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On Singularities of the Harish-Chandra Expansion of the Eisenstein Integral on Spin(4,1)

Masaichi MAMIUDA

Waseda University

Introduction.

Let G be the universal covering group of De Sitter group and $G=KAN$ be its Iwasawa decomposition. Let MAN be a minimal parabolic subgroup of G. We consider the Eisenstein integral for MAN which is defined as follows;

$$(1) \qquad E(s, v, x)=\int_K e^{(s-3/2)t(xk)}\tau_1(k(xk))v\tau_2(k^{-1})dk\ , \qquad \text{for} \quad s\in \boldsymbol{C},\ x\in G\ ,$$

where τ_i are irreducible unitary representations of K on V_i $(i=1, 2)$ and $v\in \mathscr{V}_M=\{v;$ a linear endomorphism of V_2 into V_1 with $\tau_1(m)v=v\tau_2(m)$ for all $m\in M\}$ (see, § 3, [3], [5], [15]). Then Eisenstein integral $E(s, v, a_t)$ has the series expansion on a Weyl chamber A^+, which is divided into two parts associated with the Weyl group W of the pair (G, A);

$$(2) \qquad E(s, v, a_t)=E(s, t)C_1(s)v+E(-s, t)C_W(s)v\ , \qquad \text{for} \quad t>0\ ,$$

where $E(s, t)=e^{(s-3/2)t}\sum_{k=0}^{\infty}A_k(s)e^{-kt}$. (See, Harish-Chandra [3].) The eigenvalues of Casimir operator on G is parametrized by (s, v) and the function $x\mapsto E(s, v, x)$ is an eigenfunction corresponding to (s, v). By the change of variable $y=e^{-t}$, the differential equation for $E(s, v, a_t)$ is transformed to an ordinary differential equation with a regular singular point at $y=0$. The formula (2) gives the series expansion of the solution $E(s, v, a_t)$ around the regular singular point $y=0$. As the classical theory of differential equations teaches us, the coefficients $A_k(s)$ $(k\in \boldsymbol{Z})$ satisfies a certain recurrence formula with respect to k and they are not well-defined for all $s\in \boldsymbol{C}$. So, the formula (2) is valid on a certain open dense connected subset $\mathscr{O}(\tau_1, \tau_2)$ of $\boldsymbol{C}$. But the function $s\mapsto E(s, t)$ can be extended to a meromorphic function on $\boldsymbol{C}$ with values in the space of all linear endomorphisms on $\mathscr{V}_M$ (see, § 3). In this paper, we shall say

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the formula (2) by the name of the Harish-Chandra expansion, and by the singularities of the function $s \mapsto E(s, t)$ we mean the singularities of the Harish-Chandra expansion.

The purpose of this paper is to determine the singularities of the Harish-Chandra expansion and to enumerate its singular points. So, we shall prove the next theorem (see, § 3. Theorem 3.4).

THEOREM. *For fixed $n \in \hat{M}(\tau_1, \tau_2)$ and $t>0$, the function $s \mapsto E(s, n, t) = E(s, t)v_n$ can be extended to a $\mathscr{V}_M$-valued meromorphic function on $\boldsymbol{C}$ and its singularities are all simple poles. Moreover, $E(s, n, t)$ is holomorphic on the complement of these points below;*

1) $s=k$, $k>0$ *and* $n-k \in \boldsymbol{Z}$,
2) $s=k+1/2$, $k \in \hat{M}(\tau_1, \tau_2)$ *and* $n<k$,
3) $s=k+1/2$, $0 \leq k < \min(|n_1''|, |n_2''|)$ *and* $n-k \in \boldsymbol{Z}$,
4) $s=-(k+1/2)$, $k \in \hat{M}(\tau_1, \tau_2)$ *and* $k<n$.

This theorem plays an important role in the proof of the Paley-Wiener type theorem on G ([9]). In [6], Helgason has proved the Paley-Wiener type theorem on a symmetric space $X=G/K$ for arbitrary symmetric pair (G, K) of non-compact type. His proof is based on the absence of these singular points on a certain half space for right K-invariant eigenfunctions (in our case, these correspond to the case where τ_2=trivial). In general, we can't expect this property, and so that we need more delicate and deep considerations about $E(s, t)$. Especially we need closer studies of $E(s, t)C_1(s)$.

The proof of Theorem 3.4 is long and comlicated, so we shall prove this in § 6 after establishing some preliminary results. In § 5, we determine the explicit formula of c-functions which appears in the Harish-Chandra expansion, in terms of the Gamma functions. Using this formula and Theorem 3.4, we reduce the proof of Theorem 3.1 to the case when $\mathrm{Re}(s) \leq 0$.

In § 4, we give the more precise formula of the coefficients $A_k(s)$ in the Harish-Chandra expansion. From Lemma 4.4 and the related results, we conclude that $E(s, n, t)$ is meromorphic with respect to the variable s (that is independent on $t>0$). The fundamental idea of the proof is to consider the first term of the Laurent expansion of $E(s, n, t)$ and study this as a τ-spherical eigenfunction on G^+. To do this, we need the results of § 2 and the formulas of § 4. In particular, Proposition 2.5 is the key result to prove Theorem 3.4. In this proposition, we establish the relation between the asymptotic behaviour of τ-spherical eigenfunctions and its eigenvalues. Proposition 2.5 is an extended analogue of

Trombi-Varadarajan's theorem (Theorem 7.1 of [13]).

§ 1. Notations and preliminaries.

1.1. Description of the universal convering group of De Sitter group.

As usual, $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Z}$ denote the set of real numbers, complex numbers and integers, respectively. Let $\boldsymbol{H}$ be the quaternion field and $(1, i, j, k)$ be the standard basis of $\boldsymbol{H}$ over $\boldsymbol{R}$. We write $\bar{x}$ for the conjugate quaternion of x and $|x|$ for the norm of x. Let $M_2(\boldsymbol{H})$ be the set of all 2×2 matrices with coefficients in $\boldsymbol{H}$. We write g^* for the adjoint matrix of $g \in M_2(\boldsymbol{H})$, i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$.

We use the same notations and definitions as in Chap. II, § 1 of [11]. Then the universal covering group G of De Sitter group is described as follows;

$$G=\{g \in M_2(\boldsymbol{H});\ g^*wg=w\}\ , \qquad \text{where} \quad w=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

And subgroups K, M, A, N, $\bar{N}$ of G are described as follows;

$$K=\left\{k(u, v)=\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix};\ u, v \in \boldsymbol{U}\right\}\ , \qquad \text{where} \quad \boldsymbol{U}=\{u \in \boldsymbol{H};\ |u|=1\}\ ,$$

$$M=\left\{m(u)=\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix};\ u \in \boldsymbol{U}\right\}\ ,$$

$$A=\left\{a_t=\begin{pmatrix} \mathrm{ch}t/2 & \mathrm{sh}t/2 \\ \mathrm{sh}t/2 & \mathrm{ch}t/2 \end{pmatrix};\ t \in \boldsymbol{R}\right\}\ ,$$

$$N=\left\{n_x=\begin{pmatrix} 1-x/2 & x/2 \\ -x/2 & 1+x/2 \end{pmatrix};\ x \in \boldsymbol{I}\right\}\ , \qquad \text{where} \quad \boldsymbol{I}=\{x \in \boldsymbol{H};\ \bar{x}=-x\}\ ,$$

$$\bar{N}=\left\{\bar{n}_y=\begin{pmatrix} 1-y/2 & -y/2 \\ y/2 & 1+y/2 \end{pmatrix};\ y \in \boldsymbol{I}\right\}\ .$$

So, K is a maximal compact subgroup of G and $G=KAN$ is an Iwasawa decomposition of G. Moreover, M is the centralizer of A in K and MAN is a minimal parabolic subgroup of G. The corresponding decomposition of an element g in $G=KAN$ is denoted by $g=k(g)a_{t(g)}n_{x(g)}$, and sometimes we write $k(u(g), v(g))$ for $k(g)$. When $g=\bar{n}_y$, we write $k(y)$, $u(y)$, $v(y)$, $t(y)$ for $k(\bar{n}_y)$, $u(\bar{n}_y)$, $v(\bar{n}_y)$, $t(\bar{n}_y)$, respectively.

Let $\mathfrak{g}=\{x \in M_2(\boldsymbol{H});\ X^*w+wX=0\}$, then we may regard $\mathfrak{g}$ as the Lie algebra of G, i.e., $(Xf)(g)=(d/dt)(f(g\exp(tX)))|_{t=0}$ for $f \in C^\infty(G)$ and $X \in \mathfrak{g}$.

The corresponding Lie algebras of subgroups $K, M, A, N, \bar{N}$ are denoted by $\mathfrak{k}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}, \bar{\mathfrak{n}}$ respectively. Then these are described as follows;

$$\mathfrak{k}=\left\{\begin{pmatrix} x' & 0 \\ 0 & x'' \end{pmatrix}; x', x'' \in \boldsymbol{I}\right\},$$

$$\mathfrak{m}=\left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}; x \in \boldsymbol{I}\right\},$$

$$\mathfrak{a}=\left\{\begin{pmatrix} 0 & t/2 \\ t/2 & 0 \end{pmatrix}; t \in \boldsymbol{R}\right\},$$

$$\mathfrak{n}=\left\{\begin{pmatrix} -x & x \\ -x & x \end{pmatrix}; x \in \boldsymbol{I}\right\},$$

$$\bar{\mathfrak{n}}=\left\{\begin{pmatrix} -y & -y \\ y & y \end{pmatrix}; y \in \boldsymbol{I}\right\}.$$

$1=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $w=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ constitute the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{a})$.

Let $\mathfrak{g}_c$ denote the complexification of $\mathfrak{g}$ and $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}_c$. By $\mathfrak{K}, \mathfrak{M}, \mathfrak{A}$ we denote the subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{k})$, $(1, \mathfrak{m})$, $(1, \mathfrak{a})$, respectively. Then $\mathfrak{K}, \mathfrak{M}, \mathfrak{A}$ may be regarded as the universal enveloping algebra of $\mathfrak{k}, \mathfrak{m}, \mathfrak{a}$, respectively. In general for a Lie group G_0, as usual, we regard an element D of the corresponding universal enveloping algebra $\mathfrak{G}_0$ of G_0 as a left invariant differential operator on G_0 and also as a differential operator on arbitrary open subset of G_0.

1.2. The irreducible unitary representations of M and K.

Let U, I be the subset of $\boldsymbol{H}$ which are introduced in 1.1., then U is a Lie group with a Lie algebra I and isomorphic to $SU(2)$ (cf. Chap. II, § 1 of [11]). Hence, for each nonnegative half integer n there exists the irreducible unitary represetation σ^n on V^n which is realized in Chap. III of [14]. Moreover, any irreducible representation σ of U is equivalent to σ^n for some half integer $n \geq 0$ and any two representations σ^n and $\sigma^{n'}$ $(n \neq n')$ are inequivalent. Since K is isomorphic to the group $U \times U$ and M is the diagonal subgroup of K, the sets of equivalent classes of irreducible unitary representations $\hat{M}, \hat{K}$ of M, K are parametrized respectively as follows;

$$\hat{M}=\{n; 2n \in \boldsymbol{Z} \text{ and } n \geq 0\},$$

$$\hat{K}=\{(n', n''); 2n', 2n'', n'-n'' \in \boldsymbol{Z} \text{ and } |n''| \leq n'\}.$$

More precisely,

for $\quad n \in \hat{M},\ m=m(u) \in M \qquad \sigma^n(m)=\sigma^n(u)$,

and

for $\quad (n', n'') \in \hat{K}\ ,\quad k=k(u, v) \in K\ ,\qquad \tau^{n',n''}(k)=\sigma^{n_1}(u) \otimes \tau^{n_2}(v)$,

where

$$2n_1=n'+n''\ ,\quad 2n_2=n'-n''\ .$$

We use the same symbols σ^n, $\tau^{n',n''}$ for the corresponding representations of $\mathfrak{M}$, $\mathfrak{K}$, respectively. Let

$$U_1=\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\quad U_2=\frac{1}{2}\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},\quad U_3=\frac{1}{2}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix},$$

$$U_1'=\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad U_2'=\frac{1}{2}\begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix},\quad U_3'=\frac{1}{2}\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

Then $\{U_i;\ i=1, 2, 3\}$ forms a basis of $\mathfrak{m}$ and $\{U_i, U_i';\ i=1, 2, 3\}$ forms a basis of $\mathfrak{k}$. Furthermore,

$$\omega_{\mathfrak{m}}=-\sum_1^3 U_i^2\ ,\qquad \omega_{\mathfrak{k}}=\omega_{\mathfrak{m}}-\sum_1^3 U_i''^2$$

are Casimir operators of $\mathfrak{M}$, $\mathfrak{K}$, respectively. They satisfy

$$\sigma^n(\omega_{\mathfrak{m}})=n(n+1)\sigma^n(1) \quad \text{and} \quad \tau^{n',n''}(\omega_{\mathfrak{k}})=2(n_1(n_1+1)+n_2(n_2+1))\tau^{n',n''}(1)$$

(cf. Chap. II, § 1 of [11] and Chap. III of [14]).

Let V_i be a Hilbert space with a Hermitian form $(\ ,\ )_i$ $(i=1, 2)$. For a bounded linear operator L on V_1 into V_2, let L^* denote the adjoint operator of L, i.e.,

$$(L^*v_2, v_1)_1=(v_2, Lv_1)_2 \qquad \text{for any} \quad v_i \in V_i\ (i=1, 2)\ .$$

For $\tau=\tau^{n',n''}((n', n'') \in \hat{K})$ and $\sigma^n(n \in \hat{M})$, by $\mathrm{HOM}_M(V^\tau, V^n)$, we denote the space of all linear mappings L on V^τ into V^n such that $\sigma^n(m)L=L\tau(m)$ for all $m \in M$, where V^τ is the representation space of τ.

In our case, the dimension $d(\tau, n)$ of $\mathrm{HOM}_M(V^\tau, V^n)$ equale to 0 or 1, because $\tau|_M=\sigma^{n_1} \otimes \sigma^{n_2}$ (equivalent). When $d(\tau, n)=1$, there exists an element $P_n(\tau)$ of $\mathrm{HOM}_M(V^\tau, V^n)$ such that $P_n(\tau)P_n(\tau)^*=\sigma^n(1)$. For each τ we put $\hat{M}(\tau)=\{n \in \hat{M};\ d(\tau, n)=1\}$, then we know that

$$\tau|_M=\sum_{n \in \hat{M}(\tau)} \sigma^n\ ,$$

and for $\tau=\tau^{n',n''}$

$$\hat{M}(\tau^{n',n''})=\{n \in \hat{M};\ |n''|\leq n \leq n' \text{ and } n'-n \in \boldsymbol{Z}\}$$

(see, § 8 of [14]).

§ 2. Spherical functions on G^+.

In this section we shall introduce certain function spaces of G^+ and study some properties of these function spaces.

Let $G^+=\{x \in G;\ x \notin K\}$, $A^+=\{a_t;\ t>0\}$, then G^+, A^+ are open submanifolds of G, A, respectively. M operates on the product manifold $K\times A^+\times K$ by $(k, a, k')m=(km, a, m^{-1}k')$ for $k, k' \in K$, $a \in A^+$ and $m \in M$. The following lemma is clear from Lemma 1.2 of [11].

LEMMA 2.1. *G^+ is diffeomorphic to the quotient manifold $(K\times A^+\times K)/M$.*

Let τ_1, τ_2 be two irreducible unitary representations of K on V_1, V_2 respectively. We write $\mathscr{V}$ for the space of all linear mappings of V_2 into V_1 and write $\mathscr{V}_M$ for the subspace of $\mathscr{V}$ consisting of all elements L such that $\tau_1(m)L=L\tau_2(m)$ for any $m \in M$.

DEFINITION. An infinitely differentiable function F on G^+ with values in $\mathscr{V}$ is said to be (τ_1, τ_2)-spherical (or simply, spherical) on G^+ if $F(k_1\times k_2)=\tau_1(k_1)F(x)\tau_2(k_2)$ for any $k_1, k_2 \in K$ and $x \in G^+$. Similarly, an infinitely differentiable function f on A^+ with values in $\mathscr{V}$ is said to be spherical on A^+ if $f(A^+)\subset \mathscr{V}_M$.

Fix a positive number r and let F, f be spherical functions on G^+, A^+ respectively. F (resp. f) is said to be type r if it satisfy the condition;

for each $D \in \mathfrak{G}$ (resp. $D \in \mathfrak{A}$) and any ε satisfying $0<\varepsilon<1$, there is a constant $c=c(D, F, \varepsilon)>0$ (resp. $c=c(D, f, \varepsilon)>0$) such that $|DF(k_1a_tk_2)|\leq ce^{-(r+3/2-\varepsilon)t}$ for $k_1, k_2 \in K$ and $t\geq 1$, (resp. $|Df(a_t)|\leq ce^{-(r+3/2-\varepsilon)t}$ for $t\geq 1$), where $|L|$ is the operator norm of $L \in \mathscr{V}$.

PROPOSITION 2.2. *Let F be an infinitely differentiable function on G^+ with values in $\mathscr{V}$ and let $f=F|A^+$ be the restriction of F to A^+. If F is spherical on G^+ then $f=F|A^+$ is spherical on A^+, and the mapping $F\mapsto f=F|A^+$ is a linear bijection. Furthermore, F is of type r if and only if $f=F|A^+$ is of type r.*

PROOF. Since $ma=am$ for $m \in M$, $a \in A$, the first half of assertions is easy from Lemma 2.1.

From Harish-Chandra [3] (also, Chap. 9, § 1 of [15]), we see that for each $D \in \mathfrak{G}$ there exist analytic functions $a_1, \cdots, a_n$ of $K \times A^+ \times K$ with values in $\mathrm{End}(\mathscr{V})$ and elements $D_1, \cdots, D_n$ of $\mathfrak{A}$ such that

$$c_0 = \sum_{i=1}^{n} \Big(\sup_{\substack{t \geq 1 \\ k, k' \in K}} |a_i(k, a_t, k')| \Big)$$

is bounded, and

$$(2.1) \qquad (DF)(ka_tk') = \sum_i a_i(k, a_t, k')(D_iF)(a_t) \qquad \text{for} \quad k, k' \in K, t > 0 ,$$

in particular, if $D \in \mathfrak{A}$, then

$$(2.2) \qquad (DF)(a_t) = (Df)(a_t) \qquad \text{for} \quad t > 0 \ (f = F|A^+) ,$$

where $\mathrm{End}(\mathscr{V})$ is the space of all linear endomorphisms on $\mathscr{V}$ and $|L|$ is the operator norm of $L \in \mathrm{End}(\mathscr{V})$.

If F is of type r, then

$$|(DF)(a_t)| \leq ce^{-(r+3/2-\varepsilon)t} \qquad \text{for} \quad t \geq 1 \text{ and } D \in \mathfrak{A} .$$

Hence, from (2.2) we have the estimate

$$|(Df)(a_t)| \leq ce^{-(r+3/2-\varepsilon)t} .$$

Conversely we assume that f is of type r. For each $D \in \mathfrak{G}$, let $a_1, \cdots, a_n$, $D_1, \cdots, D_n$ be the same as above and select positive constants $c_1, \cdots, c_n$ such that

$$|(D_if)(a_t)| \leq c_ie^{-(r+3/2-\varepsilon)t} \qquad \text{for} \quad t \geq 1 \ (i = 1, \cdots, n) ,$$

then from (2.1) we get

$$|(DF)(ka_tk')| \leq ce^{-(r+3/2-\varepsilon)t} \qquad \text{for} \quad k, k' \in K \text{ and } t \geq 1 ,$$

where $c = \max(c_0c_1, \cdots, c_0c_n)$. Thus, Proposition 2.2 is proved.

LEMMA 2.3. *Let V_0 be a complex Banach space with norm $|\cdot|$ and $\{a_n; n = 0, 1, 2, \cdots\}$ be a sequence of V_0. Let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for $s \in \boldsymbol{C}$ and $n \in \boldsymbol{Z}$, where Γ is the usual Gamma function. Suppose that a_n satisfy the inequality $|a_n| \leq |a_0|(c)_n/n!$ $(n = 0, 1, 2, \cdots)$ for some constant $c > 0$, then the series*

$$\sum_{n=0}^{\infty} a_n e^{-nt}$$

is absolutely convergent on $(0, \infty)$ and this convergence is uniform on $[r, \infty]$ for each $r > 0$. Moreover, when we write $f(t) = \sum_{n=0}^{\infty} a_n e^{-nt}$ $(t > 0)$, *for each integer* $i \geq 0$ *there is a constant* $c_i > 0$ *such that*

$$\left|\left(\frac{d}{dt}\right)^i f(t)\right| \leq c_i(1-e^{-t})^{-(c+i)} \qquad \text{on} \quad (0, \infty) .$$

PROOF. Since

$$\left|\left(\frac{d}{dt}\right)^i (a_n e^{-nt})\right| \leq |a_0| \left|\left(\frac{d}{dt}\right)^i (((c)_n/n!)e^{-nt})\right| ,$$

it will be sufficient to consider the case when $V_0 = \boldsymbol{C}$ and $a_n = (c)_n/n!$. In this case, our lemma is easy from the classical argument for the formula;

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} e^{-nt} = (1-e^{-t})^{-c} \qquad \text{on} \quad (0, \infty) .$$

By Leibniz formula and Lemma 2.3, we have the next proposition.

PROPOSITION 2.4. *Retain above notations and assumptions. For* $s \in \boldsymbol{C}$, *let* $f_s(t) = \exp(-(s+3/2)t)f(t)$, *then for each integer* $i \geq 0$ *there is a constant* $c_i > 0$ *such that*

$$\left|\left(\frac{d}{dt}\right)^i f_s(t)\right| \leq c_i\left(|s| + \frac{3}{2}\right)^i (1-e^{-t})^{-(c+i)} e^{-(r+3/2)t} , \qquad \text{for} \quad t > 0 ,$$

where $r = \mathrm{Re}(s)$. *In particular when* $V_0 = \mathscr{V}_M$ *and* $t > 0$, $f_s(a_t) = f_s(t)$ *is a spherical function on* A^+ *of type* r.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$ and U_1. Let $\mathfrak{Z}$ denote the center of $\mathfrak{G}$ and γ be the canonical isomorphism of $\mathfrak{Z}$ into the algebra of all polynomial functions on $\mathfrak{h}_c^*$ which is the complex dual space of $\mathfrak{h}_c$ (cf. Harish-Chandra [4]).

For each $s \in \boldsymbol{C}$ and $n \in \hat{M}$, let $\lambda_{s,n}$ be the linear function on $\mathfrak{h}_c$ with

$$\lambda_{s,n}(U_1) = -(-1)^{1/2}\left(n + \frac{1}{2}\right) , \qquad \lambda_{s,n}(H) = s , \quad \text{where} \quad H = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

So, we define the algebra homomorphism $\chi_{s,n}$ of $\mathfrak{Z}$ into $\boldsymbol{C}$ by $\chi_{s,n}(z) = \gamma(z)(\lambda_{s,n})$ for $z \in \mathfrak{Z}$.

Now we shall define certain function spaces consisting of spherical functions,

$$C^\infty(G^+, \tau) = \{f ; f \text{ is a } \tau = (\tau_1, \tau_2)\text{-spherical on } G^+\} .$$

For $s \in \boldsymbol{C}$, $n \in \hat{M}$ and $r > 0$, we define

$$C^{\infty}_{s,n}(G^+, \tau)=\{f\in C^{\infty}(G^+, \tau);\ zf=\chi_{s,n}(z)f \text{ for all } z\in\mathfrak{Z}\}$$

and

$$C^{\infty}_{s,n}(G^+, \tau, r)=\{f\in C^{\infty}_{s,n}(G^+, \tau);\ f \text{ is of type } r\}\ .$$

The following proposition, which plays an important role in our proof of the main Theorem 3.4, had proven in [10] under more general situations. The methods used in that paper are an extention of those which are used to prove the Theorem 7.1 in Trombi-Varadarajan [13].

PROPOSITION 2.5. *Retain above notations. Assume that* $s\in \boldsymbol{R}$ *satisfies* $s\neq\pm(n+1/2)$ $(n\in\hat{M}(\tau_1, \tau_2))$, *then for a given* $F\in C^{\infty}_{s,n}(G^+, \tau, r)$ $(r>0)$ $F_r=\lim_{t\to+\infty}e^{-(r+3/2)t}F(a_t)$ *exists and* $F_r\in\mathscr{V}_M$. *Moreover, if* $F_r\neq 0$ *then* r *equals one of numbers;* s, $-s$, $n+1/2$, $-(n+1/2)$.

§ 3. The Harish-Chandra expansions.

We use the notations and the definitions introduced in § 1 and § 2, and also use the notation $\mathrm{End}(V_0)$ for the space of all linear endomorphisms on a linear space V_0.

3.1. Definition of the coefficients A_k and the Theorem of Harish-Chandra.

Let

$$Y_1=2^{-1/2}(U_2'+(-1)^{1/2}U_3'),\quad Y_0=(-1)^{1/2}U_1'$$

and

$$Y_{-1}=2^{-1/2}(-U_2'+(-1)^{1/2}U_3')\ .$$

Then Y_1, Y_0, Y_{-1} are elements of $\mathfrak{k}_c$ which satisfy

$$\sum_{-1}^{1}(Y_j)^2=-\sum_{1}^{3}(U_i')^2=\omega_{\mathfrak{k}}-\omega_{\mathfrak{m}}\ .$$

Let τ_1, τ_2, $\mathscr{V}$, $\mathscr{V}_M$ be the same as in § 1 and § 2.

(3.1) For $v\in\mathscr{V}$, put

$$L_0v=\tau_1(\omega_{\mathfrak{m}})v=v\tau_2(\omega_{\mathfrak{m}})\ ,$$

$$L_1v=\sum_{-1}^{1}\tau_1(Y_j)v\tau_2(Y_{-j})=-\sum_{1}^{3}\tau_1(U_i')v\tau_2(U_i')\ ,$$

$$L_2v=\tau_1(\omega_{\mathfrak{k}}-\omega_{\mathfrak{m}})v+v\tau_2(\omega_{\mathfrak{k}}-\omega_{\mathfrak{m}})\ .$$

Then, from the definition of $\mathscr{V}_M$ and simple calculations, $\mathscr{V}_M$ is stable under the linear endomorphisms L_0, L_1, L_2. Hence, we may conclude that $L_i\in\mathrm{End}(\mathscr{V}_M)$ $(i=0, 1, 2)$.

Now we shall define the rational functions A_k $(k \in \boldsymbol{Z})$ on $\boldsymbol{C}$ with values in End$(\mathscr{V}_M)$ as follows;

(3.2) $A_k=0$ if $k<0$ and $A_0=$the identity endomorphism on $\mathscr{V}_M$,

(3.3) for $k>0$, A_k are defined by the recurrence formula below ;

$$\begin{aligned}(2ks-k^2)A_k(s)-L_0A_k(s)+A_k(s)L_0 &= 6\sum_{j\geq 1}\left(s-\frac{3}{2}+2j-k\right)A_{k-2j}(s) \\ &+4\sum_{j\geq 1}(2j-1)L_1A_{k-(2j-1)}(s) \\ &-4\sum_{j\geq 1} jL_2A_{k-2j}(s) .\end{aligned}$$

Let $E(s, v, x)$ denote the function below;

$$E(s, v, x)=\int_K \exp\left(\left(s-\frac{3}{2}\right)t(xk)\right)\tau_1(k(xk))v\tau_2(k^{-1})dk , \tag{3.4}$$

for $s \in \boldsymbol{C}$, $v \in \mathscr{V}_M$ and $x \in G$, where dk is the normalized Haar measure on K. Then the function $x \mapsto E(s, v, x)$ is a real analytic function on G with values in End$(\mathscr{V})$ and (τ_1, τ_2)-spherical on G. Moreover, the function $s \mapsto E(s, v, x)$ is holomorphic on $\boldsymbol{C}$ and for each $n \in \hat{M}(\tau_1, \tau_2)$, the function $E_{s,n}(x)=E(s, v_n, x)$ belongs to $C^{\infty}_{s,n}(G^+, \tau)$ (cf. Chap. 9 of [15], 3.2 below for v_n). The following theorem have been proved by Harish-Chandra (cf. [3], also Chap. 9 of [15]).

THEOREM 3.1 (Harish-Chandra). *There is an open connected dense subset $\mathscr{O}(\tau_1, \tau_2)$ in $\boldsymbol{C}$ and the meromorphic functions C_1, C_w on $\boldsymbol{C}$ with values in* End$(\mathscr{V}_M)$ *having the following properties;*

1) *The complement of $\mathscr{O}(\tau_1, \tau_2)$ in $\boldsymbol{C}$ is a discrete set and $A_k(k \in \boldsymbol{Z})$, C_1, C_w are all holomorphic on $\mathscr{O}(\tau_1, \tau_2)$.*

2) *Fix any compact subset B of $\mathscr{O}(\tau_1, \tau_2)$ and positive number r. Then, for each integers i, $j\geq 0$, the series*

$$\sum_{k\geq 0}\left(\frac{\partial}{\partial s}\right)^i\left(\frac{\partial}{\partial t}\right)^j(E_k(s, t)) \qquad (where\ \ E_k(s, t)=A_k(s)e^{-kt})$$

is uniformly convergent on $B\times[r, \infty)$.

3) *Put $E(s, t)=e^{(s-3/2)t}\sum_{k\geq 0}^{\infty}E_k(s, t)$, then the following equality is valid,*

$$E(s, v, a_t)=E(s, t)C_1(s)v+E(s, t)C_w(s)v , \tag{3.5}$$
$$for\ s \in \mathscr{O}(\tau_1, \tau_2),\ v \in \mathscr{V}_M \ and\ t>0 .$$

3.2. The statement of the main theorem.

Let $\hat{M}(\tau_1, \tau_2)$, $P_n(\tau_i)$ $(i=1, 2)$ be the same as in § 1. When $\hat{M}(\tau_1, \tau_2)$ is not empty, we put

$$v_n = P_n(\tau_1)^* P_n(\tau_2) \qquad \text{for} \quad n \in \hat{M}(\tau_1, \tau_2) .$$

The following Lemma is a simple application of the Schur's lemma.

LEMMA 3.2. *$\mathscr{V}_M$ contains non-zero elements if and only if $\hat{M}(\tau_1, \tau_2)$ is not empty. When $\hat{M}(\tau_1, \tau_2)$ is not empty, $\{v_n; n \in \hat{M}(\tau_1, \tau_2)\}$ forms a basis of $\mathscr{V}_M$.*

PROOF. Firstly, we note the following facts (cf. Chap. III, § 8 of [14]);

Let $P_{n,i} = P_n(\tau_i)^* P_n(\tau_i)$ $(n \in \hat{M}(\tau_i),\ i=1, 2)$, then $P_{n,i};\ n \in \hat{M}(\tau_i)$ are projections of V_i and satisfy the equalities

$$P_{n,i} P_{n',i} = \delta_{n,n'} P_{n,i} ,$$
$$\sum_{n \in \hat{M}(\tau_i)} P_{n,i} = \text{the identity endomorphism on } V_i.$$

Moreover, the subspaces $V_i^n = P_{n,i} V_i;\ n \in \hat{M}(\tau_i)$ are all stable under the linear mappings $\tau_i(m)$ $(m \in M)$. Let $\tau_i^n(m)$ be the restriction of $\tau_i(m)$ onto V_i^n $(m \in M,\ n \in \hat{M}(\tau_i))$, then τ_i^n is equivalent to σ^n as the representations of M. Now, we put

$$\mathscr{V}_M(n, n') = \{v' = P_{n,1} v P_{n',2};\ v \in \mathscr{V}_M \qquad \text{for} \quad n \in \hat{M}(\tau_1),\ n' \in \hat{M}(\tau_2)\} ,$$

then we may regard $v' \in \mathscr{V}_M(n, n')$ as a linear mapping of $V^{n'}$ into V^n such that $\sigma^n(m) v' = v' \sigma^{n'}(m)$ for all $m \in M$. Hence, by Schur's lemma, we have $\mathscr{V}_M(n, n') = \{0\}$ for $n \neq n'$. Since $P_n(\tau_i) P_n(\tau_i)^* = \sigma^n(1)$ $(n \in \hat{M}(\tau_i),\ i=1, 2)$, we obtain the relations

$$\begin{aligned} P_{n,1} v_n P_{n,2} &= P_n(\tau_1)^* P_n(\tau_1)(P_n(\tau_1)^* P_n(\tau_2)) P_n(\tau_2)^* P_n(\tau_2) \\ &= P_n(\tau_1)^* \sigma^n(1)^2 P_n(\tau_2) \\ &= P_n(\tau_1)^* P_n(\tau_2) = v_n , \qquad \text{for} \quad n \in \hat{M}(\tau_1, \tau_2) . \end{aligned}$$

Thus, $v_n \in \mathscr{V}_M(n, n)$ and

$$\begin{aligned} \operatorname{trace}(v_n v_n^*) &= \operatorname{trace}(P_n(\tau_1)^* P_n(\tau_2) P_n(\tau_2)^* P_n(\tau_1)) \\ &= \operatorname{trace}(P_{n,1}) = 2n+1 \neq 0 . \end{aligned}$$

Therefore, by Schur's lemma, we have $\mathscr{V}_M(n, n) = C v_n$, $v_n \neq 0$. But, it is easy to see that

$$\mathscr{V}_M = \sum_{n,n'} \mathscr{V}_M(n, n') \qquad \text{(direct sum)} .$$

Hence, we conclude our lemma.

From the results of § 2 and Theorem 3.1, we see that there exists an unique function $F_{s,n} \in C^\infty(G^+, \tau)$ such that $F_{s,n}(a_t)=E(s, t)v_n$ for $t>0$, $s \in \mathcal{O}(\tau_1, \tau_2)$. By the definitions of v_n, $P_n(\tau_i)$ $(i=1, 2)$, we have the equations

$$\tau_1(\omega_m)v_n = v_n\tau_2(\omega_m) = n(n+1)v_n \qquad \text{for} \quad n \in \hat{M}(\tau_1, \tau_2) .$$

Hence, from the arguments in 9.1.5. of [15], we have the next lemma.

LEMMA 3.3. *Let $F_{s,n}$ be the same as above, then $F_{s,n} \in C^\infty_{s,n}(G^+, \tau)$.*

In later sections, we write $E(s, n, t)$ for $E(s, t)v_n$. Let $\tau_i = \tau^{n'_i, n''_i}$ $(i=1, 2)$ be the same as in § 1 and assume that $\hat{M}(\tau_1, \tau_2)$ is not empty. Then we have the next result which is our main theorem.

THEOREM 3.4. *Fix $n \in \hat{M}(\tau_1, \tau_2)$ and $t>0$, then the function $s \mapsto E(s, n, t)=E(s, t)v_n$ can be extended to a $\mathcal{V}_M$-valued meromorphic function on $\boldsymbol{C}$ and its singularities are all simple poles.*

Moreover, $E(s, n, t)$ is holomorphic on the complement of these points below;

1) $s=k$, $k>0$ *and* $n-k \in \boldsymbol{Z}$,
2) $s=k+1/2$, $k \in \hat{M}(\tau_1, \tau_2)$ *and* $n<k$,
3) $s=k+1/2$, $0 \leq k < \min(|n''_1|, |n''_2|)$ *and* $n-k \in \boldsymbol{Z}$,
4) $s=-(k+1/2)$, $k \in \hat{M}(\tau_1, \tau_2)$ *and* $k<n$.

§ 4. Some properties of A_k.

In the first place, we introduce some notations. Let $I(n)=\{p;\, 2p \in \boldsymbol{Z},\ n-p \in \boldsymbol{Z}$ and $|p| \leq n\}$ for $n \in \hat{M}$ and $\{v^n_p;\, p \in I(n)\}$ be the orthonormal basis of V^n introduced in Chap. III of [14]. Let τ_i, V_i $(i=1, 2)$, $\mathcal{V}$, $\mathcal{V}_M$, etc., be the same as in § 1. By $(\ ,\ )_i$, we denote the positive definite Hermitian form on V_i such that $\tau_i(k)$ $(k \in K)$ are unitary endomorphism on V_i $(i=1, 2)$. For each $n \in \hat{M}(\tau_i)$ and $p \in I(n)$, we put $v^n_{p,i} = P_n(\tau_i)^* v^n_p$ $(i=1, 2)$. Then we know that $(v^n_{p,i}, v^n_{q,i})_i = \delta_{p,q}$ $(i=1, 2)$.

LEMMA 4.1. *For each $i=1, 2$ and $j=-1, 0, 1$ put*

$$c_{j,i}(n, p\colon n', q) = (\tau_i(Y_j)v^n_{p,i}, v^{n'}_{q,i}) \quad \text{for} \quad n, n' \in \hat{M}(\tau_i),\ p \in I(n),\ q \in I(n')$$

(see § 3 for Y_j). Then $(p-q+j)c_{j,i}(n, p\colon n', q)=0$.

PROOF. Let $U=(-1)^{1/2}U_1$. Then it is easy to see that

$$[U, Y_j] = jY_j \qquad (j=-1, 0, 1) .$$

Hence,

$$\tau_i(U)\tau_i(Y_j)-\tau_i(Y_j)\tau_i(U)-j\tau_i(Y_j)=0 .$$

But, $(\tau_i(U)v, v')_i=(v, \tau_i(U)v')_i$ for any $v, v' \in V_i$ and $\tau_i(U)v^n_{p,i}=pv^n_{p,i}$ for $n \in \hat{M}(\tau_i)$, $p \in I(n)$ $(i=1, 2)$ (cf. Chap. III of [14]). Thus,

$$(\tau_i(U)\tau_i(Y_j)v^n_{p,i}, v^{n'}_{q,i})_i=qc_{j,i}(n, p: n', q) ,$$
$$(\tau_i(Y_j)\tau_i(U)v^n_{p,i}, v^{n'}_{q,i})_i=pc_{j,i}(n, p: n', q) ,$$
$$(\tau_i(jY_j)v^n_{p,i}, v^{n'}_{q,i})_i=jc_{j,i}(n, p: n', q) .$$

Therefore,

$$(p-q+j)c_{j,i}(n, p: n', q)=0 .$$

Let (v, w) be the positive definite Hermitian form on $\mathscr{V}_M$ given by $(v, w)=\operatorname{trace}(vw^*)$ for $v, w \in \mathscr{V}_M$.

LEMMA 4.2. *Let L_0, L_1, L_2 be the same as in § 3 and assume that $\hat{M}(\tau_1, \tau_2)$ is not empty. Then, $L_0v_n=n(n+1)v_n$ and $L_2v_n=b_nv_n$ for $n \in \hat{M}(\tau_1, \tau_2)$, where b_n is a certain real number. Moreover, when we write $L_1v_n=\sum_{n'}a_{n,n'}v_{n'}$ (n' varies over the set $\hat{M}(\tau_1, \tau_2)$), we have $a_{n,n'}=0$ for $n, n' \in \hat{M}(\tau_1, \tau_2)$ with $|n-n'|>1$.*

PROOF. By definitions of v_n, L_0, L_2, we may reduce the assertions for L_0, L_2 to the calculations of the eigenvalues of ω_t, ω_m. But these eigenvalues are already given in § 1. Now we shall prove that

$$(*) \qquad (L_1v, w)=(v, L_1w) \qquad \text{for any} \quad v, w \in \mathscr{V}_M .$$

Since τ_i $(i=1, 2)$ are unitary representations, $\tau_i(Y_j)^*=\tau_i(Y_{-j})$ $(j=-1, 0, 1$ and $i=1, 2)$. Hence,

$$\begin{aligned} v(L_1w)^* &= \sum_{-1}^{1} v(\tau_1(Y_j)w\tau_2(Y_{-j}))^* \\ &= \sum_{-1}^{1} v(\tau_2(Y_j)w^*\tau_1(Y_{-j})) \\ &= \sum_{-1}^{1} v(\tau_2(Y_{-j})w^*\tau_1(Y_j)) . \end{aligned}$$

Thus,

$$\begin{aligned} (v, L_1w) &= \operatorname{trace}\Big(\sum_{-1}^{1} v(\tau_2(Y_{-j})w^*\tau_1(Y_j))\Big) \\ &= \operatorname{trace}\Big(\sum_{-1}^{1} (\tau_1(Y_j)v\tau_2(Y_{-j}))w^*\Big)=(L_1v, w) . \end{aligned}$$

Let us return to the proof of Lemma 4.2. It is easy to see that

$(v_n, v_{n'})=(2n+1)\delta_{n,n'}$. So, the assertion $(*)$ implies the formula

$$a_{n,n'}(2n'+1)=\overline{a_{n',n}}(2n+1)\ ,$$

where $\overline{a_{n',n}}$ is the conjugate complex number of $a_{n',n}$. Hence, we may consider only the case when $n'>n+1$. Since $v_n v^{n'}_{p,2}=P_n(\tau_1)^*P_n(\tau_2)P_n(\tau_2)^*v^{n'}_p=\delta_{n,n'}v^n_{p,1}$ (cf. Chap. III of [14]), we have from $L_1 v_n=\sum_n a_{n,n'}v_{n'}$ that $(L_1 v_n)(v^{n'}_{n',2})=a_{n,n'}v^{n'}_{n',1}$. By using $(v^{n'}_{n',1}, v^{n'}_{n',1})_1=1$,

$$\begin{aligned}a_{n,n'}&=((L_1 v_n)(v^{n'}_{n',2}),\ v^{n'}_{n',1})_1\\&=\sum_j(\tau_1(Y_j)v_n\tau_2(Y_{-j})v^{n'}_{n',2},\ v^{n'}_{n',1})_1\\&=\sum_{j,n'',p}c_{-j,2}(n',\ n'\colon n'',\ p)(\tau_1(Y_j)v^{n''}_{p,1},\ v^{n'}_{n',1})_1\delta_{n,n''}\\&=\sum_{j,p}c_{-j,2}(n',\ n'\colon n,\ p)c_{j,1}(n,\ p\colon n',\ n')\ ,\end{aligned}$$

where p varies on the set $I(n)$. Since $p+j\leq n+1<n'$, we have $p+j-n'\neq 0$ for $p\in I(n)$ and $j=-1, 0, 1$. Hence, we conclude from Lemma 4.1 that $c_{j,1}(n, p\colon n', n')=0$ for $p\in I(n)$ and $j=-1, 0, 1$. This implies $a_{n,n'}=0$ if $n'>n+1$.

Now we shall consider the matrix coefficients of $A_k(s)$ with respect to the basis $\{v_n\colon n\in\hat{M}(\tau_1, \tau_2)\}$ of $\mathscr{V}_M$. For $k\in\boldsymbol{Z}$ and $n, n'\in\hat{M}(\tau_1, \tau_2)$, let

$$A_k(s,\ n,\ n')=(2n'+1)^{-1}(A_k(s)v_n,\ v_{n'})\quad(s\in\mathscr{O}(\tau_1,\ \tau_2))\ . \tag{4.1}$$

Then $A_k(s, n, n')$ ($k\in\boldsymbol{Z}$ and $n, n'\in\hat{M}(\tau_1, \tau_2)$) are rational functions of s on $\boldsymbol{C}$ and

$$A_k(s)v_n=\sum_{n'\in\hat{M}(\tau_1,\tau_2)}A_k(s,\ n,\ n')v_{n'}\ .$$

Moreover, from the formula (3.3) and Lemma 4.2, we have

$$\begin{aligned}&(2sk-k^2+n(n+1)-n'(n'+1))A_k(s,\ n,\ n')\\&\quad=\sum_{j\geq 1}\Big(6\Big(s-\frac{3}{2}+2j-k\Big)-4jb_{n'}\Big)A_{k-2j}(s,\ n,\ n')\\&\quad\quad+\sum_{j\geq 1}(2j-1)\sum_p 4a_{p,n'}A_{k-(2j-1)}(s,\ n,\ p)\ ,\end{aligned} \tag{4.2}$$

where p varies on the set $\hat{M}(\tau_1, \tau_2, n')=\{p\in\hat{M}(\tau_1, \tau_2)\colon |n'-p|\leq 1\}$.

LEMMA 4.3. *For $k\in\boldsymbol{Z}$ and $n, n'\in\hat{M}(\tau_1, \tau_2)$ with $|n-n'|>k$, $A_k(s, n, n')=0$. Moreover, when $n'=n-k$ with $k>0$, we have the formula*

$$A_k(s,\ n,\ n-k)=\prod_{j=1}^{k}(2a_{n-k+j,n-k+j-1})\Big/\Big(k!\Big(s+\frac{1}{2}+n-k\Big)_k\Big)\ . \tag{4.3}$$

PROOF. When $k \leq 0$, Lemma 4.3 is clear from the definition of A_k. For $k>0$, we use the induction on k. Since $k-2j \leq k-2$ ($j=1, 2, \cdots$), $k-(2j-1) \leq k-2$ ($j=2, 3, \cdots$) and $|n-p| \geq |n-n'|-1$ ($p \in \hat{M}(\tau_1, \tau_2, n')$), we see from the induction hypothesis that $A_{k-2j}(s, n, n')=0$ ($j=1, 2, \cdots$), $A_{k-(2j-1)}(s, n, p)=0$ ($j=2, 3, \cdots$ and $p \in \hat{M}(\tau_1, \tau_2, n')$) if $|n-n'| \geq k$. Hence, when $|n-n'| \geq k$, we have from (4.2) that

$$(4.4) \qquad (2ks-k^2+n(n+1)-n'(n'+1))A_k(s, n, n') = 4 \sum_{p \in \hat{M}(\tau_1, \tau_2, n')} a_{p,n'} A_{k-1}(s, n, p) .$$

If $|n-n'|>k$, then $|n-p|>|n-n'|-1>k-1$ for all $p \in \hat{M}(\tau_1, \tau_2, n')$. Thus the induction hypothesis and (4.4) imply that

$$(2ks-k^2+n(n+1)-n'(n'+1))A_k(s, n, n')=0 \qquad \text{if} \quad |n-n'|>k .$$

Since $A_k(s, n, n')$ is a rational function, we obtain that $A_k(s, n, n')=0$ if $|n-n'|>k$. When $n'=n-k$ with $k>0$, we have $p=n'+1=n-k+1 \subset \hat{M}(\tau_1, \tau_2, n')$ and $n-(n'-j)=k+j>k-1$ ($j=0, 1$). Therefore, by the induction hypothesis and (4.4), we have the formula

$$2\left(s+\frac{1}{2}+n\right)A_k(s, n, n-k)=4a_{n-k+1,n-k}A_{k-1}(s, n, n-k+1) .$$

This implies the formula (4.3).

We define the set $S_k(n)$ ($n \in \hat{M}(\tau_1, \tau_2)$ and $k \in \boldsymbol{Z}$ with $k>0$) by

$$S_k(n)=\{(k/2)+(n'(n'+1)-n(n+1))/2k;\ n' \in \hat{M}(\tau_1, \tau_2) \text{ with } |n-n'| \leq k\} ,$$

and put $S^k(n)=\bigcup_{j=1}^k S_j(n)$. Then, we see easily from (4.2) and Lemma 4.3 that the singular points of the function $s \mapsto A_k(s)v_n$ are contained in $S^k(n)$, and from the definition of $S_k(n)$ that $S^k(n) \subset [-n-1/2+k, n+1/2+k]$. Let $\boldsymbol{C}_-=\{s \in \boldsymbol{C};\ \mathrm{Re}(s) \leq 0\}$ and put $S_k^-(n)=S_k(n) \cap \boldsymbol{C}_-$, $S_-^k(n)=S^k(n) \cap \boldsymbol{C}_-$. By $S_-(n, k)$, we denote the set of all singular points of the function $s \mapsto A_k(s)v_n$ in $\boldsymbol{C}_-$ and put $S_-=\bigcup_{n \in \hat{M}(\tau_1,\tau_2),\, k \geq 0} S_-(n, k)$. Then we have that $S_-(n, k) \subset S_-^k(n) \subset [-n, 0]$ ($k>0$, $n \geq 0$) and $S_-^k(n)=S_-^{k'}(n)$ if $k \geq k'>n+1/2$. Moreover, from (4.2) the above arguments prove that there is an integer n_0 such that

$$(s-s_0)^{n_0}A_k(s)v_n \longrightarrow o \quad \text{as} \quad s \longrightarrow s_0 \quad \text{for any} \quad k \in \boldsymbol{Z},\ s_0 \in \boldsymbol{C}_- .$$

Thus, we have the next lemma.

LEMMA 4.4. *Retain the above notations. Then* $S_-(n, k) \subset [-n, 0]$ ($k=0, 1, \cdots$ *and* $n \in \hat{M}(\tau_1, \tau_2)$) *and* S_- *is a finite set. Moreover, there is an integer* $n_0>0$ *such that*

$$(s-s_0)^{n_0}A_k(s)v_n \longrightarrow o \quad as \quad s \longrightarrow s_0$$
$$for\ any \quad k \in \boldsymbol{Z},\ s_0 \in \boldsymbol{C}_- \quad and \quad n \in \hat{M}(\tau_1, \tau_2)\ .$$

For an integer k and $s \in \mathcal{O}(\tau_1, \tau_2)$, we put

$$|A_k(s)| = \max(|A_k(s, n, n')|;\ n, n' \in \hat{M}(\tau_1, \tau_2))\ .$$

Then, for any compact subset B of $\mathcal{O}(\tau_1, \tau_2)$, there is a constant $c=c(B)>0$ such that

$$|A_k(s, n, n')| \leq (c/k)\sum_{j=0}^{k-1}|A_j(s)| \qquad \text{for} \quad s \in B,\ n, n' \in \hat{M}(\tau_1, \tau_2) \quad \text{and} \quad k>0\ .$$

(This follows from the formula (4.2).) Hence,

$$|A_k(s)| \leq (c/k)\sum_{j=0}^{k-1}|A_j(s)| \qquad \text{for} \quad s \in B \quad \text{and} \quad k>0\ .$$

LEMMA 4.5. *For arbitrary compact subset B of $\mathcal{O}(\tau_1, \tau_2)$ there is a constant $c=c(B)>0$ such that*

$$|A_k(s)| \leq (c)_k/k! \qquad for \quad k=0, 1, 2, \cdots \quad and \quad s \in B\ .$$

PROOF. Let $a_k=\sum_{j=0}^{k}|A_j(s)|$ $(k=0, 1, 2, \cdots)$, then $a_k-a_{k-1}\leq(c/k)a_{k-1}$ $(k>0)$. Hence, by the induction on k, $a_k\leq((c+k)/k)a_{k-1}$ $(k>0)$. Thus, we have from the fact; $1=a_0=|A_0(s)|$ that $a_k\leq(c+1)_k/k!$ $(k=0, 1, \cdots)$ Therefore, $|A_k(s)|\leq(c/k)a_{k-1}\leq(c)_k/k!$ $(k=0, 1, \cdots$ and $s \in B)$.

For $k \in \boldsymbol{Z}$ and $s_0 \in \boldsymbol{C}$, let $a_k(s_0, j)$ $(j \in \boldsymbol{Z})$ denote the linear endomorphism on $\mathscr{V}_M$ which appears in the Laurent expansion of $A_k(s)$ at $s=s_0$ i.e.,

$$(4.5) \qquad A_k(s)=\sum_{j=-\infty}^{+\infty} a_k(s_0, j)(s-s_0)^j\ .$$

From the Cauchy's integral formula and Lemma 4.5, we have the next corollary.

COROLLARY 4.6. *Fix $s_0 \in \boldsymbol{C}$ and integers j, k with $k\geq 0$. Then there are positive constants c_1, c_2, which are independent of j, k, such that $|a_k(s_0, j)|\leq c_1^{-j}(c_2)_k/k!$.*

§ 5. The explicit formulas of $C_1(s)$ and $C_W(s)$.

Let dy denote the Euclidean measure $dy_1dy_2dy_3$ on I (where $y=y_1i+y_2j+y_3k \in I$). Put $c_0=\int_I e^{-3t(y)}dy$, then $d\bar{n}=dy/c_0$ is the Haar measure on $\bar{N}$ such that $\int_{\bar{N}}\exp(-3t(\bar{n}))d\bar{n}=1$.
Let

$$(5.1) \qquad C(s, \tau)=\int_{\bar{N}} e^{-(s+3/2)t(\bar{n})}\tau(k(\bar{n})^{-1}w)d\bar{n}\ , \qquad \text{for} \quad s \in \boldsymbol{C} \quad \text{with} \quad \mathrm{Re}(s)>0\ ,$$

where τ is an irreducible unitary representation of K and $w=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then this integral converges absolutely and the function $s\mapsto C(s, \tau)$ can be extended to a meromorphic function with values in $\mathrm{End}(V^\tau)$ (where V^τ is the representation space of τ and $\mathrm{End}(V^\tau)$ is the space of all linear endomorphisms on V^τ) (cf. § 11 of [1] and Theorem 3 of [7]). It is easy to check that $t(m\bar{n}m^{-1})=t(\bar{n})$, $k(m\bar{n}m^{-1})=mk(\bar{n})m^{-1}$ for $\bar{n}\in\bar{N}$, $m\in M$ and $d\bar{n}$ is invariant under the adjoint action of M on $\bar{N}$. So, we have $\tau(m)C(s, \tau)=C(s, \tau)\tau(m)$ for any $m\in M$. Hence, for $n\in\hat{M}(\tau)$, by the definition of $P_n(\tau)$, there is a meromorphic function $c_n(s, \tau)$ on $\boldsymbol{C}$ such that

$$c_n(s, \tau)P_n(\tau)=P_n(\tau)C(s, \tau)\ . \tag{5.2}$$

Moreover, using the facts that $t(m\bar{n}m^{-1})=t(\bar{n})$, $k(m\bar{n}m^{-1})=mk(\bar{n})m^{-1}$ and the invariance of $d\bar{n}$, we have from Lemma 4.4 of [1] the integral formula

$$\int_K f(k)dk=\int_M\int_{\bar{N}} f(k(\bar{n})m)e^{-3t(\bar{n})}dmd\bar{n}\ , \tag{5.3}$$

where f is a continuous function on K and dm is the normalized Haar measur on M.

Put $\bar{n}(t)=a_t\bar{n}a_t^{-1}$ for $\bar{n}\in\bar{N}$ and $t\in\boldsymbol{R}$, then it is easy to see that $k(a_tk(\bar{n})m)=k(\bar{n}(t))m$, $k(k(\bar{n})m)=k(\bar{n})m$ and $t(a_tk(\bar{n})m)=t(\bar{n}(t))-t(\bar{n})+t$. Hence, from the formulas (3.4) and (5.3) we have

$$\begin{aligned} & e^{-(s-3/2)t}E(s, v, a_t) \\ &=\int_{\bar{N}} {}^{(s-3/2)t(\bar{n}(t))}e^{-(s+3/2)t(\bar{n})}\tau_1(k(\bar{n}(t)))v\tau_2(k(\bar{n})^{-1})d\bar{n} \\ &\qquad\text{for}\quad s\in\boldsymbol{C},\ t\in\boldsymbol{R}\quad\text{and}\quad v\in\mathscr{V}_M\ . \end{aligned} \tag{5.4}$$

Since $a_t\bar{n}_ya_t^{-1}=\bar{n}_e-t_y$ and $d(\bar{n}(t))=e^{-3t}d\bar{n}$, we have from (5.4)

$$\begin{aligned} & e^{(s+2/3)t}E(s, v, a_{-t}) \\ &=\int_{\bar{N}} e^{-(s+3/2)t(\bar{n}(t))}e^{(s-3/2)t(\bar{n})}\tau_1(k(\bar{n}))v\tau_2(k(\bar{n}(t))^{-1})d\bar{n} \\ &\qquad\text{for}\quad s\in\boldsymbol{C},\ t\in\boldsymbol{R}\quad\text{and}\quad v\in\mathscr{V}_M\ . \end{aligned} \tag{5.5}$$

since $\bar{n}(t)\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ as $t\to+\infty$, we see that $t(\bar{n}(t))\to 0$ and $k(\bar{n}(t))\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ as $t\to+\infty$. Hence, by using the estimates in § 11 of [1] and (5.4)-(5.5), we have

$$\lim_{t\to+\infty} e^{(-s+3/2)t}E(s, v, a_t)=\int_{\bar{N}} e^{-(s+3/2)t(\bar{n})}v\tau_2(k(\bar{n})^{-1})d\bar{n}$$

for $s \in \boldsymbol{C}$ with $\mathrm{Re}(s)>0$ and $v \in \mathscr{V}_M$, and

$$\lim_{t\to+\infty} e^{(s+3/2)t} E(s, v, a_{-t}) = \int_{\bar{N}} e^{(s-3/2)t(\bar{n})} v\tau_1(k(\bar{n}))d\bar{n}$$

for $s \in \boldsymbol{C}$ with $\mathrm{Re}(s)<0$ and $v \in \mathscr{V}_M$. But, we have from Theorem 3.1

$$\lim_{t\to+\infty} e^{(-s+3/2)t} E(s, v, a_t) = C_1(s)v \quad \text{for } s \in \mathscr{O}(\tau_1, \tau_2) \text{ with } \mathrm{Re}(s)>0,\ v \in \mathscr{V}_M ,$$

and also, since $E(s, v, a_{-t}) = E(s, v, wa_t w) = \tau_1(w)E(s, v, a_t)\tau_2(w)$,

$$\lim_{t\to+\infty} e^{(s+3/2)t} E(s, v, a_{-t}) = \tau_1(w)(C_w(s)v)\tau_2(w)$$
$$\text{for } s \in \mathscr{O}(\tau_1, \tau_2) \text{ with } \mathrm{Re}(s)<0,\ v \in \mathscr{V}_M .$$

Since τ is an unitary representation and $w=w^{-1}$, it is easy to check that

$$\int_{\bar{N}} e^{-(s+3/2)t(\bar{n})} \tau(k(\bar{n}))d\bar{n} = \tau(w)C(\bar{s}, \tau)^* \quad \text{for } s \in \boldsymbol{C} \text{ with } \mathrm{Re}(s)>0 ,$$

where $\bar{s}$ is the complex conjugate of s. Consequently, we have formulas below;

$$C_1(s)v = vC(s, \tau_2)\tau_2(w) , \qquad C_w(s)v = C(-\bar{s}, \tau_1)^* v\tau_2(w) , \tag{5.6}$$
$$\text{for } s \in \mathscr{O}(\tau_1, \tau_2) \text{ and } v \in \mathscr{V}_M .$$

By definition of v_n (see § 3) and $c_n(s, \tau)$, we obtain

$$v_n C(s, \tau_2) = c_n(s, \tau_2)v_n ,$$

$$C(-\bar{s}, \tau_1)^* v_n = \overline{c_n(-\bar{s}, \tau_1)v_n} .$$

To obtain the explicit formulas of $C_1(s)v$ and $C_w(s)v$, we need the following proposition.

PROPOSITION 5.1. *Let $\tau=\tau^{n',n''}$ be an irreducible unitary representation of K. Then for each $n \in \hat{M}(\tau)$*

$$c_n(s, \tau) = \frac{\varepsilon^{2n} 2^{-2s+3} \Gamma(2s)\Gamma(-s+3/2+n')\Gamma(-s+1/2-|n''|)}{\Gamma(-s+3/2+n)\Gamma(-s+1/2-n)\Gamma(s+3/2+n')\Gamma(s+1/2-|n''|)} , \tag{5.7}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n''\geq 0 \\ -1 & \text{if } n''<0 . \end{cases}$$

PROOF. Let σ^n ($n \in \hat{M}$) be an irreducible unitary representation of M on V^n (see § 1). For $s \in \boldsymbol{C}$ and $n \in \hat{M}$, let $\mathscr{H}^\infty_{s,n}$ be the space of all infinitely differentiable functions on G with values in V^n such that

$f(xma_tn)=e^{-(s+3/2)t}\sigma^n(m^{-1})f(x)$ for $x\in G$, $m\in M$, $t\in \boldsymbol{R}$ and $n\in N$.

Put $R_{s,n}(x)f(x')=f(x^{-1}x')$ for $f\in \mathscr{H}_{s,n}^{\infty}$, $x, x'\in G$, then $R_{s,n}$ is a representation of G on $\mathscr{H}_{s,n}^{\infty}$. For any function f in $\mathscr{H}_{s,n}^{\infty}$ satisfying $\mathrm{Re}(s)>0$, the integral

$$A_{s,n}f(x)=\int_{\bar N}f(xw\bar n)d\bar n=\int_{\bar N}e^{-(s+3/2)t(\bar n)}f(xwk(\bar n))d\bar n$$

is absolutely convergent (see § 11 of [1]). Moreover, since $wa_tw=a_{-t}$, $wmw=m$ for $t\in \boldsymbol{R}$, $m\in M$, we see that $A_{s,n}f\in \mathscr{H}_{-s,n}^{\infty}$ for $f\in \mathscr{H}_{s,n}^{\infty}$ ($\mathrm{Re}(s)>0$). and

(5.8) $\quad A_{s,n}R_{s,n}(x)f=R_{-s,n}(x)A_{s,n}f$, for $x\in G$, $f\in \mathscr{H}_{s,n}^{\infty}(\mathrm{Re}(s))>0$.

Now let τ be an irreducible unitary representation of K on V^{τ} and assume that $n\in \hat M(\tau)$. Put

$$f_{\tau,v}(x)=e^{-(s+3/2)t(x)}P_n(\tau)(\tau(k(x)^{-1})v) \quad \text{for} \quad v\in V^{\tau},\ x\in G\ ;$$

then $\mathscr{H}_{s,n}^{\infty}(\tau)=\{f_{\tau,v};\ v\in V^{\tau}\}$ is a finitely dimesional subspace of $\mathscr{H}_{s,n}^{\infty}$ which consists of all K-finite vector of type τ under $R_{s,n}$ (the Frobenius' reciprocity theorem and the branching theorem for a compact group imply this assertion). And for $f\in \mathscr{H}_{s,n}^{\infty}(\tau)$ ($\mathrm{Re}(s)>0$), it is clear that

(5.9) $$A_{s,n}f=c_n(s,\tau)f\ .$$

Now we shall prove the following assertion.

(5.10) For $\tau=\tau^{n',n''}$ and $n\in \hat M(\tau)$, let $c_n(s,\tau)=c_n(s,n',n'')$.

If $(n, n''+1)$, (n', n''), $(n'+1, n'')\in \hat K$, then

$$c_n(s,n',n''+1)\Big(s+\frac{1}{2}+n''\Big)=c_n(s,n',n'')\Big(-s+\frac{1}{2}+n''\Big),$$

$$c_n(s,n'+1,n'')\Big(s+\frac{3}{2}+n'\Big)=c_n(s,n',n'')\Big(-s+\frac{3}{2}+n'\Big).$$

Let

$$R'_{s,n}f(x)=\Big(\frac{d}{dt}\Big)(R_{s,n}(a_t)f(x))\Big|_{t=0} \quad \text{for} \quad f\in \mathscr{H}_{s,n}^{\infty},\ x\in G\ .$$

Then we see from (5.8) that

(5.11) $\quad A_{s,n}R'_{s,n}f=R'_{-s,n}A_{s,n}f$ for $f\in \mathscr{H}_{s,n}^{\infty}$ with $\mathrm{Re}(s)>0$.

In [12], Thieleker introduced a series of representations of G with parameter (λ_0, m) ([12], p. 501). The mapping $f(x)\mapsto f(x^{-1})$ ($f\in \mathscr{H}_{s,n}^{\infty}$) gives an equivalence between $R_{s,n}$ and a representation with parameter $\lambda_0=s$, $m=n$ introduced by Thieleker. Moreover, our parameter (n', n'') corresponds to $(\Lambda_{\omega_1}, \Lambda_{\omega_2})$ in Thieleker's notation ([12], p. 473). Let $P_{s,n}(\tau)f(x)=\dim(V^{\tau})\int_K \operatorname{trace}(\tau(k))f(k^{-1}x)dk$ for $f\in\mathscr{H}_{s,n}^{\infty}$, then $P_{s,n}(\tau)$ is a projection of $\mathscr{H}_{s,n}^{\infty}$ onto $\mathscr{H}_{s,n}^{\infty}(\tau)$ and (5.9) implies

$$A_{s,n}P_{s,n}(\tau)=P_{s,n}(\tau)A_{s,n}=c_n(s, \tau)P_{s,n}(\tau) ,$$

Hence, by applying Theorem 1 of [12] to $R'_{s,n}$, we have the assertion (5.10) directly from (5.8).

Therefore, using (5.10) and Lemma 5.2 below, our proposition can be proved by the induction on (n', n'').

LEMMA 5.2. *Let $c_n(s)=c_n(s, n, n)$. Then*

$$c_n(s)=s^{-2s+3}\Gamma(2s)/(\Gamma(s+3/2+n)\Gamma(s+1/2-n)) , \tag{5.12}$$

and

$$c_n(s, n, -n)=(-1)^{2n}c_n(s) .$$

PROOF. Let $\tau_n=\tau^{|n|,n}$ for a half integer n. Then the representation space of τ_n is identified with $V^{|n|}$ and $\tau_{|n|}(k)=\sigma^{|n|}(u)$, $\tau_{-|n|}(k)=\sigma^{|n|}(v)$ for $k=k(u, v)\in K$. Let $t(y)$, $u(y)$, $v(y)$ be the same as in § 1, then

$$t(y)=\log(1+|y|^2) , \qquad u(y)=v(-y)=(1-y)(1+|y|^2)^{-1/2} \quad \text{([11], p. 365).}$$

Since $w=k(1, -1)$ and $\sigma^{|n|}(-1)=(-1)^{2n}\sigma^{|n|}(1)$, we have

$$\tau_{|n|}(k(u(y), v(y))^{-1}w)=\sigma^{|n|}(u(y)^{-1}) ,$$

$$\tau_{-|n|}(k(u(y), v(y))^{-1}w)=(-1)^{2n}\sigma^{|n|}(v(y)^{-1}) .$$

Furthermore, we note that $u(y)^{-1}=u(-y)$, $v(y)^{-1}=v(-y)$ and dy is invariant under the action $y\mapsto -y$. So, by the definition of $C(s, \tau)$, we obtain

$$\begin{aligned} C(s, \tau_{|n|}) &= (-1)^{2n}C(s, \tau_{-|n|}) \\ &= c_0^{-1}\int_I (1+|y|^2)^{-(s+3/2)}\sigma^{|n|}((1-y)(1+|y|^2)^{-1/2})dy . \end{aligned}$$

for $s\in\boldsymbol{C}$ with $\operatorname{Re}(s)>0$.

Let du denote the normalized Haar measure on $\boldsymbol{U}$, then for $u_0\in\boldsymbol{U}$, $n\in\hat{\boldsymbol{M}}$, the irreducibility of σ^n implies

$$\int_U \sigma^n(uu_0u^{-1})du=f_n(u_0)\sigma^n(1) \tag{5.13}$$

where f_n is a certain continuous function on $\boldsymbol{U}$. Moreover,

$$\begin{aligned}(2n+1)f_n(u_0) &= \operatorname{trace}(f_n(u_0)\sigma^n(1)) \\ &= \int_U \operatorname{trace}(\sigma^n(uu_0u^{-1}))du \\ &= \operatorname{trace}(\sigma^n(u_0)) = C^1_{2n}\left(\frac{u_0+\bar{u}_0}{2}\right) \quad \text{(cf. [11], p. 383)},\end{aligned}$$

where C^1_{2n} is the Gegenbauer polynomial of order $2n$. Since dy is invariant under the adjoint action of U on I, by using (5.13) for $s\in C$ satisfying $\operatorname{Re}(s)>0$, we have

$$C(s, \tau_{|n|}) = c_n^{-1}\int_I (1+|y|^2)^{-(s+3/2)} C^1_{2n}((1+|y|^2)^{-1/2})dy\,\sigma^n(1)\ ,$$

where $c_n = c_0(2n+1)$ and $n\geq 0$. Since $P_{|n|}(\tau_n)=\sigma^{|n|}(1)$ and $C(s, \tau_n)=c_n(s)P_n(\tau_n)$,

$$c_n(s) = 4\pi c_n^{-1}\int_0^\infty (1+r^2)^{-(s+3/2)} C^1_{2n}((1+r^2)^{-1/2})r^2\,dr\ .$$

Let $r=\tan x$ with $0\leq x\leq \pi/2$, then

$$c_n(s) = 4\pi c_n^{-1}\int_0^{\pi/2} \sin^2 x \cos^{2s-1} x\, C^1_{2n}(\cos x)\,dx\ .$$

We know

$$\sin(2n+1)x = \sin x\, C^1_{2n}(\cos x)\ ,$$

$$\left(\frac{d}{dx}\right)(\cos^{2s} x) = (-2s)\sin x \cos^{2s-1} x\ .$$

Hence, using the integration by parts,

$$\begin{aligned}&\int_0^{\pi/2} \sin^2 x \cos^{2s-1} x\, C^1_{2n}(\cos x)dx \\ &\quad = -(2s)^{-1}\int_0^{\pi/2}\left(\frac{d}{dx}\right)(\cos^{2s} x)\sin(2n+1)x\,dx \\ &\quad = (2s)^{-1}(2n+1)\int_0^{\pi/2}\cos^{2s} x\cos(2n+1)x\,dx \qquad (\operatorname{Re}(s)>0)\ .\end{aligned}$$

But, for $s\in C$ satisfying $\operatorname{Re}(s)>0$,

$$\int_0^{\pi/2}\cos^{2s} x\cos(2n+1)x\,dx = \frac{\Gamma(2s+1)2^{-2s-1}\pi}{\Gamma(s+3/2+n)\Gamma(s+1/2-n)} \qquad \text{(cf. [8], p. 9)},$$

and by the simple calculations,

$$c_0 = \int_I e^{-3t(y)}\,dy = \pi^2/4\ .$$

Therefore, we obtain (5.12).

COROLLARY 5.3. *Retain above notations. All the zeros and poles of* $c_n(s, n', n'')$ *are simple and they are as follows;*

Zeros,

1) $s=k+1/2$, $n<k\leq n'$ *and* $n-k\in \boldsymbol{Z}$,
2) $s=k+1/2$, $0\leq k<|n''|$ *and* $n-k\in \boldsymbol{Z}$,
3) $s=-(k+1/2)$, $|n''|\leq k<n$ *and* $n-k\in \boldsymbol{Z}$,
4) $s=-(k+1/2)$, $n'<k$ *and* $n-k\in \boldsymbol{Z}$,

Poles,

5) $s=-k$, $k\geq 0$ *and* $n-k\in \boldsymbol{Z}$.

Since the distribution of zeros and poles of the Gamma function are known, this corollary is obtained from the explicit formula (5.7).

We know that $\tau^{n',n''}(w)=\varepsilon(n', n'')\tau^{n',n''}(1)$ (where $\varepsilon(n', n'')=(-1)^{2n'-2n''}$). Hence (5.6), (5.7) imply the following proposition.

PROPOSITION 5.4. *Let* $C_1(s)$, $C_w(s)$ *be meromorphic functions which appears in Theorem* 3.1, *and let* $\tau_i=\tau^{n'_i,n''_i}$ $(i=1, 2)$. *Then for each* $n\in \hat{M}(\tau_1, \tau_2)$ *and* $s\in \mathcal{O}(\tau_1, \tau_2)$,

$$C_1(s)v_n=\varepsilon(n'_2, n''_2)c_n(s, \tau_2)v_n\ , \tag{5.14}$$
$$C_w(s)v_n=\varepsilon(n'_2, n''_2)c_n(-s, \tau_1)v_n\ .$$

§ 6. Proof of Theorem 3.4.

In this section, we assume that $\hat{M}(\tau_1, \tau_2)$ is not empty. To begin with, we shall prove the following lemmas.

LEMMA 6.1. *For fixed* $n\in \hat{M}(\tau_1, \tau_2)$ *and* $t>0$, *the function* $s\mapsto E(s, n, t)=E(s, t)v_n$ *can be extended to a* $\mathscr{V}_M$*-valued meromorphic function on* $\boldsymbol{C}_-=\{s\in \boldsymbol{C};\ \mathrm{Re}(s)\leq 0\}$ *whose poles are all simple. Moreover, if* $s_0\neq -(k+1/2)$ *for* $k\in \hat{M}(\tau_1, \tau_2)$ *with* $n>k$, *then* $E(s, n, t)$ *is holomorphic at* $s=s_0$.

LEMMA 6.2. *For each* $n\in \hat{M}(\tau_1, \tau_2)$ *and* $t>0$, *the function* $s\mapsto E(s, v_n, a_t)$ *is holomorphic on* $\boldsymbol{C}$ *with values in* $\mathscr{V}_M$ *and has zeros at points below;*

1) $s=k+1/2$ *for* $n-k\in \boldsymbol{Z}$ *with* $n'_1<k\leq n'_2$, *if* $n'_1<n'_2$,
2) $s=k+1/2$ *for* $n-k\in \boldsymbol{Z}$ *with* $|n''_1|<k\leq|n''_2|$, *if* $|n''_1|<|n''_2|$,
3) $s=-(k+1/2)$ *for* $n-k\in \boldsymbol{Z}$ *with* $n'_2<k\leq n'_1$, *if* $n'_2<n'_1$,
4) $s=-(k+1/2)$ *for* $n-k\in \boldsymbol{Z}$ *with* $|n''_2|<k\leq|n''_1|$, *if* $|n''_2|<|n''_1|$.

PROOF OF LEMMA 6.1. From Lemma 4.4, we see that the functions $s\mapsto A_k(s)v_n$ ($k\in \boldsymbol{Z}$ and $n\in \hat{M}(\tau_1, \tau_2)$) are holomorphic at $s=s_0$ satisfying $\mathrm{Re}(s_0)\leq 0$ and $s_0\notin[-n, 0]$. Hence, using Lemma 2.3 and Corollary 4.6, the series

$$E(s, n, t)=e^{(s-3/2)t}\sum_{k=0}^{+\infty}A_k(s)v_n e^{-kt} \qquad (t>0) \tag{6.1}$$

is holomorphic at $s=s_0$ satisfying $\mathrm{Re}(s_0)\leq 0$ and $s_0\notin[-n, 0]$.

Let $s_0=-r\in[-n, 0]$ and consider the Laurent expansion

$$E(s, n, t)=\sum_{j=-\infty}^{+\infty}E(n, t, j)(s+r)^j \qquad (t>0)\ , \tag{6.2}$$

then the function $t\mapsto E(n, t, j)$ $(j\in \boldsymbol{Z})$ is a spherical function on A^+ (Proposition 2.4, Corollary 4.6).

By using the Cauchy's integral formula, we conclude from Theorem 3.1, Lemma 4.4 and Lemma 4.5 that there is an integer j_0 such that $E(n, t, j)=0$ if $j<j_0$, $t>0$ and $E(n, t, j_0)\neq 0$, when $E(s, n, t)\neq 0$ around $s=-r$. Moreover, we have from the definition of j_0 and (6.2) that

$$E(n, t, j_0)=e^{-(r+3/2)t}\sum_{k=0}^{+\infty}a_k(-r, j_0)v_n e^{-kt} \tag{6.3}$$

(see § 4 for $a_k(-r, j_0)$). Now we select an integer $k_0\geq 0$ such that

$$a_k(-r, j_0)=0 \quad \text{if} \quad k<k_0 \quad \text{and} \quad a_{k_0}(-r, j_0)\neq 0\ .$$

Put $a_k=a_{k+k_0}(-r, j_0)v_n$, then, from Corollary 4.6, there are constants $c_i>0$ $(i=1, 2)$ such that $|a_k|\leq c_1(c_2)_k/k!$ $(k-0, 1, 2, \cdots)$. From Lemma 3.3, we see that there is a function $F_s\in C^\infty_{s,n}(G^+, \tau)$ such that $F_s(a_t)=E(s, n, t)$ for $t>0$. Put

$$F_{j_0}(x)=\frac{1}{2\pi i}\oint F_s(x)(s+r)^{-(j_0+1)}ds \qquad \text{for} \quad x\in G^+\ . \tag{6.4}$$

Since $zF_s=\chi_{s,n}(z)F_s$ for all $z\in\mathfrak{Z}$ and the function $s\mapsto\chi_{s,n}(z)$ is a polynomial function on $\boldsymbol{C}$, we have from the definition of j_0 and (6.4) that $F_{j_0}\in C_{-r,n}(G^+, \tau)$ (by definition of F_s, the function $s\mapsto F_s(x)$ is meromorphic at $s=-r$). Moreover, from the choice of k_0 and (6.3),

$$F_{j_0}(a_t)=E(n, t, j_0)=e^{-(r+k_0+3/2)t}\sum_{k=0}^{+\infty}a_k e^{-kt}\ .$$

Hence, we have from Proposition 2.4 that F_{j_0} is of type $r+k_0$ if $r+k_0>0$. We need the next lemma.

LEMMA 6.3. *Retain above notations. If* $j_0<0$, *then* $k_0>0$ *and* $r=n-k_0+1/2$ *with* $n-k_0\in\tilde{M}(\tau_1, \tau_2)$.

PROOF. Since $A_0(s)=$ the identity endomorphism on $\mathscr{V}_M$, we get $k_0>0$. Next, since $|a_k|\leq c_1(c_2)_k/k!$ $(k=0, 1, 2, \cdots)$,

$$|e^{(r+k_0+3/2)t}F_{j_0}(a_t)-a_0|\leq|\sum_{k=1}^{+\infty}a_k e^{-kt}|\leq c_1((1-e^{-t})^{-c_2}-1) \qquad \text{for} \quad t>0 .$$

So, $\lim_{t\to+\infty}e^{(r+k_0+3/2)t}F_{j_0}(a_t)=a_0\neq 0$. Hence, Proposition 2.5 implies that $r+k_0=\pm r$ or $r+k_0=\pm(n+1/2)$. Since $k_0>0$, we have $r+k_0=n+1/2$. So, it is enough to show that $n-k_0\in\hat{M}(\tau_1, \tau_2)$. If we write $a_k(-r, j_0)v_n=\sum_{n'\in\hat{M}(\tau_1,\tau_2)}a_k(n, n')v_{n'}$, then

$$a_k=\sum_{n'\in\hat{M}(\tau_1,\tau_2)}a_{k+k_0}(n, n')v_{n'} ,$$

and

$$a_k(n, n')=\frac{1}{2\pi i}\oint A_k(s, n, n')(s+r)^{-(j_0+1)}ds .$$

Therefore, we see from (4.2) and the choice of j_0, k_0 that

$$(-2k_0r-k_0^2+n(n+1)-n'(n'+1))a_{k_0}(n, n')=0 .$$

Since $r=n-k_0+1/2$, we have

$$-2k_0r-k_0^2+n(n+1)-n'(n'+1)=(n-k_0-n')(n-k_0+n'+2) ,$$

and $n-k_0+n'+2\neq 0$ for any $n'\in\hat{M}(\tau_1, \tau_2)$. Hence $(n-k_0-n')a_{k_0}(n, n')=0$ for any $n'\in\hat{M}(\tau_1, \tau_2)$. But since $a_0=\sum_{n'}a_{k_0}(n, n')v_n\neq 0$, there is an element $n'\in\hat{M}(\tau_1, \tau_2)$ such that $n'=n-k_0\in\hat{M}(\tau_1, \tau_2)$ and $a_0=a_{k_0}(n, n-k_0)v_{n-k_0}$. This completes the proof.

Now, we return to the proof of Lemma 6.1. From (4.3), we see that the function $s\mapsto A_{k_0}(s, n, n-k_0)$ has at most simple pole at $s=-(n-k_0+1/2)$. Hence, we have from Lemma 6.3 that $j_0=-1$, $k_0>0$ and $r=n-k_0+1/2$ with $n-k_0\in\hat{M}(\tau_1, \tau_2)$ if $j_0<0$. This completes the proof of Lemma 6.1.

PROOF OF LEMMA 6.2. From Proposition 5.1, we have

$$c_n(s, \tau)c_n(-s, \tau)=c_n(s)c_n(-s) \tag{6.5}$$

for $n\in\hat{M}$ and $\tau=\tau^{n',n''}$ with $(n', n'')\in\hat{K}(n)$, where $\hat{K}(n)=\{(n', n'')\in\hat{K};\ |n''|\leq n\leq n' \text{ and } n'-n\in\boldsymbol{Z}\}$. Hence, from (3.5) and (6.5), we have

$$c_n(s, \tau_1)E(s, v_n, a_t)=c_n(s, \tau_2)E(-s, v_n, a_t) , \tag{6.6}$$

for $s\in\mathscr{O}(\tau_1, \tau_2)$, $n\in\hat{M}(\tau_1, \tau_2)$ and $t>0$. But it is clear from (3.4) that the function $s\mapsto E(s, v_n, a_t)$ is a $\mathscr{V}_M$-valued holomorphic function on $\boldsymbol{C}$. Now, (6.6) implies

$$\frac{c_n(s, \tau_1)}{c_n(s, \tau_2)}E(s, v_n, a_t)=E(-s, v_n, a_t) .$$

Noting that poles (resp. zeros) of the function $s \mapsto c_n(s, \tau_1)/c_n(s, \tau_2)$ correspond to zeros of the function $s \mapsto E(s, v_n, a_t)$ (resp. $E(-s, v_n, a_t)$) and, using Corollary 5.3, we may determine the zeros and poles of the function $s \mapsto c_n(s, \tau_1)/c_n(s, \tau_2)$. Hence, Corollary 5.3 implies Lemma 6.2.

Now, we shall prove Theorem 3.4. When $s \in \boldsymbol{C}$ satisfies $\mathrm{Re}(s) \leq 0$, Theorem 3.4 has been proved in Lemma 6.1. From (3.5) and Proposition 5.4, we have

$$E(s, n, t) = c_n(s, \tau_2)^{-1}(\varepsilon E(s, v_n, a_t) - c_n(-s, \tau_1)E(-s, n, t)) , \tag{6.7}$$

for $s \in \mathcal{O}(\tau_1, \tau_2)$, $n \in \hat{M}(\tau_1, \tau_2)$ and $t > 0$, where $\varepsilon = (-1)^{2n'_2 - n'_2}$. Hence, from Lemma 6.1 and Proposition 5.1, the function $s \mapsto E(s, n, t)$ is meromorphic on $\boldsymbol{C}_+ = \{s \in \boldsymbol{C}; \mathrm{Re}(s) > 0\}$ with values in $\mathscr{V}_M$. Noting that $\hat{M}(\tau_1, \tau_2) = \{k \in \hat{M}; \max(|n''_1|, |n''_2|) \leq n \leq \min(n'_1, n'_2)$ and $n'_i - n \in \boldsymbol{Z}$ $(i = 1, 2)\}$, we see from Corollary 5.3 that the function

$$s \longmapsto c_n(s, \tau_2)^{-1} c_n(-s, \tau_1) E(-s, n, t)$$

satisfy the conditions of Theorem 3.4 on $\boldsymbol{C}_+ = \{s \in \boldsymbol{C}; \mathrm{Re}(s) > 0\}$. Moreover, using Lemma 6.2 and Corollary 5.3, the same arguments are valid for the function

$$s \longmapsto c_n(s, \tau_2)^{-1} E(s, v_n, a_t) .$$

Hence, (6.7) implies Theorem 3.4.

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Present Address:
DEPARTMENT OF MATHEMATICS
WASEDA UNIVERSITY
NISHI-ÔKUBO, SHINJUKU-KU,
TOKYO, 160