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On the First Cohomology Group of a Minimal Set

Ippei ISHII

Keio University

Introduction.

It is one of the most important problems in the theory of topological dynamics to determine what space can be a minimal set under a continuous flow. For example, it has been conjectured that there is no minimal flow on the 3-sphere S^3. In this paper, we shall study the first cohomology of minimal sets.

It is known that the space on which an almost periodic minimal flow or a distal minimal flow exists has a non-trivial first cohomology group. However the "almost periodicity" and the "distality" are both destroyed by a time-change, while the "minimality" is invariant by a time-change. The method for calculating the first cohomology of minial sets which is exhibited in this paper is quite independent of the parametrization by the time.

In § 3 we will establish a method for calculating the first cohomology of a minimal set from certain 0-th cohomology groups. As an application of the consequence of § 3, we can get a method for deciding the first cohomology of a minimal set which forms a 3-dimensional manifold (§ 4, Theorems 1 and 2). And in § 5 we will investigate on 1-cycles of a 3-dimensional minimal set. § 1 and § 2 are preliminaries. Higher dimensional cases can be treated by the same way, but it seems to be impossible to prove the non-triviality of the first cohomology of a minimal set by our method in the case of higher dimensional manifolds. Hence we do not treat the higher dimensional case in this paper. In the case of 3-manifolds, our method seems to be useful for the proof of the non-triviality of the first cohomology of a minimal set.

In the case when the minimal set is a two dimensional manifold, using our method, we can decide the first cohomology of it completely. But it is well-known that the only two dimensional manifold admitting a minimal flow on it is the 2-torus. Therefore the results for two

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dimensional minimal sets are only stated in the appendix.

§ 1. Preliminaries.

Let (Y, ρ_t) or simply ρ_t be a flow on a compact metric space Y; i.e., ρ_t is a homeomorphism of Y for each real number t and $\rho_{t+s}=\rho_t\circ\rho_s$ for any two reals t and s. If $A\subset Y$ and $J\subset R$ (R is the set of real numbers), we write $A\cdot J$ or $(A\cdot J)_{\rho_t}$ for $\{\rho_t(y)|t\in J, y\in A\}$. A subset N on Y is said to be a *minimal set* if $\overline{\{y\}\cdot R}=N$ for any $y\in N$. Especially if Y is a minimal set, then we call (Y, ρ_t) a *minimal flow on Y*.

DEFINITION 1. A subset Σ of Y is said to be a *local section* of the flow ρ_t if it satisfies:

(i) $h:\bar{\Sigma}\times(-\mu, \mu)\to\bar{\Sigma}\cdot(-\mu, \mu)$ defined by $h(y, t)=\rho_t(y)$ is a homeomorphism for some $\mu>0$. (μ is called a *collar-size* for Σ.)

(ii) $\Sigma\cdot J$ is an open subset of Y if J is open in R.

Moreover if Σ is compact, then we call it a *global section*.

LEMMA 1. *Let (Y, ρ_t) be a minimal flow and $S=\{y_0\}\cdot Z$ $(y_0\in Y)$. If $\bar{S}\neq Y$, then $\bar{S}$ is a global section of (Y, ρ_t), where Z is the set of integers.*

PROOF. It is proved in [1] that if $\bar{S}\neq Y$, then there is a positve number r such that $\{t|\rho_t(y)\in\bar{S}\}=\{nr|n\in Z\}$ for any $y\in\bar{S}$ and $\rho_t(\bar{S})\cap\rho_s(\bar{S})=\emptyset$ for $0<t<s\leqq r$. Therefore the condition (i) in Definition 1 is satisfied by $\mu<r/2$. We shall show the condition (ii). Suppose $J=(0, \delta)$ $(\delta<r)$ and take a sequence $\{x_j\}\subset Y\backslash\bar{S}\cdot J$. Since $Y=\bar{S}\cdot(0, r]$, we can choose sequences $\{y_j\}\subset\bar{S}$ and $\{t_j\}\subset[\delta, r]$ such that $\rho_{t_j}(y_j)=x_j$. Hence we have that if $x_j\to x_0$, then $x_0=\rho_{t_0}(y_0)$ for some $t_0\in[\delta, r]$ and $y_0\in\bar{S}$. This shows that the condition (ii) holds for $J=(0, \delta)$. Evidently $\bar{S}\cdot(0, \delta)$ is homeomorphic to $\bar{S}\cdot(t, t+\delta)$ for any t, so it follows that (ii) is satisfied by any open J. This completes the proof.

LEMMA 2. *Let (Y, ρ_t) be a minimal flow and Σ be a local section. Then for each $y\in Y$ there exists a sequence $\{t_j\}$ $(j=0, \pm1, \pm2, \cdots)$ of reals such that $\delta_1<t_{j+1}-t_j<\delta_2$ for some positive constants δ_1, δ_2, and $\rho_t(y)\in\Sigma$ if and only if $t=t_j$ for some j.*

PROOF. First take δ_1 so that $\delta_1<\mu$ where μ is a collar-size for Σ. By the minimality of ρ_t, we can see that there exists a relatively dense subset L_y of R such that $\rho_t(y)\in\Sigma\cdot(-\delta_1, \delta_1)$ for $t\in L_y$ (see [2]). Hence we can take a sequence $\{t_j\}$ with the desired properties.

§ 2. A flow associated with a local section.

Throughout this section and the next, (M, ξ_t) will be a minimal flow on a compact metric space M. Let Ω be the set of all continuous functions on R with the compact-open topology, and η_t be a flow on Ω defined by $\eta_t(g)(s)=g(t+s)$.

Now take a local section Σ of (M, ξ_t) and a point $x_0 \in M$, and let $\{t_j\}$ be the sequence for x_0 as in Lemma 2. Then we can construct a uniformly continuous function f such that $f(t)>\varepsilon>0$ for some constant ε and any t, and

$$\int_{t_j}^{t_{j+1}} f(t)dt=1 \qquad (j=0, \pm 1, \pm 2, \cdots).$$

Define a flow ζ_t on $M\times\Omega$ by $\zeta_t(x, g)=(\xi_t(x), \eta_t(g))$ $(x \in M, g \in \Omega)$. Since $\overline{(\{f\}\cdot R)_{\eta_t}}$ is compact because of the uniform continuity of f, there is a compact minimal set $\tilde{M}$ of the flow ζ_t in $\overline{(\{(x_0, f)\}\cdot R)_{\zeta_t}}$, and $(\tilde{M}, \zeta_t)$ is a minimal flow. By p we denote the natural projection $p: \tilde{M}\to M$. It is easy to see that $p\circ\zeta_t=\xi_t\circ p$.

LEMMA 3. *Let Σ' be a local section such that $\Sigma'\supset\bar{\Sigma}$. Then* $\overline{p^{-1}(\Sigma)}\cup \overline{p^{-1}(\Sigma'\backslash\bar{\Sigma})}=\overline{p^{-1}(\Sigma')}$.

PROOF. Since $(\partial\Sigma\cdot R)_{\xi_t}$ does not contain an open set of M, there exists a point $x_1 \in M$ such that $(\{x_1\}\cdot R)_{\xi_t}$ has no common points with $\partial\Sigma$. Let $\tilde{x}_1=(x_1, g)$ be a point of $\tilde{M}$, then by the minimality we have $\overline{(\{\tilde{x}_1\}\cdot R)_{\zeta_t}}=\tilde{M}$. Obviously $x \in \Sigma$ or $x \in \Sigma'\backslash\bar{\Sigma}$ if $x \in \Sigma'\cap(\{x_1\}\cdot R)_{\xi_t}$, whence we obtain the consequence of the lemma.

LEMMA 4. $\overline{p^{-1}(\Sigma)}$ *is a global section of* $(\tilde{M}, \zeta_t)$.

PROOF. Let $N=\overline{(\{f\}\cdot R)_{\eta_t}}\subset\Omega$ and $F: M\times N\to R$ be a function defined by $F(x, g)=g(0)$ $(x \in M, g \in N)$. Moreover define $\sigma: M\times N\times R\to R$ by

$$\sigma(x, g, t)=\int_0^t F(\zeta_s(x, g))ds .$$

Because $F>0$, $\sigma(x, g, \cdot)$ is monotone. Hence if we define $\tilde{\zeta}_t$ by $\tilde{\zeta}_\sigma(x, g)=\zeta_t(x, g)$ where $\sigma=\sigma(x, g, t)$, then $\tilde{\zeta}_t$ is a flow on $M\times N$. Evidently $(\tilde{M}, \tilde{\zeta}_t)$ is also a minimal flow. Now let (x_1, g_1) be a point of $p^{-1}(\Sigma)$ and $\tilde{\Sigma}=\overline{(\{(x_1, g_1)\}\cdot Z)_{\tilde{\zeta}_t}}$. Then, by Lemma 1, $\tilde{\Sigma}$ is a global section of $(\tilde{M}, \tilde{\zeta}_t)$ so of $(\tilde{M}, \zeta_t)$ if $\tilde{\Sigma}\neq\tilde{M}$.

First we shall show that $p^{-1}(\Sigma)\subset\tilde{\Sigma}\subset p^{-1}(\bar{\Sigma})$. If $(x, g) \in p^{-1}(\Sigma)$, then there exists a sequence $\{(x_j, g_j)\}\subset p^{-1}(\Sigma)$ such that $(x_j, g_j)\to(x, g)$ and

$\zeta_{s_j}(x_1, g_1)=(x_j, g_j)$ for some s_j. From the definition of f, it follows that $\sigma(x_1, g_1, s_j)$ is an integer for each j, and hence that (x_j, g_j) is contained in $\tilde{\Sigma}$. This implies that $p^{-1}(\Sigma)\subset\tilde{\Sigma}$. On the other hand, if $k=\sigma(x_1, g_1, s)$ is an integer, then we have $p(\tilde{\zeta}_k(x_1, g_1))=p(\zeta_s(x_1, g_1))\in\bar{\Sigma}$. So we obtain $\tilde{\Sigma}\subset p^{-1}(\bar{\Sigma})$.

Now let us show that $\tilde{\Sigma}=\overline{p^{-1}(\Sigma)}$. If not, then, because of Lemma 3, we can take a sequence $\{x_j\}\subset\Sigma'\backslash\bar{\Sigma}$ (so $p^{-1}(x_j)\cap\tilde{\Sigma}=\emptyset$) so that there is a sequence $\{\tilde{x}_j\}$ such that $\tilde{x}_j\in p^{-1}(x_j)$ and $\tilde{x}_j$ tends to a point of $\tilde{\Sigma}$. But this contradicts with the openness of $\tilde{\Sigma}\cdot(-\mu, \mu)$. This completes the proof.

LEMMA 5. *We can choose the function f so that $p^{-1}(x)$ is totally disconnected for any $x\in M$.*

PROOF. Let $K=\sup_{x\in M}\inf\{t>0|\xi_t(x)\in\Sigma\}$ and μ be a collar-size for Σ. Define a family of functions $\{\Phi_r\}\subset\Omega$ $(\mu\leqq r\leqq K)$ so that it satisfies:

(a) $\Phi_r(t)=\varepsilon$ for $t\leqq 0$ and $t\geqq r$,

(b) $\Phi_r(t)>\varepsilon$ for $0<t<r$,

(c) $\int_0^r\Phi_r(t)dt=1$,

(d) $r\rightarrow\Phi_r$ is a homeomorphism.

where ε is a constant such that $0<\varepsilon<1/K$. For a sequence $\sigma=\{s_j\}$ such that $\mu\leqq s_{j+1}-s_j\leqq K$, define $\Psi(\sigma; t)$ to be

$$\Psi(\sigma; t)=\Phi_r(t-s_j)\qquad\text{if}\quad s_j\leqq t\leqq s_{j+1}\quad\text{and}\quad s_{j+1}-s_j=r\ .$$

For a point x of M, we denote by σ^x a sequence of reals such that $t\in\sigma^x$ if and only if $\xi_t(x)\in\Sigma$, and by σ^x_* a sequence such that $t\in\sigma^x_*$ if and only if $\xi_t(x)\in\bar{\Sigma}$. Setting $f(t)=\Psi(\sigma^{x_0}; t)$ $(x_0\in M)$, we show that $(\tilde{M}, \zeta_t)$ constructed by the same way as above has the desired property. For each point x of M, E_x denotes a family of sequences of reals $E_x=\{\sigma|\sigma^x\subset\sigma\subset\sigma^x_*\}$. Then it is easy to see that if $(x, g)\in p^{-1}(x)$, then $g(t)=\Psi(\sigma; t)$ for some $\sigma\in E_x$. On the other hand, if we define $\kappa: E_x\rightarrow 2^Z$ by

$$\kappa(\sigma)(j)=\begin{cases}0 & \text{if}\quad s_j\in\sigma\\ 1 & \text{if}\quad s_j\notin\sigma\end{cases}$$

where $\sigma^x_*=\{s_j\}$, then κ induces a homeomorphism $\{\Psi(\sigma; \cdot)|\sigma\in E_x\}\rightarrow 2^Z$. This implies that $p^{-1}(x)$ is homeomorphic to a closed subset of the Cantor set, which proves the lemma.

PROPOSITION 1. *For a minimal flow (M, ξ_t) and a local section Σ, there exists a minimal flow $(\tilde{M}, \zeta_t)$ with the following properties:*

(i) *$\tilde{M}$ is a compact metric space,*
(ii) *there is a continuous map $p: \tilde{M} \to M$ such that $p \circ \zeta_t = \xi_t \circ p$,*
(iii) *$\overline{p^{-1}(\Sigma)}$ is a global section of $(\tilde{M}, \zeta_t)$,*
(iv) *$\overline{p^{-1}(\Sigma)}$ is totally disconnected;* i.e., $\dim \overline{(p^{-1}(\Sigma))} = 0$.

PROOF. Let $\{U_j\}$ $(U_0 = \Sigma)$ be a countable base of open sets of Σ. Then each U_j is a local section. Therefore, by Lemmas 4 and 5, we can construct a minimal flow $(\tilde{M}_j, \zeta_t^{(j)})$ for each j so that

(a) $\tilde{M}_j$ is a compact metric space,
(b) there is a homomorphism $p_j: \tilde{M}_j \to M$ of flows,
(c) $\overline{p_j^{-1}(U_j)}$ is a global section of $(\tilde{M}_j, \zeta_t^{(j)})$,
(d) $p_j^{-1}(x)$ is totally disconnected for any $x \in M$.

Fix a point $x_0 \in M$ and take points a_j in $p_j^{-1}(x_0) \subset \tilde{M}_j$. Let $\tilde{M}$ be a minimal set of $(\prod \tilde{M}_j, \zeta_t)$ which is included in the orbit closure of the point $a_* = \prod a_j$, where $\zeta_t(\prod y_j) = \prod \zeta_t^{(j)}(y_j)$. Then it is clear that $\tilde{M}$ is a compact metric space and there is a homomorphism $p: \tilde{M} \to M$ of flows.

By λ_j we denote the projection $\lambda_j: \tilde{M} \to \tilde{M}_j$. It can be easily seen that we may assume that $p(a_*) = x_0$, and that $p = p_j \circ \lambda_j$ for any j if $p(a_*) = x_0$. So we assume that $p = p_j \circ \lambda_j$ for any j. Then $\tilde{\Sigma}_j = \lambda_j^{-1}\overline{(p_j^{-1}(U_j))}$ is a global section of $(\tilde{M}, \zeta_t)$. We shall show that $\tilde{\Sigma}_j$ coincides with $\overline{p^{-1}(U_j)}$. It is trivial that $p^{-1}(U_j) \subset \tilde{\Sigma}_j \subset p^{-1}(\bar{U}_j)$. Therefore, by the same reasoning used in Lemmas 3 and 4, we can see that $\tilde{\Sigma}_j$ must coincide with $\overline{p^{-1}(U_j)}$. Especially, putting $j=0$, we have that $\overline{p^{-1}(\Sigma)}$ is a global section of $(\tilde{M}, \zeta_t)$.

The only thing left to be proved is that $\dim \overline{(p^{-1}(\Sigma))} = 0$. To prove this, it must be noted that $\overline{p^{-1}(U_j)}$ is open and closed in $\overline{p^{-1}(\Sigma)}$. In fact, the closedness is clear and the openness follows from the openness of $\overline{p^{-1}(U_j)} \cdot (-\delta, \delta)$.

Let $\tilde{x}$ be an arbitrary point of $p^{-1}(\Sigma)$ and $x = p(\tilde{x})$. Since $p^{-1}(x) \subset \prod p_j^{-1}(x)$ is totally disconnected, we can find an arbitrarily small neighborhood V of $\tilde{x}$ such that $V \cap p^{-1}(x)$ is open and closed in $p^{-1}(x)$. Let $K = p^{-1}(x) \setminus V$, then we can take a neighborhood W of K so that $\bar{W} \cap \bar{V} \cap p^{-1}(x) = \emptyset$. Hence there exists a U_j such that $x \in U_j$, $\overline{W} \cap \overline{V} \cap p^{-1}(\bar{U}_j) = \emptyset$ and $W \cup V \supset p^{-1}(\bar{U}_j)$. Now it is evident that $V \cap \overline{p^{-1}(U_j)}$ is a closed and open subset of $\overline{p^{-1}(\Sigma)}$ which contains $\tilde{x}$. For a point $\tilde{x}$ in $p^{-1}(\partial \Sigma)$ there exists a neighborhood V of $\tilde{x}$ and a point $\tilde{x}_1$ in $p^{-1}(\Sigma)$ such that V is homeomorphic to some neighborhood of $\tilde{x}_1$. This implies that $\overline{p^{-1}(\Sigma)}$ is totally disconnected and this proves the proposition.

§ 3. Cohomology theory.

Let Y be any topological space. We denote by $\bar{H}^*(Y)$ the Alexander cohomology module of Y with the real coefficients.

Let Γ be a presheaf of R-module on Y and $\mathscr{U}$ be an open covering of Y. For $q \geqq 0$ define $C^q(\mathscr{U}:\Gamma)$ to be the module of functions ψ which assign to an ordered $(q+1)$-tuple $U_0, U_1, \cdots, U_q$ of elements of $\mathscr{U}$ an element $\psi(U_0, U_1, \cdots, U_q) \in \Gamma(U_0 \cap U_1 \cap \cdots \cap U_q)$. A coboundary operator $\delta: C^q(\mathscr{U};\Gamma) \to C^{q+1}(\mathscr{U};\Gamma)$ is defined by

$$(\delta\psi)(U_0, U_1, \cdots, U_{q+1}) = \sum_{j=0}^{q+1}(-1)^j \psi(U_0, \cdots, \hat{U}_j, \cdots, U_{q+1})|_{U_0 \cap U_1 \cap \cdots \cap U_{q+1}}$$

where $(U_0, \cdots, \hat{U}_j, \cdots, U_{q+1})$ denotes the q-tuple obtained by omitting U_j. The cohomology module of the cochain complex $C^*(\mathscr{U};\Gamma) = \{C^q(\mathscr{U};\Gamma), \delta\}$ is denoted by $H^*(\mathscr{U};\Gamma)$. The Čech cohomology of Y with coefficients in Γ is defined by $\check{H}^*(Y;\Gamma) = \varinjlim\{H^*(\mathscr{U}:\Gamma)\}$. For the precise definitions, see [3].

In what follows we shall investigate the cohomology of $X = M\backslash(\Sigma\cdot(-\mu, 0))_{\xi_t}$ where Σ is a local section and μ is a collar-size for Σ. In this section, p denotes the restriction of $p: \tilde{M} \to M$ to $\tilde{X} = \tilde{M}\backslash(\overline{p^{-1}(\Sigma)}\cdot(-\mu, 0))_{\zeta_t}$ where $(\tilde{M}, \zeta_t)$ is the flow constructed in Proposition 1.

Let Γ_1 and Γ_2 be presheaves on X defined by $\Gamma_1(U) = \bar{H}^0(U)$ and $\Gamma_2(U) = \bar{H}^0(p^{-1}(U))$ respectively, where U is an open subset of X. Then p induces a homomorphism $p^*: \Gamma_1 \to \Gamma_2$. Since p^* is a monomorphism, $0 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 0$ ($\Gamma_3 = \mathrm{Coker}(p^*)$) is an exact sequence. Hence, by the usual argument of the cohomology theory, we have

LEMMA 6. *There is an exact sequence*

$$0 \to \check{H}^0(X;\Gamma_1) \to \check{H}^0(X;\Gamma_2) \to \check{H}^0(X;\Gamma_3) \to \check{H}^1(X:\Gamma_1) \to \check{H}^1(X;\Gamma_2) \to \cdots .$$

Moreover we get

LEMMA 7. $\check{H}^q(X;\Gamma_1) \simeq \bar{H}^q(X)$ *and* $\check{H}^q(X;\Gamma_2) \simeq \bar{H}^q(\tilde{X})$ *for all* q.

PROOF. Since X and $\tilde{X}$ are metric spaces, they are paracompact. And $\bar{H}^q(p^{-1}(x))$ is trivial for $q > 0$ and any $x \in X$, because $p^{-1}(x)$ is totally disconnected. Hence this lemma immediately follows from the next lemma.

LEMMA 8. ([3]) *Let* $h: Y' \to Y$ *be a closed continuous map between paracompact Hausdorff spaces. Suppose* $\bar{H}^q(h^{-1}(y)) = 0$ *for all* $y \in Y$ *and* $0 < q < n$. *Let* Γ *be the presheaf on* Y *defined by* $\Gamma(U) = \bar{H}^0(h^{-1}(U))$. *Then* *there are isomorphisms* $\check{H}^q(Y;\Gamma)\simeq\bar{H}^q(Y')$ *for* $q<n$.

Consequently we obtain

PROPOSITION 2. *There is an exact sequence*

$$\check{H}^0(X;\Gamma_2)\longrightarrow\check{H}^0(X;\Gamma_3)\longrightarrow\bar{H}^1(X)\longrightarrow 0\,.$$

PROOF. Since $\overline{p^{-1}(\Sigma)}$ is a global section, it is a strong deformation retract of $\tilde{X}$. Hence $\bar{H}^1(\tilde{X})$ is isomorphic to $\bar{H}^1(\overline{p^{-1}(\Sigma)})$. On the other hand, $\bar{H}^1(\overline{p^{-1}(\Sigma)})$ is trivial, because $\dim(\overline{p^{-1}(\Sigma)})=0$. Therefore, combining Lemmas 6 and 7, we have the consequence of the proposition.

§ 4. The case of 3-manifolds.

In this section, M will be a differentiable 3-dimensional compact manifold and ξ_t will be a minimal flow on M generated by a C^1-vector field. First we introduce some notations.

NOTATIONS.

(a) For a real valued function F (not necessarily continuous) defined on a subset D of M (or $\tilde{M}$), $\hat{F}$ denotes a map $\hat{F}: D\to M$ (or $\tilde{M}$) defined by $\hat{F}(x)=\xi_{F(x)}(x)$ (or $\zeta_{F(x)}(x)$), where $(\tilde{M}, \zeta_t)$ is the flow constructed in Proposition 2.

(b) Let Σ be a local section of (M, ξ_t). Then we use the following notations.

$$\begin{aligned}
&T_\Sigma\colon M\longrightarrow R \text{ defined by } T_\Sigma(x)=\inf\{t>0\,|\,\xi_t(x)\in\bar{\Sigma}\}\,,\\
&B_\Sigma^1\subset\partial\Sigma\colon B_\Sigma^1=\{x\in\partial\Sigma\,|\,\hat{T}_\Sigma(x)\in\partial\Sigma\}\,,\\
&B_\Sigma^j\subset\partial\Sigma\colon B_\Sigma^j=\{x\in\partial\Sigma\,|\,\hat{T}_\Sigma(x)\in B_\Sigma^{j-1}\}\qquad(j=2,3,\cdots)\,,\\
&A_\Sigma^j\subset\Sigma\colon A_\Sigma^j=\{x\in\Sigma\,|\,\hat{T}_\Sigma(x)\in B_\Sigma^j\}\qquad(j=1,2,3,\cdots)\,,\\
&C_\Sigma\subset\Sigma\colon C_\Sigma=\{x\in\Sigma\,|\,\hat{T}_\Sigma(x)\in\partial\Sigma\}\,.
\end{aligned}$$

Let Σ' be a local section which is C^1-submanifold of M and Σ be an open subset of Σ' such that $\bar{\Sigma}\subset\Sigma'$ and the boundary $\partial\Sigma$ is a C^1-submanifold of Σ'. For each point $(x,t)\in\partial\Sigma\times R$ with $\xi_t(x)\in\partial\Sigma$, we can take a small piece $\gamma_{x,t}$ of $\partial\Sigma$ and a C^1-function $\omega_{x,t}\colon\gamma_{x,t}\to R$ so that $x\in\gamma_{x,t}$, $\omega_{x,t}(x)=t$ and $\hat{\omega}_{x,t}(\gamma_{x,t})\subset\Sigma'$.

DEFINITION 2. We say $\partial\Sigma$ is *transversal along the flow* at $(x,t)\in\partial\Sigma\times R$, if $\xi_t(x)\notin\partial\Sigma$ or $\hat{\omega}_{x,t}(\gamma_{x,t})$ is transversal to $\partial\Sigma$ at $\hat{\omega}_{x,t}(x)$ in Σ'.

DEFINITION 3. A local section Σ is said to be *regular* if

(a) Σ is connected,

(b) $\partial\Sigma$ consists of finitely many connected components and each component is C^1-diffeomorphic to the circle S^1,

(c) $B_\Sigma^1 \cup \hat{T}_\Sigma(B_\Sigma^1)$ intersects with every component of $\partial\Sigma$,

(d) $\partial\Sigma$ is transversal along the flow at $(x, T_\Sigma(x))$ for any $x \in \partial\Sigma$, and so A_Σ^1 is a finite set, and

(e) $A_\Sigma^j = \emptyset$ for $j \geqq 2$.

LEMMA 9. *Let Σ be a local section included in some C^1 local section Σ' and satisfying the conditions* (a), (b) *and* (c) *in Definition* 3. *Then there exists a regular local section arbitrarily close to Σ in the C^1-topology.*

PROOF. For simplicity we shall verify only the case when Σ is homeomorphic to a 2-disk. We take (r, θ) as the polar coordinate on Σ' and assume $\Sigma = \{r < r_0\}$. Then it follows from the minimality of (M, ξ_t) that for sufficiently small $\delta > 0$ there is a positive number ε with the following properties:

(1) for any θ_0 there exist continuous functions

$$G_{\theta_0}^j : D_{\theta_0} = \{(r, \theta) | r_0 - \delta < r < r_0 + \delta,\ \theta_0 - \varepsilon < \theta < \theta_0 + \varepsilon\} \longrightarrow R \quad (j = 1, 2)$$

such that $G_{\theta_0}^1 < 0$, $G_{\theta_0}^2 > 0$ and $\hat{G}_{\theta_0}^j(D_{\theta_0}) \subset \{r < r_0 - \delta\}$ $(j = 1, 2)$,

(2)

$$h : \Big(\bigcup_{x \in D_{\theta_0}} \{x\} \times [G_{\theta_0}^1(x), G_{\theta_0}^2(x)]\Big) \longrightarrow U_{\theta_0} = \{x = \xi_t(y) | y \in D_{\theta_0},\ G_{\theta_0}^1(y) \leqq t \leqq G_{\theta_0}^2(y)\}$$

defined by $h(x, t) = \xi_t(x)$ is a homeomorphism.

Fix such δ and ε, and define $\mathscr{F}$ to be a function space

$$\mathscr{F} = \{f(\theta) | f \in C^1,\ f(\theta + 2\pi) = f(\theta),\ r_0 - \delta < f(\theta) < r_0 + \delta\}\,.$$

For $f \in \mathscr{F}$, we set $\Sigma_f = \{(r, \theta) | r < f(\theta)\}$. The subspace $\mathscr{F}_{\theta_0}$ of $\mathscr{F}$ is defined as: $f \in \mathscr{F}_{\theta_0}$ if and only if $\partial\Sigma_f$ is transversal along the flow at $(x, t) \in \partial\Sigma_f \times R$ whenever $\{x\}\cdot[0, t]$ (or $\{x\}\cdot[t, 0]) \subset U_{\theta_0}$. Using the property (2) and the transversality theorem, we can see that $\mathscr{F}_{\theta_0}$ is open and dense in $\mathscr{F}$ with respect to the C^1-topology. Take finitely many numbers $\theta_1, \theta_2, \cdots, \theta_k$ so that $\bigcup_{j=1}^k D_{\theta_j} \supset \{r_0 - \delta < r < r_0 + \delta\}$, then $\bigcap_{j=1}^k \mathscr{F}_{\theta_j}$ is non-empty. Take a function f in this set, then we have that $\partial\Sigma_f$ is transversal along the flow at $(x, t) \in \partial\Sigma_f \times R$ if $(\{x\}\cdot[0, t])_{\varepsilon_t} \cap \Sigma_f = \emptyset$, and hence that $A_{\Sigma_f}^1$ is finite. Moreover it can be easily seen that for each point a of $A_{\Sigma_f}^1$ there is an open set U with properties:

(a) $U = \{\xi_t(y) | y \in S,\ 0 < t < G(y)\}$ for some $S \subset \Sigma_f$ and some continuous function $G : S \to R$,

(b) $S\cap A^1_{\Sigma_f}=\{a\}$,

(c) $\hat{G}(S)\subset\Sigma_f$.

Let $\hat{T}^j_{\Sigma_f}(a)\in\partial\Sigma_f$ for $1\leqq j\leqq n$ and $\hat{T}^{n+1}_{\Sigma_f}(a)\in\Sigma_f$. We may assume that $\hat{G}(a)=\hat{T}^{n+1}_{\Sigma_f}(a)$, and that there are continuous functions $G_j\colon S\to R$ $(j=1, 2, \cdots, n)$ such that $\hat{G}_j(a)=\hat{T}^j_{\Sigma_f}(a)$ and $\hat{G}_j(S)\subset\Sigma'$. Let γ'_j be the connected component of $\hat{G}_j(S)\cap\partial\Sigma_f$ which contains the point $\hat{G}_j(a)$, and $\gamma_j=\hat{G}_j^{-1}(\gamma'_j)$. Because γ_j's intersect transversally to each other, we may assume that $\gamma_i\cap\gamma_j=\{a\}$ for $i\neq j$. Hence we can deform f slightly to g so that g satisfies that (i) $\partial\Sigma_g=\partial\Sigma_f$ outside of U, (ii) $\bar{U}\cap A^2_{\Sigma_g}=\varnothing$ and (iii) $\partial\Sigma_g$ is transversal along the flow at $(x, t)\in\partial\Sigma_g\times R$ if $(\{x\}\cdot[0, t])_{\varepsilon_t}\cap\Sigma_g=\varnothing$. Repeating this process finitely many times, we can get a regular local section. Since δ is arbitrary small, this completes the proof.

In the following of this section, let Σ' be a C^1-local section and Σ be a regular local section whose closure is contained in Σ'.

LEMMA 10. *If A^1_Σ consists of N points, then $C_\Sigma\backslash A^1_\Sigma$ has $2N$ connected components.*

PROOF. It is evident that T_Σ is continuous on $C_\Sigma\backslash A^1_\Sigma$ and $\hat{T}_\Sigma(C_\Sigma\backslash A^1_\Sigma)=\partial\Sigma\backslash(B^1_\Sigma\cup\hat{T}_\Sigma(B^1_\Sigma))$. Hence there is a one-to-one correspondence between the components of $C_\Sigma\backslash A^1_\Sigma$ and those of $\partial\Sigma\backslash(B^1_\Sigma\cup\hat{T}_\Sigma(B^1_\Sigma))$. Since $\partial\Sigma\backslash(B^1_\Sigma\cup\hat{T}_\Sigma(B^1_\Sigma))$ has $2N$ components, also $C_\Sigma\backslash A^1_\Sigma$ has $2N$ components.

Let $A^1_\Sigma=\{a_1, a_2, \cdots, a_N\}$. We denote by $C_1, C_2, \cdots, C_{2N}$ the components of $C_\Sigma\backslash A^1_\Sigma$. Then it is easy to see the existence of a neighborhood $S_k\subset\Sigma$ of a_k $(k=1, 2, \cdots, N)$ which satisfies the following conditions:

(a) there are continuous functions $\sigma_{k,j}\colon S_k\to R$ $(j=1, 2, 3)$ such that $\hat{\sigma}_{k,j}(S_k)\subset\Sigma'$ $(j=1, 2)$, $\hat{\sigma}_{k,3}(S_k)\subset\Sigma$, and $\hat{\sigma}_{k,j}(a_k)=\hat{T}^j_\Sigma(a_k)$ $(j=1, 2, 3)$.

(b) $S_k\cap(C_\Sigma\backslash A^1_\Sigma)$ has exactly three components $\gamma_{k,j}$ $(j=1, 2, 3)$ such that $\hat{\sigma}_{k,2}(\gamma_{k,1})\subset\Sigma$, $\hat{\sigma}_{k,2}(\gamma_{k,2})\cap\bar{\Sigma}=\varnothing$ and $\hat{\sigma}_{k,2}(\gamma_{k,3})\subset\partial\Sigma$.

DEFINITION 4.

(i) For $1\leqq k\leqq N$, define integers $k(j)$ $(j=1, 2, 3, 4$ and $1\leqq k(j)\leqq 2N)$ so that $C_{k(j)}\cap\gamma_{k,j}\neq\varnothing$ $(j=1, 2, 3)$ and $\hat{T}_\Sigma(a_k)\in\bar{C}_{k(4)}$.

(ii) A $2N\times 2N$ matrix $\Lambda_\Sigma=[\lambda_1, \lambda_2, \cdots, \lambda_{2N}]$ (λ_j is a $2N$ column vector) is defined by

$$(u_1, u_2, \cdots, u_{2N})\lambda_{2k-1}=u_{k(1)}-u_{k(2)}$$
$$(u_1, u_2, \cdots, u_{2N})\lambda_{2k}=u_{k(2)}-u_{k(3)}+u_{k(4)} \qquad (k=1, 2, \cdots, 2N)\ .$$

The remainder of this section is devoted to the proof of the next theorem.

THEOREM 1. *Suppose that Σ is a regular local section with the collar-size μ and A_Σ^1 consists of N-points. Let $X=M\backslash(\Sigma\cdot(-\mu, 0))_{\xi_t}$ and $i^*: \bar{H}^1(X)\to\bar{H}^1(\Sigma)$ be the homomorphism induced by the imbedding $i: \Sigma\to X$. Then we have* $\mathrm{Ker}(i^*)\simeq R^{2N-m}$, *if* $\mathrm{rank}(\Lambda_\Sigma)=m$.

Let $\tilde{X}$, p, Γ_j etc. be the same as those in section 3. We now define a special open covering of X. First for $1\leqq k\leqq N$, we set

$$U_k^1=\{\xi_t(x)|x\in S_k,\ 0\leqq t<\sigma_{k,1}(x)-\mu/3\}\cap X$$
$$U_k^2=\{\xi_t(x)|x\in S_k,\ \sigma_{k,1}(x)-2\mu/3<t<\sigma_{k,2}(x)-\mu/3\}\cap X$$
$$U_k^3=\{\xi_t(x)|x\in S_k,\ \sigma_{k,2}(x)-2\mu/3<t\leqq\sigma_{k,3}(x)-\mu\}\cap X$$

where S_k and $\sigma_{k,j}$ are the same as before. Next for $x\in C_\Sigma\backslash A_\Sigma^1$, we choose an open neighborhood $S'_x\subset\Sigma\backslash A_\Sigma^1$ of x so that $S'_x\cap(C_\Sigma\backslash A_\Sigma^1)$ is connected and there are continuous functions $\sigma'_{x,j}: S'_x\to R$ $(j=1, 2)$ with $\hat{\sigma}'_{x,1}(S'_x)\subset\Sigma'$, $\hat{\sigma}'_{x,2}(S'_x)\subset\Sigma$ and $\hat{\sigma}'_{x,j}(x)=\hat{T}_\Sigma^j(x)$ $(j=1, 2)$, and we set

$$V_x^1=\{\xi_t(y)|y\in S'_x,\ 0\leqq t<\sigma'_{x,1}(y)-\mu/3\}\cap X$$
$$V_x^2=\{\xi_t(y)|y\in S'_x,\ \sigma'_{x,1}(y)-2\mu/3<t\leqq\sigma'_{x,2}(y)-\mu\}\cap X\ .$$

And for $x\in\Sigma\backslash C_\Sigma$, we choose an open set W_x so that there are an open neighborhood $S''_x\subset\Sigma\backslash C_\Sigma$ and a continuous function $\sigma''_x: S''_x\to R$ such that $\hat{\sigma}''_x(S''_x)\subset\Sigma$ and $\sigma''_x(x)=T_\Sigma(x)$ and W_x can be written as

$$W_x=\{\xi_t(y)|y\in S''_x,\ 0\leqq t\leqq\sigma''_x(y)-\mu\}\ .$$

It is clear that $\mathscr{U}_0=\{U_k^j\}_{\substack{1\leqq k\leqq N\\ j=1,2,3}}\cup\{V_x^j\}_{\substack{x\in C_\Sigma\backslash A_\Sigma^1\\ j=1,2}}\cup\{W_x\}_{x\in\Sigma\backslash C_\Sigma}$ is an open covering of X. We may assume, without loss of generality, that $U\cap V$ is connected for any $U, V\in\mathscr{U}_0$.

DEFINITION 5. For a $2N$ vector $u=(u_1, u_2, \cdots, u_{2N})$, we define a collection $\phi_u=\{\phi_u(U)\}_{U\in\mathscr{U}_0}$ of functions $\phi_u(U): p^{-1}(U)\to R$ as follows:

(i) if $U=U_k^1$ for some k, then $\phi_u(U)\equiv 0$,

(ii) if $U=U_k^2$ for some k, then

$$\phi_u(U)(\tilde{x})=\begin{cases}0 & \text{if}\quad \tilde{x}\in\overline{p^{-1}(U_k^{2,1})}\\ u_{k(2)} & \text{if}\quad \tilde{x}\notin\overline{p^{-1}(U_k^{2,1})}\end{cases}$$

where $U_k^{2,1}=\{\xi_t(x)|x\in S_k^1=\hat{\sigma}_{k,1}^{-1}(\hat{\sigma}_{k,1}(S_k)\cap\Sigma),\ \sigma_{k,1}(x)\leqq t<\sigma_{k,2}(x)-\mu/3\}\cap X$,

(iii) if $U=U_k^3$ for some k, then

$$\phi_u(U)(\tilde{x})=\begin{cases}0 & \text{if}\quad \tilde{x}\in\overline{p^{-1}(U_k^{3,1})}\\ u_{k(3)} & \text{if}\quad \tilde{x}\in\overline{p^{-1}(U_k^{3,2})}\\ u_{k(4)} & \text{if}\quad \tilde{x}\notin\overline{p^{-1}(U_k^{3,1})}\cup\overline{p^{-1}(U_k^{3,2})}\end{cases}$$

where $U_k^{3,1}=\{\xi_t(x)|x\in S_k^2=\hat{\sigma}_{k,2}^{-1}(\hat{\sigma}_{k,2}(S_k)\cap\Sigma),\ \sigma_{k,2}(x)\leqq t\leqq\sigma_{k,3}(x)-\mu\}$ and $U_k^{3,2}=\{\xi_t(x)|x\in S_k\backslash(\overline{S_k^1}\cup\overline{S_k^2}),\ \sigma_{k,2}(x)-2\mu/3<t\leqq\sigma_{k,3}(x)-\mu\}\cap X$,

(iv) if $U=V_x^1$ for some $x\in C_\Sigma\backslash A_\Sigma^1$, then $\phi_u(U)\equiv 0$,

(v) if $U=V_x^2$ for some $x\in C_\Sigma\backslash A_\Sigma^1$, then

$$\phi_u(U)(\tilde{y})=\begin{cases}0 & \text{if } \tilde{y}\in\overline{p^{-1}(V_x^{2,1})}\\ u_j & \text{if } \tilde{y}\notin\overline{p^{-1}(V_x^{2,1})} \text{ and } S_x'\cap C_j\neq\varnothing\end{cases}$$

where $V_x^{2,1}=\{\xi_t(y)|y\in\hat{\sigma}_{x,1}'^{-1}(\hat{\sigma}_{x,1}'(S_x')\cap\Sigma),\ \sigma_{x,1}'(y)\leqq t\leqq\sigma_{x,2}'(y)-\mu\}$,

(vi) if $U=W_x$ for some $x\in\Sigma\backslash C_\Sigma$, then $\phi_u(U)\equiv 0$.

LEMMA 11. *Each $\phi_u(U)\in\phi_u$ is a locally constant function on $p^{-1}(U)$.*

PROOF. Because $\overline{p^{-1}(\Sigma)}$ is a global section, $\overline{p^{-1}(S\cap\Sigma)}\cap p^{-1}(S)$ is open and closed in $p^{-1}(S)$ for any subset S of Σ'. Hence $\overline{p^{-1}(U_k^{j,i})}\cap p^{-1}(U_k^j)$ ($k=1, 2, \cdots, N$, $j=2, 3$, $i=1, 2$) and $\overline{p^{-1}(V_x^{2,1})}\cap p^{-1}(V_x^2)$ ($x\in C_\Sigma\backslash A_\Sigma^1$) are open and closed in $p^{-1}(U_k^j)$ and $p^{-1}(V_x^2)$ respectively. Therefore, by the definition of $\phi_u(U)$, it is a locally constant function for each $U\in\mathscr{U}_0$.

Since $\bar{H}^0(Y)$ is isomorphic to the module of locally constant functions on Y (see [3]), ϕ_u can be regarded as an element of $C^0(\mathscr{U}_0;\Gamma_2)$. Let π be the homomorphism $\Gamma_2\to\Gamma_3$. We denote by $\pi^\#$ the induced homomorphism $C^*(\mathscr{U};\Gamma_2)\to C^*(\mathscr{U};\Gamma_3)$ and by π^* the homomorphism $H^*(\mathscr{U};\Gamma_2)\to H^*(\mathscr{U};\Gamma_3)$ or $\check{H}^*(X;\Gamma_2)\to\check{H}^*(X;\Gamma_3)$.

LEMMA 12. *$\delta(\pi^\#(\phi_u))=0$ in $C^1(\mathscr{U}_0;\Gamma_3)$ if and only if $u\Lambda_\Sigma=0$.*

PROOF. Let $\langle\phi_u\rangle=\{\psi\in C^0(\mathscr{U}_0;\Gamma_2)|\pi^\#(\psi)=\pi^\#(\phi_u)\}$. Then it is clear that $\delta(\pi^\#(\phi_u))=0$ if and only if $\delta\psi\in p^\#(C^1(\mathscr{U}_0;\Gamma_1))\subset C^1(\mathscr{U}_0;\Gamma_2)$ for some $\psi\in\langle\phi_u\rangle$, where $p^\#$ is the homomorphism $C^*(\mathscr{U};\Gamma_1)\to C^*(\mathscr{U};\Gamma_2)$ induced by p. It is also evident that $\psi\in\langle\phi_u\rangle$ if and only if $\psi-\phi_u\in p^\#(C^0(\mathscr{U}_0;\Gamma_1))$, and hence that all the element $\psi\in\langle\phi_u\rangle$ can be expressed as $\psi(U)=\phi_u(U)+b_U$ for any $U\in\mathscr{U}_0$ where b_U is a real constant.

Suppose $u\Lambda_\Sigma\neq 0$. If $u_{k(2)}-u_{k(3)}+u_{k(4)}\neq 0$, then $\phi_u(U_k^2)-\phi_u(U_k^3)$ is not constant on $p^{-1}(U_k^2\cap U_k^3)$. In the case when $u_{k(1)}-u_{k(2)}\neq 0$, $\phi_u(U_k^2)-\phi_u(V_x^2)$ is not constant on $p^{-1}(U_k^2\cap V_x^2)$ for a point $x\in C_{k(1)}\cap S_k$. Therefor, noting that $\delta\psi$ is in $p^\#(C^1(\mathscr{U}_0;\Gamma_1))$ if and only if $\psi(U)-\psi(V)$ is constant on $p^{-1}(U\cap V)$ whenever $U\cap V\neq\varnothing$, we have that $\delta\psi\notin p^\#(C^1(\mathscr{U}_0;\Gamma_1))$ if $\psi(U)=\phi_u(U)+b_U$ and $u\Lambda_\Sigma\neq 0$. On the other hand, it is easy to see that $\phi_u(U)-\phi_u(V)$ is constant on $p^{-1}(U\cap V)$ for any $U, V\in\mathscr{U}_0$ if $u\Lambda_\Sigma=0$. This completes the proof.

By this lemma, $\pi^\#(\phi_u)$ represents an element of $H^0(\mathscr{U}_0;\Gamma_3)$ if $u\Lambda_\Sigma=0$. We denote by $[\phi_u]$ the element of $\check{H}^0(X;\Gamma_3)$ represented by $\pi^\#(\phi_u)$.

LEMMA 13. *Let $L=\{[\phi_u] | u\Lambda_\Sigma=0\}$. Then L is a submodule of $\check{H}^0(X;\Gamma_3)$ isomorphic to R^{2N-m} where $m=\text{rank}(\Lambda_\Sigma)$.*

PROOF. It is clear that $[\phi_u]+[\phi_v]=[\phi_{u+v}]$, so L is a submodule. In order to prove that $L\simeq R^{2N-m}$, it is sufficient to show that $[\phi_u]=0$ if and only if $u=0$. Suppose $u_j\neq 0$, and take a point $x\in C_j$. Then $\phi_u(V_x^2)+b\not\equiv 0$ for any constant b. Now it is easy to see that $\psi\neq 0$ in $C^0(\mathscr{U};\Gamma_2)$ for any refinement $\mathscr{U}$ of $\mathscr{U}_0$ if $\pi^\#(\psi)=\pi^\#(\lambda(\phi_u))$, where $\lambda: C^*(\mathscr{U}_0;\Gamma_2)\to C^*(\mathscr{U};\Gamma_2)$ is the usual homomorphism (see [3]). This implies that $[\phi_u]\neq 0$ if $u\neq 0$. The converse is trivial hence the proof is completed.

LEMMA 14. *Let θ be an element of $\check{H}^0(X;\Gamma_3)$. If $\delta^*(\theta)\in\text{Ker}(i^*)$, then there exists a $2N$-vector u such that $u\Lambda_\Sigma=0$ and $[\phi_u]-\theta\in\pi^*(\check{H}^0(X;\Gamma_2))$, where δ^* is the homomorphism $\check{H}^0(X;\Gamma_3)\to\check{H}^1(X;\Gamma_1)\simeq\bar{H}^1(X)$.*

PROOF. If $\delta^*(\theta)\in\text{Ker}(i^*)$, then there is an element $\psi'\in C^0(\mathscr{U};\Gamma_2)$ such that $\pi^\#(\psi')\in C^0(\mathscr{U};\Gamma_3)$ represents θ, where $\mathscr{U}$ is a suitable open covering of X. Consider the restriction of p onto $\overline{p^{-1}(\Sigma)}$. Then, by the same reasoning as Proposition 2, we get an exact sequence $\check{H}^0(\bar{\Sigma};\Gamma_2)\to\check{H}^0(\bar{\Sigma};\Gamma_3)\to\bar{H}^1(\bar{\Sigma})$. On the other hand, if follows from Lemma 8 that $\check{H}^0(\bar{\Sigma};\Gamma_2)\simeq\bar{H}^0(\overline{p^{-1}(\Sigma)})$. Therefore, taking a refinement if necessary, we can find a locally constant function ψ^0 on $\overline{p^{-1}(\Sigma)}$ and an element ψ of $C^0(\mathscr{U};\Gamma_2)$ such that $\pi^\#(\psi)=\pi^\#(\psi')$ and $\psi(U)=\psi^0$ on $p^{-1}(U)\cap\overline{p^{-1}(\Sigma)}$ if it is non-empty.

Now take a point $x\in C_j$. We can choose a neighborhood $S\subset\Sigma$ of x and sequences $\{t_j\}_{j=0}^n$, $\{t'_j\}_{j=0}^n$ ($t_0=0$, $t_j<t_{j+1}<t'_j<t'_{j+1}$, $t_n<T_\Sigma(x)<t'_n$) so that $S'=S\cap S'_x$ is connected and $S'\cdot[t_j, t'_j]\subset U_j$ for some $U_j\in\mathscr{U}$. Then, because $\delta\pi^\#(\psi)=0$ in $C^1(\mathscr{U};\Gamma_3)$, $\psi(U_{j+1})-\psi(U_j)$ is constant on $p^{-1}(S')\cdot[t_{j+1}, t_j]$ for each j. Therefore, since $\psi(U)$ is locally constant for any $U\in\mathscr{U}$, $\Psi(\tilde{x}, t)\equiv\psi(U_n)(\zeta_t(\tilde{x}))-\psi^0(\tilde{x})$ is a constant on $p^{-1}(S')\times[t_n, t'_n]$ which we denote by $u_j(\psi)$. We must show that this constant does not depend on the choice of a point $x\in C_j$. Let $x'\in C_j$, $\hat{T}_\Sigma(x')\in U'$ and $U'\cap U_n\neq\varnothing$. Then, because $\psi(U')-\psi(U_n)=0$ on $p^{-1}(U')\cap p^{-1}(U_n)\cap\overline{p^{-1}(\Sigma)}$ and we may assume $\psi(U')-\psi(U_n)$ is constant on $p^{-1}(U'\cap U_n)$, we have $\psi(U')-\psi(U_n)=0$ on $p^{-1}(U'\cap U_n)$. This implies that $u_j(\psi)$ is a constant determined only by j and ψ.

Setting $u(\psi)=(u_1(\psi), u_2(\psi), \cdots, u_{2N}(\psi))$, we shall show $u(\psi)\Lambda_\Sigma=0$. Take $a_k\in A_\Sigma^1$ and $U\in\mathscr{U}$ so that $\hat{T}_\Sigma(a_k)\in U$, then we have

$$\psi(U)(\zeta_\sigma(\tilde{x}))-\psi^0(\tilde{x})=\begin{cases} u_{k(1)}(\psi) & \text{if} \quad p(\tilde{x})\in(S_k\backslash\overline{S_k^1})\cap S_k^2 \\ u_{k(2)}(\psi) & \text{if} \quad p(\tilde{x})\in(S_k\backslash\overline{S_k^1})\backslash\overline{S_k^2} \end{cases} \qquad (\sigma=\sigma_{k,1}(p(\tilde{x})))$$

where S_k, S_k^j and $\sigma_{k,1}$ are those given in Definition 5. Since it is shown that $\psi(U)(\zeta_\sigma(\tilde{x}))-\psi^0(\tilde{x})$ is constant on some neighborhood of a_k, we obtain the equality $u_{k(1)}(\psi)-u_{k(2)}(\psi)=0$. In order to prove $u_{k(2)}(\psi)-u_{k(3)}(\psi)+u_{k(4)}(\psi)=0$, take an element $V\in\mathscr{U}$ so that $\hat{T}^2_\Sigma(a_k)\in V$. Then we can see that

$$\psi(V)(\zeta_\sigma(\tilde{x}))-\psi^0(\tilde{x}))=\begin{cases} u_{k(3)}(\psi) & \text{if } p(\tilde{x})\in(S_k\backslash\overline{S_k^2})\backslash\overline{S_k^1} \text{ and } \sigma=\sigma_{k,2}(p(\tilde{x}))\\ u_{k(4)}(\psi) & \text{if } p(\tilde{x})\in\hat{\sigma}_{k,1}((S_k\backslash\overline{S_k^2})\cap\overline{S_k^1}) \text{ and}\\ & \sigma=\sigma_{k,2}(\hat{\sigma}_{k,1}^{-1}(p(\tilde{x}))-\sigma_{k,1}(\hat{\sigma}_{k,1}^{-1}(p(\tilde{x}))\ .\end{cases}$$

Let U be an element of $\mathscr{U}$ containing the point $\hat{T}_\Sigma(a_k)$, and S be a subset of Σ' such that $S'=S\cap\hat{\sigma}_{k,1}(S_k\backslash\overline{S_k^2})$ is connected and $S'\cdot[t_j,t'_j]\subset U_j\in\mathscr{U}(U_0=U, U_n=V)$ for some sequences $\{t_j\}_{j=0}^n$ and $\{t'_j\}_{j=0}^n$ $(t_0=0,\ t_j<t_{j+1}<t'_j<t'_{j+1})$. Then we get

$$\psi(V)(\zeta_t(\tilde{x}))-\psi(U)(\tilde{x})=\begin{cases} u_{k(3)}(\psi)-u_{k(2)}(\psi) & \text{for } p(\tilde{x})\in S'\backslash\hat{\sigma}_{k,1}(\overline{S_k^1})\ ,\\ & t_n<t<t'_n\\ u_{k(4)}(\psi) & \text{for } p(\tilde{x})\in S'\cap\hat{\sigma}_{k,1}(S_k^1)\ ,\quad t_n<t<t'_n\ .\end{cases}$$

Since $\psi(U_{j+1})-\psi(U_j)$ is constant on $p^{-1}(S')\cdot[t_{j+1},t'_j]$, we must have $u_{k(3)}(\psi)-u_{k(2)}(\psi)=u_{k(4)}(\psi)$. This proves that $u(\psi)$ is a solution of $u\Lambda_\Sigma=0$.

Now let us verify that $[\phi_{u(\psi)}]-\theta\in\pi^*(\check{H}^0(X;\Gamma_2))$. To prove this, it is sufficient to show that $\delta(\lambda(\phi_{u(\psi)})-\psi+\chi)=0$ in $C^1(\mathscr{U};\Gamma_2)$ for some $\chi\in p^\sharp(C^0(\mathscr{U};\Gamma_1))$ where we assume that $\mathscr{U}$ is a refinement of $\mathscr{U}_0$ and λ is the usual homomorphism $C^0(\mathscr{U}_0;\Gamma_2)\to C^0(\mathscr{U};\Gamma_2)$. It is now easy to see that for each $U\in\mathscr{U}$ there is a constant b_U such that $-\lambda(\phi_{u(\psi)}(U)+\psi(U)=\psi^0\circ\hat{T}_*+b_U$ where $T_*(x)=\sup\{t<0|\zeta_t(\tilde{x})\in\overline{p^{-1}(\Sigma)}\}$. Because $\psi^0\circ\hat{T}_*$ is a locally constant function on $\tilde{X}$, $\{\psi^0\circ\hat{T}_*|_{p^{-1}(U)}\}_{U\in\mathscr{U}}\in C^0(\mathscr{U};\Gamma_2)$ is a cocycle. Hence, putting $\chi(U)\equiv b_U$, we get $\chi\in p^\sharp(C^0(\mathscr{U};\Gamma_1))$ and $\delta(\lambda(\phi_{u(\psi)})-\psi+\chi))=0$. This completes the proof.

LEMMA 15. $\pi^*(H^0(X;\Gamma_2))\cap L=\{0\}$.

PROOF. Suppose $u_j\neq0$, and take a point $x\in C_j$. Then we can choose a sequence $U_1, U_2, \cdots, U_n$ of elements of $\mathscr{U}_0$ so that $U_1=V_x^1$, $U_2=V_x^2$ and $U_i\cap U_{i+1}\cap\Sigma\neq\varnothing$ for $2\leqq i\leqq n$ $(U_{n+1}=U_1)$. From the definition of ϕ_u, it follows that $\phi(U_2)-\phi(U_1)=u_j$ and $\phi(U_{i+1})-\phi(U_i)=0$ for $2\leqq i\leqq n$, hence that $\delta\psi\neq0$ in $C^1(\mathscr{U}_0;\Gamma_2)$ for any $\psi\in\langle\phi_u\rangle$. It is now easy to see that for any refinement $\mathscr{U}$ of $\mathscr{U}_0$ there is no element $\psi\in C^0(\mathscr{U};\Gamma_2)$ such that $\delta\psi=0$ and $\pi^\sharp(\psi)=\pi^\sharp(\lambda(\phi_u))$. This proves the lemma.

PROOF OF THEOREM 1. Let α be an element of $\mathrm{Ker}(i^*)$. According to Proposition 2, we can choose an element θ of $\check{H}^0(X;\Gamma_3)$ such that

$\delta^*\theta=\alpha$. Therefore it follows from Lemma 14 that $\mathrm{Ker}(i^*)\subset\delta^*(L)$. On the other hand, it is evident that $\delta^*([\phi_u])$ is contained in $\mathrm{Ker}(i^*)$ for any element $[\phi_u]$ of L. Hence, using Proposition 2 and Lemma 13, 15, we get $\mathrm{Ker}(i^*)\simeq L/\pi^*(\check{H}^0(X;\Gamma_2))\cap L\simeq L\simeq R^{2N-m}$. This completes the proof.

As a special case, we can prove

THEOREM 2. *Let Σ be a regular local section homeomorphic to a 2-disk and A^1_Σ consist of N-points. If* $\mathrm{rank}(\Lambda_\Sigma)=m$, *then* $\bar{H}^1(M)\simeq R^{2N-m}$.

PROOF. Because $\bar{H}^1(\bar{\Sigma})$ is trivial, it follows from Theorem 1 that $\bar{H}^1(X)$ is isomorphic to R^{2N-m}. Hence it is sufficient to show that $\bar{H}^1(X)\simeq\bar{H}^1(M)$.

Consider the following Mayer-Vietoris exact sequence

$$\cdots\to\tilde{H}^0(X\cap(\overline{M\backslash X}))\to\tilde{H}^1(M)\to\tilde{H}^1(X)\oplus\tilde{H}^1(\overline{M\backslash X})\to\tilde{H}^1(X\cap(\overline{M\backslash X}))\to\cdots$$

where $\tilde{H}^*(\cdot)$ denotes the reduced singular cohomology. Because $\overline{M\backslash X}$ is homeomorphic to a 3-disk and $X\cap(\overline{M\backslash X})$ to a 2-sphere, there is an isomorphism $\tilde{H}^1(M)\simeq\tilde{H}^1(X)$. Since M and X are manifolds, we get $\bar{H}^1(X)\simeq\tilde{H}^1(X)\simeq\tilde{H}^1(M)\simeq\bar{H}^1(M)$. This completes the proof.

§ 5. 1-cycles.

Again let ξ_t be a minimal flow on a 3-dimensional manifold M, and Σ be a regular local section with the collar-size μ which is homeomorphic to a 2-disk. In this section we will investigate on 1-cycles of M.

Let $\mathscr{U}$ be an open covering of $X=M\backslash\Sigma\cdot(-\mu,0)$, and $\omega\in C^1(\mathscr{U};R)$ be a cocycle. Now we shall define the integral of ω along a circle. Let $\gamma:[a,b]\to X$ $(a<b,\ \gamma(a)=\gamma(b))$ be a closed continuous curve, and choose a partition $a=t_0<t_1<\cdots<t_k=b$ of $[a,b]$ such that there are elements U_j $(j=1,2,\cdots,k)$ of $\mathscr{U}$ such that $\gamma([t_{j-1},t_j])\subset U_j$. And define

$$I_\gamma(\omega)=\omega(U_2,U_1)+\omega(U_3,U_2)+\cdots+\omega(U_k,U_{k-1})+\omega(U_1,U_k)\ .$$

Using the cocycle condition, one can show that $I_\gamma(\omega)$ does not depend on the choice of U_j $(1\leqq j\leqq k)$, and that for another covering $\mathscr{U}'$ and a cocycle $\omega'\in C^1(\mathscr{U}';R)$ $I_\gamma(\omega)=I_\gamma(\omega')$ if ω and ω' are in the same class of $\check{H}^1(X;R)$. Moreover one can easily show that $I_\gamma(\omega)=I_{\gamma'}(\omega)$ if $[\gamma]_{H_1(X;R)}=[\gamma']_{H_1(X;R)}$. Thus we can set

$$\int_{[\gamma]_{H_1(X;R)}}[\omega]_{\check{H}^1(X;R)}=I_\gamma(\omega)\ .$$

Let A^1_Σ consist of N points, and C_j $(j=1,2,\cdots,2N)$ be the components of $C_\Sigma \backslash A_\Sigma^1$. For a point $x \in C_j$, let $\gamma'_{j,x}: [0, 1] \to X$ be a continuous curve such that $\gamma'_{j,x} \subset \bar{\Sigma}$, $\gamma'_{j,x}(0) = \hat{T}_\Sigma(x)$ and $\gamma'_{j,x}(1) = x$, and let $\gamma''_{j,x}$ be a continuous curve defined by $\gamma''_{j,x}(t) = \xi_t(x)$ $(0 \leqq t \leqq T_\Sigma(x))$. Then $\gamma_{j,x} = \gamma'_{j,x} + \gamma''_{j,x}$ is a closed curve. Because Σ is homeomorphic to a disk, $[\gamma_{j,x}]_{H_1(X;R)}$ and $[\gamma_{j,x}]_{H_1(M;R)}$ do not depend on the point $x \in C_j$ and the curve $\gamma'_{j,x}$. Hence we write $[\gamma_j]_{H_1(X;R)}$ or $[\gamma_j]_{H_1(M;R)}$ instead of $[\gamma_{j,x}]_{H_1(X;R)}$ or $[\gamma_{j,x}]_{H_1(M;R)}$ respectively.

PROPOSITION 3. *Suppose Σ is a regular section homeomorphic to a 2-disk for which A_Σ^1 consists of N points. Let C_j $(j = 1, 2, \cdots, 2N)$ be the components of $C_\Sigma \backslash A_\Sigma^1$, and $[\gamma_j]_{H_1(M,R)}$ be that defined above. If $uA_\Sigma = 0$ has a solution whose j-th component does not vanish, then $[\gamma_j]_{H_1(M;R)} \neq 0$.*

PROOF. Let $u = (u_1, u_2, \cdots, u_{2N})$ satisfy the equation $uA_\Sigma = 0$, and ϕ_u be that in Definition 5. Then we can see that

$$\int_{[\gamma_j]_{H_1(X;R)}} [\delta^*[\phi_u]]_{\check{H}^1(X;R)} = u_j \tag{1}$$

(see the proof of Lemma 15). Since Σ is homeomorphic to a disk, $H_1(X; R)$ is isomorphic to $H_1(M; R)$. Therefore (1) implies the consequence of the proposition.

PROPOSITION 4. *Under the same assumption as Proposition* 3, *$H_1(M; R)$ is spanned by $\{[\gamma_j]_{H_1(M;R)};\ j = 1, 2, \cdots, 2N\}$.*

PROOF. Let $u = (u_1, u_2, \cdots, u_{2N})$ be a solution of $uA_\Sigma = 0$. If

$$\int_{[\gamma_j]_{H_1(X;R)}} [\delta^*[\phi_u]]_{\check{H}^1(X;R)} = 0$$

for any $j = 1, 2, \cdots, 2N$, then by means of (1) we have $u = 0$. On the other hand, according to Theorem 2, for any $\omega \in \check{H}^1(X; R)$ there is a solution u of $uA_\Sigma = 0$ such that $\omega = \delta^*[\phi_u]$.

Suppose there is a closed curve γ such that $[\gamma]_{H_1(M;R)}$ is independent of $\{[\gamma_j]_{H_1(M;R)};\ 1 \leqq j \leqq 2N\}$. Because the local section Σ is homeomorphic to a disk, we may assume that $\gamma \subset X$ and $[\gamma]_{H_1(X;R)}$ is not included in the subspace spanned by $\{[\gamma_j]_{H_1(X;R)};\ 1 \leqq j \leqq 2N\}$. Therefore there is an element ω of $\check{H}^1(X; R)$ such that

$$\int_{[\gamma]_{H_1(X;R)}} \omega \neq 0$$

but

$$\int_{[\gamma_j]_{H_1(X;R)}} \omega = 0$$

for any $j=1, 2, \cdots, 2N$. It is now clear that for such ω there is no solution u of $u\Lambda_\Sigma=0$ such that $\omega=\delta^*[\phi_u]$. This is a contradiction and the proof is complete.

APPENDIX

Theorem 2 implies that, in order to prove the conjecture that there is no minimal flow on S^3, it is sufficient to show that if there is a minimal flow on S^3, then one can construct a regular local section Σ such that $\Sigma\approx D^2$ and $2N$-vectors $\lambda_1, \cdots, \lambda_{2N}$ are not linearly independent where $\Lambda_\Sigma=[\lambda_1, \cdots, \lambda_{2N}]$.

Also in the case when the dimension of minimal sets is greater than 3, we can get the results analogous to Theorems 1, 2 and Propositions 3, 4. However the matrix corresponding to Λ_Σ is not square in this higher dimensional case.

In the two dimensional case it is proved by our method that if M is a two dimensional manifold on which a minimal flow exists, then $H^1(M; R)\simeq R^2$. This gives another proof for the fact there is no minimal flow on the Klein bottle.

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Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
KEIO UNIVERSITY
HIYOSHI, YOKOHAMA 223