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Analytic Functionals with Non-compact Carrier

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Introduction.

For a compact set L in the complex number plane $\boldsymbol{C}$, we denote by $\mathcal{O}(L)$ the space of germs of holomorphic functions on L. It is well known that the space $\mathcal{O}(L)$ can be equipped with the topology of DFS space. The dual space $\mathcal{O}'(L)$ of $\mathcal{O}(L)$ is called the space of analytic functionals with carrier in L and was studied extensively by many authors. The aim of this paper is to extend the theory of analytic functionals to the case where L is not compact. For the simplicity, we suppose in this paper L is a closed strip of finite width.

In § 1 we introduce the fundamental space $Q(L; K')$ of germs of holomorphic functions. We define in § 2 a new series of spaces of holomorphic functions, which will be used to describe the complex representations of the space $Q'(L; K')$ of analytic functionals with carrier in L and of exponential type in K'. § 3 treats the Cauchy transformation and we obtain a complex representation of $Q'(L; K')$. In §§ 4 and 5, we will study the case where L is a closed right strip and that find the situation is very similar to the classical theory of analytic functionals. We will show, among others, the image of the Fourier transformation of $Q'(L; K')$ is the space $\mathrm{Exp}(\boldsymbol{R}+i(-\infty, -k_2'); L)$ of holomorphic functions of exponential type L defined on the open half plane $\boldsymbol{R}+i(-\infty, -k_2')$.

In the final section, we will treat the case where L is an entire strip $L=\boldsymbol{R}+iK$. Our space of analytic functionals $Q'(L; K')$ is a subspace of the space of Fourier ultra-hyperfunctions (Park-Morimoto [6]), which were first introduced by Sebastiaõ e Silva [8] under the name of ultra-distributions of exponential growth. If $L=\boldsymbol{R}$ and $K'=\{0\}$, then our space reduces to the space of Fourier hyperfunctions introduced by M. Sato [7] and studied by T. Kawai [2]. Using the results obtained in §§ 4 and 5, we will study the relation of two definitions of the Fourier transformation of $Q'(L; K')$, one is by the duality and other is by the

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complex method. We will find several forms of complex representation of the space $Q'(\boldsymbol{R}+iK;\ K')$.

The ideas of this paper were announced in [4] but the details have been improved (see also [5]).

§1. The fundamental space $Q(L;\ K')$.

Let L be a closed strip of finite width in the complex number plane $\boldsymbol{C}$:

$$L=A+iK\ , \qquad i=\sqrt{-1}\ , \tag{1.1}$$

A being a closed interval and K a compact interval. Let K' be a compact interval. We denote by $Q_b(L;\ K')$ the space of all continuous functions f on L holomorphic in the interior int L which satisfy the following condition:

$$\sup_{z \in L} |f(z)| \exp(H_{K'}(x)) < \infty\ , \tag{1.2}$$

where we denote $z=x+iy$ and $H_{K'}(x)$ is the supporting function of K':

$$H_{K'}(x)=\sup_{\eta \in K'} x\eta\ . \tag{1.3}$$

It is clear that the space $Q_b(L;\ K')$ endowed with the norm (1.2) is a Banach space. If $L_1 \supset L_2$ and $K'_1 \supset K'_2$, the restriction mapping

$$Q_b(L_1;\ K'_1) \longrightarrow Q_b(L_2;\ K'_2) \tag{1.4}$$

is a continuous linear injection.

We define, taking the inductive limit following mappings (1.4) as $\varepsilon \downarrow 0$ and $\varepsilon' \downarrow 0$,

$$Q(L;\ K')=\lim_{\varepsilon \downarrow 0,\ \varepsilon' \downarrow 0} \operatorname{ind}\ Q_b(L_\varepsilon;\ K'(\varepsilon'))\ , \tag{1.5}$$

where we put

$$L_\varepsilon=L+[-\varepsilon,\ \varepsilon]+i[-\varepsilon,\ \varepsilon],\ K'(\varepsilon')=K'+[-\varepsilon',\ \varepsilon']\ . \tag{1.6}$$

If L is a compact set, the space $Q(L;\ K')$ coincides with the space $\mathcal{O}(L)$ of germs of holomorphic functions on L. The space $Q(L;\ K')$ is a DFS space, being the inductive limit of the compact increasing sequence of Banach spaces:

$$Q_b(L_\varepsilon;\ K'(\varepsilon')) \longrightarrow Q_b(L_{\varepsilon_1};\ K'(\varepsilon'_1))\ , \qquad \varepsilon > \varepsilon_1 > 0\ . \tag{1.7}$$

A continuous linear functional S on the space $Q(L; K')$ is, by definition, *an analytic functional with carrier in* L *and of exponential type in* K'. We denote by $Q'(L; K')$ the dual space of $Q(L; K')$. $Q'(L; K')$ is an FS space. The value of an analytic functional $S \in Q'(L; K')$ at a testing function $f \in Q(L; K')$ will be denoted by $\langle S, f\rangle$ or by $\langle S_z, f(z)\rangle$.

By the definition of the locally convex inductive limit topology, we have

LEMMA 1.1. *A linear functional* S *on* $Q(L; K')$ *is continuous if and only if, for any* $\varepsilon>0$ *and* $\varepsilon'>0$, *there exists a non-negative constant* C *such that*

$$|\langle S, f\rangle| \leqq C \sup_{z\in L_\varepsilon} |f(z)| \exp(H_{K'}(x)+\varepsilon' |x|) \tag{1.8}$$

for any $f \in Q_b(L_\varepsilon; K'(\varepsilon'))$.

PROPOSITION 1.1. *Let* L_0 *be a compact set such that* $L_0 \subset L$. *Then the natural mapping*

$$\mathcal{O}'(L_0) \longrightarrow Q'(L; K') \tag{1.9}$$

is injective.

PROOF. By the Hahn-Banach theorem, we have only to show the image of the restriction $Q(L: K') \rightarrow \mathcal{O}(L_0)$ is dense in $\mathcal{O}(L_0)$. By the Runge theorem, we can find, for any $f \in \mathcal{O}(L_0)$, a sequence of polynomials f_n which converges uniformly to f on $L_{0,\varepsilon}$ for some $\varepsilon>0$. Then the functions $\exp(-(1/n)z^2)f_n(z) \in Q(L; K')$ converge to f uniformly on $L_{0,\varepsilon}$, hence in the topology of $\mathcal{O}(L_0)$, as $n \rightarrow \infty$. Q.E.D.

By the mapping (1.9) we may consider the space $\mathcal{O}'(L_0)$ of analytic functionals with carrier in the compact set L_0 as a subspace of our space $Q'(L; K')$ of analytic functionals.

PROPOSITION 1.2. *Let* L *be the closed strip* (1.1). *Suppose compact intervals* K_1' *and* K_2' *satisfy* $K_1' \subset K_2'$. *Then the natural mapping*

$$Q'(L; K_1') \longrightarrow Q'(L; K_2') \tag{1.10}$$

is injective.

Proof is similar to the preceding one and is omitted.

§ 2. Spaces $R^L(C\backslash L; K')$, $R^L(C; K')$ and $H^1_L(C; R^L(K'))$.

In this section we use constantly the following version of Phragmén-

Lindelöf theorem:

THEOREM 2.1. *Suppose L is a closed strip of the form* (1.1). *Suppose $0<\varepsilon<r$. Let f be a continuous function on the closed set L_r which is holomorphic in* $\operatorname{int} L_r$ *and of exponential order in the following sense*:

(2.1) *There exist non-negative numbers C and n such that $|\exp(-C|w|^n)f(w)|$ is bounded on the strip L_r.*

Then $\sup_{w\in L_r\setminus L_\varepsilon}|f(w)|\leqq M$ *implies* $\sup_{w\in L_r}|f(w)|\leqq M$.

For the proof we refer the reader to Hille [1], vol. 2, p. 393.

In order to describe the image of the Cauchy transformation of $Q'(L;\ K')$, we introduce a new series of function spaces. For positive numbers ε, r and ε' such that $\varepsilon<r$, we denote by $R_b(\overline{L_r\setminus L_\varepsilon};\ K'(\varepsilon'))$ the space of all continuous functions F on the closed set $\overline{L_r\setminus L_\varepsilon}$ which are holomorphic in $(\operatorname{int} L_r)\setminus L_\varepsilon$ and satisfy the growth condition:

$$\sup_{w\in L_r\setminus L_\varepsilon}|F(w)|\exp(-H_{K'}(u)-\varepsilon'|u|)<\infty\ , \tag{2.2}$$

where $w=u+iv$, $i=\sqrt{-1}$. For positive numbers r and ε' we denote by $R_b(L_r;\ K'(\varepsilon'))$ the space of all continuous functions on L_r which are holomorphic in $\operatorname{int} L_r$ and satisfy the growth condition:

$$\sup_{w\in L_r}|F(w)|\exp(-H_{K'}(u)-\varepsilon'|u|)<\infty\ . \tag{2.2'}$$

Equipped with the norms (2.2) and (2.2′) respectively, the spaces $R_b(\overline{L_r\setminus L_\varepsilon};\ K'(\varepsilon'))$ and $R_b(L_r;\ K'(\varepsilon'))$ are Banach spaces. By the restriction mapping, we consider the space $R_b(L_r;\ K'(\varepsilon'))$ is a subspace of the space $R_b(\overline{L_r\setminus L_\varepsilon};\ K'(\varepsilon'))$. By Theorem 2.1, $R_b(L_r;\ K'(\varepsilon'))$ is closed in $R_b(\overline{L_r\setminus L_\varepsilon};\ K'(\varepsilon'))$.

If $0<\varepsilon_1<\varepsilon<r<r_1$ and $0<\varepsilon_1'<\varepsilon'$, we can define the following commutative diagram of restriction mappings:

$$\begin{array}{lccc}
(2.3) & R_b(\overline{L_{r_1}\setminus L_{\varepsilon_1}};\ K'(\varepsilon_1')) & \longrightarrow & R_b(\overline{L_r\setminus L_\varepsilon};\ K'(\varepsilon')) \\
 & \uparrow\cup & & \uparrow\cup \\
(2.4) & R_b(L_{r_1};\ K'(\varepsilon_1')) & \longrightarrow & R_b(L_r;\ K'(\varepsilon'))\ .
\end{array}$$

The mappings (2.3) and (2.4) are compact operators.

For every fixed $\varepsilon'>0$, we form the projective limits of the spaces

$R_b(\overline{L_r\backslash L_\varepsilon};\ K'(\varepsilon'))$ and $R_b(L_r;\ K'(\varepsilon'))$ letting $r\uparrow\infty$ and $\varepsilon\downarrow 0$ following the mappings (2.3) and (2.4) respectively:

$$R_b^L(C\backslash L;\ K'(\varepsilon'))=\lim_{r\uparrow\infty,\ \varepsilon\downarrow 0}\text{proj}\ R_b(L_r\backslash L_\varepsilon;\ K'(\varepsilon'))\ , \tag{2.5}$$

$$R_b^L(C;\ K'(\varepsilon'))=\lim_{r\uparrow\infty}\text{proj}\ R_b(L_r;\ K'(\varepsilon'))\ . \tag{2.6}$$

These spaces $R_b^L(C\backslash L;\ K'(\varepsilon'))$ and $R_b^L(C;\ K'(\varepsilon'))$ are FS spaces, being the projective limits of compact decreasing sequences of Banach spaces. The space $R_b^L(C;\ K'(\varepsilon'))$ is considered as a closed subspace of the space $R_b^L(C\backslash L;\ K'(\varepsilon'))$ by restriction.

Suppose $0<\varepsilon_1'<\varepsilon'$. We have the following commutative diagram of continuous linear injections as the projective limit of the diagram (2.3)-(2.4):

$$\begin{array}{lccc} (2.7) & R_b^L(C\backslash L;\ K'(\varepsilon_1')) & \longrightarrow & R_b^L(C\backslash L;\ K'(\varepsilon')) \\ & \uparrow & & \uparrow \\ (2.8) & R_b^L(C;\ K'(\varepsilon_1')) & \longrightarrow & R_b^L(C;\ K'(\varepsilon'))\ . \end{array}$$

Passing to the quotient, we can define canonically the continuous linear mapping:

$$R_b^L(C\backslash L;\ K'(\varepsilon_1'))/R_b^L(C;\ K'(\varepsilon_1'))\longrightarrow R_b^L(C\backslash L;\ K'(\varepsilon'))/R_b^L(C;\ K'(\varepsilon'))\ . \tag{2.9}$$

The mapping (2.9) is injective because of the Phragmén-Lindelöf theorem (Theorem 2.1).

We form further the projective limits of the spaces $R_b(\overline{L_r\backslash L_\varepsilon};\ K'(\varepsilon'))$ and $R_b(L_r;\ K'(\varepsilon'))$ letting $r\uparrow\infty$, $\varepsilon\downarrow 0$ and $\varepsilon'\downarrow 0$, following the mappings (2.3) and (2.4):

$$\begin{aligned} R^L(C\backslash L;\ K')&=\lim_{r\uparrow\infty,\varepsilon\downarrow 0,\varepsilon'\downarrow 0}\text{proj}\ R_b(\overline{L_r\backslash L_\varepsilon};\ K'(\varepsilon')) \\ &=\lim_{\varepsilon'\downarrow 0}\text{proj}\ R_b^L(C\backslash L;\ K'(\varepsilon'))\ , \end{aligned} \tag{2.10}$$

$$\begin{aligned} R^L(C;\ K')&=\lim_{r\uparrow\infty,\ \varepsilon'\downarrow 0}\text{proj}\ R_b(L_r;\ K'(\varepsilon')) \\ &=\lim_{\varepsilon'\downarrow 0}\text{proj}\ R_b^L(C;\ K'(\varepsilon'))\ . \end{aligned} \tag{2.11}$$

The spaces $R^L(C\backslash L;\ K')$ and $R^L(C;\ K')$ are FS spaces. By the Phragmén-Lindelöf theorem, the space $R^L(C;\ K')$ is a closed subspace of the space $R^L(C\backslash L;\ K')$.

DEFINITION 2.1. (i) We define the space $H_L^1(C;\ R^L(K'))$ of *the cohomology classes with carrier in L and of exponential type in K'* to be the quotient space of $R^L(C\backslash L;\ K')$ by its subspace $R^L(C;\ K')$:

$$H^1_L(C;\ R^L(K'))=R^L(C\backslash L;\ K')/R^L(C;\ K')\ . \tag{2.12}$$

(ii) We define further

$$\tilde{H}^1_L(C;\ R^L(K'))=\lim_{\varepsilon'\downarrow 0}\operatorname{proj} R^L_b(C\backslash L;\ K'(\varepsilon'))/R^L_b(C;\ K'(\varepsilon'))\ , \tag{2.13}$$

where the projective limit is taken following the mappings (2.9).

As a quotient space of an FS space by its closed subspace, the space $H^1_L(C;\ R^L(K'))$ is an FS space. For a function $F\in R^L(C\backslash L;\ K')$, we will denote by $[F]$ the cohomology class represented by F. Extending the canonical mapping

$$R^L(C\backslash L;\ K')\longrightarrow R^L_b(C\backslash L;\ K'(\varepsilon'))\ , \tag{2.14}$$

we can define canonically the continuous linear mapping

$$\kappa\colon H^1_L(C;\ R^L(K'))\longrightarrow \tilde{H}^1_L(C;\ R^L(K'))\ , \tag{2.15}$$

which is injective because of the Phragmén-Lindelöf theorem.

Suppose now $f\in Q(L;\ K')$ and $F\in R^L(C\backslash L;\ K')$ are given. We can find positive numbers ε_0 and ε_0' such that $f\in Q_b(L_{\varepsilon_0};\ K'(\varepsilon_0'))$. For a positive number ε with $\varepsilon<\varepsilon_0$, consider the integral

$$\int_{\partial L_\varepsilon} f(w)F(w)dw\ . \tag{2.16}$$

The integral (2.16) converges absolutely, as we can choose, for every positive number ε' with $\varepsilon'<\varepsilon_0'$, a non-negative number C such that

$$\begin{aligned}|f(w)F(w)|&\leqq C\exp(-H_{K'}(u)-\varepsilon_0'\,|u|)\ \exp(H_{K'}(u)+\varepsilon'\,|u|)\\&\leqq C\exp(-(\varepsilon_0'-\varepsilon')\,|u|)\end{aligned}$$

for $w=u+iv\in\partial L_\varepsilon$. For every fixed ε_1 with $0<\varepsilon_1<\varepsilon_0$, the constant C can be taken uniformly in ε with $\varepsilon_1<\varepsilon<\varepsilon_0$. Therefore by the Cauchy integral theorem, the integral (2.16) does not depend on the positive number ε. For the brevity, we will write the integral (2.16) as follows:

$$\int_{\partial L_{+0}} f(w)F(w)dw\ . \tag{2.16'}$$

We have clearly the following lemma:

LEMMA 2.1. *For* $F\in R^L(C\backslash L;\ K')$ *we define a linear functional* $\operatorname{Int}(F)$ *on the space* $Q(L;\ K')$ *by*

(2.17)
$$\mathrm{Int}(F)\colon f \longmapsto -\int_{\partial L_{+0}} f(w)F(w)dw\ .$$

Then Int(F) *belongs to* $Q'(L;\ K')$. *If* F *is in the space* $R^L(\boldsymbol{C};\ K')$, *we have* $\mathrm{Int}(F)=0$.

Thanks to Lemma 2.1, passing to the quotient we can define a continuous linear mapping

(2.18)
$$\mathrm{Int}\colon H^1_L(\boldsymbol{C};\ R^L(K')) \longrightarrow Q'(L;\ K')\ .$$

A series of functions $F(w;\ \varepsilon') \in R^L_b(\boldsymbol{C}\backslash L;\ K'(\varepsilon'))$, $\varepsilon'>0$, such that for any $\varepsilon'_1<\varepsilon'$,

$$F(w;\ \varepsilon'_1)-F(w;\ \varepsilon') \in R^L_b(\boldsymbol{C};\ \varepsilon')\ ,$$

gives an element of $\tilde{H}^1_L(\boldsymbol{C};\ R^L(K'))$, which we denote by $[F(w;\ +0)]$. Suppose $f\in Q(L;\ K')$ and $[F(w;\ +0)]\in \tilde{H}^1_L(\boldsymbol{C};\ R^L(K'))$ be given. We can find positive numbers ε_0 and ε'_0 such that $f\in Q_b(L_{\varepsilon_0};\ K'(\varepsilon'_0))$. For any ε and ε' such that $0<\varepsilon<\varepsilon_0$ and $0<\varepsilon'<\varepsilon'_0$, we can define the integral

(2.19)
$$\int_{\partial L_\varepsilon} f(w)F(w;\ \varepsilon')dw\ .$$

By the Cauchy integral theorem, the integral (2.19) is independent of the numbers ε, ε' and the choise of the representative $F(w;\ \varepsilon')$ of the cohomology class $[F(w;\ +0)]$. Therefore we will write (2.19) as follows:

(2.19′)
$$\int_{\partial L_{+0}} f(w)[F(w;\ +0)]dw\ .$$

For $[F(w;\ +0)]\in \tilde{H}^1_L(\boldsymbol{C};\ R^L(K'))$, we define a linear functional $\widetilde{\mathrm{Int}}[F]$ by

(2.20)
$$\widetilde{\mathrm{Int}}[F]\colon f \longmapsto -\int_{\partial L_{+0}} f(w)[F(w;\ +0)]dw$$

on the space $Q(L;\ K')$. As $\widetilde{\mathrm{Int}}\ [F]$ is clearly continuous on every space $Q_b(L_\varepsilon;\ K'(\varepsilon'))$, it is continuous on the space $Q(L;\ K')$.

It is clear by the definition of the mappings Int and $\widetilde{\mathrm{Int}}$ that we have the following proposition:

PROPOSITION 2.1. *The following diagram is commutative*:

(2.21)
$$\begin{array}{ccc} H^1_L(\boldsymbol{C};\ R^L(K')) & \xrightarrow{\ \kappa\ } & \tilde{H}^1_L(\boldsymbol{C};\ R^L(K')) \\ & \mathrm{Int}\searrow \quad \swarrow\widetilde{\mathrm{Int}} & \\ & Q'(L;\ K') & \end{array}$$

that is,

$$\mathrm{Int} = \widetilde{\mathrm{Int}} \circ \kappa \, . \tag{2.21'}$$

It will be proved in Theorem 3.4 that all mappings in the diagram (2.21) are isomorphisms. Especially the spaces $H_L^1(C; R^L(K'))$ and $\tilde{H}_L^1(C; R^L(K'))$ are canonically isomorphic.

§ 3. The Cauchy transformation.

In this section we study the Cauchy transformation of the space $Q'(L; K')$ using the kernel $[\exp(-(w-z)^2)]/(w-z)$. By the simple calculation, we have

LEMMA 3.1. *For $w=u+iv \in C\backslash L$, the function of z, $[\exp(-(w-z)^2)]/(w-z)$ belongs to the space $Q(L; K')$. For any positive numbers ε, r and ε' with $0<\varepsilon<r$, there exists a non-negative constant C such that*

$$\sup_{z \in L_\varepsilon} \left| \frac{\exp(-(w-z)^2)}{w-z} \right| \exp(H_{K'}(x)+\varepsilon' |x|) \tag{3.1}$$
$$\leqq C \exp(H_{K'}(u)+\varepsilon' |u|) \qquad \text{for} \quad w \in L_r \backslash L_\varepsilon \, .$$

THEOREM 3.1. (The Cauchy integral formula) *Choose arbitrarily a testing function $f \in Q(L; K')$ and find positive numbers ε_0 and ε_0' such that $f \in Q_b(L_{\varepsilon_0}; K'(\varepsilon_0'))$. For any ε and ε' with $0<\varepsilon<\varepsilon_0$ and $0<\varepsilon'<\varepsilon_0'$, we have the following Cauchy integral formula:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} f(w) \frac{\exp(-(w-z)^2)}{w-z} dw \tag{3.2}$$

for $z \in \mathrm{int}\, L_\varepsilon$. The integral converges in the topology of the space $Q(L; K')$.

PROOF. Fix $z \in \mathrm{int}\, L_\varepsilon$. We put

$$L_\varepsilon(u_0) = \{w \in L_\varepsilon;\ |\mathrm{Re}\, w| \leqq u_0\} \, .$$

If $u_0>0$ is so large that $|\mathrm{Re}\, z|<u_0$, we have, by the ordinary Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial L_\varepsilon(u_0)} f(w) \frac{\exp(-(w-z)^2)}{w-z} dw \, .$$

The integral over the segments

$$\{w \in \partial L_\varepsilon(u_0);\ \mathrm{Re}\, w = \pm u_0\}$$

being convergent to 0 as $u_0 \to +\infty$, we have the formula (3.2).

In order to show the convergence of the integral (3.2) in the topology of the space $Q(L; K')$, it is sufficient to show that

$$I(u_0)=\sup_{z \in L_{\varepsilon/2}} \exp(H_{K'}(x)+\varepsilon' |x|)\left|\int_{\partial L_{\varepsilon, u_0}} f(w)\frac{\exp(-(w-z)^2)}{w-z}dw\right| ,$$

tends to 0 as $u_0 \to +\infty$, where

$$\partial L_{\varepsilon, u_0}=\{w \in \partial L_\varepsilon;\ |\mathrm{Re}\, w| \geqq u_0\} .$$

We can find, by Lemma 2.1, a constant $C_1 \geqq 0$ such that

$$I(u_0) \leqq C_1 \int_{\partial L_{\varepsilon, u_0}} |f(w)| \exp(H_{K'}(u)+\varepsilon' |u|)|dw| .$$

Because $f \in Q_b(L_{\varepsilon_0};\ K'(\varepsilon_0'))$, $0<\varepsilon<\varepsilon_0$ and $0<\varepsilon'<\varepsilon_0'$, the right hand side converges to 0 as $u_0 \to +\infty$. Q.E.D.

DEFINITION 3.1. We define *the Cauchy transformation* $\check{S}(w)$ of an analytic functional $S \in Q'(L; K')$ by the following formula:

$$\check{S}(w)=\frac{-1}{2\pi i}\left\langle S_z, \frac{\exp(-(w-z)^2)}{w-z}\right\rangle . \tag{3.3}$$

THEOREM 3.2. *The Cauchy transformation* $\check{S}(w)$ *of* $S \in Q'(L; K')$ *is a holomorphic function on* $\boldsymbol{C}\backslash L$ *and, for any positive numbers* ε, r *and* ε' *with* $0<\varepsilon<r$, *we have*

$$\sup_{w \in L_r \backslash L_\varepsilon} |\check{S}(w)| \exp(-H_{K'}(u)-\varepsilon' |u|)<\infty , \tag{3.4}$$

that is, $\check{S}$ *belongs to the space* $R^L(\boldsymbol{C}\backslash L; K')$ *defined in* § 2.

Proof is easy and left to the readers.

To $S \in Q'(L; K')$ we associate the cohomology class $[\check{S}]$ of $S \in R^L(\boldsymbol{C}\backslash L; K')$ in the quotient space $H_L^1(\boldsymbol{C}; R^L(K'))$. This mapping is also called *the Cauchy transformation* and will be denoted by $\mathscr{C}$:

$$\mathscr{C}: Q'(L; K') \longrightarrow H_L^1(\boldsymbol{C}; R^L(K')) . \tag{3.5}$$

The following theorem claims that the Cauchy transformation $\mathscr{C}$ (3.5) and the mapping Int (2.18) are inverse to each other.

THEOREM 3.3. (i) *Let* $S \in Q'(L; K')$ *and* $f \in Q(L; K')$ *be given. Then we have the following inversion formula*:

(3.6) $$\langle S, f\rangle = -\int_{\partial L_{+0}} f(w)\check{S}(w)dw = \langle \mathrm{Int}(\check{S}), f\rangle ,$$

that is,

(3.7) $$\mathrm{Int}\circ\mathscr{C} = \mathrm{id} .$$

(ii) *We have also*

(3.8) $$\mathscr{C}\circ\mathrm{Int} = \mathrm{id} ,$$

where the mappings Int *and* $\mathscr{C}$ *are defined by* (2.18) *and* (3.5) *respectively.*

PROOF. (i) can be concluded from the Cauchy integral formula (Theorem 3.1).

(ii) Let $F\in R^L(C\backslash L;\ K')$ and put $S=\mathrm{Int}(F)\in Q'(L;\ K')$. If $f\in Q_b(L_{\varepsilon_0};\ K'(\varepsilon_0'))$, we have by (i)

(3.9) $$\int_{\partial L_\varepsilon} f(w)(F(w)-\check{S}(w))dw = 0$$

for $0<\varepsilon<\varepsilon_0$. Put

$$\psi(w) = F(w) - \check{S}(w) .$$

Define

$$G(z) = \frac{1}{2\pi i}\int_{\partial L_{\varepsilon_0}} \psi(w)\ \frac{\exp(-(w-z)^2)}{w-z}dw$$

for $z\in \mathrm{int}\, L_{\varepsilon_0}$. The function $G(z)$ is clearly of exponential order in $\mathrm{int}\, L_{\varepsilon_0}$ and by (3.9) we have

$$G(w) = \psi(w) \qquad \text{for} \quad w\in(\mathrm{int}\, L_{\varepsilon_0})\backslash L_\varepsilon .$$

Therefore, if we put

$$\tilde{\psi}(w) = \begin{cases} \psi(w) & \text{for} \quad w\in C\backslash L \\ G(w) & \text{for} \quad w\in \mathrm{int}\, L_{\varepsilon_0} , \end{cases}$$

the function $\tilde{\psi}$ gives an analytic continuation of the function ψ to the whole complex plane C. From the Phragmén-Lindelöf theorem (Theorem 2.1), we conclude

$$\tilde{\psi}(w) = F(w) - \check{S}(w) \in R^L(C;\ K') ,$$

that is, $[F]=[\check{S}]$. This proves (ii). Q.E.D.

THEOREM 3.4. *The following diagram is commutative*:

(3.10)
$$\begin{array}{ccc} H^1_L(C;\ R^L(K')) & \xrightarrow{\ \kappa\ } & \tilde{H}^1_L(C;\ R^L(K')) \\ \text{Int}\searrow\nwarrow\mathscr{C} & & \swarrow\widetilde{\text{Int}} \\ & Q'(L;\ K') & \end{array}$$

Every mappings appeared in the diagram is a linear topological isomorphism.

PROOF. The commutativity is clear by Proposition 2.1 and Theorem 3.3. From the formula

(3.11) $$\widetilde{\text{Int}}\circ\kappa=\text{Int}\ ,$$

(3.12) $$\kappa\circ\mathscr{C}\circ\widetilde{\text{Int}}=\text{id}\ ,$$

we can conclude the mapping $\widetilde{\text{Int}}$ and κ are linear topological isomorphisms. Q.E.D.

We will prove a density theorem.

THEOREM 3.5. *Suppose the closed strips L_1 and L_2 of the form (1.1) satisfy $L_1\subset L_2$ and the compact intervals K_1' and K_2' satisfy $K_1'\subset K_2'$. Then the natural mapping*

(3.13) $$Q'(L_1;\ K_1')\longrightarrow Q'(L_2;\ K_2')$$

is injective.

PROOF. Thanks to Proposition 1.2, we may suppose $K_1'=K_2'=K'$. By Theorem 3.4, we have only to show the natural mapping

(3.14) $$H^1_{L_1}(C;\ R^{L_1}(K'))\longrightarrow H^1_{L_2}(C;\ R^{L_2}(K'))$$

is injective. By the Phragmén-Lindelöf theorem, we have

(3.15) $$R^{L_1}(C\backslash L_1;\ K')\cap R^{L_2}(C;\ K')=R^{L_1}(C;\ K')\ ,$$

from which concludes the injectivity of the mapping (3.14). Q.E.D.

§ 4. The ε'-cauchy transformation.

We suppose in this section L is a right half strip:

(4.1) $$L=A+iK\ ,\quad A=[a,\ +\infty),\ K=[k_1,\ k_2]\ .$$

$K'=[k_1',\ k_2']$ is a compact interval. The following lemmas are easy to prove.

LEMMA 4.1. *If L is a right half strip* (4.1), *then the space $Q_b(L_\varepsilon; K'(\varepsilon'))$ coincides with the space of all continuous functions f on L_ε which are holomorphic in* int L_ε *and satisfy*

$$\sup_{z \in L_\varepsilon} |f(z) \exp((k_2' + \varepsilon')z)| < \infty \,. \tag{4.2}$$

LEMMA 4.2. (i) *Choose arbitrarily $w \in C \backslash L$ and $\varepsilon' > 0$. Then the function* $[\exp((k_2' + \varepsilon')(w-z))]/(w-z)$ *belongs to the space $Q(L; K')$.*

(ii) *The function which associates with $w \in C \backslash L$ the function of z,* $[\exp((k_2' + \varepsilon')(w-z))]/(w-z)$ *is a $Q(L; K')$-valued holomorphic function on $C \backslash L$.*

Similarly to Theorem 3.1, we have the following Cauchy integral formula:

THEOREM 4.1. *Choose arbitrarily $f \in Q(L; K')$ and find positive numbers ε_0 and ε_0' such that $f \in Q_b(L_{\varepsilon_0}; K'(\varepsilon_0'))$. For any ε and ε' with $0 < \varepsilon < \varepsilon_0$ and $0 < \varepsilon' < \varepsilon_0'$, we have the Cauchy integral formula:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} f(w) \frac{\exp((k_2' + \varepsilon')(w-z))}{w-z} \, dw \tag{4.3}$$

for $z \in$ int L_ε. *The integral* (4.3) *converges in the topology of the space $Q(L; K')$.*

DEFINITION 4.1. For $S \in Q'(L; K')$ and $\varepsilon' > 0$, we put

$$\check{S}(w;\, \varepsilon') = \frac{-1}{2\pi i} \left\langle S_z,\, \frac{\exp((k_2' + \varepsilon')(w-z))}{w-z} \right\rangle .$$

We call $S(w; \varepsilon')$ *the ε'-Cauchy transformation* of a functional $S \in Q'(L; K')$.

From Lemma 4.2, we can conclude

PROPOSITION 4.1. *The ε'-Cauchy transformation $\check{S}(w; \varepsilon')$ of a functional $S \in Q'(L; K')$ is a holomorphic function on $C \backslash L$ and satisfies, for any $\varepsilon > 0$,*

$$\sup_{w \notin L_\varepsilon} |\check{S}(w;\, \varepsilon') \exp(-(k_2' + \varepsilon')w)| < \infty \,. \tag{4.4}$$

PROOF. By the continuity of the functional S, for any positive numbers ε and ε', we can find a non-negative number C such that

$$(4.5) \qquad |\langle S, f\rangle| \leqq C \sup_{z \in L_\varepsilon} |f(z)\exp((k_2'+\varepsilon')z)|$$

for any $f \in Q_b(L_\varepsilon;\ K'(\varepsilon'))$. If we fix $w \in \boldsymbol{C} \setminus L_{2\varepsilon}$,

$$\sup_{z \in L_\varepsilon} \left| \frac{\exp(-(k_2'+\varepsilon')z)}{w-z} \exp((k_2'+\varepsilon')z) \right| \leqq \frac{1}{\varepsilon}\,.$$

Therefore we have by (4.5)

$$\begin{aligned} &\sup_{w \notin L_{2\varepsilon}} |\check{S}(w;\ \varepsilon')\exp(-(k_2'+\varepsilon')w)| \\ &\quad = \sup_{w \notin L_{2\varepsilon}} \frac{1}{2\pi} \left| \left\langle S_z,\ \frac{\exp(-(k_2'+\varepsilon')z)}{w-z} \right\rangle \right| \leqq \frac{1}{2\pi} C \frac{1}{\varepsilon}\,. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we have shown (4.4). Q.E.D.

PROPOSITION 4.2. *Suppose $S \in Q'(L;\ K')$ and $\varepsilon' > \varepsilon'' > 0$. Then the function*

$$(4.6) \qquad F(w) = \check{S}(w;\ \varepsilon') - \check{S}(w;\ \varepsilon'')$$

can be analytically continued to an entire function of w. Further we have

$$(4.7) \qquad \sup_{w \in \boldsymbol{C}} |F(w)| \exp(-k_2'u - \varepsilon'u_+ - \varepsilon''u_-) < \infty\,,$$

where we put

$$(4.8) \qquad u_+ = \begin{cases} u & \text{for } u \geqq 0 \\ 0 & \text{for } u < 0\,, \end{cases} \qquad u_- = \begin{cases} 0 & \text{for } u \geqq 0 \\ u & \text{for } u < 0\,. \end{cases}$$

REMARK.

$$\begin{aligned} H_{[k_2'+\varepsilon'', k_2'+\varepsilon']}(u) &= \sup\{yu;\ y \in [k_2'+\varepsilon'',\ k_2'+\varepsilon']\} \\ &= (k_2'+\varepsilon')u_+ + (k_2'+\varepsilon'')u_- \\ &= k_2'u + \varepsilon'u_+ + \varepsilon''u_-\,. \end{aligned}$$

PROOF. It is clear by Proposition 4.1 that the function $F(w)$ is holomorphic in $\boldsymbol{C} \setminus L$ and

$$(4.9) \qquad \sup_{w \notin L_\varepsilon} |F(w)| \exp(-k_2'u - \varepsilon'u_+ - \varepsilon''u_-) < \infty$$

for any $\varepsilon > 0$. Now fix $w \in L_\varepsilon$ arbitrarily. The function of z

$$G(w-z) = (w-z)^{-1}\{\exp((k_2'+\varepsilon')(w-z)) - \exp((k_2'+\varepsilon'')(w-z))\}$$

is entire. Further, $G(w-z)$ belongs to the space $Q(L;\ K')$. In fact, for $z\in L_{2\varepsilon}$, $|w-z|\geqq\varepsilon$, we have

$$\begin{aligned}&|G(w-z)\exp((k_2'+\varepsilon'')z)|\\&\quad\leqq\frac{1}{\varepsilon}\{\exp((k_2'+\varepsilon')u)\exp((\varepsilon''-\varepsilon')(a-2\varepsilon))+\exp((k_2'+\varepsilon'')u)\}\ .\end{aligned}$$

By the maximum modulus principle, we have, for $w\in L_\varepsilon$,

$$\begin{aligned}&\sup_{z\in L_{2\varepsilon}}|G(w-z)\exp((k_2'+\varepsilon'')z)|\\&\quad\leqq\frac{1}{\varepsilon}\{\exp((k_2'+\varepsilon')u)\exp((\varepsilon''-\varepsilon')(a-2\varepsilon))+\exp((k_2'+\varepsilon'')u)\}\ ,\end{aligned}$$

that is, the function of z, $G(w-z)$ belongs to the space $Q_b(L_{2\varepsilon};\ K'(\varepsilon''))$. Therefore the function

$$F(w)=\frac{-1}{2\pi i}\ \langle S_z,\ G(w-z)\rangle$$

is defined for $w\in L_\varepsilon$ and we have, by the continuity of S,

$$\sup_{w\in L_\varepsilon}|F(w)|\leqq C_1\exp((k_2'+\varepsilon')u)+C_2\exp((k_2'+\varepsilon'')u)$$

for some non-negative constants C_1 and C_2. This proves the inequality (4.7). Q.E.D.

LEMMA 4.3. *If L is the right half strip* (4.1), *we have*

(4.10) $$\begin{aligned}&R_b^L(C\backslash L;\ K'(\varepsilon'))\\&\quad=\{F\in\mathcal{O}(C\backslash L);\ \sup_{w\in L_r\backslash L_\varepsilon}|F(w)|\exp(-(k_2'+\varepsilon')u)<\infty\ \textit{for any}\ r>\varepsilon>0\}\ ,\end{aligned}$$

(4.11) $$\begin{aligned}&R_b^L(C;\ K'(\varepsilon'))\\&\quad=\{F\in\mathcal{O}(C);\ \sup_{w\in L_r}|F(w)|\exp(-(k_2'+\varepsilon')u)<\infty\ \textit{for any}\ r>0\}\ .\end{aligned}$$

PROOF. The lemma results from the left boundedness of the strip L. Q.E.D.

DEFINITION 4.2. Thanks to Propositions 4.1 and 4.2 and Lemma 4.3, the ε'-Cauchy transformations $\check{S}(w;\ \varepsilon')$ of the functional $S\in Q'(L;\ K')$ define a cohomology class $[\check{S}(w;\ +0)]$ in $\tilde{H}_L^1(C;\ R^L(K'))$. We will write the mapping $S\mapsto[\check{S}(w;\ +0)]$ as follows:

(4.12) $$\tilde{\mathscr{C}}:\ Q'(L;\ K')\longrightarrow\tilde{H}_L^1(C;\ R^L(K'))$$

and call it also *the ε'-Cauchy transformation.*

From the Cauchy integral formula (Theorem 4.1), we can conclude

PROPOSITION 4.3. *Suppose L is the right half strip* (4.1). *Let* $S \in Q'(L;\ K')$ *and* $f \in Q(L;\ K')$ *be given.* $[\check{S}(w;\ +0)] \in \tilde{H}^1_L(C;\ K')$ *denotes the ε'-Cauchy transformation of S. Then we have the following inversion formula*:

$$\langle S, f\rangle = -\int_{\partial L_{+0}} f(w)[\check{S}(w;\ +0)]dw = \langle \widetilde{\mathrm{Int}}[\check{S}(w;\ +0)], f\rangle\ . \tag{4.13}$$

THEOREM 4.2. *Suppose L is the right half strip* (4.1). *Then the ε'-Cauchy transformation* $\tilde{\mathscr{C}}$ *and the operator* $\widetilde{\mathrm{Int}}$ *are inverse to each other, that is,*

(i) $\widetilde{\mathrm{Int}} \circ \tilde{\mathscr{C}} = \mathrm{id}$, (ii) $\tilde{\mathscr{C}} \circ \widetilde{\mathrm{Int}} = \mathrm{id}$.

PROOF. (i) is nothing but the preceding proposition. By (i), we have

$$\widetilde{\mathrm{Int}} \circ \tilde{\mathscr{C}} \circ \widetilde{\mathrm{Int}} = \widetilde{\mathrm{Int}}\ .$$

By Theorem 3.4, $\widetilde{\mathrm{Int}}$ is a linear topological isomorphism. Therefore we have (ii). Q.E.D.

§ 5. The Fourier transformation of $Q'(L;\ K')$.

We assume in this section as in § 4 that the strip L is a right half strip (4.1).

LEMMA 5.1. *The function* $\exp(-iz\zeta)$ *of z belongs to the space* $Q(L;\ K')$ *if and only if* $\eta = \mathrm{Im}\zeta < -k_2'$.

PROOF. As we have

$$|\exp(-iz\zeta)| = \exp(x\eta + y\xi)\ , \qquad z = x+iy,\ \zeta = \xi + i\eta\ ,$$

the lemma results from Lemma 4.1. Q.E.D.

DEFINITION 5.1. For $S \in Q'(L;\ K')$ we define *the Fourier transformation* $\tilde{S}(\zeta)$ of S as follows:

$$\tilde{S}(\zeta) = \langle S_z, \exp(-iz\zeta)\rangle\ . \tag{5.1}$$

$\tilde{S}(\zeta)$ is a function on the open half plane

$$\boldsymbol{R} + i(-\infty,\ -k_2') = \{\zeta = \xi + i\eta \in \boldsymbol{C};\ \eta < -k_2'\}\ .$$

DEFINITION 5.2. We define the space $\mathrm{Exp}(\boldsymbol{R}+i(-\infty, -k_2'); L)$ to be the space of the holomorphic functions F on the open half plane $\boldsymbol{R}+i(-\infty, -k_2')$ which are of exponential type in L in the following sense: For any positive numbers ε and ε' there exists a non-negative number C such that

$$|F(\zeta)| \leqq C \exp(h_L(\zeta)+\varepsilon|\zeta|) , \tag{5.2}$$

where $h_L(\zeta)=H_K(\xi)+a\eta$, $\zeta=\xi+i\eta$.

PROPOSITION 5.1. *The Fourier transformation* $\mathscr{F}: S \mapsto \tilde{S}$ *maps the space* $Q'(L; K')$ *into the space* $\mathrm{Exp}(\boldsymbol{R}+i(-\infty, -k_2'); L)$:

$$\mathscr{F}: Q'(L;\ K') \longrightarrow \mathrm{Exp}(\boldsymbol{R}+i(-\infty, -k_2');\ L) . \tag{5.3}$$

PROOF. Let $S \in Q'(L; K')$. By the continuity of S, we can find, for any positive numbers ε and ε', a non-negative number C such that

$$\begin{aligned} |\tilde{S}(\zeta)| &= |\langle S_z, \exp(-iz\zeta)\rangle| \\ &\leqq C \sup_{z \in L_\varepsilon} |\exp(-iz\zeta)| \exp((k_2'+\varepsilon')x) \\ &= C \exp((a-\varepsilon)(\eta+k_2'+\varepsilon'))\exp(H_K(\xi)+\varepsilon|\xi|) \end{aligned}$$

for $\eta+k_2'+\varepsilon' \leqq 0$, from which results (5.2). Q.E.D.

Suppose now $F \in \mathrm{Exp}(\boldsymbol{R}+i(-\infty, -k_2'); L)$ be given. Fix $\zeta_0=\xi_0+i\eta_0$ and $\zeta'=\xi'+i\eta'$ such that

$$\eta_0=\mathrm{Im}\, \zeta_0 < -k_2', \quad |\zeta'|=1, \quad \eta'=\mathrm{Im}\, \zeta' \leqq 0 .$$

Consider the following integral:

$$\begin{aligned} &\hat{F}(w, \zeta_0, \zeta') \\ &= \frac{1}{2\pi}\int_{\zeta_0+\boldsymbol{R}^+\zeta'} F(\tau)\exp(iw\tau)d\tau \\ &= \frac{1}{2\pi}\int_0^\infty F(\zeta_0+t\zeta')\exp(iw(\zeta_0+t\zeta'))\zeta' dt , \end{aligned} \tag{5.4}$$

where $\boldsymbol{R}^+=[0, \infty)$. By (5.2) we have for any $t \geqq 0$

$$\begin{aligned} &|F(\zeta_0+t\zeta')| \\ &\leqq C \exp((a-\varepsilon)\eta_0)\exp(t(H_K(\xi')+\varepsilon|\xi'|+(a-\varepsilon)\eta')) . \end{aligned}$$

Therefore the integral $\hat{F}(w, \zeta_0, \zeta')$ is absolutely convergent for $w \in W_\varepsilon(\zeta')$, where

$$W_\varepsilon(\zeta')=\{w\in C;\ -\mathrm{Im}\ w\zeta'+H_K(\xi')+\varepsilon|\xi'|+(a-\varepsilon)\eta'<0\}\ .$$

We put also

$$W(\zeta')=\{w\in C;\ -\mathrm{Im}\ w\zeta'+H_K(\xi')+a\eta'<0\}\ .$$

$W_\varepsilon(\zeta')$ and $W(\zeta)$ are open half planes and we have

$$W(\zeta')=\bigcup_{\varepsilon>0} W_\varepsilon(\zeta')\ .$$

For example, we have

$$\begin{aligned} &W_\varepsilon(1)=\{w;\ -v+k_2+\varepsilon<0\}\ , && W(1)=\{w;\ v>k_2\}\ ,\\ &W_\varepsilon(-i)=\{w;\ u-(a-\varepsilon)<0\}\ , && W(-i)=\{w;\ u<a\}\ ,\\ &W_\varepsilon(-1)=\{w;\ v-k_1+\varepsilon<0\}\ , && W(-1)=\{w;\ v<k_1\}\ ,\end{aligned}$$

where we put $w=u+iv$. By the definition, we have

$$\bigcup\{W_\varepsilon(\zeta');\ |\zeta'|=1,\ \mathrm{Im}\ \zeta'\leqq 0\}=C\backslash L_\varepsilon\ ,$$
$$\bigcup\{W(\zeta');\ |\zeta'|=1,\ \mathrm{Im}\ \zeta'\leqq 0\}=C\backslash L\ .$$

In particular, $\{W_\varepsilon(\zeta');\ \varepsilon>0,\ |\zeta'|=1,\ \mathrm{Im}\ \zeta'\leqq 0\}$ is an open covering of $C\backslash L$ by open half planes.

LEMMA 5.2. *The function of w, $\hat{F}(w, \zeta_0, \zeta')$ is holomorphic on the half plane $W(\zeta')$.*

PROOF. In fact, $\hat{F}(w, \zeta_0, \zeta')$ is absolutely convergent on $W_\varepsilon(\zeta')$ and is holomorphic there. Q.E.D.

LEMMA 5.3. *Suppose* $|\zeta'|=|\zeta''|=1$, $\mathrm{Im}\ \zeta'\leqq 0$ *and* $\mathrm{Im}\ \zeta''\leqq 0$. *Then we have*

$$\hat{F}(w, \zeta_0, \zeta')=\hat{F}(w, \zeta_0, \zeta'') \qquad \textit{for} \quad w\in W(\zeta')\cap W(\zeta'')\ . \tag{5.5}$$

PROOF. It is sufficient to show (5.5) for $w\in W_\varepsilon(\zeta')\cap W_\varepsilon(\zeta'')$, for every $\varepsilon>0$. The function of τ, $F(\tau)\exp(iw\tau)$ is a holomorphic function of exponential type on the sector spanned by $\zeta_0+\boldsymbol{R}^+\zeta'$ and $\zeta_0+\boldsymbol{R}^+\zeta''$ and decreases exponentially at the infinity on the boundaries $\zeta_0+\boldsymbol{R}^+\zeta'$ and $\zeta_0+\boldsymbol{R}^+\zeta''$. Therefore by the Phragmén-Lindelöf theorem, $F(\tau)\exp(iw\tau)$ decreases exponentially at the infinity uniformly in the sector. Therefore by the Cauchy integral theorem,

$$\left\{\int_{\zeta_0+\boldsymbol{R}^+\zeta'}-\int_{\zeta_0+\boldsymbol{R}^+\zeta''}\right\}F(\tau)\exp(iw\tau)d\tau=0\ ,$$

from which results the lemma. Q.E.D.

LEMMA 5.4. *Suppose* $\zeta_0=-(k_2'+\varepsilon')i$. *We define a holomorphic function* $\hat{F}(w, \zeta_0)$ *on* $C\backslash L$ *by*

$$\hat{F}(w, \zeta_0)=\hat{F}(w, \zeta_0, \zeta') \quad \text{for} \quad w\in W(\zeta') . \tag{5.6}$$

Then for any $\varepsilon>0$ *we have*

$$\sup_{w\notin L_\varepsilon}|\hat{F}(w, \zeta_0)|\exp(-(k_2'+\varepsilon')u)<\infty . \tag{5.7}$$

In particular,

$$\hat{F}(w, \zeta_0)\in R_d^L(C\backslash L;\ K'(\varepsilon')) . \tag{5.8}$$

PROOF. We have, for $w\in W(\zeta')$,

$$\hat{F}(w, \zeta_0, \zeta')=\exp((k_2'+\varepsilon')w)\frac{1}{2\pi}\int_0^\infty F(\zeta_0+t\zeta')[\exp(itw\zeta')]\zeta'dt . \tag{5.9}$$

We have, by the definition of $W_\varepsilon(\zeta')$,

$$\sup_{w\in W_\varepsilon(\zeta')}\left|\int_0^\infty F(\zeta_0+t\zeta')[\exp(itw\zeta')]\zeta'dt\right|<\infty .$$

Therefore by (5.9) the function $[\exp(-(k_2'+\varepsilon')w)]\hat{F}(w, \zeta_0)$ is bounded on $W_\varepsilon(\zeta')$, hence on $C\backslash L_\varepsilon$. From Lemma 4.3, we can conclude (5.8). Q.E.D.

LEMMA 5.5. *Suppose* $\varepsilon'>\varepsilon''>0$. *Then the function*

$$G(w)=\hat{F}(w,\ -(k_2'+\varepsilon')i)-\hat{F}(w,\ -(k_2'+\varepsilon'')i)$$

is an entire function and satisfies

$$|G(w)|\leqq C\exp(k_2'u+\varepsilon'u_++\varepsilon''u_-) ,$$

where u_+ *and* u_- *are defined by* (4.8). *In particular,*

$$G(w)\in R_b^L(C;\ K'(\varepsilon')) . \tag{5.10}$$

PROOF. The lemma results from the following formula:

$$G(w)=\frac{1}{2\pi}\int_{-(k_2'+\varepsilon')i}^{-(k_2'+\varepsilon'')i} F(\tau)\exp(iw\tau)d\tau \qquad \text{Q.E.D.}$$

DEFINITION 5.3. For $F\in \mathrm{Exp}(R+i(-\infty,\ -k_2');\ L)$, we denote

$$\hat{F}(w;\ \varepsilon')=\hat{F}(w,\ -(k_2'+\varepsilon')i) .$$

As we have (5.8) and (5.10), the functions $\hat{F}(w;\ \varepsilon')$, $\varepsilon'>0$, define a cohomolgy class $[\hat{F}(w;\ +0)]\in \tilde{H}_L^1(C;\ R^L(K'))$. We call $[\hat{F}(w;\ +0)]$ *the*

Laplace transformation of F. We will denote the mapping $F \mapsto [\hat{F}(w;\ +0)]$ by $\mathscr{L}$:

(5.11) $$\mathscr{L}: \mathrm{Exp}(\boldsymbol{R}+i(-\infty,\ -k_2');\ L) \longrightarrow \tilde{H}_L^1(\boldsymbol{C};\ R^L(K'))\ .$$

EXAMPLE. The exponential function $F(\zeta)=\exp(-iz\zeta)$ belongs to the space $\mathrm{Exp}(\boldsymbol{R}+i(-\infty,\ -k_2');\ L)$ if and only if $z \in L$. The Laplace transformation $\hat{F}(w;\ \varepsilon')$ of the exponential function $F(\tau)=\exp(-iz\tau)$ is the ε'-Cauchy kernel:

(5.12) $$F(w;\ \varepsilon')=\frac{1}{2\pi i}\frac{\exp((-k_2'+\varepsilon')(z-w))}{z-w} \qquad w \in \boldsymbol{C}\backslash L\ .$$

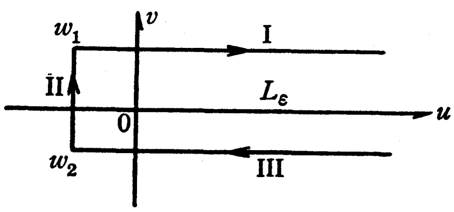
PROPOSITION 5.2. *For $F \in \mathrm{Exp}(\boldsymbol{R}+i(-\infty,\ -k_2');\ L)$, the Fourier transformation of the analytic functional $\widetilde{\mathrm{Int}}[\hat{F}(w;\ +0)]$ coincides with the original function F, that is,*

(5.13) $$\mathscr{F} \circ \widetilde{\mathrm{Int}} \circ \mathscr{L} = \mathrm{id}\ .$$

PROOF. Put $S=\widetilde{\mathrm{Int}}\circ\mathscr{L}(F)=\widetilde{\mathrm{Int}}[\hat{F}(w;\ +0)]$. By the definition of $\widetilde{\mathrm{Int}}$, we have

$$\tilde{S}(\zeta)=\int_{-\partial L_\varepsilon}[\exp(-iw\zeta)]\hat{F}(w;\ \varepsilon')dw$$

for $\mathrm{Im}\,\zeta < -k_2'-\varepsilon'$. We decompose the integral path $-\partial L_\varepsilon$ into three parts I, II and III as shown in the following figure:



$$w_1=(a-\varepsilon)+i(k_2+\varepsilon)\ ,$$
$$w_2=(a-\varepsilon)+i(k_1-\varepsilon)\ .$$

We have

(5.14) $$\int_{-\partial L_\varepsilon}=\int_{\mathrm{I}}+\int_{\mathrm{II}}+\int_{\mathrm{III}}\ .$$

We suppose for simplicity $\zeta_0=-(k_2'+\varepsilon')i$. On the integral path I, we have $\mathrm{Im}\,w>k_2$ and $\hat{F}(w;\ \varepsilon')=\hat{F}(w,\ \zeta_0,\ 1)$. Therefore for $\mathrm{Im}\,\zeta<-k_2'-\varepsilon'$, we have

$$\int_{\mathrm{I}}[\exp(-iw\zeta)]\hat{F}(w;\ \varepsilon')dw$$
$$=\int_{\mathrm{I}}[\exp(-iw\zeta)]\hat{F}(w,\ \zeta_0,\ 1)dw$$

$$=\frac{1}{2\pi}\int_{\mathrm{I}}[\exp(-iw\zeta)]\Big(\int_{\zeta_0+\boldsymbol{R}^+}F(\tau)\exp(iw\tau)d\tau\Big)dw$$

$$=\frac{1}{2\pi}\int_{\zeta_0+\boldsymbol{R}^+}F(\tau)\Big(\int_{\mathrm{I}}\exp(iw(\tau-\zeta))dw\Big)d\tau$$

$$=\frac{-1}{2\pi i}\int_{\zeta_0+\boldsymbol{R}^+}F(\tau)\,\frac{\exp(iw_1(\tau-\zeta))}{\tau-\zeta}\,d\tau\,.$$

Similarly we have, for $\operatorname{Re}\zeta\neq 0$,

$$\int_{\mathrm{II}}[\exp(-iw\zeta)]\hat{F}(w;\,\varepsilon')dw$$

$$=\int_{\mathrm{II}}[\exp(-iw\zeta)]\hat{F}(w,\,\zeta_0,\,-i)dw$$

$$=\frac{1}{2\pi i}\int_{\zeta_0-i\boldsymbol{R}^+}F(\tau)\frac{\exp(iw_1(\tau-\zeta))-\exp(iw_2(\tau-\zeta))}{\tau-\zeta}d\tau\,.$$

Finally we have, for $\operatorname{Im}\zeta<-k_2'-\varepsilon'$,

$$\int_{\mathrm{III}}[\exp(-iw\zeta)]\hat{F}(w;\,\varepsilon')dw$$

$$=\int_{\mathrm{III}}[\exp(-iw\zeta)]\hat{F}(w,\,\zeta_0,\,-1)dw$$

$$=\frac{1}{2\pi i}\int_{\zeta_0-\boldsymbol{R}^+}F(\tau)\frac{\exp(iw_2(\tau-\zeta))}{\tau-\zeta}\,d\tau\,.$$

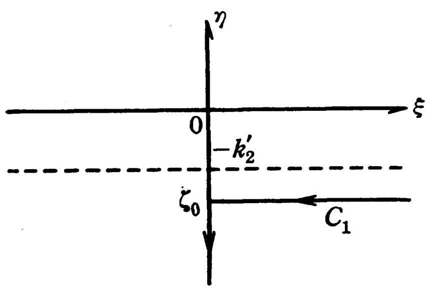
By (5.14) we have, for ζ with $\operatorname{Im}\zeta<-k_2'-\varepsilon'$ and $\operatorname{Re}\zeta\neq 0$,

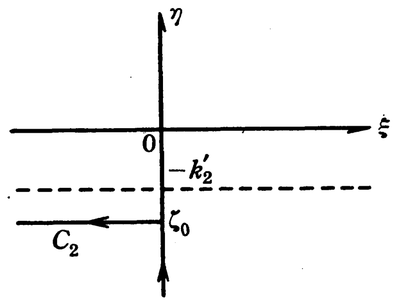
$$\tilde{S}(\zeta)=\int_{-\partial L_\varepsilon}[\exp(-iw\zeta)]\hat{F}(w;\,\varepsilon')dw$$

$$=\frac{1}{2\pi i}\int_{C_1}F(\tau)\frac{\exp(iw_1(\tau-\zeta))}{\tau-\zeta}\,d\tau$$

$$+\frac{1}{2\pi i}\int_{C_2}F(\tau)\frac{\exp(iw_2(\tau-\zeta))}{\tau-\zeta}\,d\tau\,,$$

where C_1 and C_2 are the paths depicted in the following figures:





By the Cauchy integral formula we have

$$\frac{1}{2\pi i}\int_{C_1} F(\tau)\frac{\exp(iw_1(\tau-\zeta))}{\tau-\zeta}\,d\tau = \begin{cases} F(\zeta) & \text{for } \zeta \text{ with } \operatorname{Re}\zeta>0 \text{ and } \operatorname{Im}\zeta<-k_2'-\varepsilon' \\ 0 & \text{for } \zeta \text{ with } \operatorname{Re}\zeta<0 \text{ and } \operatorname{Im}\zeta<-k_2'-\varepsilon' , \end{cases}$$

and

$$\frac{1}{2\pi i}\int_{C_2} F(\tau)\frac{\exp(iw_2(\tau-\zeta))}{\tau-\zeta}\,d\tau = \begin{cases} 0 & \text{for } \zeta \text{ with } \operatorname{Re}\zeta>0 \text{ and } \operatorname{Im}\zeta<-k_2'-\varepsilon' \\ F(\zeta) & \text{for } \zeta \text{ with } \operatorname{Re}\zeta<0 \text{ and } \operatorname{Im}\zeta<-k_2'-\varepsilon' . \end{cases}$$

Adding these two formulas, we obtain

(5.15) $$\tilde{S}(\zeta)=F(\zeta) \quad \text{for } \zeta \text{ with } \operatorname{Re}\zeta\neq 0 \text{ and } \operatorname{Im}\zeta<-k_2'-\varepsilon' .$$

$\tilde{S}$ and F are both holomorphic in the open half plane $\{\zeta;\ \operatorname{Im}\zeta<-k_2'\}$, $\tilde{S}$ and F coincide in this half plane. Q.E.D.

THEOREM 5.1. *Suppose L is the right half strip* (4.1). *Then the following diagram is commutative and every mapping in it is a linear topological isomorphism*:

(5.16) $$\begin{array}{ccc} Q'(L;\ K') & \xrightarrow{\ \mathscr{F}\ } & \operatorname{Exp}(\boldsymbol{R}+i(-\infty,\ -k_2');\ L) \\ \widetilde{\operatorname{Int}}\ \nwarrow\ \searrow\ \tilde{\mathscr{C}} & & \swarrow\ \mathscr{L} \\ & \tilde{H}_L^1(\boldsymbol{C};\ \mathscr{R}^L(K')) & \end{array}$$

PROOF. We proved in Theorem 4.2 that $\tilde{\mathscr{C}}$ and $\widetilde{\operatorname{Int}}$ are inverse to each other. $\mathscr{F}\circ\widetilde{\operatorname{Int}}\circ\mathscr{L}=\mathrm{id}$ is shown in Proposition 5.2. We have only to show $\widetilde{\operatorname{Int}}\circ\mathscr{L}\circ\mathscr{F}=\mathrm{id}$, which is equivalent to $\mathscr{L}\circ\mathscr{F}=\tilde{\mathscr{C}}$.

Let $S\in Q'(L;\ K')$ be given. Put

$$F(\zeta)=\langle S_z,\ \exp(-iz\zeta)\rangle .$$

Put $\zeta_0=-(k_2'+\varepsilon')i$. We have for $w\in W(\zeta')$,

(5.17) $$\begin{aligned} &\hat{F}(w;\ \varepsilon') \\ &=\frac{1}{2\pi}\int_{\zeta_0+\boldsymbol{R}^+\zeta'} F(\zeta)\exp(-iw\zeta)d\zeta \\ &=\frac{1}{2\pi}\int_{\zeta_0+\boldsymbol{R}^+\zeta'} \langle S_z,\ \exp(-iz\zeta)\rangle\exp(iw\zeta)d\zeta \end{aligned}$$

$$=\frac{1}{2\pi}\left\langle S_z, \int_{\zeta_0+\mathbf{R}^+\zeta'} \exp(i(w-z)\zeta)d\zeta\right\rangle$$
$$=\frac{1}{2\pi i}\left\langle S_z, \frac{\exp(-(k_2'+\varepsilon')(z-w))}{z-w}\right\rangle$$
$$=\check{S}(w; \varepsilon') .$$

The functions $\hat{F}(w; \varepsilon')$ and $\check{S}(w: \varepsilon')$ are holomorphic in the domain $\boldsymbol{C}\backslash L$. Therefore we have $\hat{F}(w; \varepsilon')=\check{S}(w; \varepsilon')$ for $w \in \boldsymbol{C}\backslash L$, which proves $\mathscr{L}\circ\widetilde{\mathscr{F}}=\mathscr{C}$. Q.E.D.

§6. The Fourier transformation of $Q'(\boldsymbol{R}+iK; K')$.

In this section we suppose the strip L is an entire strip:

$$L=\boldsymbol{R}+iK=(-\infty, \infty)+i[k_1, k_2] . \tag{6.1}$$

DEFINITION 6.1. For $f \in Q(\boldsymbol{R}+iK; K')$, we define the Fourier transformation $\mathscr{F}f=\tilde{f}$ by

$$\mathscr{F}f(\zeta)=\tilde{f}(\zeta)=\int_{-\infty}^{\infty} f(x+iy)\exp(-i\zeta(x+iy))dx . \tag{6.2}$$

It can be easily proved that the integral (6.2) converges for $\zeta \in (\boldsymbol{R}+iK')_{\varepsilon'}$ and $y \in K(\varepsilon)$ and is independent of $y \in K(\varepsilon)$ for some $\varepsilon'>0$ and $\varepsilon>0$. The Fourier transformation $\tilde{f}$ of f belongs to the space $Q(\boldsymbol{R}+iK'; -K)$. Moreover we have, by a routine argument,

THEOREM 6.1. *The Fourier transformation*

$$\mathscr{F}: Q(\boldsymbol{R}+iK; K') \longrightarrow Q(\boldsymbol{R}+iK'; -K) \tag{6.3}$$

is a linear topological isomorphism.

DEFINITION 6.2. For an analytic functional $S \in Q'(\boldsymbol{R}+iK; K')$, we define $\mathscr{F}_d S \in Q'(\boldsymbol{R}-iK'; K)$ by the formula:

$$\langle \mathscr{F}_d S, f\rangle=\langle S, \mathscr{F}f\rangle \quad \text{for } f \in Q(\boldsymbol{R}-iK'; K) . \tag{6.4}$$

We call the transformation $\mathscr{F}_d$ *the dual Fourier transformation.*

As $\mathscr{F}_d$ is the dual operator of the linear topological isomorphism (6.3), we have

COROLLARY. *The dual Fourier transformation*

$$\mathscr{F}_d: Q'(\boldsymbol{R}+iK; K') \longrightarrow Q'(\boldsymbol{R}-iK'; K) \tag{6.5}$$

is a linear topological isomorphism.

The aim of this section is to study the dual Fourier transformation (6.5) by means of the Fourier transformation defined in the preceding sections.

Let us fix a and b with $a \leqq b$ and put

$$L_+ = [a, \infty) + iK, \ L_- = (-\infty, b] + iK, \tag{6.6}$$

$$L_0 = [a, b] + iK. \tag{6.7}$$

PROPOSITION 6.1. *The sequence* (6.8) *is an exact sequence of* DFS *spaces*:

$$0 \longrightarrow Q(L;\ K') \xrightarrow{h_1} Q(L_+;\ K') \oplus Q(L_-;\ K') \xrightarrow{h_2} Q(L_0;\ K') \longrightarrow 0, \tag{6.8}$$

where we put

$$h_1(f) = (f|_{L_+}, f|_{L_-}) \quad \text{for} \quad f \in Q(L;\ K'),$$
$$h_2(f_1, f_2) = f_1|_{L_0} - f_2|_{L_0} \quad \text{for} \quad (f_1, f_2) \in Q(L_+;\ K') \oplus Q(L_-;\ K').$$

($f|_{L_+}$, $f|_{L_-}$, $f_1|_{L_0}$ *and* $f_2|_{L_0}$ *denote the restrictions to* L_+, L_- *and* L_0.)

PROOF. For brevity, we write (6.8) as follows:

$$0 \longrightarrow Q \xrightarrow{h_1} Q_+ \oplus Q_- \xrightarrow{h_2} Q_0 \longrightarrow 0. \tag{6.8'}$$

By the uniqueness of the analytic continuation, h_1 is one-to-one, i.e. (6.8′) is exact at Q. As we have clearly $\operatorname{Im} h_1 = \operatorname{Ker} h_2$, (6.8′) is exact at $Q_+ \oplus Q_-$. Therefore we have only to show h_2 is onto.

Fix $f \in Q_0 = Q(L_0;\ K') = \mathcal{O}(L_0)$ arbitrarily. Choose $\varepsilon_0 > 0$ for which f is continuous on L_{0,ε_0}. We have by the Cauchy integral formula

$$\begin{aligned} f(z)\exp(z^2) &= \frac{1}{2\pi i}\int_{\partial L_{0,\varepsilon_0}} \frac{f(w)\exp(w^2)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\left\{\int_{C_1} + \int_{C_2}\right\}\frac{f(w)\exp(w^2)}{w-z}\,dw \\ &= f_1(z) - f_2(z), \end{aligned}$$

where we put

$$C_1 = \left\{w \in \partial L_{0,\varepsilon_0},\ \operatorname{Re} w \leqq \frac{a+b}{2}\right\},$$

$$C_2 = \left\{w \in \partial L_{0,\varepsilon_0},\ \operatorname{Re} w \geqq \frac{a+b}{2}\right\}.$$

The function

$$f_j(z)=\frac{(-1)^{j-1}}{2\pi i}\int_{C_j}\frac{f(w)\exp(w^2)}{w-z}dw$$

is holomorphic in $C\backslash C_j$, for $j=1, 2$. In particular, f_1 is holomorphic in $\operatorname{int} L_{+,\varepsilon_0}$ and f_2 in $\operatorname{int} L_{-,\varepsilon_0}$. If we fix ε with $0<\varepsilon<\varepsilon_0$, f_1 is bounded on $L_{+,\varepsilon}$ and f_2 is bounded on $L_{-,\varepsilon}$. As we have

$$f(z)=\exp(-z^2)f_1(z)-\exp(-z^2)f_2(z)\ ,$$
$$\exp(-z^2)f_1(z)\in Q_+ \quad \text{and} \quad \exp(-z^2)f_2(z)\in Q_-\ ,$$

the function f belongs to $\operatorname{Im} h_2$, which proves h_2 is onto. Q.E.D.

Passing to the dual operators, we have the following

PROPOSITION 6.2. *The sequence* (6.9) *is an exact sequence of* FS *spaces*:

$$(6.9)\qquad 0 \longleftarrow Q'(L;\ K') \xleftarrow{h_1'} Q'(L_+;\ K')\oplus Q'(L;\ K') \xleftarrow{h_2'} Q'(L_0;K') \longleftarrow 0\ .$$

It is well known that the Fourier transformation

$$(6.10)\qquad \mathscr{F}: S \longmapsto \tilde{S}(\zeta)=\langle S_z, \exp(-iz\zeta)\rangle$$

establishes a linear topological isomorphism

$$(6.11)\qquad \mathscr{F}: Q'(L_0;\ K') \longrightarrow \operatorname{Exp}(C;\ L_0)\ ,$$

where

$$\operatorname{Exp}(C;\ L_0)=\{F\in \mathscr{O}(C);\ \forall\varepsilon>0,\ \exists C\geqq 0 \text{ such that } |F(\zeta)|\leqq C\exp(h_{L_0}(\zeta)+\varepsilon|\zeta|)\}\ ,$$

$$h_{L_0}(\zeta)=H_K(\xi)+b\eta_+ + a\eta_-\ ,$$

$$\eta_+=\begin{cases}\eta & \text{for } \eta\geqq 0\ ,\\ 0 & \text{for } \eta\leqq 0\ ,\end{cases} \qquad \eta_-=\begin{cases}0 & \text{for } \eta\geqq 0\ ,\\ \eta & \text{for } \eta\leqq 0\ .\end{cases}$$

We have shown in § 4 that the Fourier transformation (6.10) establishes the linear topological isomorphism

$$(6.12)\qquad F:\ Q'(L_+;\ K') \longrightarrow \operatorname{Exp}(R+i(-\infty,\ -k_2');\ L_+)\ .$$

As η is bounded from above in $R+i(-\infty, -k_2')$, we have

$$(6.13)\qquad \operatorname{Exp}(R+i(-\infty,\ -k_2');\ L_+)$$
$$=\{F\in \mathscr{O}(R+i(-\infty,\ -k_2'));\ \forall\varepsilon>0,\ \forall\varepsilon'>0,\ \exists\, C\geqq 0 \text{ such that } |F(\zeta)|\leqq C\exp(h_{L_0}(\zeta)+\varepsilon|\zeta|) \text{ for } \operatorname{Im}\zeta<-k_2'-\varepsilon'\}\ .$$

In the same way as in the case $Q'(L_+;\ K')$, we can prove the Fourier transformation (6.10) establishes the linear topological isomorphism:

$$(6.14)\qquad \mathscr{F}: Q'(L_-;\ K') \longrightarrow \mathrm{Exp}(\boldsymbol{R}+i(-k_1',\ \infty);\ L_-)\ ,$$

where

$$(6.15)\qquad \begin{aligned}&\mathrm{Exp}(\boldsymbol{R}+i(-k_1',\ \infty);\ L_-)\\ &=\{F\in\mathscr{O}(\boldsymbol{R}+i(-k_1',\ \infty));\ \forall\varepsilon>0,\ \forall\varepsilon'>0,\ \exists C\geqq 0 \text{ such that}\\ &\qquad |F(\zeta)|\leqq C\exp(h_{L_0}(\zeta)+\varepsilon\,|\zeta|)\quad \text{for}\quad \mathrm{Im}\,\zeta\geqq -k_1'+\varepsilon'\}\end{aligned}$$

Now we define

$$(6.16)\qquad \begin{aligned}&\mathrm{Exp}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK');\ L_0)\\ &=\{F\in\mathscr{O}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK'));\ \forall\varepsilon>0,\ \forall\varepsilon'>0,\ \exists\, C\geqq 0 \text{ such that}\\ &\qquad |F(\zeta)|\leqq C\exp(h_{L_0}(\zeta)+\varepsilon\,|\zeta|)\quad \text{for}\quad \mathrm{Im}\,\zeta\notin -K'(\varepsilon')\}\ .\end{aligned}$$

By the definition, we have clearly

$$(6.17)\qquad \begin{aligned}&\mathrm{Exp}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK');\ L_0)\\ &=\mathrm{Exp}(\boldsymbol{R}+i(-\infty,\ -k_2');\ L_+)\oplus\mathrm{Exp}(\boldsymbol{R}+i(-k_1',\ \infty);\ L_-)\ .\end{aligned}$$

DEFINITION 6.3. (i) We put

$$(6.18)\qquad H^1_{\boldsymbol{R}-iK'}(\boldsymbol{C};\ \mathrm{Exp}(L_0))=\mathrm{Exp}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK');\ L_0)/\mathrm{Exp}(\boldsymbol{C};\ L_0)\ .$$

(ii) We define *the Fourier transformation*

$$(6.19)\qquad F:\ Q'(\boldsymbol{R}+iK;\ K')\longrightarrow H^1_{\boldsymbol{R}-iK'}(\boldsymbol{C};\ \mathrm{Exp}(L_0))$$

in such a way that the following diagram becomes commutative:

$$(6.20)\qquad \begin{array}{ccccccc} 0\longleftarrow & Q'(\boldsymbol{R}+iK;\ K') & \overset{h_1'}{\longleftarrow} & Q_+'\oplus Q_-' & \overset{h_2'}{\longleftarrow} & Q_0' & \longleftarrow 0\\ & \big\downarrow \mathscr{F} & & \big\downarrow \mathscr{F} & & \big\downarrow \mathscr{F} & \\ 0\longleftarrow & H^1_{\boldsymbol{R}-iK'}(\boldsymbol{C};\ \mathrm{Exp}) & \longleftarrow & \mathrm{Exp}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK')) & \longleftarrow & \mathrm{Exp}(\boldsymbol{C}) & \longleftarrow 0\ , \end{array}$$

where we abbreviated $L_0=[a,\ b]+iK$:

$$\begin{gathered} H^1_{\boldsymbol{R}-iK'}(\boldsymbol{C};\ \mathrm{Exp})=H^1_{\boldsymbol{R}-iK'}(\boldsymbol{C};\ \mathrm{Exp}(L_0))\ ,\\ \mathrm{Exp}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK'))=\mathrm{Exp}(\boldsymbol{C}\backslash(\boldsymbol{R}-iK');\ L_0) \text{ and}\\ \mathrm{Exp}(\boldsymbol{C})=\mathrm{Exp}(\boldsymbol{C};\ L_0)\ . \end{gathered}$$

By the definition we have clearly

$$R^{R-iK'}(C;\ K)\supset \mathrm{Exp}(C;\ L_0) \tag{6.21}$$

$$R^{R-iK'}(C\backslash(R-iK');\ K)\supset \mathrm{Exp}(C\backslash(R-iK');\ L_0)\ , \tag{6.22}$$

where we recall $L_0=[a,\ b]+iK$. Therefore we can define canonically the mapping

$$\iota\colon H^1_{R-iK'}(C;\ \mathrm{Exp}(L_0)) \longrightarrow H^1_{R-iK'}(C;\ R^{R-iK'}(K))\ . \tag{6.23}$$

THEOREM 6.2. *Suppose* $L=R+iK$ *is the entire strip* (6.1). *Define* L_+, L_- *and* L_0 *by* (6.6) *and* (6.7). *Then the following diagram is commutative and every mapping appeared in the diagram is a linear topological isomorphism*:

$$\begin{array}{ccc} Q'(R+iK;\ K') & \xrightarrow{\mathscr{F}_d} & Q'(R-iK';\ K) \\ \downarrow{\scriptstyle \mathscr{F}} & & \nearrow \\ H^1_{R-iK'}(C;\ \mathrm{Exp}([a,\ b]+iK)) & & \mathrm{Int} \\ \downarrow{\scriptstyle \iota} & \nearrow & \\ H^1_{R-iK'}(C;\ R^{R-iK'}(K)) & & \end{array} \tag{6.24}$$

where $\mathscr{F}_d$ *is the dual Fourier transformation defined by* (6.4), $\mathscr{F}$ *is the Fourier transformation defined by* (6.20), *the mapping* ι *is defined by* (6.23) *and the mapping* Int *is defined by* (2.18).

PROOF. Choose arbitrarily $S\in Q'(R+iK;\ K')$ and $f\in Q(R-iK';\ K)$. By Proposition 6.2, we can decompose S as follows:

$$S=S^+-S^-,\quad S^+\in Q'(L_+;\ K'),\quad S^-\in Q'(L_-;\ K')\ . \tag{6.25}$$

There exists positive numbers ε' and ε such that $f\in Q_b((R-iK')_{2\varepsilon'};\ K(\varepsilon))$. The Fourier transformation $\tilde{f}(z)$ of f is given by

$$\tilde{f}(z)=\int_{-\infty-k'i}^{+\infty-k'i} f(\zeta)\exp(-i\zeta z)d\zeta\ , \tag{6.26}$$

where k' is an arbitrary number in the interval $(k_1'-2\varepsilon',\ k_2'+2\varepsilon')$. By the above remarks, we can calculate as follows:

$$\begin{aligned} \langle \mathscr{F}_d S,\ f\rangle &= \langle S,\ \tilde{f}\rangle \\ &= \left\langle S_z^+,\ \int_{-\infty-(k_2'+\varepsilon')i}^{+\infty-(k_2'+\varepsilon')i} f(\zeta)\exp(-i\zeta z)d\zeta \right\rangle \\ &\quad -\left\langle S_z^-,\ \int_{-\infty-(k_1'-\varepsilon')i}^{+\infty-(k_1'-\varepsilon')i} f(\zeta)\exp(-i\zeta z)d\zeta \right\rangle \end{aligned}$$

$$=\int_{-\infty-(k_2'+\varepsilon')i}^{+\infty-(k_2'+\varepsilon')i} f(\zeta)\tilde{S}^+(\zeta)d\zeta$$

$$-\int_{-\infty-(k_1'-\varepsilon')i}^{+\infty-(k_1'-\varepsilon')i} f(\zeta)\tilde{S}^-(\zeta)d\zeta$$

$$=-\int_{\partial(\boldsymbol{R}-iK')_{\varepsilon'}} f(\zeta)[\tilde{S}(\zeta)]d\zeta$$

$$=\langle \mathrm{Int}[\tilde{S}], f\rangle ,$$

from which results the commutativity of the diagram. We showed already the mappings $\mathscr{F}_d$, $\mathscr{F}$ and Int are linear topological isomorphisms. Therefore the mapping ι is also a linear topological isomorphism because of the commutative diagram (6.24). Q.E.D.

COROLLARY. *Suppose $\boldsymbol{R}+iK$ is the entire strip* (6.1) *and K' is a compact interval. Then we have the following several complex representations of the space of analytic functionals $Q'(\boldsymbol{R}+iK;\ K')$*:

$$Q'(\boldsymbol{R}+iK;\ K')=\tilde{H}^1_{\boldsymbol{R}+iK}(\boldsymbol{C};\ R^{\boldsymbol{R}+iK}(K')) \tag{6.27}$$
$$=H^1_{\boldsymbol{R}+iK}(\boldsymbol{C};\ R^{\boldsymbol{R}+iK}(K'))$$
$$=H^1_{\boldsymbol{R}+iK}(\boldsymbol{C};\ \mathrm{Exp}([a, b]+iK')),\ a\leqq b\ .$$

In fact, we have only to exchange the roles of $\boldsymbol{R}+iK$ and $\boldsymbol{R}-iK'$ in the theorem.

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