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On Best Approximation in Function Algebras

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In this paper we consider best approximation in function algebras and its application to projections on function algebras. After some preliminaries in § 1 we give in § 2 a characterization of continuous functions on X which have elements of best approximation in certain function algebras on a compact Hausdorff space X. In § 3 we deal with the estimation of the norms of projections on function algebras, in particular, of projections on certain function algebras on planar sets (Theorem 3.1.).

§ 1. Preliminaries.

Let E be a Banach space and let M be a closed subspace in E. For $x \in E$, a point y in M is said to be *an element of best approximation of x in M* if y satisfies that $\|x-y\| = d(x, M) = \inf\{\|x-z\| : z \in M\}$. For $x \in E$, in general, such an element y does not exist, and it is not necessary to be unique even if it exists.

We here investigate the cases of function algebras. Let A be a function algebra on a compact Hausdorff space X, i.e., let A be a closed subalgebra in $C(X)$ separating points in X and containing constant functions on X, where $C(X)$ denotes the Banach algebra of complex-valued continuous functions on X with the supremum norm. A complex Borel measure μ on X is said to *be orthogonal to A, $\mu \perp A$,* if $\int f d\mu = 0$ for each $f \in A$. Let A be a function algebra on X and let μ be a measure on X. If $\mu \perp A$, then there are a sequence $\{\mu_n\}$ of measures on X and a measure η on X such that $\mu = \sum_n \mu_n + \eta$, $\|\mu\| = \sum_n \|\mu_n\| + \|\eta\|$, $\mu_n \perp A$ $(n=1, 2, 3, \cdots)$, $\eta \perp A$, each μ_n is absolutely continuous with respect to a representing measure for a point in the maximal ideal space M_A of A, and η is completely singular, that is, η is singular with respect to every representing measure for any point in M_A. The fact is known as the decomposition theorem for orthogonal measures (for example, [2], p. 45).

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Let K be a compact subset in the complex plane C and let $P(K)$ be the uniform closure on K of the set of polynomials in z. The $P(K)$ is a function algebra on K. If the complement $C\backslash K$ of K is connected, the maximal ideal space and the Shilov boundary of $P(K)$ are K and the boundary bK of K in C respectively. Hence the restriction $P(K)|bK$ of $P(K)$ to bK coincides with $P(bK)$. Recall that if $C\backslash K$ is connected $P(K)$ agrees with $A(K)$, the function algebra consisting of the functions in $C(K)$ which are analytic on the interior K° of K, and so $P(K)|bK= P(bK)=A(K)|bK$. Suppose that K is a compact subset in C and that K° and $C\backslash K$ are both connected and the closure of K° is K. Then $A=P(K)|bK=P(bK)$ is a maximal algebra ([9], p. 297) and is also an essential algebra, i.e., if F is any proper closed subset in bK, then there is a function f in $C(bK)$ such that $f\notin A$ and $f(F)=0$. It follows that car m, the closed carrier of m, coincides with bK whenever m is a representing measure on bK for a point in K° ([1], [6]).

§ 2. Best approximation in function algebras.

Let A be a function algebra on a compact Hausdorff space X. We here assume the following condition.

(*) *If a complex measure μ on X is orthogonal to A and if it is completely singular, then* $\mu=0$ (cf. § 1).

EXAMPLES. (1) Let K be a compact subset in the complex plane C and let $R(K)$ be the uniform closure on K of the set of rational functions with poles off K. Then $R(K)$ is a function algebra on K and it satisfies (*) ([9], p. 311).

(2) Suppose that K is a compact subset in C and $C\backslash K$ is connected. Then $P(K)|bK=P(bK)$ satisfies (*).

We first characterize continuous functions which have elements of best approximation in function algebras satisfying (*).

THEOREM 2.1. *Let A be a function algebra on a compact Hausdorff space X having* (*). *Then f in $C(X)\backslash A$ has an element of best approximation in A if and only if f is of the form $g+sh$, where*

(i) *$g\in A$, $h\in C(X)$, $\|h\|=1$ and $s>0$.*

(ii) *there are a representing measure μ for some point in M_A and a function $\varphi\in L^1(\mu)$ such that $\int|\varphi|d\mu\neq 0$, $h\varphi=|\varphi|$* (a.e. μ) *and $\varphi d\mu\perp A$.*

PROOF. Let f have an element g of best approximation in A, that is, $\|f-g\|=d(f, A)=\inf\{\|f-a\|: a\in A\}$. If we put $s=\|f-g\|$, then $s>0$ because $f\in C(X)\backslash A$. Set $h=s^{-1}(f-g)$. Then $\|h\|=d(h, A)=1$. So, by

Hahn-Banach theorem and Riesz representation theorem, there is a complex measure ν on X satisfying that $\int h d\nu = ||\nu|| = 1$ and $\nu \perp A$. Since $\nu \perp A$, by the decomposition theorem for orthogonal measures (§ 1) and (*), there is a sequence $\{\mu_n\}$ of measures such that $\nu = \sum_n \mu_n$, $||\nu|| = \sum_n ||\mu_n||$, $\mu_n \perp A$ $(n=1, 2, 3, \cdots)$ and each μ_n is absolutely continuous with respect to a representing measure λ_n for a point in M_A. From this,

$$1 = \int h d\nu = \sum_n \int h d\mu_n = \left|\sum_n \int h d\mu_n\right| \leqq \sum_n \left|\int h d\mu_n\right|$$
$$\leqq ||h|| (\sum_n ||\mu_n||) = ||h|| \, ||\nu|| = 1 .$$

It implies that for each n, $\int h d\mu_n \geqq 0$ and

$$\int h d\mu_n = ||h|| \, ||\mu_n|| = ||\mu_n|| \qquad (n=1, 2, 3, \cdots) . \tag{2.1}$$

Since μ_n is absolutely continuous with respect to λ_n, there is a function $\varphi_n \in L^1(\lambda_n)$ such that $d\mu_n = \varphi_n d\lambda_n$. We see easily that

$$\int h d\mu_n = \int h\varphi_n d\lambda_n , \qquad ||\mu_n|| = \int |\varphi_n| d\lambda_n . \tag{2.2}$$

(2.1) and (2.2) also tell us that

$$\int h\varphi_n d\lambda_n = \int |\varphi_n| d\lambda_n \qquad (n=1, 2, 3, \cdots) . \tag{2.3}$$

So $h\varphi_n \geqq 0$ (a.e. λ_n) and

$$h\varphi_n = |\varphi_n| \qquad \text{(a.e. } \lambda_n) . \tag{2.4}$$

We choose n such that $||\mu_n|| \neq 0$. Put $\varphi = \varphi_n$ and $\mu = \lambda_n$. Then $h\varphi = |\varphi|$ (a.e. μ) and $\int |\varphi| d\mu \neq 0$. Since $f = g + sh$ and $\varphi d\mu \perp A$, (i) and (ii) are satisfied.

Conversely, suppose that $f = g + sh$ for g, h and s with (i) and (ii). We can assume that $\int |\varphi| d\mu = 1$ without loss of generality, since $\int |\varphi| d\mu \neq 0$. For $a \in A$, $f - a = g - a + sh$ and

$$||f-a|| \geqq \int |f-a| \, |\varphi| d\mu = \int |(f-a)\varphi| d\mu$$
$$= \int |(g-a)\varphi + sh\varphi| d\mu = \int |(g-a)\varphi + s|\varphi|| d\mu$$
$$\geqq \left| \int \{(g-a)\varphi + s|\varphi|\} d\mu \right| = s \int |\varphi| d\mu = s .$$

On the other hand, we see that $||f-g||=s$. This shows that f has g as an element of best approximation in A.

We next consider $P(bK)$ as a special case. Let K be a compact subset in C such that $C\backslash K$ is connected. Then $A=P(K)|bK$ is a Dirichlet algebra, and so any point in M_A has a unique representing measure m on bK. We denote by $H^1(m)$ the Hardy space, that is, $H^1(m)$ is the closure of A in $L^1(m)$.

COROLLARY 2.3. *Let K be a compact subset in C such that K° and $C\backslash K$ are both connected and the closure of K° is K, and let $A=P(K)|bK=P(bK)$. Then $f\in C(bK)\backslash A$ has an element of best approximation in A if and only if f is of the form $f=g+sh$, where*

(i) $g\in A$, $h\in C(bK)$, $s>0$ *and* $|h(z)|=1$ $(z\in bK)$.

(ii) *let z_0 be a fixed point in K°, then there is a function $\varphi\in H^1(m)$ such that $\int\varphi dm=0$, $\varphi\neq 0$* (a.e. m) *and $h=|\varphi|\varphi^{-1}$* (a.e. m) *for the representing measure m on bK for z_0.*

PROOF. Let $f\in C(bK)\backslash A$ have an element g of best approximation in A. Then, by Theorem 2.1, $f=g+sh$, $g\in A$, $h\in C(bK)$, $s>0$, $\int|\varphi|dm\neq 0$, $h\varphi=|\varphi|$ (a.e. m) and $\varphi dm\perp A$ for the representing measure m (on bK) of a point z of K° and for a function $\varphi\in L^1(m)$. This is because of that the only non-trivial Gleason part of A is K° (cf. [9], p. 296). The point z can be replaced by z_0 since z and z_0 are both in the same part K°. Since $\varphi dm\perp A$, $\varphi\in H^1(m)$. And $\varphi\neq 0$ on a set of positive measure. This leads us to that $\varphi\neq 0$ (a.e. m) (cf. [9], p. 291). Hence, we can write $h=|\varphi|\varphi^{-1}$ (a.e. m), and so $|h|=1$ (a.e. m). We now assert that $|h|=1$ everywhere on bK. Suppose otherwise. Then $E=\{z\in bK: |h(z)|=1\}$ is a proper closed subset in bK since h is continuous. Since $|h|=1$ (a.e. m), we have $m(bK\backslash E)=0$, and so car m does not coincides with bK. This is a contradiction because car $m=bK$ (see § 1), and it follows that $|\varphi|=1$ on bK.

REMARK 2.4. We see easily that g is uniquely determined for any f of Corollary 2.3. For, if g_1 and g_2 are two elements of best approximation of f in A, then $2^{-1}(g_1+g_2)$ is also an element of best approximation of f in A. If $f=g_1+sh_1=g_2+sh_2$, then $f=2^{-1}(g_1+g_2)+2^{-1}s(h_1+h_2)$. As in the proof of Corollary 2.3, we have $|h_1+h_2|=2$ and $|h_1|=|h_2|=1$, which yields $h_1=h_2$.

COROLLARY 2.5 (Shapiro [8]). *Let A be a disc algebra and let $f\in C(\Gamma)\backslash A$, where Γ is the unit circle in C. Then g is the element of* *best approximation of f in A if and only if f and g satisfy the following properties*:

(i) $|f(z)-g(z)|=\|f-g\|$ $(z \in \Gamma)$,

(ii) *there is a non-zero* $k \in H^1(d\theta)$ ($d\theta$: *Lebesgue measure on* Γ) *such that* $z(f(z)-g(z))k(z) \geqq 0$ (a.e. $d\theta$ *on* Γ).

PROOF. In Corollary 2.3, set $\Gamma=bK$ (K is the closed unit disc), $z_0=0$ (the origin of C) and $k=z^{-1}\varphi$.

REMARK 2.6. W. Hintzman [4] proved the existence of the element of best approximation in the disc algebra A for any polynomials in $\bar{z}$ on Γ. He also constructed a continuous function of Γ which has no element of best approximation in A ([5]).

§ 3. Projections on function algebras.

W. Rudin [7] has proved that there is no projection of $C(\Gamma)$ onto the disc algebra A. I. Glicksberg [3] extended the result to the cases of certain algebras and also proved that if A is a function algebra (in more general case) on a compact Hausdorff space X such that $A \neq C(X)$ and the Shilov boundary ∂_A of A coincides with X, and if T is a projection of $C(X)$ onto A, then $\|T\|>2$. Thereafter we deal with projections of N onto A for $A=P(bK)$ and closed linear subspaces N in $C(bK)$ containing A.

THEOREM 3.1. *Let K be a compact subset in C such that K° and $C \backslash K$ are both connected, and let $A=P(K)|bK=P(bK)$. Let N be a closed subspace in $C(bK)$ such that $N \supset A$ and let A have at least two codimension in N, that is,* $\dim N/A \geqq 2$. *If T is any projection of N onto A, then* $\|T\|>2$.

In order to prove the theorem, we need the forthcoming lemma and theorem.

LEMMA 3.2. *Let A be a function algebra on a compact Hausdorff space X and let $\partial_A=X$. Let N be a closed subspace in $C(X)$ containing A with $N \neq A$. Let T be a projection of N onto A. Then for any $f \in N$, Tf is an element of best approximation in A if and only if* $\|T\|=2$.

PROOF. Let T be a projection of N onto A. Then $\|T-I\|=\|T\|-1$ by Glicksberg ([3]), where I is the identity operator. If $\|T\|=2$, then $\|I-T\|=1$. For $f \in N$ and $g \in A$, $(I-T)(f-g)=(I-T)f=f-Tf$, and so $\|f-Tf\|=\|(I-T)(f-g)\| \leqq \|f-g\|$. This shows that Tf is an element of best approximation of f in A. Conversely, if Tf is an element of best approximation of f in A, then $\|f-Tf\|=d(f, A)\leqq\|f\|$. Since $N\neq A$, we have $\|I-T\|=1$ and $\|T\|=2$.

Now, let K be a compact subset in C such that $C\backslash K$ is connected, and let $A=P(K)|bK=P(bK)$. Let B be the set of functions in $C(bK)$ which have an element of best approximation in A and let N be a closed linear subspace in $C(bK)$ with $B\supset N\supset A$. If T is a linear operator of N to A and if for $f\in N$ Tf is an element of best approximation of f in A, then we see easily that T is a projection of N onto A with $\|T\|\leqq 2$. For, $\|Tf\|\leqq 2\|f\|$ since $\|f-Tf\|=d(f, A)\leqq\|f\|$.

THEOREM 3.3. *Let K and A be as in Theorem 3.1. Let N be a linear subspace in $C(bK)$ with $B\supset N\supsetneqq A$. If there is a projection T of N onto A such that for any $f\in N$ Tf is an element of best approximation of f in A, then A has one codimension in N.*

PROOF. We fix a function $f_0\in N\backslash A$. In order to prove the theorem, we must show that for any $f\in N\backslash A$, there is a complex number α such that $f-\alpha f_0\in A$. But if we denote by H the closure of K°, then H° and $C\backslash H$ are both connected and the closure of H° is H. It is not hard to see that the essential set for A is $bH=H\backslash H^\circ$ and $A|bH=P(bH)$ (see [6] for essential sets). By this, we can assume that the closure of K° is K without loss of generality. Now, by Corollary 2.3, we have

$$f-Tf=rh\ , \qquad f_0-Tf_0=sh_0\ , \tag{3.1}$$

where $r, s>0$, h and h_0 are in $C(bK)$ and $|h|=|h_0|=1$. Also, since $f+f_0\in N$ and $T(f+f_0)=Tf+Tf_0$,

$$f+f_0-(Tf+Tf_0)=th_1\ , \tag{3.2}$$

where $t\geqq 0$, h_1 is continuous on bK and $|h_1|=1$. If we put $p=rs^{-1}>0$, $q=ts^{-1}\geqq 0$, $g=hh_0^{-1}$, and $g_1=h_1h_0^{-1}$, then g, g_1 are in $C(bK)$ and $|g|=|g_1|=1$, and we have

$$1+pg=qg_1\ . \tag{3.3}$$

From this, we see g is constant by a simple calculation since bK is connected (for example, [2] p. 35), and so $h=ch_0$ for some constant c with $|c|=1$. By (3.1) we have that $f-crs^{-1}f_0=Tf-crs^{-1}Tf_0\in A$, which complete the proof.

PROOF OF THEOREM 3.1. If $\|T\|\leqq 2$, as in the proof of Lemma 3.2, Tf is an element of best approximation of f in A for $f\in N$. By Theorem

3.3, A has one codimension in N. This proves the theorem.

REMARK 3.4. (1) Let A be the disc algebra and let $N=\{\alpha\bar{z}+h: \alpha\in C,\ h\in A\}$. If we put $T(\alpha\bar{z}+h)=h$, then T is a projection of N onto A. Tf is the element of best approximation of f in A and $\|T\|=2$. We see that $\dim N/A=1$ and the hypothesis for codimension of Theorem 3.1 is necessary.

(2) If K° is not connected, the conclusion of Theorem 3.1 fails even if $C\backslash K$ is connected and $\dim N/A\geqq 2$. We can construct easily such an example.

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