zeta||_n) \tag{2.1}$$

where $||\zeta||_n=(|\zeta_1|^2+\cdots+|\zeta_n|^2)^{1/2}$ for $\zeta=(\zeta_i)\in\prod C$. Conversely, if K ($\subset C^n$) is a convex compact subset of $\sum C$ and if $M(\zeta)$ is an entire function and satisfies the inequality (2.1) *for any $\delta>0$, then there exists an analytic functional $T\in\mathcal{O}'(\sum C)$ carried by K such that $\tilde{T}(\zeta)=M(\zeta)$.*

Proof. If $T\in\mathcal{O}'(\sum C)$ is carried by K, by virtue of Proposition 2.11 there exists $T'\in\mathcal{O}'_n(C^n)$ such that ${}^tu_n(T')=T$ and T' is carried by K, where ${}^tu_n$ is the transpose of u_n. If we restrict $e^{\langle z,\zeta\rangle}$ to C^n, $T'(e^{\langle z',\zeta'\rangle})$ is an entire function of $\zeta'\in C^n$ by Proposition 1.12 in Chapter 2 in Martineau [13] and

$$|T'(e^{\langle z',\zeta'\rangle})|\leqq C_\delta\exp(H_K(\zeta')+\delta|\zeta'|_n)$$

holds for $\zeta' \in C^n$. Thus, we regard $T(e^{\langle z,\zeta\rangle})$ as an entire function on $\prod C$ by Proposition 2.5 and we have

$$|T(e^{\langle z,\zeta\rangle})| \leqq C_\delta \exp(H_K(\zeta)+\delta\|\zeta\|_n) \qquad \text{for} \quad \zeta \in \prod C\,.$$

Conversely, suppose that (2.1) holds for $M(\zeta)$. Then, (2.1) implies

$$|M(\zeta')| \leqq C_\delta \exp(H_K(\zeta')+\delta|\zeta'|_n) \qquad \text{for} \quad \zeta' \in C^n\,.$$

By Proposition 1.12 in Chapter 2 in [13], there exists an analytic functional $T' \in \mathcal{O}'_n(C^n)$ carried by K such that $M(\zeta')=\tilde{T}'(\zeta')$. Thus, if we set $T={}^tu_n(T') \in \mathcal{O}'(\sum C)$, then T is carried by K in view of Proposition 2.13.
Q.E.D.

Corollary 2.17. *The Fourier-Borel transformation* $\mathcal{F}: T \mapsto \tilde{T}$ *induces a topological isomorphism of* $\mathcal{O}'(\sum C)$ *onto* $\mathrm{Exp}(\prod C)$.

Proof. The injectivity follows from the fact that linear combinations of the exponential functions $e^{\langle z,\zeta\rangle}$ are dense in $\mathcal{O}(\sum C)$. It only remains to show that the Fourier-Borel transformation is continuous, so that the result follows from the above theorem and the closed graph theorem for dual Fréchet-Schwartz spaces. Let $\{K_i\}$ be an exhausting sequence of $\sum C$ which consists of convex compact sets such that $K_n \subset C^n$. Then, we have $\mathcal{O}'(\sum C) \cong \varinjlim_n \mathcal{O}'_n(K_n) \cong \varinjlim_n \mathcal{O}'_{n,B}(K_n)$, where $\mathcal{O}_{n,B}(K_n)$ denotes the space of all the functions that are holomorphic in $\mathring{K}_n$ and bounded on K_n. Since the Fourier-Borel transformation of $\mathcal{O}'_n(K_n)$ to $\mathrm{Exp}(C^n)$ is continuous, so is the Fourier-Borel transformation of $\mathcal{O}'_{n,B}(K_n)$ to $\mathrm{Exp}(C^n)$. Therefore, the Fourier-Borel transformation of $\mathcal{O}'(\sum C)$ to $\mathrm{Exp}(\prod C)$ is continuous.
Q.E.D.

Remark. It was shown in Boland-Dineen [1] that the Fourier-Borel transformation of $\mathcal{O}'(\sum C)$ onto $\mathrm{Exp}(\prod C)$ is an algebraic isomorphism.

Now, we treat the case of infinitely differentiable functions in $\sum R$.

Definition 2.18. Let Ω be an arbitrary open set in $\sum R$ (or $\sum R^2$). We denote by $\mathcal{E}(\Omega)$ the topological vector space of all infinitely differentiable functions on Ω endowed with the topology defined by the seminorms:

$$|f|_{m,K} = \sup_{|p|\leqq m} (\sup_{x\in K} |(\partial/\partial x)^p f(x)|) \qquad \text{for} \quad f \in \mathcal{E}(\Omega)\,,$$

where K runs over all compact subsets of Ω, $m=1, 2, \cdots$, $p=(p_1, p_2, \cdots)$, p_j is a non-negative integer and $|p|=p_1+p_2+\cdots$.

REMARK. We say that $f: \Omega \to C$ is infinitely differentiable if $f|_{R^n}: \Omega \cap R^n \to C$ is infinitely differentiable for any $n>0$.

Just the same as in the case of $\mathcal{O}(U)$ (Proposition 2.2), we can prove

PROPOSITION 2.19. *Let Ω be an arbitrary open set in $\sum R$. Then, we have the isomorphism $\mathscr{E}(\Omega) \simeq \varprojlim_n \mathscr{E}_n(\Omega_n)$ as topological vector spaces. Hereafter we will write $\Omega_n = \Omega \cap R^n$.*

We can easily obtain the following corollary:

COROLLARY 2.20. *$\mathscr{E}(\Omega)$ is a Fréchet-Schwartz nuclear space.*

The presheaf $\{\mathscr{E}(\Omega)\}$ defines a sheaf over $\sum R$, which is denoted by $\mathscr{E}$.

LEMMA 2.21. *Let Ω be an open set in R^{n+1}. Then, the restriction map from $\mathscr{E}_{n+1}(\Omega)$ to $\mathscr{E}_n(\Omega')$ is surjective, where $\Omega' = \Omega \cap R^n$.*

PROOF. Let $\mathscr{I}$ be the sheaf of germs of infinitely differentiable functions in R^{n+1} which vanish on R^n. Then, $\mathscr{I}$ is a sheaf of $\mathscr{E}_{n+1}$-modules, so that it is soft by Theorem 2 in Chapter A in [8]. Since $H^1(\Omega, \mathscr{I})=0$ holds, the lemma results from the following exact sequence of the sheaves on R^{n+1}:

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{E}_{n+1} \longrightarrow u^n_{n+1*}\mathscr{E}_n \longrightarrow 0 . \qquad \text{Q.E.D.}$$

PROPOSITION 2.22. *We have the isomorphism $\varinjlim_n \mathscr{E}'_n(\Omega_n) \simeq \mathscr{E}'(\Omega)$ as topological vector spaces.*

PROOF. By Proposition 2.19 and Lemma 2.21, the restriction map from $\mathscr{E}_{n+1}(\Omega_{n+1})$ to $\mathscr{E}_n(\Omega_n)$ is surjective. The restriction map from $\mathscr{E}_{n+1}(\Omega_{n+1})$ to $\mathscr{E}_n(\Omega_n)$ maps a bounded set to a relatively compact set. Therefore, from Theorem 5.13 in Komatsu [12] follows the above result. Q.E.D.

We denote by ρ_n the canonical map from $\mathscr{E}'_n(\Omega_n)$ to $\mathscr{E}'(\Omega)$. We consider the Fourier transfrom of $T \in \mathscr{E}'(\sum R)$.

DEFINITION 2.23. For any $T \in \mathscr{E}'(\sum R)$ we can find $T' \in \mathscr{E}'_n(R^n)$ such that $\rho_n(T')=T$ for some $n>0$ by Proposition 2.22. We define the support $\operatorname{Supp} T$ of $T \in \mathscr{E}'(\sum R)$ to be the support of $T' \in \mathscr{E}'_n(R^n)$.

If we regard $T \in \mathscr{E}'(\sum R)$ as $T' \in \mathscr{E}'_n(R^n)$ and as $T'' \in \mathscr{E}'_m(R^m)$ $(n \neq m)$ in two ways, the support of $T' \in \mathscr{E}'_n(R^n)$ coincides with that of $T'' \in \mathscr{E}'_m(R^m)$. Therefore, the above definition makes sense and the support of T is uniquely determined.

DEFINITION 2.24. We define the Fourier transform of $T \in \mathscr{E}'(\sum \boldsymbol{R})$ by $\langle T_x, e^{-i\langle x, \xi\rangle}\rangle$ which we denote by $\hat{T}(\xi)$.

We will prove a theorem of Paley-Wiener type.

THEOREM 2.25. *Let K be a balanced convex compact subset of $\boldsymbol{R}^n$. For $T \in \mathscr{E}'(\sum \boldsymbol{R})$ the following properties are equivalent:*

(a) *The balanced convex hull of the support of T is contained in K:*

(b) *The Fourier transform of T is extended on $\prod \boldsymbol{C}$ as an entire function $\zeta \mapsto \hat{T}(\zeta)$ and there exist an integer $m>0$ and a constant $C>0$ such that the following inequality holds for $\zeta=\xi+i\eta$ (ξ, $\eta \in \prod \boldsymbol{R}$):*

$$|\hat{T}(\zeta)| \leqq C(1+\|\zeta\|_n)^m \exp(I_K(\eta)) , \tag{2.2}$$

where $I_K(\eta)=\sup_{x\in K} |\langle x, \eta\rangle|$.

PROOF. In view of Proposition 2.22 and the property (a), there exists $T' \in \mathscr{E}_n'(\boldsymbol{R}^n)$ such that $T=T' \otimes_{j=n+1}^{\infty} \delta(x_j)$. For $T' \in \mathscr{E}_n'(\boldsymbol{R}^n)$, by the Paley-Wiener theorem, $\zeta' \mapsto \hat{T}'(\zeta')$ is an entire function on $\boldsymbol{C}^n$ and there exist an integer $m>0$ and a constant $C>0$ such that the following inequality holds:

$$|\hat{T}'(\zeta')| \leqq C(1+|\zeta'|_n)^m \exp(I_K(\eta')) \qquad \text{for} \quad \zeta' \in \boldsymbol{C}^n .$$

Therefore, we have

$$|\hat{T}(\zeta)| \leqq C(1+\|\zeta\|_n)^m \exp(I_K(\eta)) \qquad \text{for} \quad \zeta \in \textstyle\sum \boldsymbol{C} .$$

Conversely, suppose that the inequality (2.2) holds:

$$|\langle T_x, e^{-i\langle x, \zeta\rangle}\rangle| = |\hat{T}(\zeta)| \leqq C(1+\|\zeta'\|_n)^m \exp(I_K(\eta')) .$$

Let $T=T' \otimes_{j=n+1}^{\infty} \delta(x_j)$, where $T' \in \mathscr{E}_n'(\boldsymbol{R}^n)$. Then, $\hat{T}'(\zeta')=\langle T', e^{-i\langle x', \zeta'\rangle}\rangle$ is an entire function on $\boldsymbol{C}^n$ and

$$|\hat{T}'(\zeta')| \leqq C(1+|\zeta'|_n)^m \exp(I_K(\eta')) \qquad \text{for} \quad \zeta' \in \boldsymbol{R}^n ,$$

$x' \in \boldsymbol{R}^n$. Thus, by the Paley-Wiener theorem, the balanced convex hull of the support of T' is contained in K. Since $\operatorname{Supp} T=\operatorname{Supp} T'$, we proved the theorem. Q.E.D.

§ 3. Vanishing of cohomology groups.

Dineen [6] proved that $H^1(U, \mathscr{O})=0$ for any pseudo-convex open set U in a topological vector space with the finite open topology.

REMARK. An infinite dimensional vector space E is said to be endowed with the finite open topology if its topology is defined by the set of all finitely open sets, where a subset U of E is said to be finitely open if $U \cap F$ is open in F for any finite dimensional subspace F of E.

We will prove that the p-th cohomology group vanishes for $p \geqq 1$ in the case of $\sum C$ by the method remarked in the above paper.

Let D be an arbitrary open set $\sum \boldsymbol{R}$. We denote by $\mathscr{E}_D$ the restriction of the sheaf $\mathscr{E}$ to D.

PROPOSITION 3.1. *$\mathscr{E}_D$ is a fine sheaf.*

PROOF. 1. Let U, E and F be an open subset, a closed subset and a closed subset of $\sum \boldsymbol{R}$ respectively such that $U \supset F \supset \mathring{F} \supset E$. Assume without loss of generality that E contains the origin and $U \cap \boldsymbol{R}^n$ is connected in $\boldsymbol{R}^n$ for every $n>0$. Then, we will prove that there exists a C^∞-function ψ defined on $\sum \boldsymbol{R}$ such that

1) $\psi \geqq 0$,

2) $\psi = 1$ on some neighborhood of E,

3) $\operatorname{Supp} \psi \subset F$.

Put $\Delta_x(\delta_x) = \{z = (z_i) \in \sum \boldsymbol{R};\ |z_i - x_i| < \delta_{x_i}\}$, where $\delta_x = (\delta_{x,i})$, $0 < \delta_{x,i} < \infty$. Assume that $x=0$. Obviously, for every i there exists a C^∞-function ϕ_i defined on $\boldsymbol{R}$ such that

1) $\phi_i \geqq 0$,

2) $\phi_i > 0$ on the open disc with radius $\delta_{0,i}/2$, $\phi_i(0)=1$,

3) $\operatorname{Supp} \phi_i$ is contained in the open disc with radius $\delta_{0,i}$.

Put $\phi(z) = \prod_{i=1}^\infty \phi_i(z_i)$, then $\phi(z)$ is a C^∞-function defined on $\sum \boldsymbol{R}$ that satisfies the following conditions:

1) $\phi(z) \geqq 0$,

2) $\phi(z) > 0$ on $\Delta_0(\delta_0/2)$,

3) $\operatorname{Supp} \phi \subset \Delta_0(\delta_0)$.

2. Let $\{X_i\}$ be the exhausting sequence of compact subsets constructed in the proof of Proposition 1.2, replacing C by $\boldsymbol{R}$. We choose $\Delta_x(\delta_x)$ for each $x \in X_1$ as follows: For any $n>1$,

1) if $x \in X_1 \cap E$, $\Delta_x(\delta_x) \cap \boldsymbol{R}^n \subset X_n \cap F$,

2) if $x \in X_1 \cap (F \backslash E)$, $\Delta_x(\delta_x) \cap \boldsymbol{R}^n \subset X_n$ and $\Delta_x(\delta_x)$ does not intersect with both of E and ∂F,

3) if $x \in X_1 \backslash F$, $\Delta_x(\delta_x) \cap \boldsymbol{R}^n \subset X_n \cap \complement E$,

where ∂F denotes the boundary of F. Then, $\{\Delta_x(\delta_x/2) \cap \boldsymbol{R}^2\}_{x \in X_1}$ is a covering of X_1. Since X_1 is compact, a finite subcovering $\{\Delta_{x_i}(\delta_{x_i}/2) \cap \boldsymbol{R}^2\}_{i \in I_1}$ of $\{\Delta_x(\delta_x/2) \cap \boldsymbol{R}^2\}_{x \in X_1}$cover X_1. Put $V_1 = \bigcup_{i \in I_1} \Delta_{x_i}(\delta_{x_i}/2)$ and $\mathfrak{U}_1 = \{\Delta_{x_i}(\delta_{x_i})\}_{i \in I_1}$.

3. Suppose that, for any $j \leqq k$, V_j and $\mathfrak{U}_j = \{\Delta_{x_i}(\delta_{x_i})\}_{i \in I_j}$ $(x_i \in X_j)$ have been constructed with the following properties:

$$(3.1)\quad\left\{\begin{array}{ll}\text{I)} & \text{For any } \Delta_x(\delta_x)\in\mathfrak{U}_j, \text{ there exists } \varepsilon_{j-1}>0 \text{ such that for any positive integer } n>j,\\ & \quad\text{i) if } x\in(X_j-V_{j-1})\cap E, \text{ then } \Delta_x(\delta_x)\cap X_{j-1,\varepsilon_{j-1}}=\varnothing \text{ and } \Delta_x(\delta_x)\cap R^n\subset X_n\cap F,\\ & \quad\text{ii) if } x\in(X_j-V_{j-1})\cap(F\backslash E), \text{ then } \Delta_x(\delta_x)\cap X_{j-1,\varepsilon_{j-1}}=\varnothing,\ \Delta_x(\delta_x)\cap R^n\subset X_n \text{ and } \Delta_x(\delta_x) \text{ does not intersect with both of } E \text{ and } \partial F,\\ & \quad\text{iii) if } x\in(X_j-V_{j-1})\backslash F, \text{ then } \Delta_x(\delta_x)\cap X_{j-1,\varepsilon_{j-1}}=\varnothing \text{ and } \Delta_x(\delta_x)\cap R^n\subset X_n\cap\complement E,\\ \text{II)} & V_j=V_{j-1}\cup(\bigcup_{i\in I_j}\Delta_{x_i}(\delta_{x_i}/2)\cap R^{j+1})\supset X_j.\end{array}\right.$$

Then, there exists $\varepsilon_k>0$ such that $V_k\cap R^{k+1}\supset X_{k,\varepsilon_k}$. For each $x\in X_{k-1}-V_k$, we can choose $\Delta_x(\delta_x)$ which satisfies the condition I) of (3.1) with j replaced by $k+1$. Then, $\{\Delta_x(\delta_x/2)\cap R^{k+2}\}_{x\in X_{k+1}-V_k}$ is an open covering of $X_{k+1}-V_k$ in R^{k+2}. Since $X_{k+1}-V_k$ is compact, a finite subcovering $\{\Delta_{x_i}(\delta_{x_i}/2)\cap R^{k+2}\}_{i\in I_{k+1}}$ of $\{\Delta_x(\delta_x/2)\cap R^{k+2}\}_{x\in X_{k+1}-V_k}$ cover $X_{k+1}-V_k$. Put

$$V_{k+1}=V_k\cup\Big(\bigcup_{i\in I_{k+1}}\Delta_{x_i}\Big(\frac{1}{2}\delta_{x_i}\Big)\Big)\qquad\text{and}\qquad \mathfrak{U}_{k+1}=\{\Delta_{x_i}(\delta_{x_i})\}_{i\in I_{k+1}}\,.$$

Thus, by induction, we construct V_j and $\mathfrak{U}_j$ with the property (3.1) for every $j=1, 2, 3, \cdots$. Put $\mathfrak{U}=\bigcup_{i=1}^{\infty}\mathfrak{U}_i$. Then, $\mathfrak{U}$ is an open covering of U. The construction shows that, for any compact subset of U, there are only a finite number of elements of $\mathfrak{U}$ which intersect with it. Put $I_F=\{i\in\bigcup_{n=1}^{\infty}I_n;\ \Delta_{x_i}(\delta_{x_i})\cap F\neq\varnothing\}$, $I_E=\{i\in I_F;\ \Delta_{x_i}(\delta_{x_i})\cap E\neq\varnothing\}$, $\mathfrak{U}_F=\{\Delta_{x_i}(\delta_{x_i})\}_{i\in I_F}$ and $\mathfrak{U}_E=\{\Delta_{x_i}(\delta_{x_i})\}_{i\in I_E}$. Let ϕ_i be a C^∞-function constructed in the same way as in 1 subordinate to $\Delta_{x_i}(\delta_{x_i})$ $(i=1, 2, 3, \cdots)$. Thus, $\phi(z)=\sum_{i\in I_F}\phi_i(z)$ is well defined and is a positive C^∞-function. In fact, for any $x\in\sum R$ if we choose an open neighborhood U_x of x such that $U_x\cap R^n$ is relatively compact, $U_x\cap R^n$ intersects with only finitely many elements of $\mathfrak{U}_F$. Hence, $\phi(z)$ is a C^∞-function in view of the definition. Since $\{\Delta_{x_i}(\delta_{x_i}/2)\}_{i\in I_F}$ is also a covering of F, $\phi(z)$ is positive at every point of F. Set $\psi_j(z)=(\phi_j/\phi)(z)$ for $z\in F$, $\psi_j(z)=0$ for $z\in\complement F$ and $\psi=\sum_{i\in I_E}\psi_i$. $\psi(z)$ satisfies the conditions needed.

4. Let $\{U_\alpha\}$ be an arbitrary locally finite open covering of D. Since D is paracompact, there is a locally finite refinement $\{V_\alpha\}$ of $\{U_\alpha\}$ such that $\{\bar{V}_\alpha\subset U_\alpha\}$. Just the same way as above, we can choose a locally finite refinement $\{W_\alpha\}$ of $\{V_\alpha\}$ such that $\bar{W}_\alpha\subset V_\alpha$. Set $F=\bar{V}_\alpha$ and $E=\bar{W}_\alpha$, then, 1, 2 and 3 imply that there exists ψ_α subordinate to U_α. Set $\psi'=\sum_\alpha\psi_\alpha$ and $\varphi_\alpha=\psi_\alpha/\psi'$. Hence, we have a family of C^∞-functions $\{\varphi_\alpha\}$ subordinate to $\{U_\alpha\}$ such that

1) $\varphi_\alpha \geqq 0$ and $\varphi_\alpha = 0$ on $\complement U_\alpha$,

2) $\sum_\alpha \varphi_\alpha(x) = 1$ for $x \in D$.

Since the operation of multiplication by φ_α induces a sheaf homomorphism from $\mathscr{E}_D$ to $\mathscr{E}_D$, $\mathscr{E}_D$ is a fine sheaf. Q.E.D.

REMARK. Changing $\sum \boldsymbol{R}$ by $\sum \boldsymbol{R}^2$, this proposition still holds. Hereafter we identify $\sum \boldsymbol{C} \cong \sum \boldsymbol{R}^2$.

DEFINITION 3.2 [6]. Let U be an open set in $\sum \boldsymbol{C}$. For each positive integer p, we denote by $\mathscr{E}^{0,p}(U)$ the space of all forms of the following kind:

$$f = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}$$

where $f_{i_1 \cdots i_p} \in \mathscr{E}(U)$ and $z = (z_i) \in \sum \boldsymbol{C}$. For $p = 0$ we put $\mathscr{E}^{0,0}(U) = \mathscr{E}(U)$. We define the $\bar{\partial}$-operator linearly by

$$\bar{\partial} f = \sum_{i_1 < \cdots < i_p} \bar{\partial}(f_{i_1 \cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}) ,$$

where

$$\begin{aligned}
&\bar{\partial}(f_{i_1 \cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}) \\
&\quad = \sum_{j=0}^{i_1 - 1} \frac{\partial f_{i_1 \cdots i_p}}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \\
&\qquad + \sum_{k=1}^{p-1} \sum_{i_k < j < i_{k+1}} (-1)^k \frac{\partial f_{i_1 \cdots i_p}}{\partial \bar{z}_j} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k} \wedge d\bar{z}_j \wedge d\bar{z}_{i_{k+1}} \wedge \cdots \wedge d\bar{z}_{i_p} \\
&\qquad + (-1)^p \sum_{j > i_p} \frac{\partial f_{i_1 \cdots i_p}}{\partial \bar{z}_j} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \wedge d\bar{z}_j .
\end{aligned}$$

For $p \geqq 0$, $\mathscr{E}^{0,p}$ denotes the sheaf associated with the presheaf $\{\mathscr{E}^{0,p}(U)\}$.

We denote the restriction map from $\mathscr{E}^{0,p}(U)$ (resp. $\mathscr{E}^{0,p}_{n+1}(U_{n+1})$) to $\mathscr{E}^{0,p}_n(U_n)$ by

$$\sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} |_{C^n} = \sum_{i_1 < \cdots < i_p \leqq n} f_{i_1 \cdots i_p} |_{C^n} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} ,$$

where $f_{i_1 \cdots i_p} \in \mathscr{E}(U)$ (resp. $f_{i_1 \cdots i_p} \in \mathscr{E}_{n+1}(U_{n+1})$).

LEMMA 3.3 [6]. *Let U be an arbitrary pseudo-convex open set of $\sum \boldsymbol{C}$. If $g \in \mathscr{E}^{0,q+1}(U)$ and $\bar{\partial} g = 0$, then there exists $f \in \mathscr{E}^{0,q}(U)$ such that $\bar{\partial} f = g$ for each non-negative integer q.*

PROOF. For each positive integer n, it is well known that the following sequence is exact:

$$0 \longrightarrow \mathcal{O}_n(U_n) \longrightarrow \mathscr{E}_n^{0,0}(U_n) \longrightarrow \cdots \longrightarrow \mathscr{E}_n^{0,n}(U_n) \longrightarrow 0 .$$

For any integer $n \geqq q+1$, $\bar{\partial}(g|_{C^n})=0$. Owing to the exactness of the above sequence there exists $f_n \in \mathscr{E}_n^{0,q}(U_n)$ such that $\bar{\partial} f_n = g|_{C^n}$. Since $\bar{\partial}(f_{n+1}|_{C^n} - f_n) = g|_{C^n} - g|_{C^n} = 0$, there exists $h_n \in \mathscr{E}_n^{0,q-1}(U_n)$ for $q \geqq 1$ (resp. $h_n \in \mathcal{O}_n(U_n)$ for $q=0$) such that $f_{n+1}|_{C^n} - f_n = h_n$. We have $h_{n+1} \in \mathscr{E}_{n+1}^{0,q-1}(U_{n+1})$ for $q \geqq 1$ (resp. $h_{n+1} \in \mathcal{O}_{n+1}(U_{n+1})$ for $q=0$) such that $h_{n+1}|_{C^n} = h_n$ and $(f_{n+1}+h_{n+1})|_{C^n} = f_n$ for $q \geqq 1$ (resp. $(f_{n+1}+h_{n+1})|_{C^n} = f_n$ for $q=0$). In fact, the restriction map from $\mathscr{E}_{n+1}^{0,q}(U_{n+1})$ to $\mathscr{E}_n^{0,q}(U_n)$ (resp. from $\mathcal{O}_{n+1}(U_{n+1})$ to $\mathcal{O}_n(U_n)$) is surjective. Thus, a solution of the equation $\bar{\partial} f = g|_{C^n}$ on U_n can be extended to a solution on U_{n+1}. We can assume without loss of generality that $f_{n+1}|_{C^n} = f_n$. Therefore, there exists $f \in \mathscr{E}^{0,q}(U)$ such that $f|_{C^n} = f_n$ and $(\bar{\partial} f - g)|_{C^n} = 0$. We obtain $\bar{\partial} f = g$. Q.E.D.

COROLLARY 3.4. *The following sequence is exact:*

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \mathscr{E}^{0,0}(U) \longrightarrow \mathscr{E}^{0,1}(U) \longrightarrow \cdots .$$

PROPOSITION 3.5. *Let D be any open set in $\sum C$. The following exact sequence is a fine resolution of the sheaf $\mathcal{O}_D$:*

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathscr{E}_D^{0,0} \longrightarrow \mathscr{E}_D^{0,1} \longrightarrow \cdots .$$

PROOF. Since each $x \in D$ has a basis of neighborhoods consisting of pseudo-convex open sets, the exactness follows from the above corollary. In view of Proposition 3.1, the sheaf $\mathscr{E}_D^{0,p}$ is fine for each $p \geqq 0$. Q.E.D.

Now, we have the following

THEOREM 3.6. *Let U be an arbitrary pseudo-convex open set in $\sum C$. Then, we have*

$$H^p(U, \mathcal{O}) = 0 \qquad \textit{for} \quad p \geqq 1 ,$$

where $H^p(U, \mathcal{O})$ is the p-th cohomology group of U with coefficients in the sheaf $\mathcal{O}$.

PROOF. Since $0 \to \mathcal{O}_U \to \mathscr{E}_U^{0,0} \to \mathscr{E}_U^{0,1} \to \cdots$ is a fine resolution of $\mathcal{O}_U$, it can be concluded from the cohomology group theory on a paracompact space that

$$H^p(U, \mathcal{O}_U) = \frac{\operatorname{Ker}\{\Gamma(U, \mathscr{E}_U^{0,p}) \to \Gamma(U, \mathscr{E}_U^{0,p+1})\}}{\operatorname{Im}\{\Gamma(U, \mathscr{E}_U^{0,p-1}) \to \Gamma(U, \mathscr{E}_U^{0,p})\}} \qquad (p \geqq 1) .$$

Therefore, by Corollary 3.4, we get

$$H^p(U, \mathcal{O}) = H^p(U, \mathcal{O}_U) = 0 \qquad (p \geqq 1) . \qquad \text{Q.E.D.}$$

In view of Proposition 1.4, we have the following

COROLLARY 3.7. *Let K be a polynomially convex compact set in $\sum C$. Then, we have*

$$H^p(K, \mathcal{O})=0 \qquad \textit{for} \quad p\geqq 1 .$$

§ 4. Definitions of hyperfunctions and distributions.

In this section we will give definitions of hyperfunctions and distributions on $\sum R$.

Let Ω be an open set in $\sum R$. There is no continuous functions with compact support on Ω except zero. Hence, it is natural to consider the following subspace of $\mathcal{E}(\Omega)$ to define distributions on Ω. Put

$$\mathcal{D}_b(\Omega)=\{f\in\mathcal{E}(\Omega);\ \mathrm{Supp}\, f\cap R^n \text{ is compact for any } n>0\} .$$

Then, we have the canonical isomorphism

$$\mathcal{D}_b(\Omega)\simeq\varprojlim_n \mathcal{D}_n(\Omega_n)$$

as vector spaces, where $\mathcal{D}_n(\Omega_n)$ denotes the space of distributions on Ω_n. Therefore, we endow $\mathcal{D}_b(\Omega)$ with the projective limit topology $\varprojlim_n \mathcal{D}_n(\Omega_n)$. In view of Theorem 5.13 in Komatsu [12] and the above consideration, we have the following

LEMMA 4.1. *We have the isomorphism $\mathcal{D}_b'(\Omega)\simeq\varinjlim_n \mathcal{D}'(\Omega_n)$ as topological vector spaces.*

DEFINITION 4.2. We call the sheaf $\mathcal{D}_b'$ associated with the presheaf $\{\mathcal{D}_b'(\Omega)\}$ the sheaf of distributions. A section of the sheaf $\mathcal{D}_b'$ over Ω is called a distribution on Ω.

We give a definition of hyperfunctions in $\sum R$. See for example [11] for the theory of hyperfunctions in finite dimensional spaces. We define the map w_{n+1}^n from $H_{\Omega_n}^n(V_n, \mathcal{O}_n)$ to $H_{\Omega_{n+1}}^{n+1}(V_{n+1}, \mathcal{O}_{n+1})$ by $w_n(f(x'))=f(x')\otimes\delta(x_{n+1})$ for $f(x')\in H_{\Omega_n}^n(V_n, \mathcal{O}_n)$. Set $\mathcal{B}(\Omega)=\varinjlim_n H_{\Omega_n}^n(V_n, \mathcal{O}_n)$.

DEFINITION 4.3. We call the sheaf $\mathcal{B}$ associated with the presheaf $\{\mathcal{B}(\Omega)\}$ the sheaf of hyperfunctions. We call a section of the sheaf $\mathcal{B}$ over Ω a hyperfunction on Ω.

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Present Address:
DEPARTMENT OF PURE AND APPLIED SCIENCES
COLLEGE OF GENERAL EDUCATION
UNIVERSITY OF TOKYO
KOMABA, TOKYO 153