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An Analogue of Paley-Wiener Theorem on $SU(2, 2)$

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Introduction

In this paper we shall study an analogue of Paley-Wiener theorem on a connected semisimple Lie group G with finite center. Namely, we would like to characterize the Fourier transforms of τ-spherical functions with compact support on G. In the preceding papers [4], [5], we obtained some results about this problem under the assumption that the real rank of G is one. However, since the method in this case is deeply dependent on one complex variable analysis, it is useless for us to investigate the case of higher rank (but we shall use the same technique in §4). Therefore we have to think out a new method. Now we shall sketch the contents in this paper.

In §2 we shall consider the behavior of μ-functions at their singular points. Then under some assumptions we obtain a relation between the μ-functions corresponding to the cuspidal parabolic subgroups of G whose split components are not conjugate under K. Moreover, using this relation and the functional equation of the Eisenstein integral, we can obtain a key proposition which will be frequently used in the following arguments. Roughly speaking, the above assumptions are connected with an explicit imbedding theorem of discrete series as a subrepresentation of non-unitary principal series. As you know, at present these explicit forms are not obtained in general, but for the case of $G=SU(n, 1)$ $(n \in N)$ (see [8]).

Now we shall use the same notation in [4]. Let $P_0=M_0A_0N_0$ denote a minimal parabolic subgroup of G and suppose that $\alpha(\nu)=(\alpha_i^j(\nu);\ 1\leqq i\leqq n_j,\ 1\leqq j\leqq m)$ (cf. §1.3) belongs to $\mathscr{E}(\mathscr{F}_0)_*^{n(P_0)}$. Then the inverse Fourier transform is given as

$$(0.1)\qquad f(x)=\mathscr{E}_{A_0}^{-1}(\alpha) =\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}\int_{\mathscr{F}_0}\mu(\omega_j,\nu)E(P_0\colon \phi_i^j\colon \nu\colon x)\alpha_i^j(\nu)d\nu\ ,$$

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(see Theorem 1 in [4]). Now furthermore we assume that each component $\alpha_i^j(\nu)$ of α has a holomorphic extension on $(\mathscr{F}_0)_c$ which is an exponential type. Then in §3, we would like to obtain the following type theorem.

Theorem (*). Let notation be as above. Then $f(x)$ can be decomposed as

$$f(x)=F(x)+G(x)\ , \tag{0.2}$$

where $F(x)$ and $G(x)$ belong to $C_c^\infty(G, \tau)$ and the kernel of $\mathscr{E}_{A_0}$ respectively.

However, to our regret, we do not obtain a general proof of this theorem. In this paper we shall give a proof for the following two cases.

(1) The case of a real rank one.

(2) $G=SU(2, 2)$ and $\tau=(\tau^l, \tau^l)$.

We shall investigate the first in §3 and the second in §4. I believe the method in the above cases can be extended to the general case.

§1. Notation and preliminary.

1.1. Notation.

Let G be a connected semisimple Lie group with finite center and be of the real rank $R(R\in \boldsymbol{Z}, R\geqq 1)$. Let K be a maximal compact subgroup of G and θ denote the Cartan involution which is induced on G by K. Then we can define the associated Iwasawa decomposition $G=KA_0N_0$ as usual. Let M_0 be the centralizer of A_0 in K. Then it is obvious that M_0 is contained in K and $P_0=M_0A_0N_0$ is a minimal parabolic subgroup of G. Now in this paper we shall denote Lie algebras by small German letters and for any real vector space V we shall denote by V_c and V^* the complex vector space and the dual vector space of V respectively. Moreover let $\mathscr{E}(G)$ denote the set of all equivalence classes of irreducible unitary representations on G and $\mathscr{E}_2(G)$ denote the subset of $\mathscr{E}(G)$ which consists of all equivalence classes of square-integrable representations on G. For other Lie groups we shall define the same notation; $\mathscr{E}(\ )$, $\mathscr{E}_2(\ )$ as usual. Then for any equivalence class ω we shall denote by $\underline{\omega}$ an irreducible representation whose class belongs to ω.

1.2. The decomposition of $\mathscr{C}(G, \tau)$.

In this subsection we shall decompose the τ-spherical Schwartz space $\mathscr{C}(G, \tau)$ with respect to the conjugacy classes of Cartan subgroups of

G. Next arguments were developed in Harish-Chandra [3] § 27.

First let V be the finite dimensional Hilbert space satisfying the conditions in Harish-Chandra [3] § 8 and let $\tau=(\tau_1, \tau_2)$ be a unitary double representation of K on V. Then we can define the V-valued Schwartz space $\mathscr{C}(G, V)$ on G and the subspace of τ-spherical functions $\mathscr{C}(G, \tau)$ as usual. Moreover we denote by $^\circ\mathscr{C}(G, \tau)$ the subspace of τ-spherical cusp forms on G (cf. Harish-Chandra [1] § 19). Now let $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ be a complete set of θ-stable Cartan subgroups of G, no two of which are conjugate under G. Let $A_i(1\leqq i\leqq r)$ denote the vector part of Γ_i and put for each A_i

$$M_i = \bigcap_{\chi\in X(M_i^\circ)} \operatorname{Ker}|\chi| , \tag{1.1}$$

where M_i° is the centralizer of A_i in G and $X(M_i^\circ)$ is the group of all continuous homomorphisms of M_i° into the multiplicative group of real numbers. Then we denote by $P_i=M_iA_iN_i(1\leqq i\leqq r)$ the parabolic subgroup of G whose nilradical N_i is contained in N_0. Now let $\mathscr{C}_{A_i}(G, \tau)$ $(1\leqq i\leqq r)$ denote the closed subspace of $\mathscr{C}(G, \tau)$ which consists of all elements f satisfying the following condition; if Q is a parabolic subgroup of G whose split component is not conjugate to A_i under K, then

$$f^{(Q)} \sim 0 \tag{1.2}$$

(cf. Harish-Chandra [1] § 16, § 20). Then from Theorem 27.2 in Harish-Chandra [3] $\mathscr{C}(G, \tau)$ can be decomposed as follows;

$$\mathscr{C}(G, \tau)=\mathscr{C}_{A_1}(G, \tau) \oplus \mathscr{C}_{A_2}(G, \tau) \oplus \cdots \oplus \mathscr{C}_{A_r}(G, \tau) \tag{1.3}$$

where $\oplus$ is the topological direct sum. Here we note that when Γ_i is a compact Cartan subgroup of G, i.e., $A_i=\{1\}$, then $\mathscr{C}_{A_i}(G, \tau)$ coincides with $^\circ\mathscr{C}(G, \tau)$. Since A_0 is a maximal vector subgroup of G, there exists a θ-stable Cartan subgroup of G whose vector part is equal to A_0. Therefore without loss of generality we may assume that this Cartan subgroup and A_0 are equal to Γ_r and A_r respectively.

1.3. The Fourier transform on $\mathscr{C}_A(G, \tau)$.

In this subsection we shall review the results in T. Kawazoe [4]. Fix $\Gamma_i(1\leqq i\leqq r)$ and for simplicity we shall write Γ and $P=MAN$ instead of Γ_i and $P_i=M_iA_iN_i$ respectively.

First we note that A is the vector part of a θ-stable Cartan subgroup of G. Therefore P is cuspidal and $\mathscr{E}_2(M)$ is not empty (see V.S. Varadarajan [7] part II Prop. 6.23). Now we denote by $L_M={}^\circ\mathscr{C}(M, \tau_M)$

the space of τ_M-spherical cusp forms on M, where τ_M is the restriction of τ to $K_M=K\cap M$. Then for each ω in $\mathscr{E}_2(M)$ let $\mathscr{H}_\omega$ denote the smallest closed subspace of $L^2(M)$ which contains all matrix coefficients of ω and put $L_M(\omega)=(\mathscr{H}_\omega\otimes V)\cap L_M$. Then it is well known that L_M is finitely dimensional and can be decomposed as follows;

$$L_M=\sum_{\omega\in\mathscr{E}_2(M)} L_M(\omega)=\sum_{j=1}^{m}\sum_{s\in W/W(\omega_j)} L_M(s\omega_j) \tag{1.4}$$

where $W=W(A)$ is the Weyl group of (G, A) and $W(\omega_j)(1\leqq j\leqq m)$ is the subgroup of W consisting of all $s\in W$ such that $s\omega_j=\omega_j$. Here we denote by

$$\{\phi_i^j;\ 1\leqq i\leqq n_j=\dim(L_M(\omega_j))\} \tag{1.5}$$

an orthonormal basis of $L_M(\omega_j)(1\leqq j\leqq m)$ with respect to the L^2-norm on M.

Now we shall construct the Fourier transform on $\mathscr{C}_A(G, \tau)$. First put $\mathscr{F}=\mathscr{F}_A=\mathfrak{a}^*$ and

$$\hat{f}(\phi, \nu)=(c^2\gamma)^{-1}(f, E(P{:}\,\phi{:}\,\nu{:}\,.\,)) \tag{1.6}$$

for $f\in\mathscr{C}_A(G, \tau)$, $\phi\in L_M$ and $\nu\in\mathscr{F}$, where c and γ are the constants which were defined in Harish-Chandra [3] § 2, § 11 and $E(P{:}\,\phi{:}\,\nu{:}\,x)(x\in G)$ is the Eisenstein integral of G. Then from the results in Harish-Chandra [2] § 13 for fixed ϕ, $\hat{f}(\phi, \nu)$ is a rapidly decreasing function on $\mathscr{F}$, i.e., it belongs to the Schwartz space $\mathscr{C}(\mathscr{F})$ on $\mathscr{F}$. Then using these functions, we shall define the Fourier transform $\mathscr{E}_A$ of $\mathscr{C}_A(G, \tau)$ into $\mathscr{C}(\mathscr{F})^n$ $(n=n(P)=n_1+n_2+\cdots+n_m)$ as follows; for $f\in\mathscr{C}_A(G, \tau)$

$$\begin{aligned}\mathscr{E}_A(f)&=(\hat{f}(\phi_i^j, \nu);\ 1\leqq i\leqq n_j,\ 1\leqq j\leqq m)\\&=(\hat{f}(\phi_1^1, \nu), \cdots, \hat{f}(\phi_{n_1}^1, \nu), \hat{f}(\phi_1^2, \nu), \cdots, \hat{f}(\phi_{n_2}^2, \nu), \cdots\\&\qquad\cdots, \hat{f}(\phi_1^m, \nu), \cdots, \hat{f}(\phi_{n_m}^m, \nu)).\end{aligned} \tag{1.7}$$

Next we shall define the subspace of $\mathscr{C}(\mathscr{F})^n$ which becomes the image of $\mathscr{C}_A(G, \tau)$ by the above mapping $\mathscr{E}_A$. Here we note that α in $\mathscr{C}(\mathscr{F})^n$ can be written as

$$\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m), \tag{1.8}$$

where $\alpha_j(1\leqq j\leqq m)$ are elements in $\mathscr{C}(\mathscr{F})^{n_j}$. Then let $\mathscr{C}(\mathscr{F})_*^n$ denote the subspace of $\mathscr{C}(\mathscr{F})^n$ which consists of all $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ (see(1.8)) satisfying the following relations;

$$\alpha_j^t(s\nu)=\overline{{}^\circ C_{P|P}(s; s^{-1}\nu)}\alpha_j^t(\nu)\quad\text{for all }\ s\in W(\omega_j)\ \text{ and }\ \nu\in\mathscr{F}(1\leqq j\leqq m), \tag{1.9}$$

where α_j^t is the transposed vector of $\alpha_j (1 \leq j \leq m)$ and ${}^\circ C_{P|P}(s; s^{-1}\nu)$ is a unitary operator of $L_M(\omega_j)$ onto $L_M(s\omega_j)$ which was defined in Harish-Chandra [3] p. 152. Here we regard this operator as a matrix operator with respect to the orthonormal basis (1.5) and denote the complex conjugate of it by $^{-}$. Then we can easily prove that $\mathscr{C}(\mathscr{F})_*^n$ is closed in $\mathscr{C}(\mathscr{F})^n$ and obtain the following Theorem 1 (see [4] Theorem 1).

THEOREM 1. *The mapping* $\mathscr{E}_A$ *is a homoemorphism of* $\mathscr{C}_A(G, \tau)$ *onto* $\mathscr{C}(\mathscr{F})_*^n$. *Moreover the inverse mapping of* $\mathscr{E}_A$ *is given by*

$$f(x) = \sum_{j=1}^{m} |W(\omega_j)|^{-1} \sum_{i=1}^{n_j} \int_{\mathscr{F}} \mu(\omega_j, \nu) E(P: \phi_i^j: \nu: x) \hat{f}(\phi_i^j, \nu) d\nu \quad \text{for } f \in \mathscr{C}_A(G, \tau), \tag{1.10}$$

where $d\nu$ *is the Euclidean measure on* $\mathscr{F}$ *which is dual to the Haar measure* da *on* A *and* $\mu(\omega_j, \nu)(1 \leq j \leq m)$ *are* μ*-functions on* G *which were defined in Harish-Chandra* [3] § 13.

Here we note that the mapping $\mathscr{E}_A$ can be extended to a mapping of $\mathscr{C}(G, \tau)$ onto $\mathscr{C}(\mathscr{F})_*^n$ composing the projection of $\mathscr{C}(G, \tau)$ onto $\mathscr{C}_A(G, \tau)$. From now on we shall denote this extension by the same notation.

1.4. The μ-functions on G.

We keep to the notations of the subsections 1.2 and 1.3. In this subsection we shall describe the definition and some properties of the μ-functions on G. These results were obtained in Harish-Chandra [3].

Let $\Sigma(P)$ denote the set of all reduced roots of (P, A) (see Harish-Chandra [3] p. 120) and fix $\alpha \in \Sigma(P)$. Now let $\mathfrak{a}_\alpha$ be the hyperplane $\alpha = 0$ in $\mathfrak{a}$ and put

$$M_\alpha = \bigcap_{\chi \in X(M_\alpha^\circ)} \operatorname{Ker} |\chi| , \tag{1.11}$$

where M_α° is the centralizer of $\mathfrak{a}_\alpha$ in G. Moreover put ${}^*P_\alpha = M_\alpha \cap P$, ${}^*A_\alpha = M_\alpha \cap A$ and ${}^*N_\alpha = M_\alpha \cap N$ respectively. Then it is well known that ${}^*P_\alpha$ is a parabolic subgroup of M_α and its Langlands decomposition is given by

$${}^*P_\alpha = M {}^*A_\alpha^* N_\alpha . \tag{1.12}$$

Here we note that ${}^*A_\alpha$ and ${}^*\mathfrak{n}_\alpha$ can be written as

$${}^*A_\alpha = \exp \boldsymbol{R} H_\alpha \quad \text{and} \quad {}^*\mathfrak{n}_\alpha = \sum_{k \geq 1} \mathfrak{g}_{k\alpha} \tag{1.13}$$

respectively, where H_α is some element in ${}^*\mathfrak{a}_\alpha$ and $\mathfrak{g}_{k\alpha}$ is the space of all X in $\mathfrak{g}$ such that $[H, X]=k\alpha(H)X$ for all $H\in\mathfrak{a}$ (cf. Harish-Chandra [3] Lemma 2.3). Next let $P_\alpha=M_\alpha A_\alpha N_\alpha$ be the parabolic subgroup of G whose split component is equal to $A_\alpha=\exp\mathfrak{a}_\alpha$ and nilradical is contained in N_0. Here put $\mathscr{F}_\alpha=\mathfrak{a}_\alpha^*$ and ${}^*\mathscr{F}_\alpha=({}^*\mathfrak{a}_\alpha)^*$ for simplicity. Moreover we denote by $\mathscr{F}'$, $\mathscr{F}_\alpha'$ and ${}^*\mathscr{F}_\alpha'$ the set of all regular elements in $\mathscr{F}$, $\mathscr{F}_\alpha$ and ${}^*\mathscr{F}_\alpha$ respectively.

Now since $\dim {}^*A_\alpha=\mathrm{prk}(M_\alpha)+1$, there exists a number $\mu_\omega(\lambda_\alpha)>0$ for $\omega\in\mathscr{E}_2(M)$ and $\lambda_\alpha\in{}^*\mathscr{F}_\alpha'$ such that

$$\mu_\omega(\lambda_\alpha)j_{{}^*\bar{P}_\alpha|{}^*P_\alpha}(\lambda_\alpha)j_{{}^*P_\alpha|{}^*\bar{P}_\alpha}(\lambda_\alpha)=1\,, \tag{1.14}$$

(see Harish-Chandra [3] Lemma 12.1 and see § 13 for the definitions of j-functions). Now let ${}^*\nu_\alpha$ denote the restriction of $\nu\in\mathscr{F}$ on ${}^*\mathfrak{a}_\alpha$ and put

$$\mu_\alpha(\omega, \nu)=\mu_\omega({}^*\nu_\alpha) \quad \text{for} \quad \omega\in\mathscr{E}_2(M) \quad \text{and} \quad \nu\in\mathscr{F}'\,. \tag{1.15}$$

Then the μ-function on G is given by

$$\mu(\omega, \nu)=\prod_{\alpha\in\Sigma(P)}\mu_\alpha(\omega, \nu)\,. \tag{1.16}$$

Next Lemma 1 is obtained by an explicit calculation of $\mu_\omega(\lambda_\alpha)$ (cf. Harish-Chandra [3] § 28–§ 36).

LEMMA 1. *$\mu(\omega, \nu)$ extends to a meromorphic function on $\mathscr{F}_c$ and moreover there exists a number $\delta>0$ such that the following conditions hold.*

(1) *$\mu(\omega, \nu)$ is holomorphic on $\mathscr{F}_c(\delta)$.*

(2) *There exist numbers c, $r\geqq 0$ such that*

$$|\mu(\omega, \nu)|\leqq c(1+|\nu_R|)^r \quad (\nu\in\mathscr{F}_c(\delta))\,, \tag{1.17}$$

where $\nu=\nu_R+(-1)^{1/2}\nu_I(\nu_R, \nu_I\in\mathscr{F})$ and $\mathscr{F}_c(\delta)$ is the set of all $\nu\in\mathscr{F}_c$ such that $|\nu_I|\leqq\delta$.

Now let $\Sigma_0(P)$ denote the set of all $\alpha\in\Sigma(P)$ such that $\mathscr{E}_2(M_\alpha)$ is not empty, i.e., P_α is cuspidal. Here we note that for each $\alpha\in\Sigma_0(P)$ there exists a θ-stable Cartan subgroup of G whose vector part is equal to A_α. Therefore there exists a unique $i=i(\alpha)(1\leqq i\leqq r)$ such that Γ_i and A_i are conjugate to this Cartan subgroup and A_α respectively. Now we shall prove the following Lemma 2.

LEMMA 2. *When α is not in $\Sigma_0(P)$, $\mu_\alpha(\omega, \nu)$ is a polynomial on $\mathscr{F}_c$.*

In particular the singularities of $\mu(\omega, \nu)$ *depend on* $\mu_\alpha(\omega, \nu)(\alpha \in \Sigma_0(P))$.

PROOF. Suppose α is not contained in $\Sigma_0(P)$. Then from the definition of $\Sigma_0(P)$ $\mathscr{E}_2(M_\alpha)$ is empty. Therefore there exist no compact Cartan subgroups of M_α and $\operatorname{rank}(M_\alpha) > \operatorname{rank}(K_{M_\alpha})$. On the other hand the vector part of the Cartan subgroup $\Gamma \cap M_\alpha$ of M_α is equal to $^*A_\alpha$ and $\dim {}^*A_\alpha = 1$. Thus $\Gamma \cap M_\alpha$ is a fundamental Cartan subgroup of M_α. Therefore Lemma 2 is obvious from Harish-Chandra [3] Theorem 24.1. Q.E.D.

1.5. Two lemmas for the Eisenstein integrals.

Let notation be as in the preceding subsections. Now suppose that $P' = M'A'N'$ is a parabolic subgroup of G which contains $P = MAN$. Put $^*P = P \cap M'$, $^*A = A \cap M'$ and $^*N = N \cap M'$ respectively. Then we can obtain the following functional equations for the Eisenstein integrals (see Harish-Chandra [3] Lemma 17.5).

LEMMA 3. *Let notation be as above. Then for* $\phi \in L_M, \nu \in \mathscr{F}$ *and* $x \in G$,

$$E(P' : E(^*P : \phi : {}^*\nu : .) : \nu' : x) = E(P : \phi : \nu : x) , \tag{1.18}$$

where $^*\nu$ *and* ν' *are the restrictions of* ν *on* $^*\mathfrak{a}$ *and the orthogonal complement* $\mathfrak{a}'$ *of* $^*\mathfrak{a}$ *in* $\mathfrak{a}$ *respectively.*

Next let $V = C^\infty(K \times K)$. Then we can define the scalar product $(\ ,\ )$ on V and the unitary double representation τ of K on V as usual (cf. Harish-Chandra [1] §26). Now let F be a finite subset of $\mathscr{E}(K)$ and put $\alpha_F = \sum_{\delta \in F} \alpha_\delta$, where $\alpha_\delta = d(\delta)\operatorname{conj}(\chi_\delta)$ (χ_δ is the character of the class δ and $d(\delta) = \chi_\delta(1)$). Here let V_F denote the subset of V consisting of all $v \in V$ such that

$$v = \int_K \alpha_F(k)\tau(k)v dk = \int_K \alpha_F(k) v\tau(k) dk . \tag{1.19}$$

Then we can easily prove that V_F is stable under τ and its dimension is finite. Thus we can define the unitary double representation τ_F of K on V_F as the restriction of τ to V_F. From now on we fix a finite subset F of $\mathscr{E}(K)$ and write (V, τ) for the pair (V_F, τ_F).

Now let ω be in $\mathscr{E}_2(M)$ and fix it. Then the induced representation of G is given by

$$\pi^P_{\underline{\omega},\nu} = \operatorname{Ind}^G_{MAN}(\underline{\omega} \otimes e^\nu \otimes 1) \quad \text{for} \quad \nu \in \mathscr{F}_c \tag{1.20}$$

(cf. Harish-Chandra [3] §4). Let $\mathfrak{H}_{\underline{\omega}} = \mathfrak{H}^P_{\underline{\omega},\nu}$ denote the representation space of $\pi^P_{\underline{\omega},\nu}$. Now put

$$P_F=\int_K \alpha_F(k)\pi^P_{\underline{\omega},\nu}(k)dk \quad\text{and}\quad \mathfrak{H}^F_{\underline{\omega}}=P_F(\mathfrak{H}_{\underline{\omega}}) . \tag{1.21}$$

Then we obtain the following Lemma 4 (cf. Harish-Chandra [3] Lemma 7.1 and Theorem 7.1)

LEMMA 4. *For each T in* $\mathrm{End}(\mathfrak{H}^F_{\underline{\omega}})$ *we can associate a ψ_T in $L_M(\omega)$ such that the mapping; $T\mapsto\psi_T\cdot d^{1/2}_\omega$ is a linear isometry of* $\mathrm{End}(\mathfrak{H}^F_{\underline{\omega}})$ *with Hilbert-Schmidt norm onto $L_M(\omega)$ with L^2-norm, where d_ω is the formal degree of the class ω. Moreover these T and ψ_T satisfy the following relation;*

$$E(P:\psi_T:\nu:x)(1;1)=\mathrm{tr}(\pi^P_{\underline{\omega},\nu}(x)T) \quad \textit{for} \quad x\in G \quad \textit{and} \quad \nu\in\mathcal{F}_c , \tag{1.22}$$

where we restrict $\pi^P_{\underline{\omega},\nu}$ on $\mathfrak{H}^F_{\underline{\omega}}$ and regard it as an endomorphism of $\mathfrak{H}^F_{\underline{\omega}}$.

Last we recall that the Eisenstein integral satisfies the following inequality (see Harish-Chandra [2] Lemma 17.1). For $\phi\in L_M$, there exist numbers $r\geqq 0$ and $c>0$ such that

$$|E(P:\phi:\nu:x)|_V<c\Xi(x)|(\nu, x)|^r e^{c_0|\nu_I|\sigma(x)} \tag{1.23}$$

for all $\nu\in\mathcal{F}_c$ and $x\in G$, where $|\cdot|_V$ is the norm in V and c_0 is a constant which does not depend on ϕ, ν, x (for the definitions of c_0, $\Xi(x)$, $\sigma(x)$, $|(\nu, x)|$ see Harish-Chandra [1] § 10 and [2] § 17).

1.6. The parametrization of ν in $\mathcal{F}_c$.

We keep to the notations in the preceding subsections. Suppose that the parabolic rank of P, i.e., $\dim A$ is $p(1\leqq p\leqq R)$. Then it is obvious that $\dim A_\alpha=p-1$ and $\dim{}^*A_\alpha=1$. Therefore using the dual form $(\ ,\ )$ of the Cartan-Killing form on $\mathfrak{a}_0$, we can choose an orthogonal basis $\{e^\alpha_1, e^\alpha_2, \cdots, e^\alpha_p\}$ of $\mathcal{F}$ such that $\{e^\alpha_1, e^\alpha_2, \cdots, e^\alpha_{p-1}\}$ (resp. $\{e^\alpha_p\}$) is an orthogonal basis of $\mathcal{F}_\alpha$ (resp. ${}^*\mathcal{F}_\alpha$). Now for simplicity we put $x^\alpha_i=(\nu, e^\alpha_i)/(e^\alpha_i, e^\alpha_i)(1\leqq i\leqq p)$ for $\nu\in\mathcal{F}$. Then $\nu=\nu_\alpha+{}^*\nu_\alpha$, where ν_α and ${}^*\nu_\alpha$ are the restrictions of ν on $\mathfrak{a}_\alpha$ and ${}^*\mathfrak{a}_\alpha$, can be written as

$$\nu=\sum_{i=1}^{p} x^\alpha_i e^\alpha_i,\ \nu_\alpha=\sum_{i=1}^{p-1} x^\alpha_i e^\alpha_i \quad\text{and}\quad {}^*\nu_\alpha=x^\alpha_p e^\alpha_p . \tag{1.24}$$

Moreover without loss of generality we may identify $\nu\in\mathcal{F}_c$, $\nu_\alpha\in(\mathcal{F}_\alpha)_c$ and ${}^*\nu_\alpha\in({}^*\mathcal{F}_\alpha)_c$ with $(x^\alpha_1, x^\alpha_2, \cdots, x^\alpha_p)\in \boldsymbol{C}^p$, $(x^\alpha_1, x^\alpha_2, \cdots, x^\alpha_{p-1})\in\boldsymbol{C}^{p-1}$ and $(x^\alpha_p)\in\boldsymbol{C}$ respectively. Now let $C^{\alpha\beta}=(C^{\alpha\beta}_{ij})\,(1\leqq i, j\leqq p)$ denote the $p\times p$-matrix which transforms $(e^\beta_1, e^\beta_2, \cdots, e^\beta_p)^t$ to $(e^\alpha_1, e^\alpha_2, \cdots, e^\alpha_p)^t$, where α, $\beta\in$

$\Sigma(P)$ and $(.)^t$ is the transposed vector of (.). Here we note that for each $\alpha, \beta \in \Sigma(P)$ $C^{\alpha\beta}$ belongs to $GL(p, \boldsymbol{R})$ and moreover satisfies; $(C^{\alpha\beta})^{-1} = C^{\beta\alpha}$ and $(C^{\alpha\beta}_{p1}, C^{\alpha\beta}_{p2}, \cdots, C^{\alpha\beta}_{pp}) \neq (0, 0, \cdots, 0, 1)(\alpha \neq \beta)$.

§ 2. The key proposition.

In this section we shall prove two lemmas and one proposition, which will be used in § 4 under some assumptions.

First fix $P_k = M_k A_k N_k (1 \leqq k \leqq r)$ and for simplicity we shall write $P = MAN$ instead of P_k. Suppose α is in $\Sigma_0(P)$ and fix it. Then we can define the cuspidal parabolic subgroup $P_\alpha = M_\alpha A_\alpha N_\alpha$ of G and the parabolic subgroup $^*P_\alpha = M^* A^*_\alpha N_\alpha$ of M_α as in § 1.4 respectively. Then it is obvious that P_α contains P and is conjugate to $P_i (i = i(\alpha))$ under K (see § 1.4). From now on we assume that $\phi \in L_M(\omega)$ $(\omega \in \mathscr{E}_2(M))$ and $^*\nu^\circ_\alpha \in (^*\mathscr{F}_\alpha)_c$ satisfy the following condition;

$$E(^*P_\alpha\colon \phi\colon {}^*\nu^\circ_\alpha\colon m) \in L_{M_\alpha}(\sigma) \quad (m \in M_\alpha) \tag{2.1}$$

for some $\sigma = \sigma(\phi, {}^*\nu^\circ_\alpha) \in \mathscr{E}_2(M_\alpha)$.

Now we shall apply the arguments in § 1.5 for the pair (M_α, M) instead of (G, M) and define the induced representation of M_α as follows;

$$\pi^{*P_\alpha}_{\underline{\omega}, {}^*\nu^\circ_\alpha} = \mathrm{Ind}^{M_\alpha}_{M^* A_\alpha {}^* N_\alpha}(\underline{\omega} \otimes e^{{}^*\nu^\circ_\alpha} \otimes 1) . \tag{2.2}$$

Then it is obvious that this representation is non-unitary principal series of M_α and from Lemma 1.4 its matrix coefficients can be written as

$$E(^*P_\alpha\colon \psi\colon {}^*\nu^\circ_\alpha\colon m)(1{:}1) \quad (m \in M_\alpha) \tag{2.3}$$

for $\psi \in L_M(\omega)$. Therefore $E(^*P_\alpha\colon \phi\colon {}^*\nu^\circ_\alpha\colon m)(1{:}1)$ is a finite linear combinations of these matrix coefficients of $\pi^{*P_\alpha}_{\underline{\omega}, {}^*\nu^\circ_\alpha}$. On the other hand from (2.1) $E(^*P_\alpha\colon \phi\colon {}^*\nu^\circ_\alpha\colon m)(1{:}1)$ belongs to $\mathscr{H}_\sigma$, in particular, to $L^2(M_\alpha)$. Therefore without loss of generality we may assume that $\underline{\sigma}$ is infinitesimally equivalent to a subrepresentation of $\pi^{*P_\alpha}_{\underline{\omega}, {}^*\nu^\circ_\alpha}$. Here we denote this relation by

$$\underline{\sigma} < \pi^{*P_\alpha}_{\underline{\omega}, {}^*\nu^\circ_\alpha} . \tag{2.4}$$

Next we shall apply the arguments in § 1.5 for the pair (G, M_α) instead of (G, M) and obtain the following relations;

$$\pi^{P_\alpha}_{\underline{\sigma}, \nu_\alpha} = \mathrm{Ind}^G_{M_\alpha A_\alpha N_\alpha}(\underline{\sigma} \otimes e^{\nu_\alpha} \otimes 1) \tag{2.5}$$

$$<\mathrm{Ind}^G_{M_\alpha A_\alpha N_\alpha}(\mathrm{Ind}^{M_\alpha}_{M^* A^*_\alpha N_\alpha}(\underline{\omega}\otimes e^{*\nu^\circ_\alpha}\otimes 1)\otimes\nu_\alpha\otimes 1)$$
$$\cong \mathrm{Ind}^G_{MAN}(\underline{\omega}\otimes e^{\nu_\alpha+{}^*\nu^\circ_\alpha}\otimes 1)=\pi^P_{\underline{\omega},\nu^\circ}$$

and

$$\pi^{\bar{P}_\alpha}_{\underline{\sigma},\nu_\alpha}=\mathrm{Ind}^G_{M_\alpha A_\alpha \bar{N}_\alpha}(\underline{\sigma}\otimes e^{\nu_\alpha}\otimes 1)$$
$$<\mathrm{Ind}^G_{M_\alpha A_\alpha \bar{N}_\alpha}(\mathrm{Ind}^{M_\alpha}_{M^* A^*_\alpha N_\alpha}(\underline{\omega}\otimes e^{*\nu^\circ_\alpha}\otimes 1)\otimes\nu_\alpha\otimes 1)$$
$$\cong \mathrm{Ind}^G_{P'}(\underline{\omega}\otimes e^{\nu_\alpha+{}^*\nu^\circ_\alpha}\otimes 1)=\pi^{P'}_{\underline{\omega},\nu^\circ}\,, \tag{2.6}$$

where $\nu^\circ=\nu_\alpha+{}^*\nu^\circ_\alpha(\nu_\alpha\in\mathscr{F}_\alpha)$, $\bar{P}_\alpha=M_\alpha A_\alpha \bar{N}_\alpha(\bar{N}_\alpha=\theta(N_\alpha))$ and $P'=MA^*N_\alpha\bar{N}_\alpha$. Then using these relations, we can obtain the following Lemma 5.

LEMMA 5. *Let notation be as above. Then for* $\nu_\alpha\in\mathscr{F}''_\alpha$,

$$\mu(\sigma,\nu_\alpha)=c\prod_{\gamma\in\Sigma(P)-\{\alpha\}}\mu_\gamma(\omega,\nu^\circ)\quad(\nu^\circ=\nu_\alpha+{}^*\nu^\circ_\alpha)\,, \tag{2.7}$$

where $\mathscr{F}''_\alpha$ *is the set of all* $\nu_\alpha\in\mathscr{F}_\alpha$ *such that the both sides of* (2.7) *are well-defined and* c *is a constant which does not depend on* ν_α.

PROOF. Let $\mathfrak{H}^P_\omega=\mathfrak{H}^P_{\underline{\omega},\nu^\circ}$ and $\mathfrak{H}^\alpha_\sigma=\mathfrak{H}^\alpha_{\underline{\sigma},\nu_\alpha}$ denote the representation spaces of $\pi^P_{\underline{\omega},\nu^\circ}$ and $\pi^{P_\alpha}_{\underline{\sigma},\nu_\alpha}$ respectively. Put $\bar{N}=\theta(N)$ and $\bar{P}=MA\bar{N}$. Now we shall recall the results about intertwining operators of the induced representations; $\pi^P_{\underline{\omega},\nu^\circ}$ of G and $\pi^{P_\alpha}_{\underline{\sigma},\nu_\alpha}$ of M_α, which were obtained in Harish-Chandra [3] § 5. Here we shall use the same notation in it. Then we have the following relation;

$$J_{P|\bar{P}}(\nu)=J_{P|P'}(\nu)J_{P'|\bar{P}}(\nu)\,, \tag{2.8}$$

where ν is an element in $\mathscr{F}_c$ on which these three intertwining operators are well-defined (cf. Harish-Chandra [3] Lemma 5.2). Now from the relations (2.5) and (2.6), without loss of generality we may assume that $\mathfrak{H}^{P_\alpha}_{\underline{\sigma}}$ and $\mathfrak{H}^{\bar{P}_\alpha}_{\underline{\sigma}}$ are closed subspaces of $\mathfrak{H}^P_{\underline{\omega}}$ and $\mathfrak{H}^{P'}_{\underline{\omega}}$ respectively. Here we note that the intertwining operators $J_{P|P'}(\nu)(\nu\in\mathscr{F})$ of $\mathfrak{H}^{P'}_{\underline{\omega}}$ onto $\mathfrak{H}^P_{\underline{\omega}}$ and $J_{P_\alpha|\bar{P}_\alpha}(\nu_\alpha)(\nu_\alpha\in\mathscr{F}_\alpha)$ of $\mathfrak{H}^{\bar{P}_\alpha}_{\underline{\sigma}}$ onto $\mathfrak{H}^{P_\alpha}_{\underline{\sigma}}$ can be written as

$$J_{P|P'}(\nu)(h)=\gamma^{-1}_{\bar{P}|P'}\int_{N_\alpha}h(\bar{n}x)d\bar{n}\quad(h\in\mathfrak{H}^{P'}_{\underline{\omega}})\,, \tag{2.9}$$

$$J_{P_\alpha|\bar{P}_\alpha}(\nu_\alpha)(h')=\gamma^{-1}_{\bar{P}_\alpha|\bar{P}_\alpha}\int_{N_\alpha}h'(\bar{n}x)d\bar{n}\quad(h'\in\mathfrak{H}^{\bar{P}_\alpha}_{\underline{\sigma}})\,, \tag{2.10}$$

respectively (see Harish-Chandra [3] § 5 for notations). Moreover we note that $\gamma_{P|P'}$ and $\gamma_{P_\alpha|\bar{P}_\alpha}$ don't depend on $\nu\in\mathscr{F}$ and $\nu_\alpha\in\mathscr{F}_\alpha$ respectively. Therefore we can easily prove that the restriction of $J_{P|P'}(\nu^\circ)$ to $\mathfrak{H}^P_{\underline{\omega}}$ is well-defined and coincides with $J_{P_\alpha|\bar{P}_\alpha}(\nu_\alpha)$ up to a scalar multiplication.

Now we shall rewrite these relations about J-functions in terms of small j-functions (see § 1.4 (1.14) and (1.16)). Then using the definition of j-functions and the fact that the μ-functions are independent of a finite subset F in $\mathscr{E}(K)$, we can obtain the following calculation;

$$
\begin{aligned}
(2.11)\quad \mu(\sigma, \nu_\alpha)^{-1} &= j_{P_\alpha|\bar{P}_\alpha}(\nu_\alpha) j_{\bar{P}_\alpha|P_\alpha}(\nu_\alpha) \\
&= j_{P_\alpha|\bar{P}_\alpha}(\nu_\alpha) j_{P_\alpha|\bar{P}_\alpha}(\nu_\alpha)^* \\
&= j_{P|P'}(\nu^\circ) j_{P|P'}(\nu^\circ)^* \times c \quad (c = \gamma_{P_\alpha|\bar{P}_\alpha}^{-1} \cdot \gamma_{P|P'}) \\
&= \lim_{\mathscr{F}'' \ni \nu \to \nu^\circ} j_{P|\bar{P}}(\nu) j_{P'|\bar{P}}(\nu)^{-1} j_{P'|\bar{P}}(\nu)^{*-1} j_{P|\bar{P}}(\nu)^* \times c \ ,
\end{aligned}
$$

where $\nu_\alpha \in \mathscr{F}_\alpha''$ and $\mathscr{F}'' = \mathscr{F}_\alpha'' + ({}^*\mathscr{F}_\alpha)_c$. Here we note that P' and P are adjacent parabolic subgroups of G, and moreover α is the unique root in $\Sigma(P') \cap \Sigma(P)$ (cf. Harish-Chandra [3] p. 120). Therefore from Lemma 13.1 in Harish-Chandra [3] we have

$$
(2.12)\qquad j_{P'|P}(\nu)^{-1} j_{P'|P}(\nu)^{*-1} = \mu_\alpha(\omega, \nu)
$$

where ν is in $\mathscr{F}_c$ on which these functions are well-defined. Thus from (2.11) and (2.12) we have for $\nu_\alpha \in \mathscr{F}_\alpha''$

$$
\begin{aligned}
(2.13)\quad \mu(\sigma, \nu_\alpha)^{-1} &= \lim_{\mathscr{F}'' \ni \nu \to \nu^\circ} \mu_\alpha(\omega, \nu) j_{P|\bar{P}}(\nu) j_{P|\bar{P}}(\nu)_*^{-1} \times c \\
&= \lim_{\mathscr{F}'' \ni \nu \to \nu_\circ} \mu_\alpha(\omega, \nu) \mu(\omega, \nu)^{-1} \times c \\
&= c \prod_{\gamma \in \Sigma(P) - \{\alpha\}} \mu_\gamma(\omega, \nu^\circ)^{-1} \ ,
\end{aligned}
$$

(see (1.16) in § 1.4 and Remark 2 in below). Therefore Lemma 5 was proved. Q.E.D.

REMARK 1. Under the assumption that (2.1) holds, we can easily prove that ${}^*\nu_\alpha^\circ$ is in $(-1)^{1/2}{}^*\mathscr{F}_\alpha$, and ${}^\circ x_p^\alpha({}^*\nu_\alpha^\circ = {}^\circ x_p^\alpha e_p^\alpha)$ is in $(-1)^{1/2}\boldsymbol{Z}$ by using the facts about infinitesimally characters of induced representations and discrete series for M_α.

Here let $L_\delta(\delta \in \boldsymbol{R})$ denote the line in the complex plane C whose imaginary part equals to $(-1)^{1/2}\delta$ and for $\underline{\delta} = (\delta_1, \delta_2, \cdots, \delta_p) \in \boldsymbol{R}^p$ we put $\mathscr{F}_i^{\underline{\delta}} = \{x_i^\alpha e_i^\alpha;\ x_i^\alpha \in L_i\}$, where $p = \dim A$ and $L_i = L_{\delta_i}$. Then for simplicity put

$$
\mathscr{F}^{\underline{\delta}} = \prod_{i=1}^{p} \mathscr{F}_i^{\underline{\delta}}\ , \quad \mathscr{F}_\alpha^{\underline{\delta}} = \prod_{i=1}^{p-1} \mathscr{F}_i^{\underline{\delta}} \quad \text{and} \quad {}^*\mathscr{F}_\alpha^{\underline{\delta}} = \mathscr{F}_p^{\underline{\delta}} \quad \text{respectively}\ .
$$

REMARK 2. Let ν be in $\mathscr{F}_c$ and regard it as a vector $(x_1^\alpha, x_2^\alpha, \cdots, x_p^\alpha)$ in C^p (see § 1.6). Then for $\gamma \in \Sigma(P) - \{\alpha\}$, x_p^γ can be written as

(2.14) $$x_p^\gamma=(x_1^\alpha, x_2^\alpha, \cdots, x_p^\alpha)(C_{1p}^{\alpha\gamma}, C_{2p}^{\alpha\gamma}, \cdots, C_{pp}^{\alpha\gamma})^t .$$

Now put $\nu^\circ=\nu_\alpha+{}^*\nu_\alpha^\circ$ ($\nu_\alpha\in\mathscr{F}_\alpha^\delta$ and ${}^*\nu_\alpha^\circ$ be as above). Then using the above Remark 1 and the facts that $(C_{1p}^{\alpha\gamma}, C_{2p}^{\alpha\gamma}, \cdots, C_{pp}^{\alpha\gamma})$ belongs to $\boldsymbol{R}^p$ and is not equal to $(0, 0, \cdots, 0, 1)$(see § 1.6), we prove that for a suitable $\underline{\delta}$, x_p^γ is not in $(-1)^{1/2}\boldsymbol{Z}$. On the other hand each singularity ${}^*\nu_\gamma^\circ={}^\circ x_p^\gamma e_p^\gamma$ of $\mu_\gamma(\omega, \nu)$ belongs to $(-1)^{1/2}{}^*\mathscr{F}_\gamma$ and in particular ${}^\circ x_p^\gamma$ to $(-1)^{1/2}\boldsymbol{Z}$. Therefore we can prove that ν° is not a singularity of $\mu_\gamma(\omega, \nu)(\gamma\in\Sigma(P)-\{\alpha\})$ for a suitable $\underline{\delta}\in\boldsymbol{R}^p$.

COROLLARY 1. *Let ${}^*\nu_\alpha^\circ$ be as above. Then there exists a sufficiently small $\underline{\delta}$ in $\boldsymbol{R}^p$ such that for $\nu_\alpha\in\mathscr{F}_\alpha^\delta$,*

(2.15) $$\mu(\sigma, \nu_\alpha)=c\prod_{\gamma\in\Sigma(P)-\{\alpha\}}\mu_\gamma(\omega, \nu^\circ) ,$$

where $\nu^\circ=\nu_\alpha+{}^*\nu_\alpha^\circ$.

PROOF. Since the singularities of $\mu_\beta(\omega, \nu)$ $(\beta\in\Sigma_0(P))$ are discrete, this corollary is obvious from Lemma 1 in § 1 and the similar arguments in the above Remark 2. Q.E.D.

Now let ${}^*\nu_\alpha^\circ$ and $\sigma=\sigma(\phi, {}^*\nu_\alpha^\circ)$ be as above and put $\psi(m)=E({}^*P_\alpha\colon\phi\colon{}^*\nu_\alpha^\circ\colon m)$ $(m\in M_\alpha)$ (see (2.1)). Then the following Lemma 6 is obvious from Lemma 3 in § 1.5.

LEMMA 6.

(2.16) $$E(P_\alpha\colon\psi\colon\nu_\alpha\colon x)=E(P\colon\phi\colon\nu^\circ\colon x)\quad(\nu_\alpha\in(\mathscr{F}_\alpha)_c \text{ and } x\in G) .$$

Therefore using these two lemmas, we obtain the following key proposition.

PROPOSITION 1. *Let notation be as above. Suppose that $\alpha(\nu_\alpha)$ belongs to $\mathscr{C}(\mathscr{F}_\alpha)$ and moreover has a holomorphic extension on $(\mathscr{F}_\alpha)_c$ which is an exponential type. Then we have*

(2.17) $$\int_{\mathscr{F}_\alpha^\delta}\prod_{\gamma\in\Sigma(P)-\{\alpha\}}\mu_\gamma(\omega, \nu_\alpha+{}^*\nu_\alpha^\circ)E(P\colon\phi\colon\nu_\alpha+{}^*\nu_\alpha^\circ\colon x)\alpha(\nu_\alpha)d\nu_\alpha$$

belongs to $\mathscr{C}_{A_\alpha}(G, \tau)=\mathscr{C}_{A_{\xi(\alpha)}}(G, \tau)$, where $\underline{\delta}$ is the same as in Corollary 1 and $d\nu_\alpha$ is the restriction of $d\nu$ on $\mathfrak{a}_\alpha$.

PROOF. From the results in Harish-Chandra [3] § 26, we can obtain that

$$(2.18) \qquad \int_{\mathscr{F}_\alpha} \mu(\sigma, \nu_\alpha)E(P_\alpha: \psi: \nu_\alpha: x)\alpha(\nu_\alpha)d\nu_\alpha$$

belongs to $\mathscr{C}_{A_\alpha}(G, \tau)=\mathscr{C}_{A_{i(\alpha)}}(G, \tau)$. Here we note that $\mu(\sigma, \nu_\alpha)$ is holomorphic on $(\mathscr{F}_\alpha)_c(\delta)$ (see Lemma 1 in § 1.4 for the case of (P_α, σ) instead of (P, ω)) and moreover $E(P_\alpha: \psi: \nu_\alpha: x)\alpha(\nu_\alpha)$ is holomorphic on $(\mathscr{F}_\alpha)_c$. Therefore using (1.17), (1.23) and the fact that $\alpha(\nu_\alpha)$ is an exponential type, we can obtain that (2.18) is equal to

$$(2.19) \qquad \int_{\mathscr{F}_\alpha^\delta} \mu(\sigma, \nu_\alpha)E(P_\alpha: \psi: \nu_\alpha: x)\alpha(\nu_\alpha)d\nu_\alpha$$

by Cauchy's integral theorem. Thus Proposition 1 is obvious from Corollary 1 and Lemma 6. Q.E.D.

Now suppose $\dim A=p(1\leqq p\leqq R)$ and put $S_k=\{i;\ 1\leqq i\leqq r$ such that $\dim A_i\leqq k\}$. Here we shall define the closed subspace $\mathscr{C}_k(G, \tau)$ of $\mathscr{C}(G, \tau)$ by

$$(2.20) \qquad \bigoplus_{i\in S_k} \mathscr{C}_{A_i}(G, \tau)\ .$$

In particular, we note that $\mathscr{C}_{R-1}(G, \tau)$ coincides with the kernel of $\mathscr{C}_{A_0}$.

§ 3. The decomposition of $f(x)$ and an analogue of Paley-Wiener theorem.

We keep to the notations in the preceding sections. For simplicity we shall denote the minimal parabolic subgroup $P_0=M_0A_0N_0$ of G by $P=MAN$ in this section. Suppose that $(\alpha_i^j(\nu);\ 1\leqq i\leqq n_j,\ 1\leqq j\leqq m)$ is in $\mathscr{C}(\mathscr{F})_*^n$ (see § 1.3) and each $\alpha_i^j(\nu)$ has a holomorphic extension on $\mathscr{F}_c$ with an exponential type. Now put

$$(3.1) \qquad f(x)=\sum_{j=1}^{m} |W(\omega_j)|^{-1} \sum_{i=1}^{n_j}\int_{\mathscr{F}} \mu(\omega_j, \nu)E(P: \phi_i^j: \nu: x)\alpha_i^j(\nu)d\nu$$

(cf. (1.10)). In this section we would like to prove the following type theorem.

THEOREM(*) *$f(x)$ can be decomposed as follows.*

$$(3.2) \qquad f(x)=F(x)+G(x)\ ,$$

where $F(x)$ and $G(x)$ belong to $C_c^\infty(G, \tau)$ and $\mathscr{C}_{R-1}(G, \tau)$ respectively.

However we don't have a general proof of this theorem. Here we shall prove for the case of a real rank one, i.e., $\dim A=1$, and for the case of $G=SU(2, 2)$.

(1) The case of a real rank one.

First we shall consider the following condition (see [4]).

(c1) If there exists a relation such that

$$\sum_{i,j,t} C_{i,j,t}\left(\frac{d^{m_t}}{d\nu^{m_t}}\right)_{|\nu=\nu_t} E(P: \phi_i^j: \nu: x)=0 , \tag{3.3}$$

where m_t is a non-negative integer, $\nu_t \in \mathscr{F}_c$ and $C_{i,j,t} \in C$, then

$$\sum_{i,j,t} C_{i,j,t}\left(\frac{d^{m_t}}{d\nu^{m_t}}\right)_{|\nu=\nu_t} \alpha_i^j(\nu)=0 . \tag{3.4}$$

Then we have obtained the following theorem in [4].

THEOREM 3. *Let notation be as above and suppose $\alpha_i^j(\nu)$ satisfy the condition* (c1). *Then $f(x)$ can be decomposed as follows.*

$$f(x)=F(x)+G(x) , \tag{3.5}$$

where $F(x) \in C_c^\infty(G, \tau)$ and $G(x) \in \mathscr{C}_0(G, \tau)={}^\circ\mathscr{C}(G, \tau)$ ($R=1$ in this case).

Next we shall consider the following condition about $\phi \in L_M(\omega)$ ($\omega \in \mathscr{E}_2(M)$).

(c2) Let $\nu=\nu^\circ \in \mathscr{F}_c$ be one of singularities of $\Phi(\nu: a)C_{P|P}(s; s^{-1}\nu)^{*^{-1}}\phi(1)$ ($a \in A^+$ and $s \in W=W(G, A)$) on $\mathscr{F}+(-1)^{1/2}CL(\mathscr{F}^+)$, where $\mathscr{F}^+=\{\nu \in \mathscr{F};\ \nu(H_\alpha)>0$ for all positive roots α of $(G, A)\}$. Then $\Phi(s\nu: a)C_{P|P}(s; \nu)^{*^{-1}}\phi(1)$ is holomorphic at $\nu=\nu^\circ$ and the following (3.7) is valid.

$$E(P: \phi: s^{-1}\nu^\circ: x) \text{ belongs to } L_G(\sigma) \text{ for some } \sigma \in \mathscr{E}_2(G) \tag{3.7}$$

(see G. Warner [9] Chap. 9.1 for the definitions of these operators).

REMARK 3. Here we note that $\mathfrak{Z}$-eigenvalues of the matrix coefficients of discrete series for G are real and regular. Therefore from the relation (3.7), we can easily prove that $\nu^\circ \in (-1)^{1/2}\mathscr{F}$ and $\mathrm{Im}(\nu^\circ)\neq 0$. In particular $\Phi(\nu: a)C_{P|P}(s; s^{-1}\nu)^{*^{-1}}\phi(1)$ ($s \in W$ and $a \in A_0^+$) is holomorphic on $\mathscr{F}$. Moreover, we note that there exists $b_0 \in (-1)^{1/2}\mathscr{F}^+$ such that the above function is holomorphic on $\mathrm{Im}(\nu)\geqq b_0$.

EXAMPLE. In the case of $G=SU(n, 1)$ ($n \in N$) and $\tau=(\tau_l, \tau_l)$, where $\tau_l (l \in Z)$ is a one-dimensional representation of K on C, these assumptions hold (see N. Wallach [8]).

Theorem 4. *Suppose that all* $\phi_i^j (1 \leqq i \leqq n_j,\ 1 \leqq j \leqq m)$ *satisfy the condition* (c2). *Then* $f(x)$ *can be decomposed as* (3.5).

Proof. First we put

$$f(x) = \int_{\mathscr{F}} \mu(\omega, \nu) E(P: \phi: \nu: x) \alpha(\nu) d\nu , \tag{3.8}$$

where ϕ satisfies the above condition (c2) and $\alpha(\nu) \in \mathscr{C}(\mathscr{F})$ has a holomorphic extension on $\mathscr{F}_c$ with an exponential type. Now we recall the Harish-Chandra expansion of the Eisenstein integral, i.e., for $a \in A^+$ and $\nu \in \Gamma'$,

$$\begin{aligned} E(P: \phi: \nu: a) &= \sum_{s \in W} \Phi(s\nu: a) C_{P|P}(s; \nu) \phi(1) e^{-\rho_0(\log(a))} \\ &= \sum_{s \in W} \Phi'(s\nu: a) C_{P|P}(s; \nu) \phi(1) e^{\{(-1)^{1/2} s\nu - \rho_0\}(\log(a))} , \end{aligned} \tag{3.9}$$

and

$$\Phi'(\nu: a) \longrightarrow 1 \text{ when } \alpha(\log(a)) \longrightarrow \infty \text{ for all } \alpha \in \Delta^+ \tag{3.10}$$

(see G. Warner [9] Chap. 9, Theorems 9.1.4.1 and 9.1.5.1). Therefore we can rewrite $f(a)$ $(a \in A^+)$ as follows;

$$\begin{aligned} f(a) &= \int_{\mathscr{F}} \mu(\omega, \nu) \sum_{s \in W} \Phi(s\nu: a) C_{P|P}(s; \nu) \phi(1) e^{-\rho_0(\log(a))} \alpha(\nu) d\nu \\ &= \sum_{s \in W} c^2 \int_{\mathscr{F}} \Phi(s\nu: a) C_{P|P}(s; \nu)^{*-1} \phi(1) e^{-\rho_0(\log(a))} \alpha(\nu) d\nu \\ &= \sum_{s \in W} c^2 \int_{\mathscr{F}} \Phi(\nu: a) C_{P|P}(s; s^{-1}\nu)^{*-1} \phi(1) e^{-\rho_0(\log(a))} \alpha(s^{-1}\nu) d\nu . \end{aligned} \tag{3.11}$$

Here we used the following relation (3.12) and the fact that the integrand is holomorphic on $\mathscr{F}$ (see Remark 3).

$$\mu(\omega, \nu) C_{P|P}(s; \nu)^* C_{P|P}(s; \nu) = c^2 \tag{3.12}$$

for $s \in W$ and $\nu \in \mathscr{F}'$ (see Harish-Chandra [3] Lemma 17.1).

Next we note that the dimension of $\mathscr{F}$ is equal to one and the singular points of $\Phi(\nu: a) C_{P|P}(s; s^{-1}\nu)^{*-1} \phi(1)$ $(a \in A^+$ and $s \in W)$ are discrete and finite on $\mathscr{F}^+ (-1)^{1/2} CL(\mathscr{F}^+)$. Thus using the residue theorem, we can change the integral line $\mathscr{F}$ to $\mathscr{F} + b_0$ as follows.

$$\begin{aligned} f(a) = \sum_{s \in W} c^2 \int_{\mathscr{F} + b_0} & \Phi(\nu: a) C_{P|P}(s; s^{-1}\nu)^{*-1} \phi(1) e^{-\rho_0(\log(a))} \alpha(s^{-1}\nu) d\nu \\ & + \sum \mathrm{Res}\{\Phi(\nu: a) C_{P|P}(s; s^{-1}\nu)^{*-1} \phi(1) \alpha(s^{-1}\nu)\} c^2 e^{-\rho_0(\log(a))} , \end{aligned} \tag{3.13}$$

where $\sum$ Res is the residua corresponding to the finite singular points of the integrand of (3.11) on $0<\mathrm{Im}(\nu)<b_0$. For simplicity we shall denote the first term of (3.13) by $I(a)$ ($a\in A^+$). Then using the same method in the proof of Theorem 2 in [4] and the fact that the integrand of (3.11) is holomorphic on $\mathrm{Im}(\nu)\geqq b_0$, we can easily prove that $I(a)=0$ for a sufficiently large $a\in A^+$. Thus from the Cartan decomposition $G=KCL(A^+)K$ of G and the τ-sphericalness of $f(x)$ we can extend $I(a)$ to a compactly supported function $F(x)$ on G.

Next let $\nu=\nu^\circ\in\mathscr{F}_c$ be one of singularities of the integrand of (3.11) on $0<\mathrm{Im}(\nu)<b_0$. Then from the condition (c2) we have

$$(3.14)\qquad E(P:\phi:s^{-1}\nu^\circ:x)\in{}^\circ\mathscr{C}(G,\tau)\,.$$

In particular it belongs to $L^2(G,\tau)$ and thus, from (3.9) and (3.10) we can obtain

$$(3.15)\qquad E(P:\phi:s^{-1}\nu^\circ:a)=\Phi(\nu^\circ:a)C_{P|P}(s;s^{-1}\nu^\circ)\phi(1)\quad(a\in A^+\text{ and } s\in W)\,.$$

Therefore using these facts, we have for $a\in A^+$

$$(3.16)\qquad \begin{aligned}&\operatorname*{Res}_{\nu=\nu^\circ}\{\Phi(\nu:a)C_{P|P}(s;s^{-1}\nu)^{*-1}\phi(1)\alpha(s^{-1}\nu)\}\\ &=\operatorname*{Res}_{\nu=\nu^\circ}\{\mu(\omega,\nu)\Phi(\nu:a)C_{P|P}(s;s^{-1}\nu)\phi(1)\alpha(s^{-1}\nu)\}\\ &=\operatorname*{Res}_{\nu=\nu^\circ}\{\mu(\omega,\nu)\}\Phi(\nu^\circ:a)C_{P|P}(s;s^{-1}\nu^\circ)\phi(1)\alpha(s^{-1}\nu)\\ &=\operatorname*{Res}_{\nu=\nu^\circ}\{\mu(\omega,\nu)\}E(P:\phi:s^{-1}\nu^\circ:a)\alpha(s^{-1}\nu^\circ)\,.\end{aligned}$$

Hence we can easily extend $\mathrm{Res}_{\nu=\nu^\circ}\{\Phi(\nu:a)C_{P|P}(s;s^{-1}\nu)^{*-1}\phi(1)\alpha(s^{-1}\nu)\}c^2e^{-\rho_0(\log(a))}$ to a function in ${}^\circ\mathscr{C}(G,\tau)$. Therefore the second term of (3.13) extends to a function $G(x)\in{}^\circ\mathscr{C}(G,\tau)$. Then $f(x)$ can be written as

$$(3.17)\qquad f(x)=F(x)+G(x)\,,$$

where $F(x)$ and $G(x)$ belong to $C_c^\infty(G,\tau)$ and ${}^\circ\mathscr{C}(G,\tau)$ respectively. This completes the proof of Theorem 4. Q.E.D.

(2) The case of $G=SU(2,2)$.

First we shall consider the following two conditions, which are natural extensions of the conditions (c1) and (c2) in the case of a real rank one.

(C1) We shall use the parametrization of $\nu\in\mathscr{F}_c$ in §1.6, i.e., $\nu=(x_1^\alpha,x_2^\alpha)\in C^2(\alpha\in\Sigma(P))$. If there exists a relation such that

$$(3.18) \qquad \sum_{i,j,t,s} C_{i,j,t,s} \frac{\partial^{m_t+n_s}}{(\partial x_1^\alpha)^{m_t}(\partial x_2^\alpha)^{n_s}}\Big|_{\nu=\nu_{t,s}^\circ} E(P: \phi_i^j: \nu: x)=0 ,$$

where m_t, n_s are non-negative integers, $\nu_{t,s}^\circ \in \mathscr{F}_c$ and $C_{i,j,t,s} \in \boldsymbol{C}$, then

$$(3.19) \qquad \sum_{i,j,t,s} C_{i,j,t,s} \frac{\partial^{m_t+n_s}}{(\partial x_1^\alpha)^{m_t}(\partial x_2^\alpha)^{n_s}}\Big|_{\nu=\nu_{t,s}^\circ} \alpha_i^j(\nu)=0 .$$

(C2) Let $\nu=\nu^\circ \in \mathscr{F}_c$ be one of singular points of $\Phi(\nu: a)C_{P|P}(s; s^{-1}\nu)^{*-1}$ $\phi(1)$ ($a \in A^+$ and $s \in W$) on $\mathscr{F}+(-1)^{1/2}CL(\mathscr{F}^+)$. Then there exists a $\beta \in \Sigma_0(P)$ such that

$$(3.20) \qquad E({}^*P_\beta: \phi: {}^*\nu_\beta^\circ: m) \text{ belongs to } L_{M_\beta}(\sigma) \text{ for some } \sigma \in \mathscr{E}_2(M_\beta) ,$$

where ${}^*\nu_\beta^\circ$ is the restriction of ν° on ${}^*\mathfrak{a}_\beta$. Here we note that in this case ${}^*\nu_\beta^\circ \in (-1)^{1/2}{}^*\mathscr{F}_\beta$ and moreover there exists a $b_0 \in (-1)^{1/2}\mathscr{F}^+$ such that $\Phi(\nu: a)C_{P|P}(s; s^{-1}\nu)^{*-1}\phi(1)$ is holomorphic on $\mathrm{Im}(\nu) \geqq b_0$.

To obtain a proof of Theorem(*) for the case of $\tau=(\tau^l, \tau^l)$, where τ^l ($l \in \boldsymbol{Z}$) is a one-dimensional representation of K on $\boldsymbol{C}$, we assume the above condition (C1). Moreover we have to calculate the explicit forms of c-and μ-functions. We shall give these calculations and a proof of Theorem(*) in the following § 4.

§ 4. $G=SU(2, 2)$ and $\tau=(\tau^l, \tau^l)$.

4.1. The Cartan decomposition.

$$\mathfrak{g}=\left\{\begin{pmatrix} A & C \\ {}^t\bar{C} & B \end{pmatrix} ; \; A, B \text{ skew Hermitian of order 2, } \mathrm{Tr}A+\mathrm{Tr}B=0, \; C \text{ arbitrary}\right\} ,$$

$$\mathfrak{k}=\left\{\begin{pmatrix} A & \\ & B \end{pmatrix} ; \; A, B \text{ skew Hermitian of order 2, } \mathrm{Tr}A+\mathrm{Tr}B=0\right\} ,$$

$$\mathfrak{p}=\left\{\begin{pmatrix} & C \\ {}^t\bar{C} & \end{pmatrix} ; \; C \text{ arbitrary} \right\} .$$

4.2. The conjugacy classes of Cartan subalgebras.

$$\mathfrak{h}_0=\left\{\begin{pmatrix} u & & h_1 & \\ & -u & & h_2 \\ h_1 & & u & \\ & h_2 & & -u \end{pmatrix} ; \; u \in \sqrt{-1}\boldsymbol{R}, \; h_1, h_2 \in \boldsymbol{R}\right\} ,$$

$$\mathfrak{h}_1=\left\{\begin{pmatrix} u_1 & & h & \\ & u_2 & & \\ h & & u_1 & \\ & & & -2u_1-u_2 \end{pmatrix};\quad u_1,\, u_2\in\sqrt{-1}\boldsymbol{R},\ h\in\boldsymbol{R}\right\},$$

$$\mathfrak{h}_2=\left\{\begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & u_3 & \\ & & & -u_1-u_2-u_3 \end{pmatrix};\quad u_1,\, u_2,\, u_3\in\sqrt{-1}\boldsymbol{R}\right\}.$$

4.3. A minimal parabolic subalgebra.

$$\mathfrak{m}_0=\left\{\begin{pmatrix} u & & & \\ & -u & & \\ & & u & \\ & & & -u \end{pmatrix};\quad u\in\sqrt{-1}\boldsymbol{R}\right\},$$

$$\mathfrak{a}_0=\left\{\begin{pmatrix} & & h_1 & \\ & & & h_2 \\ h_1 & & & \\ & h_2 & & \end{pmatrix};\quad h_1,\, h_2\in\boldsymbol{R}\right\},$$

$$\mathfrak{n}_0=\left\{\begin{pmatrix} a & c+d & -a & c-d \\ -\bar{c}-\bar{d} & -b & \bar{c}+\bar{d} & b \\ a & c+d & -a & c-d \\ \bar{c}-\bar{d} & -b & -\bar{c}+\bar{d} & b \end{pmatrix};\quad a,\, b\in\sqrt{-1}\boldsymbol{R},\ c,\, d\in\boldsymbol{C}\right\}.$$

4.4. The root system of $(\mathfrak{g}, \mathfrak{h}_0)$.

Let $e_i(1\leqq i\leqq 4)$ be the linear form on $\mathfrak{h}_0$ defined by

$$\begin{aligned} e_1(H)&=-\sqrt{-1}u+h_1\\ e_2(H)&=\sqrt{-1}u+h_2\\ e_3(H)&=-\sqrt{-1}u-h_1\\ e_4(H)&=\sqrt{-1}u-h_2 \end{aligned}\qquad \text{for } H=\begin{pmatrix} u & & h_1 & \\ & -u & & h_2 \\ h_1 & & u & \\ & h_2 & & -u \end{pmatrix}\in\mathfrak{h}_0\,.$$

Then the root system $\mathscr{R}$ of $\mathfrak{g}$ is given by

$$\mathscr{R}=\{\pm(e_i-e_j);\ 1\leqq i<j\leqq 4\}\,.$$

4.5. The root system of $(\mathfrak{g}, \mathfrak{a}_0)$.

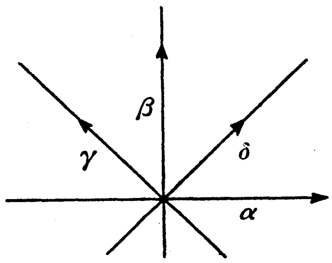
Now we shall denote the restriction of $e_i(1\leqq i\leqq 4)$ to $\mathfrak{a}_0$ by the same notation. Then the restricted roots system of Σ is expressed as follows.

$$\begin{aligned}\Sigma=\Sigma(P_0)&=\{\pm(e_1-e_2),\ \pm(e_1-e_3),\ \pm(e_1-e_4),\ \pm(e_2-e_4)\}\\ &=\{\pm\alpha_2,\ \pm(\alpha_1+2\alpha_2),\ \pm(\alpha_1+\alpha_2),\ \pm\alpha_1\}\ ,\end{aligned}$$

where $\alpha_1=e_2-e_4$, $\alpha_2=e_1-e_2$. Now put $c_{ij}=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)(1\leqq i,\ j\leqq 2)$. Then the Cartan matrix (c_{ij}) is given by

$$\begin{pmatrix} 2 & -1\\ -2 & 2\end{pmatrix}.$$

For simplicity we put $\alpha=\alpha_1$, $\beta=\alpha_1+2\alpha_2$, $\gamma=\alpha_2$ and $\delta=\alpha_1+\alpha_2$ respectively. Then the relation between these roots can be written as follows.



4.6. $*\mathfrak{a}_\varepsilon,\ \mathfrak{a}_\varepsilon,\ \mathfrak{m}_\varepsilon,\ \mathfrak{g}_\varepsilon(\varepsilon=\alpha,\beta,\gamma,\delta)$.

$$*\mathfrak{a}_\gamma=\left\{\begin{pmatrix} & & a & \\ & & & -a\\ a & & & \\ & -a & & \end{pmatrix};\ a\in\boldsymbol{R}\right\}\qquad \mathfrak{a}_\gamma=\left\{\begin{pmatrix} & & a & \\ & & & a\\ a & & & \\ & a & & \end{pmatrix};\ a\in\boldsymbol{R}\right\}$$

$$\mathfrak{m}_\gamma=\left\{\begin{pmatrix} a & c & v & u\\ -\bar{c} & -a & \bar{u} & -v\\ v & u & a & c\\ \bar{u} & -v & -\bar{c} & -a\end{pmatrix};\quad a\in\sqrt{-1}\boldsymbol{R},\ v\in\boldsymbol{R},\ c,\ v\in\boldsymbol{C}\right\}$$

$$\mathfrak{g}_\gamma=\left\{\begin{pmatrix} & a & & a\\ -\bar{a} & & \bar{a} & \\ & a & & a\\ \bar{a} & & -\bar{a} & \end{pmatrix};\quad a\in\boldsymbol{C}\right\}$$

$$*\mathfrak{a}_\delta=\left\{\begin{pmatrix} & & a & \\ & & & a\\ a & & & \\ & a & & \end{pmatrix};\ a\in\boldsymbol{R}\right\}\qquad \mathfrak{a}_\delta=\left\{\begin{pmatrix} & & a & \\ & & & -a\\ a & & & \\ & -a & & \end{pmatrix};\ a\in\boldsymbol{R}\right\}$$

$$\mathfrak{m}_\delta=\left\{\begin{pmatrix} a & c & v & u \\ -\bar{c} & a & -\bar{u} & v \\ v & -u & -a & -c \\ \bar{u} & v & \bar{c} & -a \end{pmatrix};\quad a\in\sqrt{-1}\boldsymbol{R},\ v\in\boldsymbol{R},\ c,\ u\in\boldsymbol{C}\right\}$$

$$\mathfrak{g}_\delta=\left\{\begin{pmatrix} & a & & -a \\ -\bar{a} & & \bar{a} & \\ & a & & -a \\ -\bar{a} & & \bar{a} & \end{pmatrix};\quad a\in\boldsymbol{C}\right\}$$

$${}^*\mathfrak{a}_\alpha=\left\{\begin{pmatrix} & & & \\ & & & a \\ & & & \\ & a & & \end{pmatrix};\quad a\in\boldsymbol{R}\right\}\qquad \mathfrak{a}_\alpha=\left\{\begin{pmatrix} & & a & \\ & & & \\ a & & & \\ & & & \end{pmatrix};\quad a\in\boldsymbol{R}\right\}$$

$$\mathfrak{m}_\alpha=\left\{\begin{pmatrix} a & & & \\ & b & & v \\ & & a & \\ \bar{v} & & -2a & -b \end{pmatrix};\quad a,\ b\in\sqrt{-1}\boldsymbol{R},\ v\in\boldsymbol{C}\right\}$$

$$\mathfrak{g}_\alpha=\left\{\begin{pmatrix} & & & \\ & & -a & a \\ & & & \\ & & -a & a \end{pmatrix};\quad a\in\sqrt{-1}\boldsymbol{R}\right\}$$

$${}^*\mathfrak{a}_\beta=\left\{\begin{pmatrix} & & a & \\ & & & \\ a & & & \\ & & & \end{pmatrix};\quad a\in\boldsymbol{R}\right\}\qquad \mathfrak{a}_\beta=\left\{\begin{pmatrix} & & & \\ & & & a \\ & & & \\ & a & & \end{pmatrix};\quad a\in\boldsymbol{R}\right\}$$

$$\mathfrak{m}_\beta=\left\{\begin{pmatrix} a & v & & \\ & b & & \\ \bar{v} & -2b & -a & \\ & & b & \end{pmatrix};\quad a,\ b\in\sqrt{-1}\boldsymbol{R},\ v\in\boldsymbol{C}\right\}$$

$$\mathfrak{g}_\beta=\left\{\begin{pmatrix} a & & -a & \\ & & & \\ a & & -a & \\ & & & \end{pmatrix};\quad a\in\sqrt{-1}\boldsymbol{R}\right\}.$$

Here we note that

$$M_\gamma \cong M_\delta SL(2, \boldsymbol{C}), \quad M_\alpha \cong M_\beta \cong U(1, 1) \cong SU(1, 1) \times \boldsymbol{T}.$$

4.7. One-dimensional representations of K.

For each integer l the one-dimensional representation τ^l of K on $\boldsymbol{C}$ is given as follows.

$$\tau^l(k) = (\det V)^l \quad \text{for} \quad k = \begin{pmatrix} U & \\ & V \end{pmatrix} \in K,$$

where $U, V \in U(2)$ and $\det U \det V = 1$.

4.8. The restriction of τ^l to $M_\varepsilon(\varepsilon = 0, \alpha, \beta, \gamma, \delta)$.

Put $K_\varepsilon = K \cap M_\varepsilon(\varepsilon = 0, \alpha, \beta, \gamma, \delta)$ and denote the restriction of $\tau^l (l \in \boldsymbol{Z})$ to K_ε by τ^l_ε. Then

$$\tau^l_0(k) = 1 \qquad \text{for} \quad k = \begin{pmatrix} e^{\sqrt{-1}\theta} & & & \\ & e^{-\sqrt{-1}\theta} & & \\ & & e^{\sqrt{-1}\theta} & \\ & & & e^{-\sqrt{-1}\theta} \end{pmatrix} (\theta \in \boldsymbol{R}) \in K_0,$$

$$\tau^l_\alpha(k) = e^{\sqrt{-1}l(\theta+\phi)} \quad \text{for} \quad k = \begin{pmatrix} e^{\sqrt{-1}\theta} & & & \\ & e^{-\sqrt{-1}(2\theta+\phi)} & & \\ & & e^{\sqrt{-1}\theta} & \\ & & & e^{\sqrt{-1}\phi} \end{pmatrix} (\theta, \phi \in \boldsymbol{R}) \in K_\alpha,$$

$$\tau^l_\beta(k) = e^{\sqrt{-1}l(\theta+\phi)} \quad \text{for} \quad k = \begin{pmatrix} e^{-\sqrt{-1}(2\phi+\theta)} & & & \\ & e^{\sqrt{-1}\phi} & & \\ & & e^{\sqrt{-1}\theta} & \\ & & & e^{\sqrt{-1}\phi} \end{pmatrix} (\theta, \phi \in \boldsymbol{R}) \in K_\beta,$$

$$\tau^l_\gamma(k) = (\pm 1)^l \qquad \text{for} \quad k = \begin{pmatrix} U & \\ & U \end{pmatrix} (U \in U(2), \det U^2 = 1) \in K_\gamma,$$

$$\tau^l_\delta(k) = \det U^{-l} \qquad \text{for} \quad k = \begin{pmatrix} U & \\ & U^{-1} \end{pmatrix} (U \in U(2)) \in K_\delta.$$

Here we note that τ^l_α and τ^l_β can be written as $\tau^l_\alpha = \lambda_l \zeta_l$ and $\tau^l_\beta = \lambda_l \zeta_l$, where λ_l is a representation of $SO(2)$ (a maximal compact subgroup of $SL(2, \boldsymbol{R})$) and ζ_l is a representation of $\boldsymbol{T}$ respectively.

4.9. Explicit forms of c-functions.

Put $\tau = (\tau^l, \tau^l)$ $(l \in \boldsymbol{Z})$. Here we shall calculate the explicit forms of c-functions; $C_{P_0|P_0}(1; \nu)$ and $C_{*P_\varepsilon|*P_\varepsilon}(1; {}^*\nu_\varepsilon)(\varepsilon = \alpha, \beta, \gamma, \delta)$. For simplicity we shall denote these functions by $C^l_0(\nu)$ and $C^l_\varepsilon(\nu)(\varepsilon = \alpha, \beta, \gamma, \delta)$ respectively and moreover use the parametrization of ν in §1.6 and put $x^{\varepsilon}=x_2^{\varepsilon}$ ($\varepsilon=\alpha, \beta, \gamma, \delta$). Then

$$C_{\varepsilon}^{l}(\nu)=\frac{\Gamma\left(\frac{\sqrt{-1}x^{\varepsilon}}{2}\right)\Gamma\left(\frac{\sqrt{-1}x^{\varepsilon}}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\sqrt{-1}x^{\varepsilon}}{2}+\frac{l+1}{2}\right)+\Gamma\left(\frac{\sqrt{-1}x^{\varepsilon}}{2}+\frac{1-l}{2}\right)} \quad \text{for} \quad \varepsilon=\alpha, \beta,$$

$$C_{\varepsilon}^{l}(\nu)=\frac{1}{x^{\varepsilon}} \quad \text{for} \quad \varepsilon=\gamma, \delta$$

(see Y. Muta [6] and G. Warner [9] Chap. 9). Here we note that $x^{\gamma}=x^{\beta}/2-x^{\alpha}/2$ and $x^{\delta}=x^{\alpha}/2+x^{\beta}/2$. Thus $C_0^l(\nu)$ is given as follows.

$$\begin{aligned}C_0^l(\nu)&=C_{\alpha}^l(\nu)C_{\beta}^l(\nu)C_{\gamma}^l(\nu)C_{\delta}^l(\nu)\\ &=\Gamma\left(\frac{\sqrt{-1}x^{\alpha}}{2}\right)\Gamma\left(\frac{\sqrt{-1}x^{\alpha}}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\sqrt{-1}x^{\beta}}{2}\right)\Gamma\left(\frac{\sqrt{-1}x^{\beta}}{2}+\frac{1}{2}\right)\\ &\quad\times\left\{\pi\Gamma\left(\frac{\sqrt{-1}x^{\alpha}}{2}+\frac{l+1}{2}\right)\Gamma\left(\frac{\sqrt{-1}x^{\alpha}}{2}+\frac{1-l}{2}\right)\Gamma\left(\frac{\sqrt{-1}x^{\beta}}{2}+\frac{1+l}{2}\right)\right.\\ &\quad\left.\times\Gamma\left(\frac{\sqrt{-1}x^{\beta}}{2}+\frac{1-l}{2}\right)\left(\left(\frac{x^{\beta}}{2}\right)^2-\left(\frac{x^{\alpha}}{2}\right)^2\right)\right\}^{-1}.\end{aligned}$$

4.10. Explicit forms of μ-functions.

For simplicity we denote $\mu(1, \nu)$ and $\mu_{\varepsilon}(1, {}^*\nu_{\varepsilon})$ ($\varepsilon=\alpha, \beta, \gamma, \delta$), where 1 is the trivial representation of $K_0=M_0$, by $\mu_0^l(\nu)$ and $\mu_{\varepsilon}^l(\nu)$ respectively. Then using the relation; $\mu_{\varepsilon}^l(\nu)C_{\varepsilon}^l(\nu)C_{\varepsilon}^l(-\nu)=1$($\varepsilon=0, \alpha, \beta, \gamma, \delta$), we can obtain

$$\mu_{\varepsilon}^l(\nu)=\pi\left(\frac{x^{\varepsilon}}{2}\right)\text{th}\,\pi\left(\frac{x^{\varepsilon}}{2}+\frac{\sqrt{-1}l}{2}\right) \quad \text{for} \quad \varepsilon=\alpha, \beta,$$

$$\mu_{\varepsilon}^l(\nu)=(x^{\varepsilon})^2 \quad \text{for} \quad \varepsilon=\gamma, \delta,$$

$$\mu_0^l(\nu)=\pi^2\left(\frac{x^{\alpha}}{2}\right)\text{th}\,\pi\left(\frac{x^{\alpha}}{2}+\frac{\sqrt{-1}l}{2}\right)\left(\frac{x^{\beta}}{2}\right)\text{th}\,\pi\left(\frac{x^{\beta}}{2}+\frac{\sqrt{-1}l}{2}\right)\left(\left(\frac{x^{\beta}}{2}\right)^2-\left(\frac{x^{\alpha}}{2}\right)^2\right)^2,$$

4.11. Proof of Theorem(*).

First we note that $L_{M_0}={}^{\circ}\mathscr{C}(M_0, 1)\cong C$ and moreover $n=n(P_0)=1$. Thus we can choose an orthonormal base $\{1\}=\{1_l\}$ in L_{M_0}. Therefore

$$\mathscr{C}(\mathscr{F}_0)_*^n=\mathscr{C}(\mathscr{F}_0)_*=\{\alpha\in\mathscr{C}(\mathscr{F}_0)\,;\ \alpha(s\nu)=\alpha(\nu) \ \text{ for all } s\in W_0=W(A_0) \ \text{ and } \ \nu\in\mathscr{F}_0\},$$

and the mapping $\mathscr{C}_{A_0}$ is an homoemorphism of $\mathscr{C}_{A_0}(G, \tau)$ onto $\mathscr{C}(\mathscr{F}_0)_*$.

Now let $\alpha(\nu)$ be in $\mathscr{C}(\mathscr{F}_0)_*$ which has a holomorphic extension on $(\mathscr{F}_0)_c$ with an exponential type. Here we put

(4.1) $$f(x)=\int_{\mathscr{F}_0}\mu_0^l(\nu)E(P_0\colon 1\colon \nu\colon x)\alpha(\nu)d\nu \quad (x\in G) .$$

Then we obtain the following Theorem 5.

THEOREM 5. *Suppose that* $\alpha(\nu)$ *satisfies the condition* (C1). *Then* $f(x)$ *can be decomposed as follows.*

(4.2) $$f(x)=F(x)+G(x) ,$$

where $F(x)\in C_c^\infty(G, \tau)$ *and* $G(x)\in \mathscr{C}_1(G, \tau)$ (*note that* $R=2$ *in this case*).

PROOF. First we recall a relation between the matrix coefficients of discrete series and non-unitary principal series on $SL(2, \boldsymbol{R})$ (cf. [10]). Then using this relation and the explicit forms of c- and μ-functions, we can easily prove that the poles $\nu=\nu^\circ$ of $C_0^l(\nu)^{*-1}$ satisfy the condition (C2) in §3. Here we note that from the recurrence definition of $\Phi(\nu\colon a)$ ($a\in A_0^+$) (cf. Appendix (3)), $\Phi(\nu\colon a)$ has no poles on $(-1)^{1/2}CL(\mathscr{F}^+)$ and is holomorphic on $\mathscr{F}$. Moreover, from the explicit form of c- and μ-functions, $C_0^l(\nu)^{*-1}$ is also holomorphic on $\mathscr{F}$. Therefore by the same calculation in the case of a real rank one, we obtain that

(4.3) $$\begin{aligned} f(a)&=\sum_{s\in W_0} c^2\int_{\mathscr{F}_0}\Phi(\nu\colon a)C_0^l(\nu)^{*-1}e^{-\rho_0(\log(a))}\alpha(s^{-1}\nu)d\nu \\ &=|W_0|c^2\int_{\mathscr{F}_0}\Phi(\nu\colon a)C_0^l(\nu)^{*-1}e^{-\rho_0(\log(a))}\alpha(\nu)d\nu \quad (a\in A_0^+) .\end{aligned}$$

Here we shall denote $x_1^\alpha(=x_2^\beta)$ and $x_2^\alpha(=x_1^\beta)$ by x and y respectively and put

(4.4) $$\begin{aligned} I(x, y\colon a)&=\Phi(\nu\colon a)C_0^l(\nu)^{*-1}e^{-\rho_0(\log(a))}\alpha(\nu) \\ &=\mu_0^l(\nu)\Phi(\nu\colon a)C_0^l(\nu)e^{-\rho_0(\log(a))}\alpha(\nu) \quad (\nu=xe_1^\alpha+ye_2^\alpha \text{ and } a\in A_0^+)\end{aligned}$$

for simplicity. Moreover we shall regard it as a meromorphic function of ν on $(\mathscr{F}_0)_c$. Here we note that the Weyl group of (G, A_0) consists of the following transformations;

(4.5) $$\begin{array}{ll} s_0\colon (x, y)\longrightarrow (x, y) & s_4\colon (x, y)\longrightarrow (y, x) \\ s_1\colon (x, y)\longrightarrow (x, -y) & s_5\colon (x, y)\longrightarrow (-y, x) \\ s_2\colon (x, y)\longrightarrow (-x, y) & s_6\colon (x, y)\longrightarrow (y, -x) \\ s_3\colon (x, y)\longrightarrow (-x, -y) & s_7\colon (x, y)\longrightarrow (-y, -x) .\end{array}$$

Thus, $|W_0|=8$. For an integer p we put $z_p=\sqrt{-1}(2p-1)$ (resp. $2\sqrt{-1}p$) when l is even (resp. odd). Moreover let $v_t=v_t(p)$ $(0\leqq t\leqq T=T(p))$ denote the singular points of $\mathrm{Res}_{x=z_p} I(x, y: a)(a \in A_0^+)$ on $0 \leqq \mathrm{Im}(y) \leqq \sqrt{-1}l$ and m_t denote its order. In particular we put $v_0=0$ (if $v_0=0$ is not a pole, it is not necessary to consider the residue at this point). Let b_0 be as in the condition (C2) in §3. In this case $(\sqrt{-1}l, \sqrt{-1}l)<b_0 \leqq (\sqrt{-1}(l+1), \sqrt{-1}(l+1))$.

Now we shall change the integral line $\mathscr{F}_0$ of (4.3) to $\mathscr{F}_0+b_0$. Then using the residue theorem repeatly, we obtain that for $a \in A_0^+$ and a sufficiently small δ in $(-1)^{1/2}\boldsymbol{R}^+$,

$$
\begin{aligned}
(4.6) \qquad f(a) &= 8c^2 \int_{\mathscr{F}_0} I(x, y: a) d\nu \\
&= 8c^2 \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} I(x, y: a) dx dy \\
&= 8c^2 \int_{\boldsymbol{R}+b_1} \int_{\boldsymbol{R}+b_2} I(x, y: a) dx dy \\
&\quad + 8c^2 \left\{ \int_{\boldsymbol{R}} \sum_{p=1}^{l'} \operatorname*{Res}_{y=z_p} I(x, y: a) dx + 8c^2 \int_{\boldsymbol{R}-\delta} \sum_{q=1}^{l'} \operatorname*{Res}_{x=z_q} I(x, y: a) dy \right\} \\
&\quad - 8c^2 \sum_{q=1}^{l'} \sum_{t=0}^{T(q)} \operatorname*{Res}_{y=v_t} \left(\operatorname*{Res}_{x=z_q} I(x, y: a) \right),
\end{aligned}
$$

where $b_i=(b_0, e_i^\alpha)$ $(i=1, 2)$ and $l'=[l/2]$. For simplicity we shall denote these three terms by $I_1^f(a)$, $I_2^f(a)$ and $I_3^f(a)$ respectively.

(I_1^f) First we note that $I(x, y: a)$ is a holomorphic function on $\mathrm{Im}(x) \geqq b_1$ and $\mathrm{Im}(y) \geqq b_2$, and moreover satisfies the following Lemma 7.

LEMMA 7. *Fix $a \in A_0^+$ and suppose that ν is sufficiently distant from the singularities of $\Phi'(\nu: a)C_0^l(\nu)^{*-1}$. Then there exist numbers c, $r>0$ such that*

$$
(4.7) \qquad |\Phi'(\nu: a)C_0^l(\nu)^{*-1}| < c(1+|\nu|)^r .
$$

PROOF. This Lemma is obvious from G. Warner [9] Chap. 9 and the explicit form of $C_0^l(\nu)$. Q.E.D.

Moreover we note that $\alpha(\nu)$ is a holomorphic function on $(\mathscr{F}_0)_c$ with an exponential type. Therefore using the above lemma and the method of the classical Paley-Wiener theorem on an Euclidean space, we can prove that for a sufficiently large $a \in A_0^+$, $I_1^f(a)=0$. Hence from the Cartan decomposition $G=KCL(A_0^+)K$ and the fact that $f(x)$ is a τ-spherical function on G, we can easily prove that $I_1^f(a)$ extends to a compactly supported function $I_1^f(x)$ on G.

($^\circ I_2^f$) From the results in §2 we obtain that for a safficiently small $\delta \in (-1)^{1/2}\boldsymbol{R}^+$,

$$R(p: g)=\int_{R+\delta} \operatorname*{Res}_{y=z_p} \mu_0^l(x, y)E(P_0: 1: x, y: g)\alpha(x, y)dx \qquad (g \in G \quad \text{and} \quad 1 \leqq p \leqq l') \tag{4.8}$$

belongs to $\mathscr{C}_1(G, \tau)$. Thus for $a \in A_0^+$

$$\begin{aligned} R(p: a) &= \int_{R+\delta} \operatorname*{Res}_{y=z_p} \mu_0^l(x, y)E(P_0: 1: x, y: a)\alpha(x, y)dx \\ &= \sum_{s \in W_0} \int_{R+\delta} \operatorname*{Res}_{y=z_p} \mu_0^l(x, y)\Phi(s(x, y): a)C_0^l(s(x, y)) \\ &\qquad \times \alpha(x, y)dx \times e^{-\rho_0(\log(a))} . \end{aligned} \tag{4.9}$$

Here we note that $\Phi((x, y): a)(x \in \boldsymbol{R}$ or $y \in \boldsymbol{R})$ has at most simple poles and $e^{-\rho_0(\log(a))}R(p: a)(a \in A_0^+)$ is a rapily decreasing function on A_0^+. Therefore noting (3.9) and (3.10) in § 3, we can easily obtain that

$$\begin{aligned} R(p: a) &= 2\int_{R} \operatorname*{Res}_{y=z_p} \{\mu_0^l(x, y)\}\Phi(x, z_p: a)C_0^l(x, z_p)\alpha(x, z_p)dx \times e^{-\rho_0(\log(a))} \\ &\quad + \int_{R\pm\delta} \operatorname*{Res}_{y=z_p} \{\mu_0^l(y, x)\}\Phi(z_p, x: a)C_0^l(z_p, x)\alpha(z_p, x)dx \times e^{-\rho_0(\log(a))} \\ &= 2\int_{R} \operatorname*{Res}_{y=z_p} I(x, y: a)dx + 2\int_{R-\delta} \operatorname*{Res}_{x=z_p} I(x, y: a)dy - \operatorname*{Res}_{y=0}\{\operatorname*{Res}_{x=z_p} I(x, y: a)\} . \end{aligned} \tag{4.10}$$

Here we used the following relation;

$$\mu_0^l(x, y) = \mu_0^l(s(x, y)) \quad \text{for} \quad s \in W_0 . \tag{4.11}$$

Thus $I_2^f(a)$ can be written as

$$\begin{aligned} I_2^f(a) &= 8c^2 \sum_{p=1}^{l'} \left\{ \int_{R} \operatorname*{Res}_{y=z_p} I(x, y: a)dx + \int_{R-\delta} \operatorname*{Res}_{x=z_p} I(x, y: a)dy \right\} \\ &= 4c^2 \sum_{p=1}^{l'} R(p: a) + 8c^2 \sum_{p=1}^{l'} \operatorname*{Res}_{y=0} \{\operatorname*{Res}_{x=z_p} I(x, y: a)\} \quad (a \in A_0^+) . \end{aligned} \tag{4.12}$$

For simplicity we denote these terms by ${}^\circ I_2^f$ and R_2^f respectively. Therefore using the τ-sphericalness of $f(x)$ and the Cartan decomposition, we can easily prove that ${}^\circ I_2^f(a)$ extends to a function ${}^\circ I_2^f(x)$ on G which belongs to $\mathscr{C}_1(G, \tau)$.

($R_2^f + I_3^f$) Here we note that from (4.10) and the definition of R_2^f $R_2^f(a) + I_3^f(a)$ can be written as follows.

$$\begin{aligned} R_2^f(a) + I_3^f(a) &= -8c^2 \sum_{q=1}^{l'} \sum_{t=1}^{T(q)} \operatorname*{Res}_{y=v_t} \{\operatorname*{Res}_{x=z_q} I(x, y: a)\} \\ &= -8c^2 \sum_{q=1}^{l'} \sum_{t=1}^{T(q)} \sum_{m=0}^{m_t-1} U(m, t, q: a) \left(\frac{d}{dy}\right)^m_{|y=v_t} \alpha(z_q, y) , \end{aligned} \tag{4.13}$$

where $U(m, t, q: a)(0 \leqq m \leqq m_t - 1,\ 1 \leqq t \leqq T(q),\ 1 \leqq q \leqq l')$ is a $(m+1)$-th Laurent coefficient of $\mathrm{Res}_{x=z_q} I(x, y: a)$ at $y = v_t$.

Now we put

$$(4.14)\qquad E(m, t, q: g) = \left(\frac{d}{dy}\right)^m_{|y=v_t} E(P_0: 1: z_q, y: g) \quad (g \in G)$$

for $0 \leqq m \leqq m_t - 1$, $1 \leqq t \leqq T(q)$, $1 \leqq q \leqq l'$ and $e_k = E(m^k, t^k, q^k:.)(1 \leqq k \leqq \gamma)$, where $0 \leqq m^k \leqq m_t - 1$, $1 \leqq t^k \leqq T(q^k)$, $1 \leqq q^k \leqq l'$, denote a maximal linearly independent subset of the above functions (4.14). Then $E(m, t, q: g)$ can be written as

$$(4.15)\quad E(m, t, q: g) = \sum_{k=1}^{\gamma} A(m, t, q; k) e_k(g) \quad (g \in G \quad \text{and} \quad A(m, t, q; k) \in C) .$$

Here we note that from the condition (C1) and (4.15) $\alpha(\nu)$ satisfies the following relation;

$$(4.16)\qquad \left(\frac{d}{dy}\right)^m_{|y=v_t} \alpha(z_q, y) = \sum_{k=1}^{\gamma} A(m, t, q; k) \left(\frac{d}{dy}\right)^{m^k}_{|y=v_{t^k}} \alpha(z_{q^k}, y)$$

for all m, t, q. For simplicity we put $(d/dy)^{m^k}_{|y=v_{t^k}} \alpha(z_{q^k}, y) = A_k (1 \leqq k \leqq \gamma)$. Since $e_k (1 \leqq k \leqq \gamma)$ are real analytic functions on G, we can choose $h_k \in C_c^\infty(G, \tau)(1 \leqq k \leqq \gamma)$ such that

$$(4.17)\qquad (h_i, e_j) = \delta_{ij} \quad (1 \leqq i, j \leqq \gamma) .$$

Now we put

$$(4.18)\qquad g(x) = f(x) - \sum_{k=1}^{\gamma} A_k \bar{h}_k(x) \quad (x \in G) ,$$

where $\bar{h}_k = \mathscr{C}_{A_0}^{-1}(\mathscr{C}_{A_0}(h_k))(1 \leqq k \leqq \gamma)$ (note that $h_k' = h_k - \bar{h}_k$ belongs to $\mathscr{C}_1(G, \tau)$). Then

$$(4.19)\qquad \begin{aligned} \hat{g}(\nu) = \hat{g}(1; \nu) &= (g, E(P_0: 1: \nu:.)) \\ &= \alpha(\nu) - \sum_{k=1}^{\gamma} A_k \hat{\bar{h}}_k(\nu) . \end{aligned}$$

Therefore $\hat{g}(\nu)$ belongs to $\mathscr{C}(\mathscr{F}_0)_*$ and moreover has a holomorphic extension on $(\mathscr{F}_0)_c$ which is an exponential type (note that each h_k has a compact support). Now using the same arguments as before, we can prove that $I_1^{\bar{g}}(x)$ and ${}^\circ I_2^{\bar{g}}(x)$ belong to $C_c^\infty(G, \tau)$ and $\mathscr{C}_1(G, \tau)$ respectively. Now we shall prove that $R_2^{\bar{g}} + I_3^{\bar{g}} = 0$. To do this we note that

$$(4.20)\quad \left(\frac{d}{dy}\right)^m_{|y=v_t} \hat{\bar{g}}(z_p, y) = \left(\frac{d}{dy}\right)^m_{|y=v_t} \alpha(z_p, y) - \sum_{k=1}^{\gamma} A_k \left(\frac{d}{dy}\right)^m_{|y=v_t} \hat{\bar{h}}_k(z_p, y)$$

for $0 \leqq m \leqq m_t - 1$, $1 \leqq t \leqq T(q)$, $1 \leqq p \leqq l'$. By the way since $h_k \in C_c^\infty(G, \tau)$ $(1 \leqq k \leqq \gamma)$, we have

$$\begin{aligned}(4.21)\quad \left(\frac{d}{dy}\right)^m_{|y=v_t} \hat{\bar{h}}_k(z_p, y) &= \left(\frac{d}{dy}\right)^m_{|y=v_t} (\bar{h}_k, E(P_0\colon 1\colon x, y\colon.))_{x=z_p} \\ &= \left(\frac{d}{dy}\right)^m_{|y=v_t} (h_k, E(P_0\colon 1\colon x, y\colon.))_{x=z_p} \\ &= \left(h_k, \left(\frac{d}{dy}\right)^m_{|y=v_t} E(P_0\colon 1\colon z_p, y\colon.)\right) = (h_k, E(m, t, p\colon.)) \\ &= A(m, t, p; k)\ .\end{aligned}$$

Therefore from (4.16) the above equation (4.20) is equal to 0. Thus $R_2^{\bar{g}}(a) + I_3^{\bar{g}}(a) = 0$ for $a \in A_0^+$ (cf. (4.13)) and $R_2^f(a) + I_3^f(a) = \sum_{k=1}^{\gamma} A_k(R_2^{\bar{h}_k}(a) + I_3^{\bar{h}_k}(a))$.

Now using these facts, we can obtain that

$$\begin{aligned}(4.22)\quad f(x) &= g(x) + \sum_{k=1}^{\gamma} A_k \bar{h}_k(x) \\ &= \{I_1^{\bar{g}}(x) + {}^\circ I_2^{\bar{g}}(x) + g'(x)\} + \left\{\sum_{k=1}^{\gamma} A_k(h_k(x) - h_k'(x))\right\} \\ &= \left\{I_1^{\bar{g}}(x) + \sum_{k=1}^{\gamma} A_k h_k(x)\right\} + \left\{{}^\circ I_2^{\bar{g}}(x) + g'(x) - \sum_{k=1}^{\gamma} A_k h_k'(x)\right\}\ ,\end{aligned}$$

where $g' = g - \bar{g} \in \mathscr{C}_1(G, \tau)$. Then the first term of (4.22) belongs to $C_c^\infty(G, \tau)$ and the second term belongs to $\mathscr{C}_1(G, \tau)$. Thus this decomposition is desired. Q.E.D.

Next Corollary is an analogue of Paley-Wiener theorem.

COROLLARY 2. *Let $\alpha(\nu)$ be in $\mathscr{C}(\mathscr{F}_0)_*$ and have a holomorphic extension on $(\mathscr{F}_0)_c$ which is an exponential type. Now suppose that $\alpha(\nu)$ satisfies the condition* (C1) *in* §3. *Then there exists $F \in C_c^\infty(G, \tau)$ such that*

$$(4.23)\qquad \mathscr{C}_{A_0}(F) = \alpha\ .$$

PROOF. Corollary is obvious from Theorem 5 and the fact that $\mathscr{C}_1(G, \tau)$ is the kernel of $\mathscr{C}_{A_0}$. Q.E.D.

Appendix

In this appendix we shall obtain the results in § 2 for the case of $G=\mathrm{SU}(2, 2)$ and $\tau=(\tau^l, \tau^l)$ by a direct calculation of c- and μ-functions.

First we recall that the Eisenstein integral can be expanded as

$$(1)\qquad \begin{aligned} E(P_0\colon 1\colon x, y\colon a) &= \sum_{s\in W_0} \Phi'_G(s(x, y)\colon a)C_0^l(s(x, y))e^{\sqrt{-1}s(x\ y)-\rho_0(\log(a))} \\ E({}^*P_\alpha\colon 1\colon x\colon a_\alpha) &= \sum_{t\in {}^*W} \Phi'_{M_\alpha}(tx\colon a_\alpha)C_\alpha^l(tx)e^{(\sqrt{-1}tx-\rho_\alpha)(\log(a_\alpha))}, \end{aligned}$$

where ρ_0 (resp. ρ_α) is the half of the sum of all positive roots Δ^+ of (G, A_0) (resp. $(M_\alpha, {}^*A_\alpha)$) and $a\in A_0^+$, $a_\alpha\in {}^*A_\alpha^+$. Moreover (x, y) (resp. x in the second equation) belongs to a suitable dense subset $\Gamma'_G(c)$ in C^2 (resp. $\Gamma'_{M_\alpha}(c)$ in C). Here we note that the definitions of Φ'_G and Φ'_{M_α} are given as follows;

$$(2)\qquad \Phi'_G(x, y\colon a)=\sum_{\lambda\in L} \Gamma_\lambda(\sqrt{-1}(x, y)-\rho_0)e^{-\lambda(\log(a))} \quad (a\in A_0^+),$$

where L is the set of all $\lambda\in \mathscr{F}_0$ of the form $\lambda=m_1\alpha+m_2\gamma$, the $m_i(1\leq i\leq 2)$ being the non-negative integers and $\Gamma_\lambda(\nu)$ is defined by the following recurrence relation (here we note $\dim\tau^l=1$);

$$\Gamma_0(\nu)=0\,,\quad \Gamma_\lambda(\nu)=0 \quad \text{for} \quad \lambda\notin L\,,$$

$$(3)\qquad \begin{aligned} &\{2(\lambda, \nu)-(\lambda, \lambda-2\rho_0)\}\Gamma_\lambda(\nu) \\ &=2\sum_{\alpha\in P_+}\sum_{n\geq 1} \{(\tilde{\alpha}, \nu)-(\tilde{\alpha}, \lambda-2n\tilde{\alpha})\}\Gamma_{\lambda-2n\tilde{\alpha}}(\nu) \\ &\quad +8\sum_{\alpha\in P_\perp}\sum_{n\geq 1} (2n-1)\tau^l(Y_\alpha)\tau^l(Y_{-\alpha})\Gamma_{\lambda-(2n-1)\tilde{\alpha}}(\nu) \\ &\quad -8\sum_{\alpha\in P_+}\sum_{n\geq 1} n\{\tau^l(Y_\alpha Y_{-\alpha})+\tau^l(Y_\alpha Y_{-\alpha})\}\Gamma_{\lambda-2n\tilde{\alpha}}(\nu) \end{aligned}$$

(P_+ is the set of all positive roots α of $(\mathfrak{g}, \mathfrak{h}_0)$ whose restriction $\tilde{\alpha}$ on $\mathfrak{a}_0$ does not vanish, and $Y_{\pm\alpha}=(X_\alpha\pm\theta(X_\alpha))/2$). $\Phi'_{M_\alpha}(x\colon a_\alpha)(a_\alpha\in {}^*A_\alpha^+)$ is defined by the same way for M_α. Then $\Gamma'_G(c)$ and $\Gamma'_{M_\alpha}(c)$ are the set of all (x, y) and x on which the above recurrence relation is well-defined (see [9] 9.1.4 for the detail).

Here we note that from the results about discrete series for $SL(2, \boldsymbol{R})$ (see [10]), $E({}^*P_\alpha\colon 1\colon z_p\colon a_\alpha)$ $(1\leq p\leq l')$ can be written as

$$(4)\qquad E({}^*P_\alpha\colon 1\colon z_p\colon a_\alpha)=\Phi'_{M_\alpha}(z_p\colon a)C_\alpha^l(z_p)e^{(\sqrt{-1}z_p-\rho_\alpha)(\log(a_\alpha))}(a_\alpha\in {}^*A_\alpha^+)\,.$$

For simplicity we denote it by $\psi(a_\alpha)$. Then we can easily prove that $\psi(a_\alpha)$ extends to a function $\psi(m)$ on M which belongs to $L_{M_\alpha}(\sigma)$ for some $\sigma\in\mathscr{E}_2(M_\alpha)$. Moreover from the functional equation of the Eisenstein integral we have

$$(5) \qquad E(P_\alpha: \psi: y: g) = E(P_0: 1: z_p, y: g) \quad (g \in G) .$$

Now we shall calculate the constant terms of the both sides of (5). Then from the left we have

$$(6) \qquad E_{P_\alpha}(P_\alpha: \psi: y: a_\beta m) = \sum_{s \in W_\alpha} C_{P_\alpha | P_\alpha}(s; y)\psi(m) e^{\sqrt{-1}y \log(a_\beta)} ,$$

where $W_\alpha = W(G, A_\alpha)$. Here we note that

$$(7) \qquad \lim_{a_\beta \underset{P_\alpha}{\to} \infty} \Phi'_G(x, y: a_\beta a_\alpha)$$

converges to a function which does not depend on y and a_β, where $(x, y) \in \Gamma'_G(c)$ and $a_\beta \in {}^*A^+_\beta$, $a_\alpha \in {}^*A^+_\alpha$ (see [1] §21 for the definition of $a_\beta \underset{P_\alpha}{\to} \infty$). This fact is obvious from the definition of Φ'_G and the recurrence relation of $\Gamma_\lambda(\nu)$. From now on we denote it by $\Phi(x: a_\alpha)$.

Therefore, using the explicit form of the Weyl group of (G, A_0) and the fact that the left side of (5) (thus, the right side of (5)) satisfies the weak inequality (see [1]§21), we can obtain that from (1) and (5)

$$(8) \qquad \lim_{a_\beta \underset{P_\alpha}{\to} \infty} \Big| e^{\rho_\beta(\log(a_\beta))} E(P_\beta: \psi: y: a_\beta a_\alpha) - \Phi(z_p: a_\alpha) C^l_0(z_p, y) e^{\sqrt{-1}y(\log(a_\beta))} e^{(\sqrt{-1}z_p - \rho_\alpha)(\log(a_\alpha))} - \Phi(z_p: a_\alpha) C^l_0(z_p, -y) e^{-\sqrt{-1}y(\log(a_\beta))} e^{(\sqrt{-1}z_p - \rho_\alpha)(\log(a_\alpha))} \Big| = 0 .$$

Since the constant terms is unique, we have the following relation comparing (6) and (8).

$$(9) \qquad C_{P_\alpha | P_\alpha}(1; y)\psi(a_\alpha) = \Phi(z_p: a_\alpha) C^l_0(z_p, y) e^{(\sqrt{-1}z_p - \rho_\alpha)(\log(a_\alpha))} \quad (a_\alpha \in {}^*A^+_\alpha) .$$

However, since the μ-function satisfies the following relation;

$$(10) \qquad \mu(\sigma, y) \| C_{P_\alpha | P_\alpha}(1; y)\psi \|^2 = c^2 \|\psi\|^2 \quad (y \in \mathscr{F}_\alpha{}') ,$$

we can obtain that from (4) and (9)

$$(11) \qquad \mu(\sigma, y) | C^l_0(z_p, y) |^2 = c | C^l_\alpha(z_p) |^2 \quad (y \in \mathscr{F}_\alpha{}') ,$$

where c is a constant which does not depend on y. Therefore we have

$$(12) \qquad \mu(\sigma, y) = c \mu_\beta(y) \mu_\gamma(y - z_p) \mu_\delta(y + z_p) \quad (y \in \mathscr{F}_\alpha{}') .$$

This is the desired relation.

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