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On the Values of Eisenstein Series

Dedicated to Professor Yukiyoshi Kawada on his 60th Birthday

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Introduction.

The main purpose of the present paper is to settle the theorem of v. Staudt-Clausen for 'normalized Hurwitz-Herglotz function' $H_s(\tau; u, v)$ in the singular case (i.e. the case τ is imaginary quadratic and $u, v \in \boldsymbol{Q}$):

$$H_s(\tau; u, v)=\frac{s!}{12\sqrt{\Delta^s}}\sum_{(m,n)\in Z^2}{}' \frac{e^{2\pi i(mu+nv)}}{(m\omega_1+n\omega_2)^s}$$

where $\tau=\omega_2/\omega_1$ and $\Delta=\Delta(\omega_1, \omega_2)$ is the usual discriminant function for Weierstrass' $\wp$-function with periods ω_1, ω_2.

The result is, roughly speaking, that the 'theorem of v. Staudt-Clausen' is of the same type as Herglotz except for an algebraic additive term whose denominator is divisible by at most prime factors of a finite number of integers given in the respective case.

Here note that in $\boldsymbol{Q}(\sqrt{-1})$, for example, $H_s(\sqrt{-1}; u, v)$ does not vanish and has an additive contribution mentioned above to v. Staudt-Clausen even for $s\not\equiv 0 \pmod 4$, while $H_s(\sqrt{-1}; 0, 0)$, the Hurwitz-Herglotz number, vanishes for $s\not\equiv 0 \pmod 4$.

Further it should be noted that as a byproduct of our theory, an interesting identity is obtained from modular transformation formula for function W_λ (see **2.2**).

In the final part, we add some comment on Ramanujan's formula for series of Lambert type.

§ 1. Kronecker's function K.

1.1. Let w, τ be complex variables and Im τ be positive. We define

$$(1.1)\qquad \vartheta_1(w, \tau)=\sum_{n=-\infty}^{\infty} e^{\pi i(n+1/2)^2\tau+2\pi i(n+1/2)(w-1/2)}$$

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and for $\omega_1, \omega_2 \in \boldsymbol{C}$, $\tau=\omega_2/\omega_1$, $z \in \boldsymbol{C}$ and $u, v \in \boldsymbol{R}$,

$$G(z;\omega_1,\omega_2;u,v)=-i\,\frac{\eta(\tau)^3\vartheta_1(u\tau-v+(z/\omega_1);\tau)}{\vartheta_1(z/\omega_1;\tau)\vartheta_1(u\tau-v;\tau)}\,. \tag{1.2}$$

Here η is the so-called Dedekind η-function:

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{2\pi i m\tau})\,.$$

Then our starting point is the following function K which is extensively investigated by Kronecker:

$$K(z;\omega_1,\omega_2;u,v)=\frac{2\pi i}{\omega_1}e^{2\pi u i z/\omega_1}G(z;\omega_1,\omega_2;u,v)\,. \tag{1.3}$$

Kronecker proved the following

THEOREM K (Kronecker). *For* $0<u<1$,

$$K(z;\omega_1,\omega_2;u,v)=\lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M}\frac{e^{-2\pi i(mu+nv)}}{z+m\omega_1+n\omega_2}\,. \tag{1.4}$$

Here the sum on the right is called Eisenstein sum by A. Weil [15]. For short, we write

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M}=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\,.$$

1.2. For $1>|y|>|q|$, we put

$$F(q,x,y)=\sum_{n=-\infty}^{\infty}\frac{y^n}{q^n x-1}\,. \tag{1.5}$$

Then two main points to get Theorem K are as follows:

The first. For $0<u, v<1$, we have

$$F(q,x,y)=G(z;\omega_1,\omega_2;u,v) \tag{1.6}$$

with

$$x=e^{2\pi i z/\omega_1},\quad y=e^{2\pi i(u\tau-v)}\quad\text{and}\quad q=e^{2\pi i\tau}\,. \tag{1.7}$$

Therefore we have

$$K(z;\omega_1,\omega_2;u,v)=\frac{2\pi i}{\omega_1}x^u F(q,x,y)\,.$$

The second. For $0<u<1$, we have

$$\frac{e^{2\pi iuz}}{e^{2\pi iz}-1}=\frac{1}{2\pi iz}+\frac{1}{2\pi i}\sum_{m=1}^{\infty}\left\{\frac{e^{-2\pi imu}}{z+m}+\frac{e^{2\pi imu}}{z-m}\right\}. \tag{1.8}$$

This holds for $u=0$ up to the additive constant $-1/2$ on the right.

As Kronecker put a stress, we represent (1.5) in a symmetric form:

$$F(q, x, y)=1-\frac{1}{1-x}-\frac{1}{1-y}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{mn}(x^{-m}y^{-n}-x^{m}y^{n}), \tag{1.9}$$

which is vaild for $1>|x|>|q|$. Thus we have

$$F(q, x, y)=\sum_{n=-\infty}^{\infty}\frac{x^{n}}{q^{n}y-1}. \tag{1.10}$$

REMARK. Kronecker's formula (1.8) is an easy consequence of Dirichlet's formula [3] in the Fourier analysis:

$$\lim_{n\to\infty}\sum_{k=-n}^{n}\int_{\rho}^{\rho+1}F(z)e^{2k(v-z)\pi i}dz$$
$$=\begin{cases}1/2\lim(F(v+\delta)+F(v-\delta)) & \text{for } \rho<v<\rho+1,\\ 1/2\lim(F(\rho+\delta)+F(\rho+1-\delta)) & \text{for } v=\rho \text{ or } \rho+1\end{cases}$$

where $F(v)$ is a function defined on $[\rho, \rho+1]$ with some conditions.

In fact, take $\rho=0$, $F(v)=e^{2vw\pi i}$ with arbitrary w. This is the original proof by Kronecker. Another proof of (1.8) can be seen in Siegel [13].

1.3. We can easily derive the transformation formulas for K and G under the full modular group Γ since we know the transformation formulas for η and ϑ_1.

For $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$, we put

$$\begin{aligned}\omega_1^*&=d\omega_1+c\omega_2, & \omega_2^*&=b\omega_1+a\omega_2\\ u^*&=du+cv, & v^*&=bu+av.\end{aligned} \tag{1.11}$$

Then we have the following

THEOREM 1. *Assume* $c>0$.

(i) $G(z;\omega_1^*,\omega_2^*;u^*,v^*)=(c\tau+d)e^{2\pi icz(u\tau-v)/(c\tau+d)\omega_1}G(z;\omega_1,\omega_2.u,v)$

(ii) $K(z;\omega_1^*,\omega_2^*:u^*,v^*)=K(z;\omega_1,\omega_2;u,v)$.

Now put $u=0$, $v\neq 0$. Taking $\sigma=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ in (ii) of the above theorem, we have

$$K(z;\omega_1,\omega_2;0,v)=K(z;\omega_2,-\omega_1;v,0).$$

Thus, in what follows, we use this right hand side for the meaning of Eisenstein sum in Theorem K in the case $u=0$, $v\neq 0$:

$$\text{(1.12)}\qquad K(z;\omega_1,\omega_2;0,v)=K(z;\omega_2,-\omega_1;v,0)$$
$$=\lim_{M\to\infty}\sum_{m=-M}^{M}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{e^{-2\pi inv}}{z+m\omega_1+n\omega_2}\,.$$

This will be a supply of Theorem K. Further we have

$$K(z;\omega_1,\omega_2;0,v)=K(z;\omega_2,-\omega_1;v,0)=x_0^vF(q_0,x_0,y_0)\,,$$

where $x_0=e^{2\pi iz/\omega_2}$, $y_0=e^{2\pi i(-v/\tau)}$ and $q_0=e^{2\pi i(-1/\tau)}$. This is a supply to (1.7). (See Weil [15] p. 71, 72.)

§ 2. Normalized Herglotz-Hurwitz function.

2.1. First we recall Bernoulli's case. Denote by $B_s(u)$, $0\leqq u<1$, the Bernoulli polynomial. Its generating function is

$$\text{(2.1)}\qquad te^{ut}/(e^t-1)$$

and $B_s(u)$ is defined by

$$\text{(2.2)}\qquad \frac{te^{ut}}{e^t-1}=\sum_{s=0}^{\infty}B_s(u)\frac{t^s}{s!}\,.$$

On the other hand, from (1.8), we get

$$\text{(2.3)}\qquad \frac{te^{ut}}{e^t-1}=1+t\sum_{n=1}^{\infty}\left\{\frac{e^{2\pi inu}}{t-2\pi in}+\frac{e^{-2\pi inu}}{t+2\pi in}\right\}.$$

Then developing the right hand side with respect to t and comparing it with (2.2), we have

$$\text{(2.4)}\qquad B_s(u)=-s!\sum_{n=-\infty}^{\infty}{}'\frac{e^{2\pi inu}}{(2\pi in)^s}\,.$$

Here $\sum'$ means the sum except for $n=0$.

The formula (2.4) is fundamental in representing values of Dirichlet L-function at integral arguments in terms of generalized Bernoulli numbers in the sense of Leopoldt.

2.2. Now we want to consider an analogy lying between Herglotz-Hurwitz's case and Bernoulli's. We view (1.12) corresponds to (2.3). In order to get a formula corresponding to (2.4), we expand $G(z;\omega_1,\omega_2;u,v)$,

$0<u, v<1$, in power series with respect to z: we have

$$-\frac{1}{1-x}=\frac{1}{2\pi iz}+\sum_{\lambda=0}^{\infty}B_{\lambda+1}\left(\frac{2\pi i}{\omega_1}\right)^{\lambda}\frac{z^{\lambda}}{(\lambda+1)!}\ .$$

(2.5) $$\frac{d^{\lambda}}{dz^{\lambda}}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{mn}(x^{-m}y^{-n}-x^{m}y^{n})\right)_{z=0}=\left(\frac{2\pi i}{\omega_1}\right)^{\lambda}R_{\lambda}(\tau;u,v)$$

where x, y and q are the same as in (1.7) and

(2.6) $$R_{\lambda}(\tau;u,v)=\sum_{m=1}^{\infty}\left\{(-1)^{\lambda}\frac{m^{\lambda}q^{m}y^{-1}}{1-q^{m}y^{-1}}-\frac{m^{\lambda}q^{m}y}{1-q^{m}y}\right\}\ ,$$
$$\lambda=0,1,2,\cdots$$

Inserting this in the formula (1.6) combined with (1.9) and writing G as

(2.7) $$G(z;\omega_1,\omega_2;u,v)=\frac{\omega_1}{2\pi iz}+\sum_{\lambda=0}^{\infty}\frac{W_{\lambda}(\tau;u,v)}{\omega_1^{\lambda}}\frac{z^{\lambda}}{\lambda!}$$

we get the following

PROPOSITION 1. *Assume* $0<u, v<1$. *Put*

$$R_{\lambda}(\tau;u,v)=\sum_{m=1}^{\infty}\left\{(-1)^{\lambda}\frac{m^{\lambda}q^{m}y^{-1}}{1-q^{m}y^{-1}}-\frac{m^{\lambda}q^{m}y}{1-q^{m}y}\right\}$$

for $\lambda=0,1,2,\cdots$ *with* $q=e^{2\pi i\tau}$, $y=e^{2\pi i(u\tau-v)}$, *then*

(i) $W_0(\tau;u,v)=1-\dfrac{1}{1-e^{2\pi i(u\tau-v)}}+B_1+R_0(\tau;u,v)$

(ii) $W_{\lambda}(\tau;u,v)=(2\pi i)^{\lambda}\left\{\dfrac{B_{\lambda+1}}{\lambda+1}+R_{\lambda}(\tau;u,v)\right\}\ .$

Next, put

(2.8) $$K(z;\omega_1,\omega_2;u,v)=\frac{1}{z}+\sum_{\lambda=0}^{\infty}\frac{S_{\lambda+1}(\omega_1,\omega_2;u,v)}{\lambda+1}\frac{z^{\lambda}}{\lambda!}\ .$$

Then using (2.7) and Proposition 1, the series expansion of $e^{2\pi iuz/\omega_1}$ with respect to z, we have

THEOREM 2.

$$S_{\lambda+1}(\omega_1,\omega_2;u,v)=\left(\frac{2\pi iu}{\omega_1}\right)^{\lambda+1}+\frac{2\pi i(\lambda+1)}{\omega_1^{\lambda+1}}\sum_{\substack{\lambda=p+q\\ p,q\geq 0}}\binom{\lambda}{p}(2\pi iu)^{q}W_p(\tau;u,v)$$

$$=\left(\frac{2\pi i}{\omega_1}\right)^{\lambda+1}\left\{(\lambda+1)u^{\lambda}\left(1-\frac{1}{1-e^{2\pi i(u\tau-v)}}\right)+B_{\lambda+1}(u)\right.$$
$$\left.+(\lambda+1)\sum_{\substack{\lambda=p+q\\ p,q\geq 0}}\binom{\lambda}{p}u^{q}R_{p}(\tau;u,v)\right\}.$$

Further using the series expansion of the right hand side of the formula in Theorem K, we get the following

THEOREM 3.

$$(-1)^{\lambda}(\lambda+1)!\sum_{(m,n)\in Z^2}{}'\frac{e^{-2\pi i(mu+nv)}}{(m\omega_1+n\omega_2)^{\lambda+1}}=S_{\lambda+1}(\omega_1,\omega_2;u,v).$$

The left hand side is a sort of Eisenstein sum. When $u=0$, the sum means

$$\lim_{M\to\infty}\sum_{m=-M}^{M}\lim_{N\to\infty}\sum_{n=-N}^{N}.$$

This theorem is an analogy to (2.4).

We define

$$G_4=G_4(\omega_1,\omega_2)=\sum_{(m,n)\in Z^2}{}'\frac{1}{(m\omega_1+n\omega_2)^4}60,$$

$$G_6=G_6(\omega_1,\omega_2)=\sum_{(m,n)\in Z^2}{}'\frac{1}{(m\omega_1+n\omega_2)^6}120$$

and

(2.9) $$\Delta=\Delta(\omega_1,\omega_2)=G_4^3-27G_6^2.$$

In accordance with Herglotz [7], we call the following the "normalized Herglotz-Hurwitz function":

(2.10) $$H_{\lambda+1}(\tau;u,v)=\frac{(-1)^{\lambda}}{\sqrt[12]{\Delta^{\lambda+1}}}S_{\lambda+1}(\omega_1,\omega_2;u,v).$$

This is an analogy to Bernoulli polynomial (2.4).

REMARK. In (2.10), put formally $u=v=0$. Then $H_{\lambda+1}$ becomes $C_{(\lambda+1)/2}$ in the notation of Herglotz [7], which is called the normalized coefficient of $\wp$-function by successors (see below).

2.3. *Theory of Herglotz.* We shall quote here some results of Herglotz [7], in the range of later necessity.

As usual, $\wp(z; \omega_1, \omega_2)$ is Weierstrass' elliptic function with ω_1, ω_2 as periods. Its series expansion is

$$\wp(z; \omega_1, \omega_2) = z^{-2} + 3s_2 z^2 + 5s_4 z^4 + \cdots\cdots$$

where

$$s_{2\lambda} = S_{2\lambda}(\omega_1, \omega_2; 0, 0)/(2\lambda)!$$

Herglotz defined

$$C_\lambda = \frac{(2\lambda)!}{\sqrt[6]{\Delta^\lambda}} s_{2\lambda}$$

and derived the theorem of v. Staudt-Clausen for C_λ. This is a generalization of Hurwitz [8] on the theorem of v. Staudt-Clausen for coefficients of series expansion of the lemniscate function.

Let j, γ_2, γ_3 be functions as in Weber [14]; namely

$$j = 2^6 3^3 G_4^3/\Delta \tag{2.11}$$

$$\gamma_2 = \sqrt[3]{j} = 2^2 3 G_4/\sqrt[3]{\Delta} \tag{2.12}$$

$$\gamma_3 = \sqrt{j - 1728} = 2^3 3^3 G_6/\sqrt{\Delta}\ . \tag{2.13}$$

Then Herglotz showed

$$C_\lambda = (-1)^{\lambda+1} \frac{\gamma_2^h \gamma_3^k j^m}{6} + \sum_p \frac{A_p^{2\lambda/(p-1)}}{p} + \gamma_2^h \gamma_3^k G_\lambda(j) \tag{2.14}$$

where $G_\lambda(j) \in \boldsymbol{Z}[j]$, $\lambda = 6m + 2h + 3k$, $h = 0, 1, 2$, $k = 0, 1$, p runs over all primes such that $p \geqq 5$, $(1/2)(p-1)|\lambda$ and A_p is the coefficient of x of the "Multiplikator Gleichung"

$$x^{p+1} - A_1 x^p + \cdots\cdots - A_p x + (-1)^{(1/2)(p-1)} p = 0$$

satisfied by

$$x = \sqrt[12]{\Delta(\omega_1/p, \omega_2)/\Delta(\omega_1, \omega_2)}\ .$$

It is known that

$$A_k \in \boldsymbol{Z}[j, \gamma_2, \gamma_3]\ .$$

Now let K be a field of the ring of complex multiplications of $\wp(z; \omega_1, \omega_2)$. Then exact formulas of v. Staudt-Clausen for C_λ are given individually for quadratic fields K with class number 1.

(1) *The case* $K=\boldsymbol{Q}(\sqrt{-1})$.
Put

$$\omega_1=\varpi_{(4)}=2\int_0^1\frac{dx}{\sqrt{1-x^4}},\quad \omega_2=\varpi_{(4)}\sqrt{-1}\ .$$

In this case, $j=1728$, $\gamma_2=12$, $\gamma_3=0$, $G_4=4$ and $G_6=0$. Then

$$C_{2n}\in\boldsymbol{Q},\ A_p\in\boldsymbol{Z}\ \text{and}\ C_{2n+1}=0\ ,$$

and the theorem of v. Staudt-Clausen is of the form:

$$C_{2n}=\sum_p\frac{A_p^{4n/(p-1)}}{p}+J_{2n} \tag{2.15}$$

with $J_{2n}\in\boldsymbol{Z}$, $p\geqq5$, $p\equiv1 \pmod 4$ and $p-1|4n$. Note that $C_\lambda=0$ for $\lambda\not\equiv0 \pmod 4$.

(2) *The case* $K=\boldsymbol{Q}(\rho)$; $\rho^2+\rho+1=0$.
Put

$$\omega_1=\varpi_{(6)}=2\int_0^1\frac{dx}{\sqrt{1-x^6}}\ ,\ \omega_2=\varpi_{(6)}\rho\ .$$

Then $j=0$, $\gamma_2=0$, $\gamma_3=24\sqrt{-3}$, $G_4=0$ and $G_6=4$. In this case,

$$\sqrt{-3}^{3n}C_{3n}\in\boldsymbol{Q}\ ,\quad 3^{(1/2)(p-1)}A_p=a_p\in\boldsymbol{Z}\quad\text{and}\quad C_{3n\pm1}=0\ .$$

Then the theorem of v. Staudt-Clausen takes the form

$$\sqrt{-3}^{3n}C_{3n}=\sum_p\frac{a_p^{6n/(p-1)}}{p}+J_{3n} \tag{2.16}$$

with $J_{3n}\in\boldsymbol{Z}$, $p\geqq5$, $p\equiv1 \pmod 3$ and $p-1|6n$. Also note that

$$C_\lambda=0\ \text{for}\ \lambda\not\equiv0 \pmod 6\ .$$

(3) *The case* $K=\boldsymbol{Q}(\sqrt{-2})$.
Put

$$\omega_1=1,\ \omega_2=\sqrt{-2}\ .$$

Then $j=8000$, $\gamma_2=20$, $\gamma_3=56\sqrt{2}$. Hence

$$\sqrt{2}^{\lambda}C_\lambda\in\boldsymbol{Q}\quad\text{and}\quad\sqrt{2}^{(1/2)(p-1)}A_p=a_p\in\boldsymbol{Z}.$$

Now the theorem of v. Staudt-Clausen is

$$\sqrt{2}^{\lambda} C_{\lambda}=\frac{1}{3}+\sum_{p} \frac{a_{p}^{2\lambda/(p-1)}}{p}+J_{\lambda} \tag{2.17}$$

with $J_\lambda \in \boldsymbol{Z}$, $p \geqq 5$, $p \equiv 1, 3 \pmod 8$ and $p-1 \mid 2\lambda$.

(4) *The case* $K=\boldsymbol{Q}(\sqrt{-m})$ with $m=7, 11, 19, 43, 67, 163$.

Put

$$\omega_1=1,\ \omega_2=\frac{1}{2}(-3+\sqrt{-m}) .$$

Then j, γ_2 and $\gamma_3\sqrt{-m}$ are in $\boldsymbol{Z}$, $\sqrt{-m}^{\lambda} C_\lambda \in \boldsymbol{Q}$ and $\sqrt{-m}^{(1/2)(p-1)} A_p = a_p \in \boldsymbol{Z}$. The theorem of v. Staudt-Clausen is of the form

$$\sqrt{-m}^{\lambda} C_{\lambda}=c_{\lambda}+\sum_{p} \frac{a_{p}^{2\lambda/(p-1)}}{p}+J_{\lambda} \tag{2.18}$$

where $J_\lambda \in \boldsymbol{Z}$, $p \geqq 5$, $\left(\frac{p}{m}\right)=1$, $p-1 \mid 2\lambda$ and $c_\lambda = 1/2,\ (-1)^{\lambda}/3$ for $m=7, 11$ and $=0$ otherwise.

We quote values of γ_2, γ_3 for the aforementioned m from Weber [14].

m	7	11	19	43	67	163
γ_2	-15	-32	-96	-960	-5280	-640320
$\frac{\gamma_3}{\sqrt{-m}}$	27	56	216	4536	15624	40133016

§ 3. Transformation formulas.

3.1. For $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we define ω_1^*, ω_2^*, u^*, v^* as in (1.11). Inserting the series expression (2.8) into the both hands of the formula in (ii) of Theorem 1 and comparing coefficients of z^λ, we have the following.

PROPOSITION 2.

$$S_{\lambda+1}(\omega_1^*, \omega_2^*; u^*, v^*)=S_{\lambda+1}(\omega_1, \omega_2; u, v)$$
$$\lambda+1=0, 1, 2, \cdots$$

3.2. ω_1^*, ω_2^*, $u^* v^*$ being the same as in (1.11) for $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we put

$$\tau^*=\frac{\omega_2^*}{\omega_1^*}=\frac{a\tau+b}{c\tau+d} .$$

THEOREM 4. *Assume* $c>0$.

(i) $0<u^*<1$.

$$W_\lambda(\tau^*; u^*, v^*)=\frac{(2\pi i)c^{\lambda+1}}{\lambda+1}(u\tau-v)^{\lambda+1}$$
$$+\sum_{\substack{\lambda=p+q\\ p,q\geq 0}}\binom{\lambda}{p}W_p(\tau; u, v)(2\pi ic)^q(u\tau-v)^q(c\tau+d)^{p+1}$$

(ii) $u^*<0$ *or* >1 *and* $u^*\notin \mathbf{Z}$.

$$\frac{(-[u^*])^{\lambda+1}(2\pi i)^\lambda}{\lambda+1}+\sum_{\substack{\lambda=p+q\\ p,q\geq 0}}\binom{\lambda}{p}W_p(\tau^*; \{u^*\}, v^*)(-2\pi i[u^*])^q$$
$$=\frac{(2\pi i)^\lambda c^{\lambda+1}}{\lambda+1}(u\tau-v)^{\lambda+1}+\sum_{\substack{\lambda=p+q\\ p,q\geq 0}}\binom{\lambda}{p}W_p(\tau; u, v)(2\pi ic)^q(u\tau-v)^q(c\tau+d)^{p+1},$$

where $[u^*]$ *is the largest integer smaller than* u^* *and* $\{u^*\}=u^*-[u^*]$.

PROOF. The transformation formula (i) is obtained by comparing coefficients of z^λ of the series expansion of the both hands of Theorem 1 (i). For the case (ii), observe that

$$G(z; \omega_1^*, \omega_2^*; u^*, v^*)=e^{-2\pi iz[u^*]/\omega_1^*}G(z; \omega_1^*\omega_2^*; u^*, v^*)\ .$$

Then apply the same consideration as in the case (i).

3.3. As an application of Theorem 4, we can derive an interesting identity.

Take $a=d=0$, $c=-b=1$, $u=v=1/2$, $\tau=\sqrt{-1}$. Then $u^*=v^*=1/2$ and $\tau^*=\sqrt{-1}$. Namely

$$\sigma=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\quad \text{leaves}\quad \left(\sqrt{-1}\,,\frac{1}{2}\,,\frac{1}{2}\right)\quad \text{invariant}\ .$$

Hence we can compute the first three W_λ's by Theorem 4. In fact, we have

$$W_0\left(\sqrt{-1}\,;\frac{1}{2}\,,\frac{1}{2}\right)=-\frac{1}{2}\,,$$

(3.1) $$W_1\left(\sqrt{-1}\,;\frac{1}{2}\,,\frac{1}{2}\right)=\frac{1}{2}\pi i\,,$$

(3.2) $$W_2\left(\sqrt{-1}\,;\frac{1}{2}\,,\frac{1}{2}\right)=\pi^2/6\ .$$

Then using (3.1) and Proposition 1, we can easily derive the following

PROPOSITION 3.

$$\sum_{m=1}^{\infty}\frac{2m-1}{e^{(2m-1)\pi}+1}=\frac{1}{24}.$$

This identity also follows from (3.2). Further we have

$$\sum_{(m,n)\in Z'}{}' \frac{e^{-2\pi i((1/2)m+(1/2)n)}}{(m+ni)^{\lambda}}=0 \qquad \text{for } \lambda\not\equiv 0 \pmod 4$$

by Proposition 2.

§ 4. Theorem of v. Staudt-Clausen.

4.1. Let f be a positive integer. Let μ, ν be integers with $0\leqq\mu, \nu<f$ and u, v rational numbers whose reduced common denominator is f.

For every pair (μ, ν), we define

$$g_k^{(\mu,\nu)}=g_k^{(\mu,\nu)}(u, v; \omega_1, \omega_2)=\sum_{\substack{m\equiv\mu \pmod f\\ n\equiv\nu \pmod f}}\frac{e^{-2\pi i(mu+nv)}}{(m\omega_1+n\omega_2)^k}.$$

We note that the formula (7) of p. 18 of Weil [15] depends on (5) of p. 17. In our case, extra term of (5) of p. 17 disappears by p. 71 of Weil [15]. Thus we have

$$S_k(u, v; \omega_1, \omega_2)=k!(-1)^{k-1}\sum_{0\leqq\mu,\nu<f} g_k^{(\mu,\nu)}, \qquad k\geqq 1. \tag{4.1}$$

For Weierstrass' $\wp$-function,

$$\wp'^2=4\wp^3-G_4\wp-G_6,$$

holds with

$$G_4=60 f^{-4}g_4^{(0,0)}(u, v; \omega_1, \omega_2)$$
$$G_6=140 f^{-6}g_6^{(0,0)}(u, v; \omega_1, \omega_2).$$

From this, it follows

$$2\wp''=12\wp^2-G_4. \tag{4.2}$$

Then putting

$$C_{\mu,\nu}=C_{\mu,\nu}(\omega_1, \omega_2)=\wp((\mu\omega_1+\nu\omega_2)/f; \omega_1, \omega_2)$$

for $(\mu, \nu)\neq(0, 0)$, we have the following

THEOREM 5.*

(I) *The case* $(\mu, \nu)\neq(0, 0)$. *Put* $\varepsilon_{\mu,\nu}=e^{-2\pi i(\mu u+\nu v)}$.

(i) $12f^4\varepsilon_{\mu,\nu}g_4^{(\mu,\nu)}=12C_{\mu,\nu}^2-G_4$

(ii) $f^2g_5^{(\mu,\nu)}=2C_{\mu,\nu}g_3^{(\mu,\nu)}$

(iii) *For* $k\geqq 4$,

$$(k+1)k(k-1)g_{k+2}^{(\mu,\nu)}=6\varepsilon_{\mu,\nu}\sum_{\substack{p+q=k-2\\ p\geqq 1, q\geqq 1}}(p+1)(q+1)g_{p+2}^{(\mu,\nu)}g_{q+2}^{(\mu,\nu)} +12f^{-2}C_{\mu,\nu}(k-1)g_k^{(\mu,\nu)} .$$

(II) *The case* $(\mu, \nu)=(0, 0)$.

(i) $g_k^{(0,0)}=0$ *for odd* k.

(ii) *For* $k\geqq 4$,

$$(2k+1)(2k-1)(k-3)g_{2k}^{(0,0)}=3\sum_{p=2}^{k-2}(2p-1)(2k-2p-1)g_{2p}^{(0,0)}g_{2(k-p)}^{(0,0)} .$$

The part (II) is well known. In the case (I), we note that

$$g_3^{(\mu,\nu)}=-2f^{-3}\varepsilon_{\mu,\nu}D_{\mu,\nu}$$

with

$$D_{\mu,\nu}=D_{\mu,\nu}(\omega_1, \omega_2)=\wp'((\mu\omega_1+\nu\omega_2)/f; \omega_1, \omega_2) .$$

PROOF FOR THE CASE (I).

Put

$$\wp_{\mu,\nu}=\wp_{\mu,\nu}(z; u, v; \omega_1, \omega_2)=\sum_{\substack{m\equiv\mu \ (\mathrm{mod}\ f)\\ n\equiv\nu \ (\mathrm{mod}\ f)}}\left\{\frac{e^{2\pi i(mu+nv)}}{(z-m\omega_1-n\omega_2)^2}-\frac{e^{2\pi i(mu+nv)}}{(m\omega_1+n\omega_2)^2}\right\} .$$

Futher put

$$z_{\mu,\nu}=f^{-1}(z-\mu\omega_1-\nu\omega_2) .$$

Then we have

$$\begin{aligned}\wp_{\mu,\nu}&=e^{2\pi i(\mu u+\nu v)}\sum_{\substack{m\equiv 0 \ (\mathrm{mod}\ f)\\ n\equiv 0 \ (\mathrm{mod}\ f)}}\left\{\frac{1}{(z-\mu\omega_1-\nu\omega_2-m\omega_1-n\omega_2)^2}-\frac{1}{(\mu\omega_1+\nu\omega_2-m\omega_1-n\omega_2)^2}\right\}\\ &=e^{2\pi i(\mu u+\nu v)}\sum{}'\left\{\frac{1}{(z-(\mu\omega_1+\nu\omega_2)-(mf\omega_1+nf\omega_2))^2}-\frac{1}{(mf\omega_1+nf\omega_2)^2}\right.\\ &\qquad\left.-\frac{1}{((\mu\omega_1+\nu\omega_2)-(mf\omega_1+nf\omega_2))^2}+\frac{1}{(mf\omega_1+nf\omega_2)^2}\right\}\\ &=f^{-2}e^{2\pi i(\mu u+\nu v)}(\wp(z_{\mu,\nu}; \omega_1, \omega_2)-C_{\mu,\nu}) .\end{aligned}$$

* This theorem was announced in the preconference in 1975 for "International Symposium on Algebraic Number Theory, 1976, at Kyoto."

Then applying of (4.2) to $\wp(z_{\mu,\nu})$ yields

(4.3) $$2f^4\varepsilon_{\mu,\nu}\wp''_{\mu,\nu}(z; u, v; \omega_1, \omega_2)=12(f^2\varepsilon_{\mu,\nu}\wp_{\mu,\nu}(z; u, v; \omega_1, \omega_2)+C_{\mu,\nu})^2-G_4 .$$

Taking into account the series expansion

$$\wp_{\mu,\nu}=\sum_{k=1}(k+1)g_{k+2}^{(\mu,\nu)}(u, v; \omega_1, \omega_2)z^k$$

and inserting this to (4.3), we get

$$2f^4\varepsilon_{\mu,\nu}\sum_{k=2}k(k+1)(k-1)g_{k+2}^{(\mu,\nu)}(u, v; \omega_1, \omega_2)z^{k-2}$$
$$=12(f^2\varepsilon_{\mu,\nu}\sum_{k=1}(k+1)g_{k+2}^{(\mu,\nu)}(u, v; \omega_1, \omega_2)z^k+C_{\mu,\nu})^2-G_4 .$$

Finally compare the coefficients of z^k of both hands to get (I) of the Theorem 5.

THEOREM 6. *Assume $(\mu, \nu)\neq(0, 0)$. Then we have*

$$f^s(s-1)!\ g_s^{(\mu,\nu)}(u, v; \omega_1, \omega_2)=\sum_{\substack{s=2a+3b+4c\\ a,b,c\geqq 0}}A_{(a,b,c)}^{(\mu,\nu)}C_{\mu,\nu}^aD_{\mu,\nu}^bG_4^c$$

with $A_{(a,b,c)}^{(\mu,\nu)}=A_{(a,b,c;\, u,v;\, \omega_1,\omega_2)}^{(\mu,\nu)}$ in $\boldsymbol{Q}(\zeta_f)$, the field of f-th roots of unity. More precisely, the numerator of A is an integer in $\boldsymbol{Q}(\zeta_f)$ and the denominator of A is most powers of 2. In particular, for $K=\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\rho)$, A is an integer in $\boldsymbol{Q}(\zeta_f)$.

PROOF. From Theorem 5, (I), (i),

$$f^4\,3!\ g_4^{(\mu,\nu)}=3!\ \varepsilon_{\mu,\nu}^{-1}C_{\mu,\nu}^2-\frac{1}{2}G_4\varepsilon_{\mu,\nu}^{-1} .$$

Hence our theorem holds for $s=4$. We observe that in particular, for $K=\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\rho)$, 2 of the denominator disappears because of $G_4=4$ for $K=\boldsymbol{Q}(\sqrt{-1})$ and $G_4=0$ for $K=\boldsymbol{Q}(\rho)$.

From (I), (ii), we have

$$f^5\,4!\ g_5^{(\mu,\nu)}=-2^2 4!\ C_{\mu,\nu}D_{\mu,\nu}\varepsilon_{\mu,\nu}^{-1}$$

and so the theorem holds for $s=5$.

Now assume that the theorem holds for $s\leqq k+1$. Then by (iii), we have

$$f^{k+2}(k+1)!\ g_{k+2}^{(\mu,\nu)}=6f^{k+2}\varepsilon_{\mu,\nu}\sum\frac{(p+1)(q+1)(k+1)!}{(k+1)k(k-1)}g_{p+2}^{(\mu,\nu)}g_{q+2}^{(\mu,\nu)}$$
$$+12f^k\frac{(k+1)!\ (k-1)}{(k+1)k(k-1)}C_{\mu,\nu}g_k^{(\mu,\nu)} .$$

The general term inside $\sum$ is

$$\frac{(k-2)!}{p!\,q!} f^{p+2}(p+1)!\; g_{p+2}^{(\mu,\nu)} f^{q+2}(q+1)!\; g_{q+2}^{(\mu,\nu)}$$

and the last term is equal to

$$f^k(k-1)!\; g_k^{(\mu,\nu)}\ .$$

Hence by induction we get the theorem.

4.2. Let K be an imaginary quadratic field and w the number of roots of 1 in K. Hence $w=4, 6, 2$ for $K=\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\rho)$ and otherwise, respectively.

After Weber, we define τ-functions as follows:

$$\tau(z;\,\omega_1,\,\omega_2)=(-1)^{1/2w}(\wp(z;\,\omega_1,\,\omega_2))^{1/2w}G^{(w)}$$

with

$$G^{(2)}=G^{(2)}(\omega_1,\,\omega_2)=2^7 3^5 G_4 G_6/\Delta\ ,$$
$$G^{(4)}=G^{(4)}(\omega_1,\,\omega_2)=2^8 3^4 G_4^2/\Delta\ ,$$
$$G^{(6)}=G^{(6)}(\omega_1,\,\omega_2)=2^9 3^6 G_6/\Delta\ .$$

Let $\mathfrak{f}$ be an integral ideal in K and f be the smallest integer divisible by $\mathfrak{f}$. Let ω_1, ω_2 be a basis of an ideal in K with $\operatorname{Im}\omega_2/\omega_1>0$. Thus $K=\boldsymbol{Q}(\tau)$, $\tau=\omega_2/\omega_1$. j is algebraic.

We put

$$\tau_{\mu,\nu}=\tau_{\mu,\nu}(\omega_1,\,\omega_2)=\tau((\mu\omega_1+\nu\omega_2)/f;\,\omega_1,\,\omega_2)\ .$$

Then it is known that these f-division values of τ are algebraic numbers whose denominators are at most divisible by prime factors of f (cf. Hasse [6]).

4.3. *L*-functions.

Let K be an imaginary quadratic field and $\mathfrak{f}$ an integral ideal of K. Let χ be a primitive ray-class character mod $\mathfrak{f}$. Denote by λ a Grössen character of the type

$$\lambda((\beta))=(\beta/|\beta|)^{-e}\ ,\quad (\beta,\,\mathfrak{f})=1\ .$$

Then $e\equiv 0 \pmod{w}$.

We consider L-function

$$L(s,\,\chi\cdot\lambda)=\sum_{(\mathfrak{a},\mathfrak{f})=1}\frac{\chi\cdot\lambda(\mathfrak{a})}{(N\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over all non-zero integral ideals coprime to $\mathfrak{f}$. Since K is imaginary quadratic, the Vorzeichen character attached to χ is identical: namely we have

$$\chi((\beta))=\chi(\beta)\ , \qquad (\beta, \mathfrak{f})=1$$

where χ on the right is the character of residue classes mod $\mathfrak{f}$ attached to χ on the left. We extend χ so that

$$\chi(\beta)=0 \qquad \text{for} \quad (\beta, \mathfrak{f})\neq 1\ .$$

Let $\mathfrak{d}$ be the different of K. Let $\mathfrak{q}$ be an ideal such that $(\mathfrak{q}, \mathfrak{f})=1$, belonging to the inverse class of $\mathfrak{f}^{-1}\mathfrak{d}^{-1}$. Then there exists an element γ of K such that

$$(\gamma)=\mathfrak{q}\mathfrak{f}^{-1}\mathfrak{d}^{-1}\ .$$

We fix γ once for all. The Gaussian sum is defined by

$$T_\chi=\sum_{\alpha \bmod \mathfrak{f}} \bar{\chi}(\alpha)e^{2\pi\sqrt{-1}S(\alpha\gamma)}\ .$$

where α runs over the complete set of representatives mod $\mathfrak{f}$ and S denotes the trace from K to $\boldsymbol{Q}$.

Let $\mathfrak{b}_B$ be an ideal from a ray class B mod $\mathfrak{f}$ and $\omega_1(B)$, $\omega_2(B)$ a fixed basis of $\mathfrak{b}_B$ such that $\mathrm{Im}(\omega_2(B)/\omega_1(B))>0$.

We put, for $\beta\in\mathfrak{b}_B$,

$$S(\beta)=mu_B+nv_B$$

with

$$u_B=S(\gamma\omega_1(B))\ , \qquad v_B=S(\gamma\omega_2(B))\ .$$

The rational numbers u_B, v_B have the reduced common denominator f. Then following Siegel's computation ([13]), we have, for $s>1$, $s=1/2e$,

$$L(s, \chi\cdot\lambda)=T_\chi^{-1}w_\mathfrak{f}^{-1}\sum_B \overline{\chi\cdot\lambda}(\mathfrak{b}_B)\left(\frac{2y_BN(\omega_1(B))}{\sqrt{|d|}}\right)^s S_{2s}(u_B, v_B; \omega_1(B), \omega_2(B))$$

where $w_\mathfrak{f}$ is the number of roots of 1 congruent to 1 mod $\mathfrak{f}$, B runs over all ray-classes mod $\mathfrak{f}$, y_B is the imaginary part of $\omega_2(B)/\omega_1(B)$ and d is the discriminant of K.

Thus "the theorem of v. Staudt-Clausen for L" is reduced to that for $H_{2s}(\omega_2(B)/\omega_1(B), u_B, v_B)$.

4.4. We shall establish the theorem of v. Staudt-Clausen for

$$f^{\lambda}H_{\lambda}(\tau;\, u,\, v)$$

according to what is K.

(1) $K=\boldsymbol{Q}(\sqrt{-1})$. In this case $w=4$, and we take $\tau=\sqrt{-1}$, $\omega_1=1$, $\omega_2=\sqrt{-1}$. Then

$$G_4=4\varpi_{(4)},\ G_6=0$$

with

$$\varpi_{(4)}=2\int_0^1\frac{dx}{\sqrt{1-x^4}}\,.$$

Put

$$C'_{\mu,\nu}\varpi^2_{(4)}=C_{\mu,\nu}$$

to get

$$C'^2_{\mu,\nu}=\tau_{\mu,\nu}/2^6 3^4\,.$$

Let $\mathfrak{K}_{(4)}$ be the field generated by $\{\tau_{\mu,\nu}\}$ over $K(j)$. Denoting by h the denominator of $\tau_{\mu,\nu}$, we have

$$(2^3 3^2 h C'_{\mu,\nu})^2=h^2\tau_{\mu,\nu}\,.$$

Thus $2^3 3^2 h C'_{\mu,\nu}$ is an integer belonging to a quadratic extension $\mathfrak{K}^*_{(4),\mu,\nu}$ of $\mathfrak{K}_{(4)}$. Further, since h is divisible by at most prime factors of f, we see that the denominator of $C'_{\mu,\nu}$ is divisible by prime factors of, at most 2, 3, f.

Now put

$$D_{\mu,\nu}=D'_{\mu,\nu}\varpi^3_{(4)}\,.$$

Then we have

$$D'^2_{\mu,\nu}=4\,C'^3_{\mu,\nu}-4\,C'_{\mu,\nu}\,.$$

From this it follows that $2^9 3^6 h^3 D'_{\mu,\nu}$ is an integer in a quadratic extension of $\mathfrak{K}^{**}_{(4),\mu,\nu}$ of $\mathfrak{K}^*_{(4),\mu,\nu}$ and the denominator of $D'_{\mu,\nu}$ is divisible by at most prime factors of 2, 3, f.

Observe that

$$\wp'(\sqrt{-1}\,z)=\wp(z)\,.$$

Then taking into account the addition formula for $\wp$-function, we see that every $C_{\mu,\nu}$, $(\mu,\nu)\neq(0,0)$ can be written rationally (over K) by

$$\xi_{(4)}=\wp(\varpi_{(4)}/f;\ \varpi_{(4)},\ \varpi_{(4)}\sqrt{-1})$$

and

$$\eta_{(4)}=\wp'(\varpi_{(4)}/f;\ \varpi_{(4)},\ \varpi_{(4)}\sqrt{-1})\ .$$

Therefore, for every $(\mu, \nu)\neq(0, 0)$, we have

$$\mathfrak{K}^{*}_{(4),\mu,\nu}=\mathfrak{K}_{(4)}(\xi_{(4)}),\ \mathfrak{K}^{**}_{(4),\mu,\nu}=\mathfrak{K}_{(4)}(\xi_{(4)},\ \eta_{(4)})\ .$$

We denote by ζ_f a primitive f-th root of 1. Then combining the above result with Theorem 6, we have, by (4.1), the following

PROPOSITION 4. *For* $K=\boldsymbol{Q}(\sqrt{-1})$, $\omega_1=1$, $\omega_2=\sqrt{-1}$,

$$\frac{f^s(s-1)!\sum'_{\mu,\nu} g_s^{(\mu,\nu)}(u,\ v;\ 1,\ \sqrt{-1})}{\varpi^s_{(4)}}\in\mathfrak{K}_{(4)}(\xi_{(4)},\ \eta_{(4)},\ \zeta_f)$$

and the denominator of this algebraic number is divisible by, at most prime factors of 2, 3, f.

Here $\sum'$ means the sum except for $(\mu, \nu)=(0, 0)$.

Now we can derive the theorem of v. Staudt-Clausen for f^sH_s in the present case, using Proposition 4 and Herglotz's result.

We have

$$\begin{aligned} f^sH_s&=\frac{f^s(-1)^{s-1}}{\sqrt[12]{\Delta^s}}S_s(u,\ v;\ 1,\ \sqrt{-1})\\ &=\frac{f^s s!}{\sqrt[12]{\Delta^s}}\,g_s^{(0,0)}(u,\ v;\ 1,\ \sqrt{-1})+\frac{f^s s!}{\sqrt[12]{\Delta^s}}\sum'_{\mu,\nu} g_s^{(\mu,\nu)}(u,\ v;\ 1,\ \sqrt{-1})\ . \end{aligned}$$

In this case, $g_s^{(0,0)}=0$ for $s\not\equiv0 \pmod 4$. For $s\equiv0 \pmod 4$, put $s=4\lambda$. Then

$$\frac{f^s s!}{\sqrt[12]{\Delta^s}}\,g_s^{(0,0)}=C_{2\lambda}$$

in Herglotz notation. Further we know

$$\sqrt[12]{\Delta}=\sqrt{2}\,\varpi_{(4)}$$

(note that we employ here inhomogeneous notation). Summing up, we have the following, as an analogy to v. Staudt-Clausen,

THEOREM 7. *Assume* $K=\boldsymbol{Q}(\sqrt{-1})$, $\omega_1=1$ *and* $\omega_2=\sqrt{-1}$.

(i) *For* $s \not\equiv 0 \pmod 4$,

$$f^s\sqrt{2^s}H_s(\sqrt{-1};u,v) \in \mathfrak{K}_{(4)}(\xi_{(4)},\eta_{(4)},\zeta_f)$$

and the denominator of $f^s\sqrt{2^s}H_s$ *is divisible by at most prime factors of* 2, 3, f.

(ii) *For* $s \equiv 0 \pmod 4$,

$$f^sH_s(\sqrt{-1};u,v)=\sum_p \frac{A_p^{s/(p-1)}}{p}+T_s^{(4)}$$

where $T_s^{(4)}$ *is a number in* $\mathfrak{K}_{(4)}(\xi_{(4)},\eta_{(4)},\zeta_f)$ *with the denominator divisible by at most prime factors of* 2, 3, f *and* p *is prime such that*

$$p \geqq 5\,, \qquad p\equiv 1 \pmod 4\,, \qquad p-1\,|\,s\,.$$

A_p *is the same as in* 2.3.

In particular, we consider the case $u=v=1/2$. In this case, every $D_{\mu,\nu}=0$. Hence in Theorem 6, only terms with $b=0$ appear. Since $G_4=4$, we see that $f^s(s-1)!\,g_s$ is a linear combination of $C_{\mu,\nu}(=0, 1, \text{ or } -1)$ with integral coefficients in $Q(\zeta_f)$. Thus the theorem of v. Staudt-Clausen is of the same type as Herglotz, up to an additive constant in $Q(\sqrt{-1},\zeta_f)$.

(2) $K=Q(\rho)$, $\omega_1=1$, $\omega_2=\rho$, $w=6$.
Put

$$\varpi_{(6)}=2\int_0^1 \frac{1}{\sqrt{1-x^6}}\,dx\,.$$

Then $G_4=0$, $G_6=4\varpi_{(6)}^6$. Putting $C'_{\mu,\nu}\varpi_{(6)}^2=C_{\mu,\nu}$, we have

$$C'^3_{\mu,\nu}=\tau_{\mu,\nu}/2^7 3^3\,.$$

Let $\mathfrak{K}_{(6)}$ be the field generated by $\{\tau_{\mu,\nu}\}$ over $K(j)$. Then as in the case of $K=Q(\sqrt{-1})$, we see that $C'_{\mu,\nu}$ belongs to a cubic extension $\mathfrak{K}^*_{(6),\mu,\nu}$ of $\mathfrak{K}_{(6)}$ and the denominator of $C'_{\mu,\nu}$ is divisible by at most prime factors of 2, 3, f.

If we put

$$D'_{\mu,\nu}\varpi_{(6)}^3=D_{\mu,\nu}\,,$$

then

$$D'^2_{\mu,\nu}=4C'^3_{\mu,\nu}-4\,.$$

From this we see that $D'_{\mu,\nu}$ is a number of a quadratic extension $\mathfrak{K}^{**}_{(6)}$

of $\mathfrak{K}_{(6)}$ and its denominator is divisible by, at most prime factors of 2, 3, f. Also as in the case of $K=\boldsymbol{Q}(\sqrt{-1})$, we see that

$$\mathfrak{K}^{*}_{(6),\mu,\nu}\cdot\mathfrak{K}^{**}_{(6),\mu,\nu}=\mathfrak{K}_{(6)}(\xi_{(6)},\eta_{(6)})$$

for every $(\mu, \nu)\neq(0, 0)$ with

$$\xi_{(6)}=C'_{1,0}=\wp(\varpi_{(6)}/f;\ \varpi_{(6)},\ \varpi_{(6)}\rho)$$

and

$$\eta_{(6)}=D'_{1,0}=\wp'(\varpi_{(6)}/f;\ \varpi_{(6)},\ \varpi_{(6)}\rho)\ .$$

Then by (4.1), Theorem 6, we have

PROPOSITION 5. *Assume* $K=\boldsymbol{Q}(\rho)$, $\omega_1=1$, $\omega_2=\rho$. *Then*

$$\frac{f^{s}(s-1)!\sum'_{\mu,\nu} g_s^{(\mu,\nu)}(u,v;1,\rho)}{\varpi^{s}_{(6)}}\in\mathfrak{K}_{(6)}(\xi_{(6)},\eta_{(6)},\zeta_f)$$

and the denominator of this algebraic number is divisible by at most prime factors of 2, 3, f.

Now we shall derive the theorem of v. Staudt-Clausen for f^sH_s in the present case.

We have

$$\begin{aligned}f^sH_s&=\frac{f^s(-1)^{s-1}}{\sqrt[12]{\Delta^s}}S_s(u,v;1,\rho)\\&=\frac{f^s s!}{\sqrt[12]{\Delta^s}}g_s^{(0,0)}(u,v;1,\rho)+\frac{f^s s!}{\sqrt[12]{\Delta^s}}\sum'_{\mu,\nu} g_s^{(\mu,\nu)}(u,v;1,\rho)\ .\end{aligned}$$

Here it is known that $g_s^{(0,0)}=0$ for $s\not\equiv 0 \pmod 6$. For $s\equiv 0 \pmod 6$ put $s=6\lambda$. Then

$$\frac{f^s s!}{\sqrt[12]{\Delta^s}}g_s^{(0,0)}=C_{3\lambda}$$

in Herglotz notation.

Observing that

$$\sqrt[12]{\Delta}=e^{-\pi i/12}\sqrt[3]{2}\,\sqrt[3]{3}\,\varpi_{(6)}$$

in the inhomogeneous notation, we have the following Theorem 8 as an analogy to v. Staudt-Clausen.

THEOREM 8. *Assume* $K=\boldsymbol{Q}(\rho)$, $\omega_1=1$, $\omega_2=\rho$.

(1) *For* $s\not\equiv 0 \pmod 6$,

$$f^s(e^{-\pi i/12}\sqrt[3]{2}\,\sqrt[4]{3})^s H_s(\rho;\, u,\, v)\in \mathfrak{K}_{(6)}(\xi_{(6)},\, \eta_{(6)},\, \zeta_f)$$

and the denominator of this number is divisible by at most prime factors of 2, 3, f.

(ii) *For* $s\equiv 0 \pmod 6$,

$$f^s(-3)^{(1/4)s}H_s(\rho;\, u,\, v)=\sum_p \frac{a_p^{s/(p-1)}}{p}+T_s^{(6)}$$

where $p>5$, $p\equiv 1 \pmod 3$, $p-1|s$ *and* $T_s^{(6)}\in \mathfrak{K}_{(6)}(\xi_{(6)},\, \eta_{(6)},\, \zeta_f)$ *and the denominator of the above number is divisible by at most prime factors of* 2, 3, f. *For* a_p, *see* **2.3**, *case* (2).

In particular, put $u=v=1/2$. Then every $D_{\mu,\nu}=0$. Since $G_4=0$, only terms for $b=c=0$ appear in the formula of Theorem 6. Thus

$$2^s(s-1)!\sum_{\mu,\nu}{}' g_s^{(\mu,\nu)}$$

is a linear combination of $C_{\mu,\nu}=1$, ρ, ρ^2 with integral coefficients in $\boldsymbol{Q}(\zeta_f)$. Hence in this case, the theorem of v. Staudt-Clausen is of the same type as Herglotz up to an integral additive constant belonging to $\boldsymbol{Q}(\zeta_f,\, \rho)$.

Now let K be a quadratic field other than $\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\rho)$. Let ω_1, ω_2 be a basis of an integral ideal in K. Put $\tau=\omega_2/\omega_1$. Hence $K=\boldsymbol{Q}(\tau)$. In Theorem 6, further assume s is even. Then b is even and we put $b=2b'$. We have

$$\begin{aligned}
f^s H_s(\tau;\, u,\, v)&=\frac{f^s(-1)^{s-1}}{\sqrt[12]{\Delta^s}}S_s(u,\, v;\, \omega_1,\, \omega_2)\\
&=\frac{f^s s!}{\sqrt[12]{\Delta^s}}g_s^{(0,0)}(u,\, v;\, \omega_1,\, \omega_2)+\frac{f^s s!}{\sqrt[12]{\Delta^s}}\sum_{\mu,\nu}{}' g_s^{(\mu,\nu)}(u,\, v;\, \omega_1,\, \omega_2)\\
&=C_{(1/2)s}+\frac{s}{\sqrt[12]{\Delta^s}}\sum_{\substack{s=2a+3b+4c\\ a,b,c\geqq 0}} A^{(\mu,\nu)}(a,\, b,\, c)C_{\mu,\nu}^a D_{\mu,\nu}^b G_4^c\\
&=(-1)^{(1/2)s+1}\frac{\gamma_2^h\gamma_3^k j^m}{6}+\sum_p \frac{A_p^{s/(p-1)}}{p}+\gamma_2^h\gamma_3^k G_{(1/2)s}(j)
\end{aligned}$$

$$+\frac{s}{\sqrt[12]{\Delta^s}}\sum_{\mu,\nu}{}'\sum_{\substack{(1/2)s=a+3b'+2c\\ a,b',c\geqq 0}} A^{(\mu,\nu)}(a,\, b',\, c)\frac{(-1)^a 2^{3b'+2c}3^{s+c}\tau_{\mu,\nu}^a(-\tau_{\mu,\nu}^3+3\gamma_2^3\gamma_3^2\tau_{\mu,\nu}-2\gamma_2^3\gamma_3^4)^{b'}}{\gamma_2^{(1/2)s-3c}\gamma_3^{(1/2)s-2c}}$$

where

$G_{(1/2)s}(j) \in \boldsymbol{Z}[j]$, $\frac{1}{2}s=6m+2h+3k$, $h=0, 1, 2$, $k=0, 1$, p is a prime, $p\geqq 5$, $p-1|s$, and A is in $\boldsymbol{Q}(\zeta_f)$ and the denominator of A is at most powers of 2. Here the last equality is obtained by a straight forward calculation under the use of (2.11, 12, 13) and the definition of $\tau_{\mu,\nu}$. The above formula can be viewed as v. Staudt-Clausen for $K=\boldsymbol{Q}(\tau)$. We shall give more precise form of v. Staudt-Clausen for K with class number 1.

(3) $K=\boldsymbol{Q}(\sqrt{-2})$, $\omega_1=1$, $\omega_2=\sqrt{-2}$, $w=2$.

Let $\mathfrak{K}_{(2)}$ be the field generated by $\{\tau_{\mu,\nu}\}$ over $K(j)$. Put

$$C'_{\mu,\nu}\frac{\Delta^{1/6}}{\sqrt{2}}=C_{\mu,\nu}\,,\qquad D'_{\mu,\nu}\left(\frac{\Delta}{2}\right)^{1/4}=D_{\mu,\nu}\,.$$

Then $C'_{\mu,\nu}\in\mathfrak{K}_{(2)}$ and $D'_{\mu,\nu}\in\mathfrak{K}_{(2)}(\eta_{(2)})$ for every $(\mu,\nu)\neq(0,0)$, where $\eta_{(2)}$ is a value of $\wp'$ at f-division point of one fundamental period. $\eta_{(2)}$ is quadratic over $\mathfrak{K}_{(2)}$. We note that $g_s^{(0,0)}=0$ for $s\not\equiv 0 \pmod 2$, $j=8000$, $\gamma_2=20$ and $\gamma_3=56\sqrt{2}$.

THEOREM 9. *Assume* $K=\boldsymbol{Q}(\sqrt{-2})$, $\omega_1=1$, $\omega_2=\sqrt{-2}$.

(i) *For* $s\not\equiv 0 \pmod 2$, $(s=2a+3b+4c,\ a, b, c\geqq 0)$

$$f^s 2^{(1/4)(s-b)}H_s(\sqrt{-2};u,v)\in\mathfrak{K}_{(2)}(\eta_{(2)},\zeta_f)$$

and the denominator of this number is divisible by at most prime factors of 2, 3, 5, 7, f.

(ii) *For* $s\equiv 0 \pmod 2$

$$f^s 2^{(1/4)s}H_s(\sqrt{-2};u,v)=\frac{1}{3}+\sum_p\frac{a_p^{s/(p-1)}}{p}+T_s^{(2)}$$

where $p\geqq 5$, $p\equiv 1, 3 \pmod 8$, $p-1|s$ *and* $T_s^{(2)}\in\mathfrak{K}_{(2)}(\eta_{(2)},\zeta_f)$. *The denominator of* $T_s^{(2)}$ *is divisible by at most prime factors of* 2, 3, 5, 7, f.

For a_p, see case (3) of 2.3.

(4) $K=\boldsymbol{Q}(\sqrt{-m})$, $m=7, 11, 19, 43, 67, 163$.

Let $\mathfrak{K}_{(m)}$ be the field generated by $\{\tau_{\mu,\nu}\}$ over $K(j)$. Put

$$\gamma'_3=\sqrt{-m}\,\gamma_3,\quad C'_{\mu,\nu}=-\tau_{\mu,\nu}/2^2 3\,\gamma_2\gamma'_3,\quad D'_{\mu,\nu}\left(\frac{\Delta^{1/2}}{\sqrt{-m}}\right)^{1/2}=D_{\mu,\nu}\,.$$

Then $C'_{\mu,\nu}\in\mathfrak{K}_{(m)}$ and $D'_{\mu,\nu}\in\mathfrak{K}_{(m)}(\eta_{(m)})$ for every $(\mu,\nu)\neq(0,0)$ where $\eta_{(m)}$ is a value of $\wp'$ at f-division point of one fundamental period. $\eta_{(m)}$ is quadratic over $\mathfrak{K}_{(m)}$.

THEOREM 10. *Assume* $K=\boldsymbol{Q}(\sqrt{-m})$, $\omega_1=1$, $\omega_2=(1/2)(1+\sqrt{-m})$, $m=7$, 11, 19, 43, 67, 163.

(i) *For* $s\not\equiv 0 \pmod 2$, $(s=2a+3b+4c,\ a, b, c\geqq 0)$

$$f^s(-m)^{(1/4)(s-b)}H_s\Big(\frac{1}{2}\big(1+\sqrt{-m}\big);\ u,\ v\Big)\in \mathfrak{K}_{(m)}(\eta_{(m)},\ \zeta_f)$$

and the denominator of this number is divisible by at most prime factors of 2, 3, γ_2, γ_3', m, f.

(ii) *For* $s\equiv 0 \pmod 2$,

$$f^s(-m)^{(1/4)s}H_s\Big(\frac{1}{2}\big(1+\sqrt{-m}\big);\ u,\ v\Big)=h_s+\sum_p \frac{a_p^{s/(p-1)}}{p}+T_s^{(m)}$$

where $p\geqq 5$, $\left(\frac{p}{m}\right)=1$, $p-1|s$, $h_s=1/2$, $(-1)^{(1/2)s}/3$ *for* $m=7$, 11 *and* $=0$ *otherwise and*

$$T_s^{(m)}\in \mathfrak{K}_{(m)}(\eta_{(m)},\ \zeta_f)$$

The denominator of $T_s^{(m)}$ *is divisible by at most prime factors of* 2, 3, γ_2, γ_3', m, f.

For a_p, see the case (4) of 2.3.

By the table given in 2, 3, we can make the following table of prime factors of $\gamma_2\gamma_3'$ in $\boldsymbol{Q}$.

m	7	11	19	43	67	163
$p\mid\gamma_2\gamma_3'$	3, 5	2, 7	2, 3	2, 3, 5, 7	2, 3, 5, 7, 11, 31	2, 3, 5, 7, 11, 19, 23, 29, 127

§ 5. Numerical computations. Examples.

5.1. (1°) Take $u=v=1/2$, $\omega_1=\varpi_{(4)}$, $\omega_2=\varpi_{(4)}\sqrt{-1}$. Then $f=2$, $G_4=4$, $G_6=0$, $\Delta=4^3$ and by Proposition 2.

$$S_\lambda=0 \qquad \text{for } \lambda\not\equiv 0 \pmod 4\ .$$

In general, we have

$$\wp'^2=4(\wp-e_1)(\wp-e_2)(\wp-e_3)$$

with 2-division values

$$e_1=\wp\Big(\frac{1}{2}\omega_1;\ \omega_1,\ \omega_2\Big),\quad \omega_1+\omega_2+\omega_3=0\ .$$

In our case,

$$C_{1,0}=1,\quad C_{0,1}=-1,\quad C_{1,1}=0,\quad D_{1,0}=D_{0,1}=D_{1,1}=0\,.$$

Now the values of $\varepsilon_{\mu,\nu}$ are given as

$$\varepsilon_{1,0}=-1,\quad \varepsilon_{0,1}=-1,\quad \varepsilon_{1,1}=1\,.$$

Therefore by Theorem 5,

$$g_4^{(1,0)}=g_4^{(0,1)}=-\frac{1}{24}\,,\quad g_4^{(1,1)}=-\frac{1}{48}\,.$$

Further we have

$$g_4^{(0,0)}=\frac{24}{15}\,,$$

which is given by Hurwitz formula

$$\sum_{(m,n)\in Z^2}{}' \frac{1}{(m+n\sqrt{-1})^{4k}}=\frac{(2\varpi_{(4)})^{4k}}{(4k)!}E_{4k} \tag{5.0}$$

with the so-called Hurwitz number $E_{4k}\in \boldsymbol{Q}$. Thus we get

$$S_4=-4!\left(\frac{16}{15}-\frac{5}{48}\right) \text{ and } 2^4H_4\left(\sqrt{-1};\ \frac{1}{2}\,,\ \frac{1}{2}\right)=2^5\left(\frac{16}{5}-\frac{5}{16}\right).$$

In 3.3, we cannot compute $W_3(\sqrt{-1};1/2,1/2)$ from Theorem 4. This value is given in the present context, by Theorem 5. In fact

$$W_3\left(\sqrt{-1}\,;\ \frac{1}{2}\,,\ \frac{1}{2}\right)=\left(\frac{16}{5}-\frac{5}{16}\right)\frac{\varpi_{(4)}^4}{\pi}\sqrt{-1}-\frac{\pi^3}{8}\sqrt{-1}\,.$$

From this we have, for example,

$$\sum_{m=1}^{\infty}\frac{m^3+(m-1)^3}{e^{(2m-1)\pi}+1}=\frac{1}{64}+\frac{1}{120}-\frac{\varpi_{(4)}^4}{8\pi^4}\left(\frac{16}{5}-\frac{5}{16}\right).$$

(2°) ω_1 and ω_2 being the same as above, we take $u=v=(1/3)$. Then $f=3$. Put

$$C_{1,0}=\xi_1,\quad C_{1,1}=\xi_2,\quad D_{1,0}=\eta_1,\quad D_{1,1}=\eta_2\,.$$

Then

$$\eta_i^2=4\xi_i^3-4\xi_i\,,\quad i=1,2$$

and the addition theorem gives

$$\xi_2=-\sqrt{-1}\eta_1^2/8\xi_1^2 .$$

Now

$$\varepsilon_{1,0}=\varepsilon_{0,1}=\varepsilon_{2,2}=\rho^2,\quad \varepsilon_{1,1}=\varepsilon_{2,0}=\varepsilon_{0,2}=\rho,\quad \varepsilon_{1,2}=\varepsilon_{2,1}=1 ,$$
$$C_{1,0}=-C_{0,1}=C_{2,0}=-C_{0,2}=\xi_1,\quad C_{1,1}=-C_{1,2}=-C_{2,1}=C_{2,2}=\xi_2 ,$$
$$D_{1,0}=-\sqrt{-1}D_{0,1}=-D_{2,0}=\sqrt{-1}D_{0,2}=\eta_1 ,$$
$$D_{1,1}=\sqrt{-1}D_{1,2}=-\sqrt{-1}D_{2,1}=-D_{2,2}=\eta_2 .$$

Thus we get

$$3^4H_4\left(\sqrt{-1}\,;\frac{1}{3},\frac{1}{3}\right)=\frac{2}{5}+3!(-2\xi_1^2+\xi_2^2+1)=\frac{2}{5}+3!\left(-2\xi_1^2-\frac{\eta_1^4}{64\xi_1^4}+1\right)$$

$$3^3H_3\left(\sqrt{-1}\,;\frac{1}{3},\frac{1}{3}\right)=3!(\rho^2-\rho)(\eta_2-\eta_1(1+\sqrt{-1}))$$
$$=3!(\rho^2-\rho)\left(\frac{\sqrt{-1}\eta_1^6}{2^7\xi_1^6}+\frac{\sqrt{-1}\eta_1^2}{2\xi_1^2}+\eta_1(1+\sqrt{-1})\right) .$$

In this case, $j=2^63^3$ and $\tau_{\mu,\nu}\in \boldsymbol{Q}(C_{\mu,\nu})$. Hence

$$K(j)=K \quad\text{and}\quad \mathfrak{K}_{(4)}=K(\xi_1) .$$

Since we have

$$\xi_1^2=\tau_{1,0}/2^63^4 ,$$

the number $3!(-2\xi_1^2+\xi_2^2+1)$ on the right of 3^4H_4 belongs to $\mathfrak{K}_{(4)}$ and its denominator is divisible by at most prime factors of 2, 3. Further we see that

$$3^3H_3\left(\sqrt{-1}\,;\frac{1}{3},\frac{1}{3}\right)$$

belongs to $\mathfrak{K}_{(4)}(\eta_1)$, a quadratic extension of $\mathfrak{K}_{(4)}$.

5.2. (1°) Take $\omega_1=\varpi_{(6)}$, $\omega_2=\varpi_{(6)}\rho$, $u=v=1/3$. Then $f=3$. Since $\sigma=\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$ fixes $(\rho, 1/3, 1/3)$, we see that

$$S_\lambda(1/3, 1/3; \varpi_{(6)}, \varpi_{(6)}\rho)=0 \quad \text{if} \quad \lambda\not\equiv 0 \pmod 3) \tag{5.1}$$

by Proposition 2.

Put

$$C_{1,0}=\xi_1,\quad D_{1,0}=\eta_1,\quad C_{2,1}=\xi_2,\quad D_{2,1}=\eta_2\ .$$

Then

$$\eta_i^2=4\xi_i^3-4,\quad i=1,2 \tag{5.2}$$

and by the addition theorem,

$$-\xi_2=(\xi_1^3-4)/3\rho\xi_1^2\ . \tag{5.3}$$

Further we have

$$\begin{aligned}
&\varepsilon_{1,0}=\varepsilon_{0,1}=\varepsilon_{2,2}=\rho^2,\quad \varepsilon_{1,1}=\varepsilon_{2,0}=\varepsilon_{0,2}=\rho,\quad \varepsilon_{2,1}=\varepsilon_{1,2}=1\ ,\\
&C_{1,2}=\rho^2C_{0,1}=\rho C_{1,1}=C_{2,0}=\rho^2C_{0,2}=\rho C_{2,0}=\xi_1,\quad C_{2,1}=C_{1,2}=\xi_2\ ,\\
&D_{1,0}=D_{0,1}=-D_{1,1}=-D_{2,0}=-D_{0,2}=D_{2,2}=\eta_1,\quad D_{2,1}=D_{1,2}=\eta_2\ ,\\
&g_3^{(0,0)}=0,\ g_3^{(1,0)}=g_3^{(0,1)}=g_3^{(2,2)}=-(2/3^3)\rho^2\eta_1\ ,\\
&g_3^{(1,1)}=g_3^{(2,0)}=g_3^{(0,2)}=(2/3^3)\rho\eta_1,\ g_3^{(2,1)}=g_3^{(1,2)}=-(2/3^3)\eta_2\ ,\\
&g_4^{(0,0)}=0,\ g_4^{(1,0)}=g_4^{(0,2)}=(\rho/3^4)\xi_1^2,\ g_4^{(0,1)}=g_4^{(1,1)}=\xi_1^2/3^4\ ,\\
&g_4^{(2,0)}=g_4^{(2,2)}=(\rho^2/3^4)\xi_1^2\ ,\quad g_4^{(1,2)}=g_4^{(2,1)}=\xi_2^2/3^4\ .
\end{aligned}$$

From this, we have

$$0=S_4=(2/3^4)\xi_2^2\quad\text{and}\quad \xi_2=0\ .$$

Namely

$$\wp((2\varpi_{(6)}+\varpi_{(6)}\rho)/3;\ \varpi_{(6)},\ \varpi_{(6)}\rho)=\wp((\varpi_{(6)}+2\varpi_{(6)}\rho)/3;\ \varpi_{(6)},\ \varpi_{(6)}\rho)=0\ .^* \tag{5.4}$$

Further from (5.2), (5.3), we have

$$\begin{aligned}
\eta_2^2&=\wp'((2\varpi_{(6)}+\varpi_{(6)}\rho)/3;\ \varpi_{(6)},\ \varpi_{(6)}\rho)^2\\
&=\wp'((\varpi_{(6)}+2\varpi_{(6)}\rho)/3;\ \varpi_{(6)},\ \varpi_{(6)}\rho)^2=-4\ ,
\end{aligned} \tag{5.5}$$

$$\xi_1^3=\wp(\varpi_{(6)}/3;\ \varpi_{(6)},\ \varpi_{(6)}\rho)^3=4\ , \tag{5.6}$$

$$\eta_1^2=\wp'(\varpi_{(6)}/3;\ \varpi_{(6)},\ \varpi_{(6)}\rho)^2=12\ . \tag{5.7}$$

Thus we get

$$2(-3^3)^{1/4}H_3(\rho;\ 1/3,\ 1/3)=2\cdot 3!\,[3(\rho-\rho^2)\eta_1-2\eta_2] \tag{5.8}$$

and this is an integer in $\boldsymbol{Q}(\rho,\sqrt{-1})$.

Moreover, computation of the values $g_6^{(\mu,\nu)}$ by Theorem 5, shows

* This also follows from the general relation $\sum_{\mu,\nu}\wp\left(\frac{\mu\omega_1+\nu\omega_2}{f};\ \omega_1,\omega_2\right)=0.$ ([9])

$$3^6 3\sqrt{-3}H_6(\rho;\ 1/3,\ 1/3)=\frac{2^2 3^{14}}{7}-2^8 3^2-2^2 3^4\xi_1^3+2^6 3^3\eta_1^2 .$$

Hence by (5.6), (5.7), we get

$$3^3 3\sqrt{-3}\ H_6(\rho;\ 1/3,\ 1/3)=\frac{2^2 3^{14}}{7}+2^4\cdot 3^2\cdot 7\cdot 17 .$$

By the way, the 'Teilungsgleichung' for $\tau_{\mu,\nu}$ in our case, becomes

$$X^2(X^3-2^9 3^9)^2 .$$

(2°) Since $\sigma=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$ leaves $(-\rho^2;\ 1/3,\ 2/3)$ invariant, we can compute some of W_λ by Theorem 4 and in fact

$$W_0(-\rho^2;\ 1/3,\ 2/3)=-1/3\ ,\quad W_1(-\rho^2;\ 1/3,\ 2/3)=\pi\sqrt{-1}/3^2 .$$

Put

$$\gamma=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} .$$

γ sends $(-\rho^2,\ 1/3,\ 2/3)$ to $(-\rho^2;\ 2/3,\ 1/3)$. Then by Theorem 4, we have

$$W_0(-\rho^2;\ 2/3,\ 1/3)=-2/3\ ,\quad W_1(-\rho^2,\ 2/3,\ 1/3)=2^2\pi\sqrt{-1}/3^2 .$$

We view $\gamma\colon \tau\to\tau^*,\ u\to u^*,\ v\to v^*$ in Theorem 4 to get

(5.9) $$W_2(-\rho^2;\ 2/3,\ 1/3)+W_2(-\rho^2;\ 1/3,\ 2/3)=2^2\pi^2/3^2 .$$

On the other hand, we can compute $W_2(-\rho^2;\ 1/3,\ 2/3)$ by (5.8) and Theorem 2. In fact we have

$$W_2(-\rho^2;\ 1/3,\ 2/3)=\frac{2^2\pi^2}{3^3}-\frac{2\sqrt{-1}}{3^2}\,[3(\rho-\rho^2)\eta_1-2\eta_2]\,\frac{\varpi_{(6)}^3}{\pi} .$$

We transform (5.9) to R_2 by Proposition 1 to get

$$\begin{aligned}\frac{2^2\pi^2}{3^3}&=(2\pi\sqrt{-1})^2[R_2(-\rho^2;\ 1/3,\ 2/3)+R_2(-\rho^2;\ 2/3,\ 1/3)]\\ &=-(2\pi\sqrt{-1})^2\sum_{m=1}^{\infty}\left\{\frac{2m-1}{e^{2\pi i\rho^2(m-1/3)-\pi i/3}+1}+\frac{2m-1}{e^{2\pi i\rho^2(m-2/3)+\pi i/3}+1}\right\} .\end{aligned}$$

Therefore

$$\sum_{m=1}^{\infty}\left\{\frac{2m-1}{e^{2\pi i\rho^2(m-1/3)-\pi i/3}+1}+\frac{2m-1}{e^{2\pi i\rho^2(m-2/3)+\pi i/3}+1}\right\}=\frac{1}{9} .$$

In the same way, we get

$$\sum_{m=1}^{\infty}\left\{\frac{1}{(-1)^{m+1}e^{\sqrt{3}(m-2/3)\pi}+1}-\frac{1}{(-1)^{m}e^{\sqrt{3}(m-1/3)\pi}+1}\right\}=\frac{1}{6},$$

from the value of $W_0(-\rho^2; 1/3, 2/3)$.

§ 6. Comments on Ramanujan's formula.

6.1. Ramanujan obtained

$$\sum_{n=1}^{\infty}\frac{n^{13}}{e^{2n\pi}-1}=\frac{1}{24}. \tag{6.1}$$

Compare this with our Proposition 3. The identity (6.1) is an easy consequence of the following formula:

$$\alpha^k\left(\frac{1}{2}\zeta(1-2k)+\sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\alpha}-1}\right) =(-\beta)^k\left(\frac{1}{2}\zeta(1-2k)+\sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\beta}-1}\right) \tag{6.2}$$

where $\alpha\beta=\pi^2$, $k>1$ ($k\in \boldsymbol{Z}$). (Ramanujan [12], Berndt [1]).

From (6.2) we obtain values of

$$\sum_{n=1}^{\infty}\frac{n^{l}}{e^{2n\pi}-1} \tag{6.3}$$

for $l\equiv 1 \pmod 4$. In fact

$$\sum_{n=1}^{\infty}\frac{n^{4k+1}}{e^{2n\pi}-1}=\frac{B_{4k+2}}{8k+4}. \tag{6.4}$$

But we cannot derive values of (6.3) for $l\equiv -1 \pmod 4$.

The formula (6.1) is regained by Watson. Hardy, in [5], gave two proofs (essentially the same) of (6.2) without mentioning the following expression of Eisenstein series by Lambert series ($k\geqq 2$):

$$\sum_{(m,n)\in Z^2}{}'\frac{1}{(m\omega_1+n\omega_2)^{2k}}=\frac{(-1)^k 2(2\pi)^{2k}}{(2k-1)!\ \omega_1^{2k}}\left\{-\frac{B_{2k}}{4k}+\sum_{m=1}^{\infty}\frac{m^{2k-1}}{e^{-2\pi i m\omega_2/\omega_1}-1}\right\}. \tag{6.5}$$

This can be obtained by considering the expression of $E(z;\omega_1,\omega_2)$ in a line of Eisenstein (Weil [15]) and also by Lipschitz's formula.

From (6.5), we easily get

$$\frac{1}{\omega_1^{*2k}}\left\{-\frac{B_{2k}}{4k}+\sum_{m=1}^{\infty}\frac{m^{2k-1}}{e^{-2\pi i m\omega_2^*/\omega_1^*}-1}\right\}=\frac{1}{\omega_1^{2k}}\left\{-\frac{B_{2k}}{4k}+\sum_{m=1}^{\infty}\frac{m^{2k-1}}{e^{-2\pi i m\omega_2/\omega_1}-1}\right\}$$

and (6.2) is a consequence of this.

Further, (6.5) is better than (6.2) because we can compute values of (6.3) even for $l\equiv -1 \pmod 4$: in fact we have, by combining Hurwitz formula (5.0) with (6.4),

$$\sum_{m=1}^{\infty}\frac{m^{4k-1}}{e^{2\pi m}-1}=\frac{1}{8k}\left\{\left(\frac{\varpi_{(4)}}{\pi}\right)^{4k}E_{4k}+B_{4k}\right\}. \tag{6.6}$$

In the same way, we also get

$$\sum_{n=1}^{\infty}\frac{n^{6k-1}}{e^{2n\pi i/\rho}-1}=\frac{1}{12k}\left\{(-1)^{3k}\left(\frac{\varpi_{(6)}}{\pi}\right)^{6k}M_k+B_{6k}\right\}$$

and

$$\sum_{n=1}^{\infty}\frac{n^{6k\pm 2-1}}{e^{2n\pi i/\rho}-1}=\frac{B_{6k\pm 2}}{12k\pm 4},$$

where M_k is rational and is defined by

$$\sum_{(m,n)\in Z^2}{}'\frac{1}{(m+n\rho)^{6k}}=\frac{(2\varpi_{(6)})^{6k}}{(6k)!}M_k.$$

The last formula is obtained from the power series expansion of

$$\wp(z;\varpi_{(6)},\varpi_{(6)}\rho)$$

with respect to z.

6.2. As for Proposition 3, Professor Bruce C. Berndt has kindly informed me* that the more general formula holds:

$$\sum_{m=1}^{\infty}\frac{(2m-1)^{4N+1}}{e^{(2m-1)\pi}+1}=\frac{1}{4}(2^{4N+1}-1)\frac{B_{4N+2}}{2N+1}, \qquad N\geqq 0, \tag{6.7}$$

which first appeared in Glaisher (Mess. Math. 18, (1889), 1-84) and he recently found some new reciprocity theorems and several identities which contains (6.7).

Now as in (5.0), Hurwitz number E_{2k} is defined by

$$\sum_{(m,n)\in Z^2}{}'\frac{1}{(m+n\sqrt{-1})^{2k}}=\frac{(2\varpi_{(4)})^{2k}}{(2k)!}E_{2k}.$$

Then

$$E_{2k}=0 \quad \text{for odd } k.$$

* A letter dated on November 23, 1977.

(Hurwitz used the notation E_k for our E_{4k}.)

Consider

$$g_k^{(1,1)} = g_k^{(1,1)}(0, 0; 1, \sqrt{-1}) = \sum_{\substack{m\equiv 1 \\ n\equiv 1}(\bmod 2)} \frac{1}{(m+n\sqrt{-1})^k}$$

Every term of the right hand side of $g_k^{(1,1)}$ in Theorem 6 contains $D_{1,1}$ for odd k and $D_{1,1}=0$. Hence

$$g_k^{(1,1)}=0 \quad \text{for odd } k .$$

Also we can easily show

$$g_{2k}^{(1,1)}=0 \quad \text{for odd } k .$$

Since $C_{1,1}=0$, we have, by Theorem 5, (I),

$$g_4^{(1,1)} = \frac{(2\varpi_{(4)})^4}{4!}\left(-\frac{1}{2^5}\right)$$

and inductively

$$\frac{g_{4k}^{(1,1)}}{\varpi_{(4)}^{4k}} \in \boldsymbol{Q} .$$

We define rational numbers $E_{2k}^{(1,1)}$ by

(6.8)
$$g_{2k}^{(1,1)} = \frac{(2\varpi_{(4)})^{2k}}{(2k)!} E_{2k}^{(1,1)} .$$

$E_{2k}^{(1,1)}$ is to be called as "2-division Hurwitz number" and we have

$$E_{2k}^{(1,1)}=0 \quad \text{for odd } k .$$

For example,

$$E_4^{(1,1)} = -\frac{1}{2^5}, \quad E_8^{(1,1)} = \frac{3^2}{2^9}, \quad E_{12}^{(1,1)} = -\frac{3^4\cdot 7}{2^{13}} .$$

We shall generalize (6.7) to and prove the following

THEOREM 11.

$$\sum_{m=1}^{\infty} \frac{(2m-1)^{2k-1}}{e^{(2m-1)\pi}+1} = \frac{3\cdot 2^{2k-3}E_{2k} - 4\cdot 2^{2k-3}E_{2k}^{(1,1)}}{k}\left(\frac{\varpi_{(4)}}{\pi}\right)^{2k} + \frac{1}{4}(2^{2k-1}-1)\frac{B_{2k}}{k}, \quad k\geqq 1 .$$

PROOF. For short, put

$$G_{2k}(\tau)=(2k)!^{-1}S_{2k}(1,\tau;0,0)=\sum_{(m,n)\in Z^2}{}' \frac{1}{(m+n\tau)^{2k}}.$$

Put

$$A_j(\tau)=\sum_{n=1}^{\infty}\frac{n^j}{e^{-2\pi\sqrt{-1}n\tau}-1}$$

and

$$F_j(\tau)=\sum_{n=1}^{\infty}\frac{(2n-1)^j}{e^{-2\pi\sqrt{-1}(2n-1)\tau}+1}.$$

Then by (6.5), we have

$$A_j(\tau)=(G_{2k}(\tau)-2\zeta(2k))\frac{(2k-1)!}{2(2\pi\sqrt{-1})^{2k}} \tag{6.9}$$

with $j=2k-1$ and we can easily show

$$F_j(\tau)=A_j(\tau)-2(2^{j-1}+1)A_j(2\tau)+2^{j+1}A_j(4\tau). \tag{6.10}$$

Since

$$G_{2k}\left(\frac{1}{2}\sqrt{-1}\right)=(-4)^k G_{2k}(2\sqrt{-1})$$

holds, we can derive, by (6.9) and (6.10),

$$F_{2k-1}\left(\frac{1}{2}\sqrt{-1}\right)=\frac{(2k-1)!}{2(2\pi\sqrt{-1})^{2k}}\{((-1)^k+1)2^{2k}G_{2k}(2\sqrt{-1}) \\ -2(2^{2k-2}+1)G_{2k}(\sqrt{-1})+(2-2^{2k})\zeta(2k)\}. \tag{6.11}$$

Note that

$$G_{2k}(\sqrt{-1})=0 \quad \text{for odd } k.$$

Hence for $k=2N+1$, $N>0$, follows

$$F_{4N+1}\left(\frac{1}{2}\sqrt{-1}\right)=\frac{(4N+1)!}{2(2\pi\sqrt{-1})^{4N+2}}(2-2^{4N+2})\zeta(4N+2)$$

from (6.11). This, together with Proposition 3, shows Theorem 11 for $k=2N+1$, $N\geqq 0$.

Now consider the case $k=2N$. Then (6.11) becomes

$$F_{4N-1}\left(\frac{1}{2}\sqrt{-1}\right)=\frac{(4N-1)!}{2(2\pi\sqrt{-1})^{4N}}\{2^{4N+1}G_{4N}(2\sqrt{-1})$$
$$-2(2^{4N-2}+1)G_{4N}(\sqrt{-1})+(2-2^{4N})\zeta(4N)\}\,. \tag{6.12}$$

Hence the problem is to compute $G_{4N}(2\sqrt{-1})$.

We can represent $G_{4N}(2\sqrt{-1})$ as

$$G_{4N}(2\sqrt{-1})=2^{-4N}G_{4N}(\sqrt{-1})+g_{4N}^{(1,0)}(0,\,0;\,1,\,\sqrt{-1})\,.$$

Observing that $g_{4N}^{(1,0)}=g_{4N}^{(0,1)}$ holds, we have

$$G_{4N}(\sqrt{-1})=2^{-4N}G_{4N}(\sqrt{-1})+2g_{4N}^{(1,0)}+g_{4N}^{(1,1)}\,.$$

Then it follows from (6.8),

$$G_{4N}(2\sqrt{-1})=\frac{(2\varpi_{(4)})^{4N}}{2(4N)!}((1+2^{-4N})E_{4N}-E_{4N}^{(1,1)})\,.$$

Insert this to (6.12). Then a straightforward calculation gives our Theorem 11, for $k=2N$, $N>1$.

References

[1] B. C. BERNDT, Modular transformations and generalizations of several formulas of Ramanujan, The Rocky Mountain J. of Math., **7** (1977), 147-189.

[2] R. M. DAMERELL, L-functions of elliptic curves with complex multiplication I, Acta Arith., (1970) 287-301.

[3] L. DIRICHLET, Sur la covergence des séries trigonometriques qui servent a représenter une fonction arbitraire entre des limites donnees, Werke, 117-132.

[4] G. EISENSTEIN, Beiträge zur Theorie der elliptischen Functionen, Werke I, 299-478.

[5] G. H. HARDY, A formula of Ramanujan, J. London Math. Soc., **3** (1928), 238-240.

[6] H. HASSE, Neue Bergründung der komplexen Multiplikation I, Crelle, **157** (1927), 115-139.

[7] G. HERGLOTZ, Über die Entwicklungskoeffizienten der Weierstrassschen $\wp$-Funktion, Sitzung. Berichte, Leipzig, (1922), 269-289.

[8] A. HURWITZ, Über die Entwicklungskoeffizienten der lemiskatischen Funktionen, M.A., **51** (1899), 342-373.

[9] C. JORDAN, Cours d'Analyse II, Chapt. VII.

[10] L. KRONECKER, Zur Theorie der elliptischen Funktionen, Werke IV, 311-318, 347-495, V, 3-184.

[11] L. KRONECKER, Bemerkungen über die Darstellung von Reihen durch Integrale, Werke V, 329-342.

[12] S. RAMANUJAN, Notebooks of Srinivasa Ramanujan, Tata, Bombay, 1957.

[13] C. L. SIEGEL, Lectures on advanced analytic number theory, Tata, Bombay, 1964.

[14] H. WEBER, Lehrbuch der Algebra III, Chelsea (Originally 1908).

[15] A. WEIL, Elliptic functions according to Eisenstein and Kronecker, Springer, Ergebnisse, 1976.

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