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Invariants of Finite Abelian Groups Generated by Transvections

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Introduction

Let k be a field of characteristic p and G be a finite subgroup of $GL(V)$ where V is a k-space. Then G acts naturally on the symmetric algebra $k[V]$ of V. In the case of $p=0$ or $(|G|, p)=1$, it is well known (e.g., [1], [4]) that the invariant subring $k[V]^G$ is a polynomial ring if and only if G is generated by pseudo-reflections in $GL(V)$.

Suppose that $p\,|\,|G|$. We have classified in [2] finite irreducible groups G such that $k[V]^G$ are polynomial rings under certain conditions. In this paper we try to classify the modular representations of finite abelian groups with regular rings of invariants. Our main result is the following

THEOREM. *Let G be an abelian p-subgroup of $GL(V)$ which is realizable on F_p. If* $\dim V^G \leqq 2$ *or* $\dim V^{*G} \leqq 2$, *then the following conditions are equivalent*:

(1) *$k[V]^G$ is a polynomial ring.*

(2) *There exist couples $(W_i, G_i)(1 \leqq i \leqq m)$ which satisfy the following*

(i) *$G_i (1 \leqq i \leqq m)$ are subgroups of G and $G = \bigoplus_{i=1}^m G_i$.*

(ii) *$W_i (1 \leqq i \leqq m)$ are 1-dimensional subspaces of V and $V = V^G \oplus \bigoplus_{i=1}^m W_i$.*

(iii) *Each $V^G \oplus W_i$ is a kG_i-submodule $(1 \leqq i \leqq m)$ of V and $W_i \subseteq V^{G_j}$ if $i \neq j$.*

Furthermore we will give some remarks concerned with singular loci of the rings $k[V]^G$ and examples of abelian p-groups generated by transvections with the invariant subrings which are not Macaulay rings. It should be noted that there are finite irreducible subgroups G of $GL(V)$

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generated by reflections which contain no transvections such that $k[V]^G$ are not Macaulay rings ([3]).

§1. Preliminaries.

An element σ of $GL(V)$ is said to be a pseudo-reflection if $\dim(1-\sigma)V=1$. A pseudo-reflection σ is called a transvection if $\sigma|_{(1-\sigma)V}=1$ and a reflection if $\sigma|_{(1-\sigma)V}=-1$.

First, we quote the following well known result:

THEOREM 1.1 (J.-P. Serre [4]). *If $k[V]^G$ is a polynomial ring, then G is generated by pseudo-reflections.*

Without specifying, k stands for the algebraic closure of $\boldsymbol{F}_p$.

PROPOSITION 1.2. *Let G be a finite abelian group generated by pseudo-reflections in $GL(V)$ and let G_1 denote the p-part of G. Then $k[V]^G$ is a polynomial ring if and only if $k[V]^{G_1}$ is a polynomial ring.*

PROOF. Let G_2 be the p'-part of G. It is easy to show that there are kG_i-submodules $V_i (i=1, 2)$ of V satisfying $V=V_1\oplus V_2$ and $V_i \subseteq V^{G_j}$ $(i\neq j)$. Then we have $k[V]^G=k[V_1]^{G_1}\otimes_k k[V_2]^{G_2}$. Since $k[V_i]^{G_i}(i=1, 2)$ are noetherian graded algebras, the assertion follows.

Hence we have only to treat the case where G is an abelian p-group. A kG-module V is said to be trivial if G acts trivially on V.

PROPOSITION 1.3. *If G is an abelian p-group generated by pseudo-reflections in $GL(V)$, then V/V^G is a trivial kG-module.*

PROOF. We show this by induction on $\dim V$. Since $\operatorname{Im}(G\to GL(V/V^G))$ is generated by pseudo-reflections, $V/V^G/[V/V^G]^G$ is a trivial kG-module by the assumption of induction. Set $\Omega=\{\sigma\in G: \sigma$ is a pseudo-reflection acting trivially on $V/V^G\}$, $H=\langle\Omega\rangle$ and $U=\Psi^{-1}([V/V^G]^G)$, where $\Psi: V\to V/V^G$ is the canonical epimorphism. Since H is generated by pseudo-reflections, by (3.2) of [2] we can show $\dim U-\dim U^H=\dim[V/V^G]^G$. Assume that V/V^G is not trivial. Then there is a pseudo-reflection $\tau\in G$ which does not belong to Ω. Since $\tau\sigma=\sigma\tau$ for $\sigma\in\Omega$, the equality $\dim U-\dim U^H=\dim[V/V^G]^G$ implies that $\tau\in\Omega$, which is a contradiction.

NOTATION 1.4. (V, G) stands for a couple of a finite group G and a kG-module V such that V/V^G is a nonzero trivial kG-module. The dimension of a couple (V, G) is defined to be $\dim V-\dim V^G$.

For a couple (V, G), let $\rho: G\to GL(V)$ be the representation of G afforded by the kG-module V. Then we put

$d(V, G) = \min\{\dim W^G\colon W$ is a $G/\mathrm{Ker}\,\rho$-faithful kG-submodule of $V\}$

and

$d^*(V, G) = \min\{\dim W^{*G}\colon W$ is a $G/\mathrm{Ker}\,\rho$-faithful kG-submodule of $V\}$.

We denote by $Q(V, G)$ the ring $\mathrm{Im}\,(k[V]^G/\langle V^G\rangle \cap k[V]^G \xrightarrow[\text{can.}]{} k[V/V^G])$.

DEFINITION 1.5. We say that a couple (V, G) decomposes (to couples), if there are couples $(V^G \oplus W_i, G_i)(1 \leqq i \leqq m)$ which satisfy the following conditions:

(i) $G_i (1 \leqq i \leqq m)$ are subgroups of G and $G = \bigoplus_{i=1}^m G_i$.

(ii) $W_i (1 \leqq i \leqq m)$ are nonzero subspaces of V and $V = V^G \oplus \bigoplus_{i=1}^m W_i$.

(iii) Each $V^G \oplus W_i$ is a kG_i-module and $W_i \subseteqq V^{G_j}$ if $i \neq j$.

PROPOSITION 1.6. *If a couple (V, G) decomposes to 1-dimensional couples, then $k[V]^G$ is a polynomial ring.*

PROOF. This follows easily from (3.5) of [2].

We now extend a result of [2].

LEMMA 1.7. *Let (V, G) be a couple with $V^G = kX$ and let $Y_i \in V$ $(1 \leqq i \leqq m)$ such that $V = kX \oplus \bigoplus_{i=1}^m kY_i$. For integers $t_i \in N (1 \leqq i \leqq m)$, set $R = k[X, Y_1^{p^{t_1}}, \cdots, Y_m^{p^{t_m}}]$. Then R^G is a polynomial ring and we can construct a system of fundamental invariants of R^G effectively.*

PROOF. Since the affine space $A^m(k)$ acts transitively on the set of maximal ideals of R containing the ideal RX, the normality of R^G implies that R^G is a polynomial ring.

Now let us make a system of fundamental invariants. We may assume that V is G-faithful and $t_1 = t_2 = \cdots = t_{m_1} < t_{m_1+1} = \cdots = t_{m_2} < \cdots < t_{m_{n-1}+1} = \cdots = t_{m_n}$. Put $V_1 = kX^{p^{t_1}} \oplus \bigoplus_{i=1}^{m_1} kY_i^{p^{t_1}}$. The natural action of G on V_1 defines $\rho_1\colon G \to GL_{m_1}(k)$. Since G is a vector group over F_p, we know $G = \mathrm{Ker}\,\rho_1 \oplus G_1$ where $G_1 \cong \rho_1(G)$. For an element $j \in N$, put $V_1^{p^j} = kX^{p^{t_1+j}} \oplus \bigoplus_{i=1}^{m_1} kY_i^{p^{t_1+j}}$ and let $\rho_1^{p^j}\colon G \to GL_{m_1}(k)$ be the matrix representation afforded by the natural action of G on $V_1^{p^j}$. Then clearly $\mathrm{Ker}\,\rho_1^{p^j} = \mathrm{Ker}\,\rho_1$ and $G_1 \cong \rho_1^{p^j}(G)$. Set $V_{1i} = V_1^{p^{s_i}} \oplus \bigoplus_{j=m_{i-1}+1}^{m_i} kY_j^{p^{t_{m_i}}}$ where $s_i = t_{m_i} - t_{m_1}$ $(2 \leqq i \leqq n)$, then each V_{1i} is a kG_1-module. Replacing the elements $Y_j^{p^{t_{m_i}}} \in R$ $(m_{i-1}+1 \leqq j \leqq m_i;\ 2 \leqq i \leqq n)$, we may suppose that each kG_1-submodule $kX^{p^{t_{m_i}}} \oplus \bigoplus_{j=m_{i-1}+1}^{m_i} kY_j^{p^{t_{m_i}}}$ is trivial. As in the proof of (3.4) in [2], we can find homogeneous polynomials f_i $(1 \leqq i \leqq m_1)$ which satisfy $k[V_1]^{G_1} = k[X^{p^{t_{m_1}}}, f_1, \cdots, f_{m_1}]$. Hence it follows that $R^{G_1} = k[X, f_1, \cdots, f_{m_1}]$ $[Y_j^{p^{t_{m_i}}}\colon m_{i-1}+1 \leqq j \leqq m_i,\ 2 \leqq i \leqq n]$. We continue this procedure for the graded polynomial subalgebra $R_1=k[X][Y_j^{p^{t_{m_i}}}: m_{i-1}+1\leqq j\leqq m_i, 2\leqq i\leqq n]$ of R with the action of the group Ker ρ_1. Since G is finite, we finally get a system of fundamental invariants of R.

When a group G acts on a ring R as automorphisms, for a prime ideal $\mathfrak{p}$ of R, $I_G(\mathfrak{p})$ is defined to be the inertia group at $\mathfrak{p}$.

PROPOSITION 1.8. *Let G be a finite subgroup of $GL(V)$. If $k[V]^G$ is a polynomial ring, then, for any prime ideal $\mathfrak{p}$ of $k[V]$, $k[V]^{I_G(\mathfrak{p})}$ is a polynomial ring.*

PROOF. Put $\mathfrak{p}^*=\langle V\cap\mathfrak{p}\rangle$ and $H=I_G(\mathfrak{p})$, then $H=I_G(\mathfrak{p}^*)$. For maximal ideals $\mathfrak{M}_i(i=1, 2)$ containing the ideal $\mathfrak{p}^*$, we have $(k[V]^H)_{\mathfrak{M}_1\cap k[V]^H}\cong(k[V]^H)_{\mathfrak{M}_2\cap k[V]^H}$. Suppose that $k[V]^H$ is not a polynomial ring, then clearly $(k[V]^H)_{\mathfrak{M}_+\cap k[V]^H}$ is not a regular local ring, where $\mathfrak{M}_+=\bigoplus_{i>0}k[V]_{(i)}$. Since the singular locus of $k[V]^H$ is closed with respect to the Zariski topology of Spec $k[V]^H$, $(k[V]^H)_{\mathfrak{p}^*\cap k[V]^H}$ is not a regular local ring. But the local homomorphism $(k[V]^G)_{\mathfrak{p}^*\cap k[V]^G}\to(k[V]^H)_{\mathfrak{p}^*\cap k[V]^H}$ is étale, and so $(k[V]^H)_{\mathfrak{p}^*\cap k[V]^H}$ is a regular local ring. This is a contradiction.

As a consequence of Lemma 1.7 and Proposition 1.8, we have

PROPOSITION 1.9. *Let (V, G) be a couple. If $k[V]^G$ is a polynomial ring, then we can construct a regular system of homogeneous parameters of $Q(V, G)$ effectively.*

PROOF. Let $0=W_0\subsetneqq W_1\subsetneqq\cdots\subsetneqq W_d=V^G$ be an ascending chain of subspaces satisfying dim $W_i/W_{i-1}=1(1\leqq i\leqq d=\dim V^G)$. Set

$$R_0=k[V]\ ,\quad R_1=R_0^{I_G(\langle W_1\rangle)}/W_1R_0^{I_G(\langle W_1\rangle)}\ ,\quad R_2=R_1^{I_G(\langle W_2\rangle)}/W_2R_1^{I_G(\langle W_2\rangle)},\ \cdots\ ,$$

and

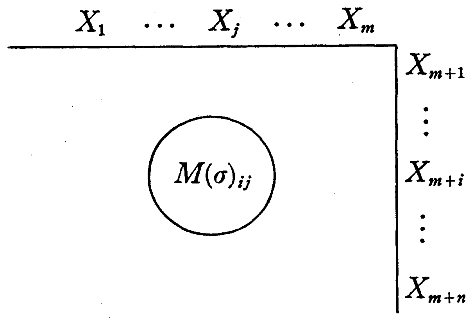
$$R_d=R_{d-1}^G/W_dR_{d-1}^G\ .$$

Then clearly $R_d=R_{d-1}^G/W_dR_{d-1}^G\underset{\text{can.}}{\cong}Q(V, G)$. For it follows from Proposition 1.8 that $R^{I_G(\langle W_i\rangle)}(1\leqq i\leqq d)$ are polynomial rings. Hence, by Lemma 1.7, we can make a regular system of homogeneous parameters of $Q(V, G)$ inductively.

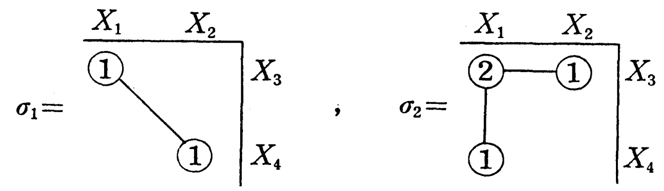
NOTATION 1.10. Let (V, G) be a couple such that $k[V]^G$ is a polynomial ring. And let $0=W_0\subsetneqq W_1\subsetneqq\cdots\subsetneqq W_d=V^G$ be an ascending chain of subspaces satisfying dim $W_i/W_{i-1}=1(1\leqq i\leqq d)$. As in the proof of Proposition 1.9, we can determine the ring $Q(V, G)$ inductively. We denote the algorithm $A(W_1, \cdots, W_d)$ of the computation of $Q(V, G)$ as stated above.

For convenience we introduce graphic representations of certain groups.

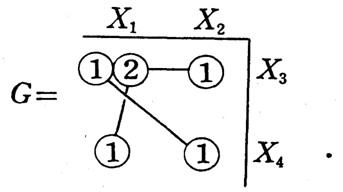
NOTATION 1.11. Let an element σ of $GL(V)$ have the form $\begin{bmatrix} 1_m & 0 \\ M(\sigma) & 1_n \end{bmatrix}$ on the basis $\{X_1, \cdots, X_m, X_{m+1}, \cdots, X_{m+n}\}$ of V. Let us define a graph of σ. When $M(\sigma)_{ij} \neq 0$, we represent this by a circle of (i, j)-position in the following table:



and we write $M(\sigma)_{ij}$ in it. Connect these circles by line segments. For group G generated by elements $\sigma = \begin{bmatrix} 1_m & 0 \\ M(\sigma) & 1_n \end{bmatrix}$, we can define a graph of G, projecting all graphs of independent generators on a table. For example, let $V = \bigoplus_{i=1}^{4} kX_i$, $\sigma_i = \begin{bmatrix} 1_2 & 0 \\ M(\sigma_i) & 1_2 \end{bmatrix}$ $(i=1, 2)$ and $G = \langle \sigma_1, \sigma_2 \rangle$, where $M(\sigma_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M(\sigma_2) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Then corresponding graphic representations are



and



§2. The case of dim $V^G = 2$.

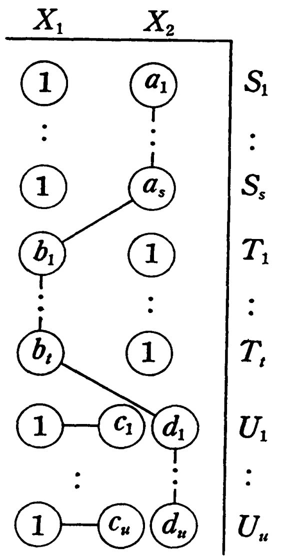
Let G be an abelian p-subgroup of $GL(V)$ which is realizable on F_p.

Suppose that G is generated by pseudo-reflections, then the kG-module V defines a couple (V, G) from Proposition 1.3.

LEMMA 2.1. *If* $\dim V^G=2$, $\dim V\geqq 4$ *and* $k[V]^G$ *is a polynomial ring, then the couple* (V, G) *is decomposable.*

PROOF. We may assume $k=\boldsymbol{F}_p$ and $\dim V\geqq 4$. Suppose that (V, G) is indecomposable. Then it is easy to see that $|G|>p^d$, where $d=d^*(V, G)$.

Step 1. If $|G|=p^{d+1}$, then there is no minimal prime ideal $\mathfrak{p}$ satisfying $|I_G(\mathfrak{p})|=p^d$. So G is conjugate to

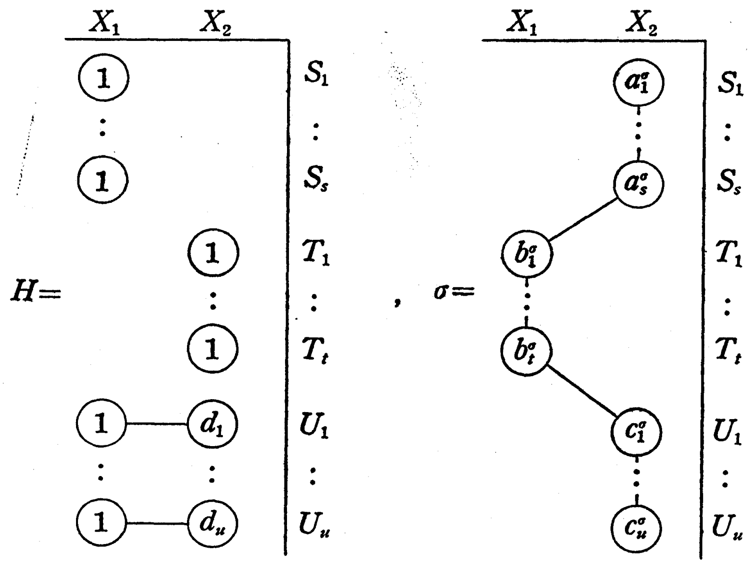


where $V=kX_1\oplus kX_2\oplus\bigoplus_{i=1}^{s} kS_i\oplus\bigoplus_{i=1}^{t} kT_i\oplus\bigoplus_{i=1}^{u} kU_i$, $st\geqq 1$, $c_i\neq 0$ and $u\geqq 0$. If $a_1=0$, then (V, G) is decomposable. Hence $a_i(1\leqq i\leqq s)$ and $b_i(1\leqq i\leqq t)$ never vanish. Taking polynomials $\{f_i, f_i', g_i, g_i'\}$,

$$Q(V, G)=\begin{cases} k[\bar{S}_1^{p^2}, \bar{f}_2, \cdots, \bar{f}_s, \bar{T}_1^p, \cdots, \bar{T}_t^p, \bar{g}_1, \cdots, \bar{g}_u]; \\ \qquad \text{by the algorithm } A(kX_1, kX_1\oplus kX_2) \\ k[\bar{S}_1^p, \cdots, \bar{S}_s^p, \bar{T}_1^{p^2}, \bar{f}_2', \cdots, \bar{f}_t', \bar{g}_1', \cdots, \bar{g}_u']; \\ \qquad \text{by the algorithm } A(kX_2, kX_1\oplus kX_2) \end{cases},$$

which is a contradiction.

Step 2. Suppose $|G|>p^{d+1}$. Then G contains the following



where $st\geqq 1$, $d_i\neq 0$ and $u\geqq 0$. Replacing a basis of V^G, we may assume $|I_G(\langle X_1\rangle)|=p^s$. Since G is a vector group over F_p, $G=H\oplus\bigoplus_{i=1}^m\langle\sigma_i\rangle$. If $b_i^{\sigma_1}=0(1\leqq i\leqq t)$ and $c_i^{\sigma_1}=0(1\leqq i\leqq u)$, then (V, G) is decomposable. Let $R_1=k[\bar{S}_1, \cdots, \bar{S}_s, \bar{U}_1, \cdots, \bar{U}_u]$ and $R_2=k[\bar{T}_1, \cdots, \bar{T}_t]$. The group G contains H, and so we can easily see that $Q(V, G)=R_1'\otimes_k R_2'$ where $R_i'(i=1, 2)$ are graded polynomial subalgebras of R_i. By the algorithm $A(kX_2, kX_1\oplus kX_2)$, the polynomial ring $R_2'=k[h_1, h_2, \cdots, h_t]$ satisfies $h_1=\bar{T}_{i_0}^{p^2}$ for some i_0. On the other hand, since $Q(V, I_G(\langle X_1\rangle))=k[\bar{S}_1^p, \cdots, \bar{S}_s^p]$, we know $Q(V, G)\ni\bar{T}_{i_0}^p$. This is a contradiction.

We now establish

Theorem 2.2. *Let G be an abelian p-subgroup of $GL(V)$ which is realizable on F_p. If* $\dim V^G=2$, *then the following conditions are equivalent:*

(1) *$k[V]^G$ is a polynomial ring.*

(2) *The kG-module V defines the couple (V, G) which decomposes to 1 dimensional couples.*

Proof. From Proposition 1.6, it suffices to show that (1) implies (2). Hence suppose that $k[V]^G$ is a polynomial ring and that (V, G) decomposes to couples $(V^G\oplus W_i, G_i)(1\leqq i\leqq m)$ which are indecomposable. Say $\dim W_1\geqq 2$. Since we may identify $Q(V, G)$ with $Q(V^G\oplus W_1, G_1)\otimes_k Q(V^G\oplus W_2, G_2)\otimes_k\cdots\otimes_k Q(V^G\oplus W_m, G_m)$, the fact that $\dim W_1\geqq 2$ conflicts with Lemma 2.1. Therefore (V, G) decomposes to 1 dimensional couples.

§3. The case of dim $V^{*G}=2$.

In this section we prove the following

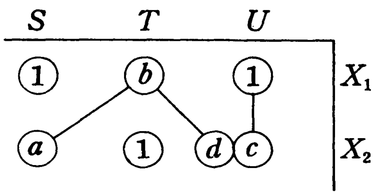
THEOREM 3.1. *Let G be an abelian p-subgroup of $GL(V)$ which is realizable on F_p. If* dim $V^{*G}=2$, *then the following conditions are equivalent*:

(1) $k[V]^G$ *is a polynomial ring.*

(2) *The kG-module V defines the couple (V, G) which decomposes to 1-dimensional couples.*

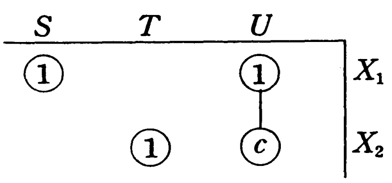
Let us give with

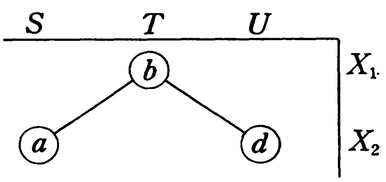
LEMMA 3.2. *Let* $V=kS\oplus kT\oplus kU\oplus \bigoplus_{i=1}^{2} kX_i$ *and*

$G=$ 

where $\{a, b, c, d\}\subseteq F_p^*$. *Then* $k[V]^G$ *is not a polynomial ring.*

PROOF. Assume that $k[V]^G$ is a polynomial ring, then $k[V]^G=k[S, T, U, f_1, f_2]$ for some homogeneous polynomials $f_i=X_i^{p^2}+Sf_{i1}+Tf_{i2}+Uf_{i3}(i=1, 2)$. Put

$H=$ 

, $\sigma=$ 

,

$$Y_1(X_1)=X_1^p-S^{p-1}X_1\ , \quad Y_2(X_2)=X_2^p-T^{p-1}X_2\ ,$$

$$Z_1(X_1)=Y_1(X_1)^p-Y_1(U)^{p-1}Y_1(X_1)\ , \quad Z_2(X_2)=Y_2(X_2)^p-c^{p-1}Y_2(U)^{p-1}Y_2(X_2)\ ,$$

and

$$B(X_1, X_2)=cY_1(X_1)(U^{p-1}-T^{p-1})-Y_2(X_2)(U^{p-1}-S^{p-1})\ .$$

Then $k[V]^H=k[S, T, U, Z_1(X_1), Z_2(X_2), B(X_1, X_2)]$. We may suppose

$$f_1+Z_1=\sum_{(2p-1)j_1+j_2+j_3+j_4=p^2\ j_1>0} g[j_1, j_2, j_3, j_4]B(X_1, X_2)^{j_1}S^{j_2}T^{j_3}U^{j_4}\ ,$$

and so

$$(f_1+Z_1)^\sigma|_{S,U,X_i=0}=\sum_{(2p-1)j_1+j_3=p^2\ j_1>0} g[j_1,\,0,\,j_3,\,0](-cbT^{2p-1})^{j_1}T^{j_3}\ .$$

From this,

$$(g[1,\,0,\,p^2-(2p-1),\,0]-1)ib+\sum_{2\leqq j\leqq[p^2/(2p-1)]} g[j,\,0,\,p^2-(2p-1)j,\,0](ib)^j=0$$

$(1\leqq i\leqq p-1)$. Hence

$$g[j,\,0,\,p^2-(2p-1)j,\,0]=\begin{cases}1\ ; & \text{if}\quad j=1\\ 0\ ; & \text{if}\quad 2\leqq j\leqq\left[\dfrac{p^2}{2p-1}\right].\end{cases}$$

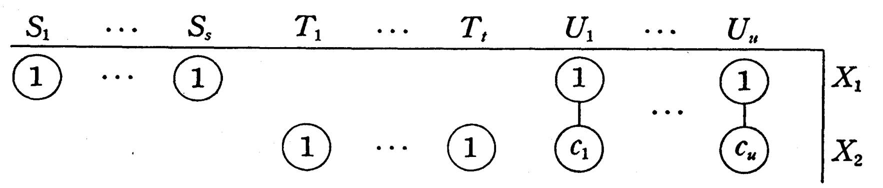
Calculating $(f_1+Z_1)^\sigma|_{U,X_i=0,T=1}$,

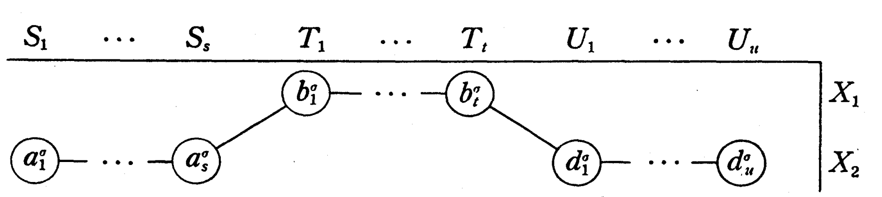
$$\sum_{(2p-1)j_1+j_2+j_3=p^2,j_1,j_2>0} g[j_1,\,j_2,\,j_3,\,0]\{cbS^{p-1}-cb+aS^{2p-1}-aS^p\}^{j_1}S^{j_2}$$
$$=bS^{p-1}-\frac{a}{c}S^p+\frac{a}{c}S^{2p-1}-bS^{p(p-1)}\ .$$

But we can show that $g[j,\,t,\,p^2-t-(2p-1)j,\,0]=0$ $(j>1)$ by induction on t. This gives a contradiction.

PROOF. Let us begin the proof of Theorem 3.1. Since G is realizable on $\boldsymbol{F}_p$, we may suppose $k=\boldsymbol{F}_p$. G is generated by pseudo-reflections in $GL(V)$, then V/V^G is a trivial kG-module by Proposition 1.6. We prove the theorem by induction on $d=d(V,G)$. So we assume that, for a subspace W of $d(W,G)<d$, (W,G) decomposes to 1-dimensional couples if and only if $k[W]^G$ is a polynomial ring.

Now we assume that $k[V]^G$ is a polynomial ring and the couple (V,G) is indecomposable. Let $V=\bigoplus_{i=1}^s kS_i\oplus\bigoplus_{i=1}^t kT_i\oplus\bigoplus_{i=1}^u kU_i\oplus\bigoplus_{i=1}^2 kX_i$, and let

$H=$  ,

$\sigma=$  .

Step 1. *"If there is a minimal prime ideal* $\mathfrak{p}$ *of* $k[V]$ *satisfying* $|I_G(\mathfrak{p})|=p^2$, *then* (V,G) *is decomposable."* Put $N=I_G(\mathfrak{p})$. Then we can find a homogeneous prime ideal $\mathfrak{p}'$ such that $N\times I_G(\mathfrak{p}')=G$. Since $k[V]^{I_G(\mathfrak{p}')}$ is a polynomial ring and $d(V, I_G(\mathfrak{p}'))<d$, $(V, I_G(\mathfrak{p}'))$ decomposes to 1-dimensional couples. Hence, it follows that (V, G) is decomposable.

Step 2. Suppose $|G|=p^d$. From Step 1, we identify G with H. Since (V, G) is indecomposable, then we have $stu\geqq 1$ and $c_i\neq 0(1\leqq i\leqq u)$. Put $\mathfrak{p}=\langle S_1, T_1, U_1\rangle$, then

$$k[V]^{I_G(\mathfrak{p})}=k[S_1, T_1, U_1, X_1, X_2]^{I_G(\mathfrak{p})}[S_2, \cdots, S_s, T_2, \cdots, T_t, U_2, \cdots, U_u]$$

is a polynomial ring. We can directly show that

$$k[S_1, T_1, U_1, X_1, X_2]^{I_G(\mathfrak{p})}_{(p)}|_{S_1,T_1,U_1=0}=0 .$$

This contradicts the fact that $|I_G(\mathfrak{p})|=p^3$.

Step 3. If $|G|=p^{d+1}$, then we may assume that $\langle H, \sigma\rangle=G$ for some element σ. Clearly $st\geqq 1$ and $c_i\neq 0(1\leqq i\leqq u)$. Say $a_1^\sigma=0$. Put $\mathfrak{p}=\langle S_2, S_3, \cdots, S_s, T_1, \cdots, T_t, U_1, \cdots, U_u\rangle$ and $N=I_G(\mathfrak{p})$. Then $d(V, N)<d$ and $k[V]^N$ is a polynomial ring. From the assumption of induction, (V, N) is decomposable. Since $|N|=p^{d(V,N)+1}$, we can find a minimal prime ideal $\mathfrak{q}$ of $k[V]$ satisfying $|I_N(\mathfrak{q})|=|I_G(\mathfrak{q})|=p^2$. By Step 1, this is a contradiction. It follows that $a_i^\sigma\neq 0(1\leqq i\leqq s)$, $b_i^\sigma\neq 0(1\leqq i\leqq t)$ and $d_i^\sigma\neq 0$ $(1\leqq i\leqq u)$. Suppose $u>0$. By Lemma 3.2, we know $stu>1$. Say $s>1$. Set $W=\bigoplus_{i=2}^s kS_i\oplus\bigoplus_{i=1}^t kT_i\oplus\bigoplus_{i=1}^u kU_i$, then $d(W\oplus\bigoplus_{i=1}^2 kX_i, I_G(\langle W\rangle))=u-1$ and $|I_G(\langle W\rangle)|=p^{u-1}$. From Step 2,

$$k[V]^{I_G(\langle W\rangle)}=k\left[W\oplus\bigoplus_{i=1}^2 kX_i\right]^{I_G(\langle W\rangle)}[S_1] .$$

Hence we conclude $u=0$. Let us make a regular system of nomogeneous parameters of $Q(V, G)$. Then

$$Q(V, G)=\begin{cases} k[\bar{X}_1^{p^{s+1}}, \bar{X}_2^{p^t}]\ ; & \text{by the algorithm } A(kS_1, \cdots, \bigoplus_{i=1}^s kS_i\oplus kT_1, \cdots, V^G) \\ k[\bar{X}_1^{p^s}, \bar{X}_2^{p^{t+1}}]\ ; & \text{by the algorithm } A(kT_1, \cdots, \bigoplus_{i=1}^t kT_i\oplus kS_1, \cdots, V^G) , \end{cases}$$

which is a contradiction.

Step 4. Finally we suppose $|G|>p^{d+1}$ and $G\supseteqq H$. Then $G=H\oplus\bigoplus_{i=1}^m\langle\sigma_i\rangle$. Put $F=\{\sigma_1, \cdots, \sigma_m\}$,

$$F(S_j)=\{\sigma_i\in F\colon a_j^{\sigma_i}\neq 0\}\quad (1\leqq j\leqq s) ,$$

$$F(T_j)=\{\sigma_i \in F\colon b_j^{\alpha_i}\neq 0\} \quad (1\leqq j\leqq t)\ ,$$

and

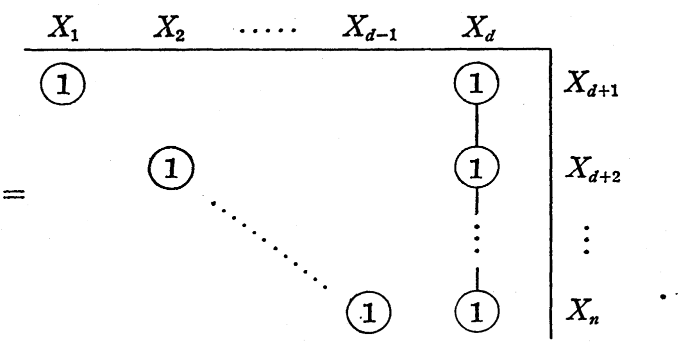
$$F(U_j)=\{\sigma_i \in F\colon d_j^{\alpha_i}\neq 0\} \quad (1\leqq j\leqq u)\ .$$

Replacing the set F, we may assume that $|F(S_1)|=1$. Set $W=\bigoplus_{i=2}^{s} kS_i\oplus \bigoplus_{i=1}^{t} kT_i\oplus \bigoplus_{i=1}^{u} kU_i$ and $N=I_G(\langle W\rangle)$, then $d(W\oplus \bigoplus_{i=1}^{2} kX_i, N)<d$. By the assumption of induction, we know that the couple $(W\oplus \bigoplus_{i=1}^{2} kX_i, N)$ decomposes to 1-dimensional couples. Since $|F(S_1)|=1$ and $|G|>p^{d+1}$, $|N|\geqq p^{d(W\oplus \oplus_{i=1}^{2} kX_i, N)}$. This inequality and the decomposition of $(W\oplus \bigoplus_{i=1}^{2} kX_i, N)$ imply that there is an element $f\in W$ such that $|I_N(fk[W\oplus \bigoplus_{i=1}^{2} kX_i])|=p^2$. Hence we conclude that $|I_G(fk[V])|=|I_N(fk[V])|=|I_N(fk[W\oplus \bigoplus_{i=1}^{2} kX_i])|=p^2$. From Step 1, this is a contradiction.

§4. Some remarks.

For an abelian p-group G generated by transvections, it has not been known whether $k[V]^G$ is a Macaulay ring. We give some examples of G such that $k[V]^G$ are not Macaulay rings.

EXAMPLE 4.1. Let $V=\bigoplus_{i=1}^{n} kX_i (n=2d-1,\ d\geqq 1)$ and

$G=$ 

Then $k[V]^G$ is a Macaulay ring if and only if $d\leqq 3$. (Of course the groups G are elementary abelian p-groups generated by transvections.)

OUTLINE OF THE PROOF. When $d\leqq 3$, the assertion is clear. For $d\geqq 4$, we may treat the case of $d=4$. Let $R=k[V]$ and assume that R^G is a Macaulay ring.

Step 1. *"The ring R^G/X_1R^G is normal."* If not, we readily find a homogeneous prime ideal $\mathfrak{p}$ of height 2 containing X_1R such that R^G is singular at $\mathfrak{p}\cap R^G$. For the prime ideal $\mathfrak{p}^*=\langle V^G\cap \mathfrak{p}\rangle$, we know $I_G(\mathfrak{p})=I_G(\mathfrak{p}^*)$ and that $R^{I_G(\mathfrak{p})}$ is not regular. Hence $\mathfrak{p}^*=\mathfrak{p}$. But it can be shown that $R^{I_G(\langle W\rangle)}$ is a polynomial ring for any 2-dimensional subspace W of V^G containing kX_1.

Step 2. "$\langle X_1, X_2\rangle_{R^G}=\langle X_1, X_2\rangle_R \cap R^G$." From Step 1, we see that the canonical map $\varphi: R^G/X_1R^G \to [R^{I_G(\langle X_1\rangle)}/X_1R^{I_G(\langle X_1\rangle)}]^{G/I_G(\langle X_1\rangle)}$ is an isomorphism. Since $R^{I_G(\langle X_1\rangle)}/X_1R^{I_G(\langle X_1\rangle)}$ is a polynomial ring, for $\mathfrak{m}=\langle X_1, X_2\rangle_R$, $\mathfrak{m}\cap R^{I_G(\langle X_1\rangle)}=\langle X_1, X_2\rangle_{R^{I_G(\langle X_1\rangle)}}$. Through the isomorphism φ, we get $\mathfrak{m}\cap R^G \bmod X_1R^G=[\mathfrak{m}\cap R^{I_G(\langle X_1\rangle)}/X_1R^{I_G(\langle X_1\rangle)}]^{G/I_G(\langle X_1\rangle)}\subseteq R^G/X_1R^G$.

Step 3. "*The ring* $R^G/\langle X_1, X_2\rangle_R\cap R^G$ *satisfies the Serre condition* R_1." As in the proof of Step 1, it suffices to show that $R^{I_G(\langle W\rangle)}$ is a polynomial ring for any 3 dimensional subspace W of V^G containing $kX_1\oplus kX_2$. This follows from direct computation.

Step 4. From Step 2 and Step 3, the ring $R^G/\langle X_1, X_2\rangle_R\cap R^G$ is normal. Hence we have

$$R^G/\langle X_1, X_2\rangle_R\cap R^G \cong [R^{I_G(\langle X_1, X_2\rangle)}/\langle X_1, X_2\rangle_R\cap R^{I_G(\langle X_1, X_2\rangle)}]^{G/I_G(\langle X_1, X_2\rangle)} .$$

Put $q=\langle X_1, X_2\rangle_R$ and $H=I_G(\langle X_1, X_2, X_3\rangle)$. Then we have an exact cohomology sequence $0\to H^1(G/H, \mathfrak{q}^H)\to H^1(G/H, R^H)$ of cohomology groups. Denote by σ a generator of $I_G(\langle X_4\rangle)$ and define N to be $1+\sigma+\cdots+\sigma^{p-1}$. Clearly $(1-\sigma)\mathfrak{q}^H_{(p)}=0$ and $(1-\sigma)R^H\cap \operatorname{Ker}(N)\cap \mathfrak{q}^H=(1-\sigma)\mathfrak{q}^H$. Let $U_1(X_5)=X_5^p-X_1^{p-1}X_5$, $U_2(X_6)=X_6^p-X_2^{p-1}X_6$, and $\chi=U_1(X_5)-U_2(X_6)$. Then $\chi\in R^H_{(p)}$ and $(1-\sigma)\chi=U_1(X_4)-U_2(x_4)\in \mathfrak{q}^H$. Since $(1-\sigma)\chi\in R^G$, we have $(1-\sigma)\chi\in (1-\sigma)R^H_{(p)}\cap \operatorname{Ker}(N)\cap \mathfrak{q}^H$. But $(1-\sigma)\chi\neq 0$, which is a contradiction.

We can give another proof of Example 4.1, computing local cohomology groups at a prime ideal. But the above proof seems to be helpful to get many examples like Example 4.1 for the spaces V of lower dimensions.

For a ring R, we denote by Sing R the singular locus of R.

REMARK 4.2. Let G be an abelian p-group generated by pseudo-reflections in $GL(V)$. If $\operatorname{codim}\operatorname{Sing} k[V]^G=\dim V^G$, then the following conditions are equivalent:

(1) $k[V]^G$ is a Gorenstein ring.

(2) There is a subspace W of V^G with $\operatorname{codim}_{V^G} W=2$ such that $k[V]^G/\langle W\rangle\cap k[V]^G$ is normal.

PROOF. Let $0=W_0\subsetneqq W_1\subsetneqq\cdots\subsetneqq W_d=W\subsetneqq V^G$ be an ascending chain of subspaces satisfying $\dim W_i/W_{i-1}=1(1\leqq i\leqq d)$ and $\operatorname{codim}_{V^G} W=2$. Set $I_i=I_G(\langle W_i\rangle)$, $G_i=G/I_i(1\leqq i\leqq d)$ and $R=k[V]$. Let us prove that (2) implies (1). If $k[V]^G/\langle W_{i+1}\rangle\cap k[V]^G$ is normal for some $i<d$, then, since $\operatorname{codim}\operatorname{Sing} R^G=\dim V^G$, there is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \langle W_i\rangle^G & \longrightarrow & R^G & \longrightarrow & [R^{I_i}/\langle W_i\rangle^{I_i}]^{G_i} & \longrightarrow & \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \langle W_{i+1}\rangle^G & \longrightarrow & R^G & \longrightarrow & [R^{I_{i+1}}/\langle W_{i+1}\rangle^{I_{i+1}}]^{G_{i+1}} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & [\langle W_{i+1}\rangle^{I_i}/\langle W_i\rangle^{I_i}]^{G_i} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Hence, we can show that $R^G/\langle W_i\rangle^G \cong [R^{I_i}/\langle W_i\rangle^{I_i}]^{G_i}$ is normal and $[\langle W_{i+1}\rangle^{I_i}/\langle W_i\rangle^{I_i}]^{G_i}$ is monogene as an R^G-module. And so $\langle W\rangle \cap R^G = \langle W\rangle_{R^G}$. Since the ring $R^{I_d}/\langle W\rangle \cap R^{I_d}$ is a polynomial ring, we can take elements $Y_i \in V (1 \leqq i \leqq m = \dim V - \dim V^G)$ and integers $t_i \in N$ such that the canonical map

$$\varphi\colon R^{I_d}/\langle W\rangle \cap R^{I_d} \longrightarrow k[V^G/W][\bar{Y}_1^{p^{t_1}}, \cdots, \bar{Y}_m^{p^{t_m}}]$$

is an isomorphism where $\bar{Y}_i = Y_i \bmod W$. The group G_d acts naturally on $R' = k[V^G/W][\bar{Y}_1^{p^{t_1}}, \cdots, \bar{Y}_m^{p^{t_m}}]$ and the isomorphism φ is compatible with the action of G_d. On the other hand, the affine space $A^m(k)$ acts transitively on the set of maximal ideals of R' containing the ideal $\langle V^G/W\rangle_{R'}$. Since the action of $A^m(k)$ on $\operatorname{Spec} R'$ is commutative with the one of G_d on R', R'^{G_d} is a Macaulay ring by the openness of Macaulay loci. Hence R^G is a Gorenstein ring. Conversely, assume that R^G is a Gorenstein ring. Then we can show inductively $R^G/\langle W_i\rangle \cap R^G \cong [R^{I_i}/\langle W_i\rangle \cap R^{I_i}]^{G_i}$ and $\langle W_{i-1}\rangle \cap R^G = \langle W_{i-1}\rangle_{R^G} (1 \leqq i \leqq d)$. Therefore $R^G/\langle W\rangle \cap R^G$ is normal.

In relation with Remark 4.2, we would like to describe the indecomposable couple (V, G) which satisfies the condition $\operatorname{codim} \operatorname{Sing} k[V]^G = \dim V^G > 2$. But we know the following

REMARK 4.3. Let G be an abelian p-group generated by pseudoreflections in $GL(V)$ which is realizable on F_p. Suppose that $k[V]^G$ is not a polynomial ring and $\dim V^{*G} = 2$. If $\dim V^G > 2$, then $\operatorname{codim} \operatorname{Sing} k[V]^G < \dim V^G$.

This can be shown by improving the method stated in the proof of Lemma 3.2. We omit the proof of Remark 4.3.

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