TOKYO J. MATH.
VOL. 1, No. 1, 1978

On the Volume Elements on an Expansive Set

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In [6], J. Moser proved that the group $\mathscr{D}(M)$ of all C^∞-diffeomorphisms of a compact connected C^∞-manifold M with $\partial M=\emptyset$ acts transitively on the space $\mathscr{V}$ of all C^∞-volume elements with total volume one, where the action is of course given by the pullback φ^*dV for $\varphi\in\mathscr{D}(M)$ and $dV\in\mathscr{V}$.

Moreover the mapping $\Phi: \mathscr{D}(M)\to\mathscr{V}$ given by $\Phi(\varphi)=\varphi^*dV$ for any fixed $dV\in\mathscr{V}$ defines a structure of principal fibre bundle with the fibre $\mathscr{D}_{dV}(M)=\{\varphi\in\mathscr{D}(M); \varphi^*dV=dV\}$, where the topologies are given by the C^∞-topology. Since $\mathscr{V}$ is convex, the above principal bundle turns out to be trivial, and hence $\mathscr{D}(M)$ is homeomorphic to $\mathscr{D}_{dV}(M)\times\mathscr{V}$ (cf. [8], [1], [9]). Especially, $\mathscr{D}_{dV}(M)$ is homotopically equivalent with $\mathscr{D}(M)$.

The purpose of this note is to show that a little weaker theorem holds for a wider class of compact sets, i.e., orientable expansive sets with nonvoid connected interior $'S$ such that $S=\overline{'S}$. Namely, in such a compact set S, the inclusion $i: \mathscr{D}_{dV}(S)\to\mathscr{D}(S)$ gives a *weak* homotopy equivalence.

§1. Preliminaries and the precise statement of the theorem.

Let N be an n-dimesional smooth (C^∞-) manifold and S a compact subset of N. By T'_S we denote the restriction of the tangent bundle T_N onto S. Functions, vector fields (sections of T'_S) or p-forms (sections of the exterior product $\bigwedge^p T'_S$) are said to be *smooth* if they can be extended smoothly on a neighborhood of S in N. A smooth vector field u on S is called a *strictly tangent vector field on* S if the integral curves of an extension $\tilde{u}$ of u with initial points in S are contained in S for $-\infty<t<\infty$. This property for u does not depend on the choice of extension $\tilde{u}$. By $\Gamma(T_S)$, we denote the totality of smooth strictly tangent vector fields on S. As it will be proved in the next section, $\Gamma(T_S)$ is a Lie algebra under the usual Lie bracket product and a $\Gamma(1_S)$-module, where $\Gamma(1_S)$ is the ring of all C^∞-functions on S.

Received January 26, 1978

A mapping φ of S onto S is said to be a *C^∞-diffeomorphism* of S if φ can be extended to a C^∞-diffeomorphism of a neighborhood of S onto another one of S. The group of all C^∞-diffeomorphisms of S will be denoted by $\mathscr{D}(S)$. A compact subset S of N will be called an *expansive set*, if for each $x \in S$ there is $X_x \in \Gamma(T_S)$ such that $X_x(x)=0$ and there is an extension $\tilde{X}_x$ of X_x with the following property (P):

(P) The eigenvalues of the linear part of $\tilde{X}_x$ at x are real and positive.

We call such X_x an *expansive vector field on* S at x. Remark at first that for any $x \in {}'S$, there is an expansive vector field on S at x. Therefore the above condition for expansive sets is only related to the shape of $S-{}'S$. However, if $S \neq \overline{{}'S}$, then the property (P) may depend on the choice of extension X_x. A compact cornered manifold is an important example of expansive set. Moreover a subset such as $\{(x, y) \in \boldsymbol{R}^2; x^3-y^2 \geqq 0, x^2+y^2 \leqq 1\}$ is an expansive set. An expansive vector field at the origin is given by

$$\frac{1}{3}x\frac{\partial}{\partial x}+\frac{1}{2}y\frac{\partial}{\partial y}$$

multiplied by an appropriate cut off function.

Note that if $\overline{{}'S} \neq S$, then it is rather hard to define the C^∞-topology on $\Gamma(T_S)$, $\Gamma(1_S)$ or $\mathscr{D}(S)$. So, for the simplicity we assume $\overline{{}'S}=S$ throughout this note. Under this condition, $\mathscr{D}(S)$ is a topological group in the C^∞-topology. Now, assume that S has an orientable neighborhood in N, and let $\tilde{\mathscr{V}}_S$ be the totality of C^∞-volume forms on S with the C^∞-topology. If S is an expansive set, then it is not hard to see that S is a measurable set by every $dV \in \tilde{\mathscr{V}}_S$. (See § 3.) Let $\mathscr{V}_S=\left\{dV \in \tilde{\mathscr{V}}_S; \int_S dV=1\right\}$. Then $\mathscr{V}_S$ is a closed convex subset of $\tilde{\mathscr{V}}_S$.

Let δ^{k+1} be the unit closed disk in $\boldsymbol{R}^{k+1}$ with the origin 0 as the center, and σ^k the boundary of δ^{k+1}. The statement to be proved in this note is as follows:

THEOREM. *Let S be a compact expansive set in N with orientable neighborhood and with nonvoid connected interior ${}'S$ such that $\overline{{}'S}=S$. For an arbitrary k, let $h: \sigma^k \to \mathscr{V}_S$ be a continuous mapping. Then there is a continuous mapping $H: \delta^{k+1} \to \mathscr{D}(S)$ such that $H(0)=$*identity *and $H(q)^* dV_0=h(q)$ for $q \in \sigma^k$, where dV_0 is a prescribed element in $\mathscr{V}_S$.*

Apply the above theorem to the case $k=0$, and we have that the group $\mathscr{D}(S)$ acts transitively on $\mathscr{V}_S$. Moreover it is not hard to see the following:

COROLLARY. *Let* $\mathscr{D}_{dV_0}(S)=\{\varphi\in\mathscr{D}(S);\ \varphi^*dV_0=dV_0\}$. *Notations and assumptions being as above, the inclusion* $i: \mathscr{D}_{dV_0}(S)\to\mathscr{D}(S)$ *gives a weak homotopy equivalence,* i.e., $\pi_*(\mathscr{D}^0_{dV_0}(S))\to\pi_*(\mathscr{D}(S))$ *is an isomorphism.*

REMARK. If $\mathscr{D}_{dV_0}(S)$ and $\mathscr{D}(S)$ are ANR (absolute neighborhood retract), then Theorem 15 in [11] shows that the above i gives a homotopy equivalence. However, there is no simple method to prove $\mathscr{D}_{dV_0}(S)$ or $\mathscr{D}(S)$ is ANR.

For the later use, we shall define notions of smoothness of mappings in the remainder of this section. Let W be an open subset of a C^∞-manifold N'. A mapping $\varphi: W\to\mathscr{D}(S)$ is said to be *smooth* if there are open neighborhoods U, U' of $W\times S$ in $W\times N$, and a smooth diffeomorphism $\tilde{\Phi}: U\to U'$ such that (a) Φ can be written in the form $\Phi(w, x)=(w, \tilde{\varphi}(w)(x))$, and (b) $\tilde{\varphi}(w): U_w\to U'_w$ is an extension of $\varphi(w)$, where $U_w=U\cap(\{w\}\times N)$, $U'_w=U'\cap(\{w\}\times N)$. Let T be a compact subset of N'. A mapping $\psi: T\to\mathscr{D}(S)$ is said to be *smooth* if ψ can be extended to a smooth mapping of an open neighborhood of T into $\mathscr{D}(S)$. We denote by $\mathscr{M}(T, \mathscr{D}(S))$ the totality of smooth mappings of T into $\mathscr{D}(S)$. If T is an interval $[a, b]$, ψ is called a *smooth arc* in $\mathscr{D}(S)$.

Let E be a C^∞-vector bundle on N and E'_S the restriction of E onto S. By $\Gamma(E'_S)$ we denote the space of all C^∞-sections of E'_S. A mapping $\psi: T\to\Gamma(E'_S)$ is said to be *smooth*, if there are neighborhoods W_T, V_S of T, S respectively in N', N and a mapping $\tilde{\psi}$ of W_T into $\Gamma(E'_S)$ which extends ψ such that $\tilde{\psi}(w)(x)$ is smooth with respect to $(w, x)\in W_T\times V_S$. By $\mathscr{M}(T, \Gamma(E'_S))$ we denote the totality of smooth mappings of T into $\Gamma(E'_S)$.

Let E_1, E_2 are C^∞-vector bundles on N and $F=E_1\otimes E_2^*$. Then there is a natural evaluation mapping $ev: \mathscr{M}(T, \Gamma(F'_S))\times\mathscr{M}(T, \Gamma(E'_{2,S}))\to\mathscr{M}(T, \Gamma(E'_{1,S}))$, defined by $ev(A, v)(t)(x)=A(t)(x)v(t)(x)$, where $t\in T$ and $x\in S$. Let $GL(E)$ be the bundle of the fibre isomorphisms of E onto itself, and $GL(E'_S)$ its restriction to S. The space $\mathscr{M}(T, \Gamma(GL(E'_S)))$ is defined by the same manner as above. The group inversion defines naturally a mapping i of $\mathscr{M}(T, \Gamma(GL(E'_S)))$ onto itself. Now, assume that $\overline{T}=T$, $\overline{S}=S$. Then, the C^∞-topologies can be well-defined on $\mathscr{M}(T, \mathscr{D}(S))$, $\mathscr{M}(T, \Gamma(E'_S))$ and $\mathscr{M}(T, \Gamma(GL(E'_S)))$ by regarding each element as a mapping from $T\times S$. The following continuity lemma is easy to prove:

LEMMA 1.1. *Notations and assumptions being as above, ev and i are continuous in the C^∞-topology.*

A system $\{\boldsymbol{E}, \boldsymbol{E}^k, k \geqq 0\}$ (k's are integers) is called an *ILB-system* if each $\boldsymbol{E}^k$ is a Banach space, $\boldsymbol{E}^{k+1}$ is bounded-linearly and densely imbedded in $\boldsymbol{E}^k$ and $\boldsymbol{E}$ is the intersection of all $\boldsymbol{E}^k$ with the inverse limit topology (cf. [10] Chap. I). By $\mathscr{M}^r(T, \boldsymbol{E}^k)$ we denote the space of all C^r-mappings of T into $\boldsymbol{E}^k$, where the smoothness should be understood by the same manner as in the case of C^∞. Since $'\overline{T}=T$, $\mathscr{M}^r(T, \boldsymbol{E}^k)$ is a Banach space in the C^r-uniform topology, and $\mathscr{M}^r(T, \boldsymbol{E}^{k+1})$ is bounded-linearly and densely imbedded in $\mathscr{M}^r(T, \boldsymbol{E}^k)$. Let $\mathscr{M}(T, \boldsymbol{E})$ be the intersection of all $\mathscr{M}^k(T, \boldsymbol{E}^k)$ with the inverse limit topology. Then $\{\mathscr{M}(T, \boldsymbol{E}), \mathscr{M}^k(T, \boldsymbol{E}^k), k \geqq 0\}$ is an ILB-system. An element of $\mathscr{M}(T, \boldsymbol{E})$ will be called a *smooth* mapping of T into $\boldsymbol{E}$. Let $\{\boldsymbol{F}, \boldsymbol{F}^k, k \geqq 0\}$ be another ILB-system. A linear mapping $A: \boldsymbol{E} \to \boldsymbol{F}$ is said to be *order* r if A can be extended to a continuous mapping $A: \boldsymbol{E}^{k+r} \to \boldsymbol{F}^k$ for every k such that $k+r$, $k \geqq 0$. We denote by $L_r(\boldsymbol{E}, \boldsymbol{F})$ the linear space of all linear mappings of order r, and by L_r^k its completion by the norm $|||A|||_k = \max\{||A||_i; \max(0, r) \leqq i \leqq k\}$, where $||A||_i$ is the operator norm of $A: \boldsymbol{E}^{i+r} \to \boldsymbol{F}^i$. Obviously, $\{L_r(\boldsymbol{E}, \boldsymbol{F}), L_r^k, k \geqq \max(r, 0)\}$ is an ILB-system. Therefore one can define the space $\mathscr{M}(T, L_r(\boldsymbol{E}, \boldsymbol{F}))$. Let $GL_r(\boldsymbol{E}, \boldsymbol{F})$ be the totality of $A \in L_r(\boldsymbol{E}, \boldsymbol{F})$ such that A can be extended to a continuous bijection of $\boldsymbol{E}^{k+r}$ onto $\boldsymbol{F}^k$ for every k such that $k+r$, $k \geqq 0$. A mapping $\varphi: T \to GL_r(\boldsymbol{E}, \boldsymbol{F})$ is said to be *smooth*, if $\varphi: T \to L_r(\boldsymbol{E}, \boldsymbol{F})$ is smooth. The following lemma is not hard to prove:

LEMMA 1.2. *Notations and assumptions being as above,*

$$ev: \mathscr{M}(T, L_r(\boldsymbol{E}, \boldsymbol{F})) \times \mathscr{M}(T, \boldsymbol{E}) \longrightarrow \mathscr{M}(T, \boldsymbol{F})$$
$$i: \mathscr{M}(T, GL_r(\boldsymbol{E}, \boldsymbol{F})) \longrightarrow \mathscr{M}(T, GL_{-r}(\boldsymbol{F}, \boldsymbol{E}))$$

are continuous, where $ev(A, u)(t) = A(t)u(t)$ *and* $(iA)(t) = A(t)^{-1}$.

§2. The group $\mathscr{D}(S)$ and the Lie algebra $\Gamma(T_S)$.

Let S be a compact subset of N. Without loss of generality, one may assume that N is a compact manifold with C^∞ boundary ∂N such that $S \cap \partial N = \emptyset$. Since N itself is an expansive set, the group $\mathscr{D}(N)$ is defined by the same manner as above. $\mathscr{D}(N)$ is a strong ILB-Lie group (cf. [9]) with the Lie algebra $\Gamma(T_N)$, where $\Gamma(T_N)$ is the totality of C^∞-vector fields $\tilde{u}$ on N such that $\tilde{u} | \partial N$ are tangent vector fields on ∂N (cf. [9] II.4 or [10] Chap. 8, §7). We denote by $\mathscr{D}_S(N)$ the group $\{\tilde{\varphi} \in \mathscr{D}(N); \tilde{\varphi}(S) = S\}$, and by $\mathscr{D}_{S,0}(N)$ the group $\{\tilde{\varphi} \in \mathscr{D}_S(N); \tilde{\varphi}(x) = x$ for every $x \in S\}$. $\mathscr{D}_{S,0}(N)$ is a closed normal subgroup of $\mathscr{D}_S(N)$. Let $\Gamma_S(T_N)$ be the totality of $\tilde{u} \in \Gamma(T_N)$ such that $\exp t\tilde{u} \in \mathscr{D}_S(N)$ for $-\infty < t < \infty$, where $\exp t\tilde{u}$ is the one parameter subgroup generated by $\tilde{u}$. Since

$\mathscr{D}_S(N)$ is a closed subgroup of $\mathscr{D}(N)$, $\Gamma_S(T_N)$ is a closed Lie subalgebra of $\Gamma(T_N)$ (cf. 1.4.1 Theorem [9]). Set $\Gamma_{S,0}(T_N)=\{\tilde{u}\in\Gamma_S(T_N);\ \tilde{u}|S\equiv 0\}$. It is clear that $\Gamma_{S,0}(T_N)=\{\tilde{u}\in\Gamma(T_N);\ \exp t\tilde{u}\in\mathscr{D}_{S,0}(N)$ for $-\infty<t<\infty\}$. Therefore, $\Gamma_{S,0}(T_N)$ is a closed Lie subalgebra of $\Gamma(T_N)$ and in fact a closed ideal of $\Gamma_S(T_N)$. We denote by Γ_S the factor space $\Gamma_S(T_N)/\Gamma_{S,0}(T_N)$. Let $\Gamma(1_S)$ be the ring of all C^∞-functions on S.

LEMMA 2.1. *Γ_S can be canonically identified with $\Gamma(T_S)$, and $\Gamma(T_S)$ is a $\Gamma(1_S)$-module.*

PROOF. Let $u\in\Gamma(T_S)$. Then, u can be extended to a smooth vector field $\tilde{u}$ on N such that $\tilde{u}\equiv 0$ on a neighborhgod of ∂N. Thus, $\tilde{u}\in\Gamma_S(T_N)$. Let $\pi\tilde{u}=\hat{u}$, where $\pi:\Gamma_S(T_N)\to\Gamma_S$ is the canonical projection. It is clear that $\hat{u}$ depends only on u, and the mapping $u \rightsquigarrow \hat{u}$ is injective. The converse is trivial. Thus, $\Gamma(T_S)$ is a Lie algebra. The bracket product defined on $\Gamma(T_S)$ is obviously the usual Lie bracket product. Looking at every integral curve, we get easily the second assertion.

By the above result, we can make $\Gamma(T_S)$ a topological Lie algebra by the factor space topology. If $\overline{'S}=S$, then it coincides with the C^∞-topology. So, we assume $\overline{'S}=S$ in the remainder of this section. Let $\mathscr{N}$ be a basis of neighborhoods of the identity of $\mathscr{D}(S)$ in the C^∞-topology. For any $W\in\mathscr{N}$, we denote by W_0 the points in W which can be joined to the identity by piecewise smooth arcs in W. Set $\mathscr{N}_0=\{W_0;\ W\in\mathscr{N}\}$. Then, $\mathscr{N}_0$ satisfies the axioms of a basis of neighborhoods of the identity of a topological group, hence by $\mathscr{N}_0$ one can define a topology on $\mathscr{D}(S)$, making $\mathscr{D}(S)$ a topological group. This topology will be called LPSAC-topology, where LPSAC means "Locally-Piecewise-Smooth-Arcwise Connected". (See also [9] p. 13.)

LEMMA 2.2. *Let $\mathscr{D}_S$ be the factor group $\mathscr{D}_S(N)/\mathscr{D}_{S,0}(N)$ Then, $\mathscr{D}_S$ can be canonically identified with an open subgroup of $\mathscr{D}(S)$ in* LPSAC-*topology.*

PROOF. Evidently, $\mathscr{D}_S$ can be canonically imbedded in $\mathscr{D}(S)$. Thus, for the proof we have only to show that $\mathscr{D}_S$ contains the identity component of $\mathscr{D}(S)$ in LPSAC-topology. Let φ_t, $t\in[0, 1]$ be a piecewise smooth arc joining φ_1 and the identity φ_0. By definition, there is a division $0=t_0<t_1<\cdots<t_m=1$ of $[0, 1]$ such that φ_t, $t\in[t_i, t_{i+1}]$, are smooth arc in $\mathscr{D}(S)$. Hence, there is an extension $\tilde{\varphi}_t$ of φ_t on each $[t_i, t_{i+1}]$. Define $\tilde{u}_t$ by $(d/dt)\tilde{\varphi}_t=\tilde{u}_t\tilde{\varphi}_t$. Then, $\tilde{u}_t$ is a C^∞-vector field defined on a neighborhood U_t of S for every $t\in[t_i, t_{i+1}]$. Since $[t_i, t_{i+1}]\times S$ is compact, $V=\bigcap_{t\in[t_i,t_{i+1}]}U_t$ is a neighborhood of S. One may assume $\bar{V}\cap\partial N=\varnothing$

without loss of generality. Hence multipying a C^∞-function g on N such that $\operatorname{supp} g \subset V$ and $g \equiv 1$ on a neighborhood of S, $\tilde{v}_t = g\tilde{u}_t$ is a smooth vector field on N. Solve the equation $(d/dt)\tilde{\psi}_t = \tilde{v}_t\tilde{\psi}_t$ on $[t_i, t_{i+1}]$ with the initial condition $\tilde{\psi}_{t_i}$ which is obtained by solving the same equation on $[t_{i-1}, t_i]$, where we set $\tilde{\psi}_0 =$ identity. Then, $\tilde{\psi}_t \equiv \tilde{\varphi}_t$ on some neighborhood of S. Since $\tilde{\psi}_t \in \mathscr{D}_S(N)$ for $t \in [0, 1]$ and $\tilde{\psi}_t|S = \varphi_t$, we get $\varphi_t \in \mathscr{D}_S$ for $t \in [0, 1]$.

For any $\tilde{\varphi} \in \mathscr{D}(N)$, $\tilde{u} \in \boldsymbol{\Gamma}(T_N)$, we define $Ad(\tilde{\varphi})\tilde{u}$ by $d/dt|_{t=0}\tilde{\varphi}\cdot\exp t\tilde{u}\cdot\tilde{\varphi}^{-1}$. Then by a simple computation, we see

$$(Ad(\tilde{\varphi})\tilde{u})(x) = d\tilde{\varphi}u(\tilde{\varphi}^{-1}(x)) .$$

Now, recall the definition of $\boldsymbol{\Gamma}_S(T_N)$. If $\tilde{\varphi} \in \mathscr{D}_S(N)$, then obviously

$$(1) \qquad Ad(\tilde{\varphi})\boldsymbol{\Gamma}_S(T_N) = \boldsymbol{\Gamma}_S(T_N) .$$

Let $\mathscr{D}_S(N)_0$ be the identity component of $\mathscr{D}_S(N)$ in LPSAC-topology. Then, by the same proof as in Lemma 2.2 [2], we see that every orbit $\mathscr{D}_S(N)_0(x)$ of $x \in N$ is a C^∞-immersed submanifold of N. Moreover, if $x \in S$, then

$$(2) \qquad \mathscr{D}_S(N)_0(x) = \mathscr{D}(S)_0(x)$$

where $\mathscr{D}(S)_0$ is the identity component of $\mathscr{D}(S)$ in LPSAC-topology. Therefore we get the following:

LEMMA 2.3. *S is a disjoint union of connected C^∞-immersed submanifolds $S_\lambda: \lambda \in \Lambda$. Each S_λ is an orbit $\mathscr{D}(S)_0$.*

Note that if $u \in \boldsymbol{\Gamma}(T_S)$, then $u|S_\lambda$ is a smooth tangent vector field on S_λ for each $\lambda \in \Lambda$. Since every $u \in \boldsymbol{\Gamma}(T'_S)$ can be extended to a complete vector field $\tilde{u}$ on N, we have easily the following:

COROLLARY 2.4. *A smooth vector field u on S is a strictly tangent vector field on S if and only if $u(x) \in T_xS_\lambda$ for any $x \in S$, where S_λ is the orbit which contains x and T_xS_λ is the tangent space of S_λ at x.*

Let $\Lambda_r = \{\lambda \in \Lambda;\ \dim S_\lambda \leqq r\}$, and let $S^{(r)} = \bigcup_{\lambda \in \Lambda_r} S_\lambda$. In general, the structure of $S^{(r)}$ is very complicated. However, if there is an orbit S_μ with $\dim S_\mu \geqq 1$, we can see a local product structure of S at every $x \in S_\mu$. To do this, we need at first the following:

LEMMA 2.5. *Let $\tilde{\mathfrak{A}}$ be a Lie subalgebra of $\boldsymbol{\Gamma}(T_N)$. Suppose there is $\tilde{u} \in \boldsymbol{\Gamma}(T_N)$ such that*

(a) $Ad(\exp t\tilde{u})\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}$

(b) $\int_a^b Ad(\exp t\tilde{u})\tilde{v}\,dt \in \tilde{\mathfrak{A}}$ *for any* $\tilde{v} \in \tilde{\mathfrak{A}}$ *and* $-\infty < a \leqq b < \infty$.

Suppose $\tilde{u}$ *does not vanish at* $p \in N$. *Then, there is a relatively compact open neighborhood* V_p *of* p *in* N *satisfying the following: For any* $\tilde{w} \in \tilde{\mathfrak{A}}$ *with* $\operatorname{supp}\tilde{w} \subset V_p$, *the integral*

$$I(\tilde{w}) = \int_0^{2c} Ad(\exp t\tilde{u})\tilde{w}\,dt \qquad (\in \tilde{\mathfrak{A}} \text{ by (b)})$$

satisfies $[u, I(\tilde{w})] \equiv \tilde{w}$ *on* V_p, *where* $c > 0$ *depends only on* V_p.

The proof is seen in Lemma 1.3 [2]. However, it should be remarked that if $\tilde{u} = \partial/\partial x^1$ on a local coordinate system, then

$$I(\tilde{w}) = \sum_{i=1}^{n} \int_0^{2c} \tilde{w}^i(x^1 - t, x^2, \cdots, x^n)dt\, \partial/\partial x^i, \tag{3}$$

where $\tilde{w} = \sum_{i=1}^{n} \tilde{w}^i \partial/\partial x^i$. Remark also that if $\tilde{\mathfrak{A}}$ is closed in the C^∞-topology, then the assumed property (b) is obtained by (a). Thus, by (1), we can apply this lemma for $\tilde{u} \in \boldsymbol{\Gamma}_S(T_N)$, replacing $\tilde{\mathfrak{A}}$ by $\boldsymbol{\Gamma}_S(T_N)$.

Note that $\boldsymbol{\Gamma}_S(T_N)$ is an $\boldsymbol{\Gamma}(1_N)$-module (cf. Lemma 2.1). For a vector field $\tilde{u}$ defined on an open subset of N, we denote by $\tilde{u} \in_{\text{loc}} \boldsymbol{\Gamma}_S(T_N)$ if a suitable extension of $\tilde{u}$ is contained in $\boldsymbol{\Gamma}_S(T_N)$, and we use the notation $\in_{\text{loc}}$ throughout this note.

PROPOSITION 2.6. *Suppose* $\dim S_\mu = r \geqq 1$. *Then for every point* $p \in S_\mu$, *there is a* C^∞-*local coordinate system* $(x^1, \cdots, x^n)$ *on a neighborhood* U *of* p *in* N *such that* $\partial/\partial x^i \in_{\text{loc}} \boldsymbol{\Gamma}_S(T_N)$ *for* $1 \leqq i \leqq r$.

PROOF. Since S_μ is an orbit of $\mathscr{D}(S)_0$, the tangent space T_pS_μ is given by $\boldsymbol{\Gamma}_S(T_N)(p)$. Since $r \geqq 1$, there is $v_1 \in \boldsymbol{\Gamma}_S(T_N)$ such that $v_1(p) \neq 0$. By a suitable choice of a local coordinate system $(x^1, \cdots, x^n)$, we may assume that $v_1 \equiv \partial/\partial x^1$ on that coordinate neighborhood. Moreover, we may assume without loss of generality that $\partial/\partial x^1|_p, \cdots, \partial/\partial x^r|_p$ span the tangent space T_pS_μ. Since $\boldsymbol{\Gamma}_S(T_N)$ is an $\boldsymbol{\Gamma}(1_N)$-module, one can find r vector fields $v_1, \cdots, v_r \in_{\text{loc}} \boldsymbol{\Gamma}_S(T_N)$ such that

$$\begin{cases} v_1 = \partial/\partial x^1 \\ v_i = \partial/\partial x^i + \sum_{j=r+1}^{n} g_i^j(x^1, \cdots, x^n)\partial/\partial x^j \qquad (2 \leqq i \leqq r). \end{cases} \tag{4}$$

Now, assume that there is an integer s $(1 \leqq s \leqq r)$ such that

$$\begin{cases} v_i = \partial/\partial x^i \qquad (1 \leqq i \leqq s) \\ v_j = \partial/\partial x^j + \sum_{k=r+1}^{n} g_j^k(x^s, \cdots, x^n)\partial/\partial x^k \qquad (s+1 \leqq j \leqq r) \end{cases} \tag{5}$$

on a neighborhood U of p. Replacing $\tilde{u}$ in Lemma 2.5 by v_s, we choose a neighborhood V_p of p in U with the same property as in Lemma 2.5. Let $f(x^1, \cdots, x^{s-1})h(x^s, \cdots, x^n)$ be a C^∞-function on N such that $\operatorname{supp} fh \subset V_p$ and $f \equiv 1$, $h \equiv 1$ on neighborhoods V_1, V_2 of zeros of $\boldsymbol{R}^{s-1}$, $\boldsymbol{R}^{n-s+1}$ respectively. Set

$$u_j = v_j - I(fh[\partial/\partial x^s, v_j]), \qquad s+1 \leqq j \leqq r,$$

and we see that $[\partial/\partial x^s, u_j] \equiv 0$ on $V_1 \times V_2 \subset V_p$. Moreover, since I is merely an integration (cf. (3)), $[\partial/\partial x^i, u_j] \equiv 0$ on $V_1 \times V_2$ for every i, j such that $1 \leqq i \leqq s-1$, $s+1 \leqq j \leqq r$. Thus, $u_{s+1}, \cdots, u_r$ do not depend on the variable x^s on $V_1 \times V_2$. Therefore by a suitable change of variables $x^{s+1}, \cdots, x^n$, one may assume that $u_{s+1} \equiv \partial/\partial x^{s+1}$ on a neighborhood of p, hence one has vector fields

$$\begin{cases} v_i' = \partial/\partial x^i & (1 \leqq i \leqq s+1) \\ v_j' = \partial/\partial x^j + \sum_{k=r+1}^{n} g_j'^k(x^{s+1}, \cdots, x^n)\partial/\partial x^k & (s+2 \leqq j \leqq r) \end{cases} \tag{6}$$

on a neighborhood of p such that $v_1', \cdots, v_r' \in {}_{\mathrm{loc}}\boldsymbol{\Gamma}_S(T_N)$. Thus, by induction we obtain the desired result.

Let $(x^1, \cdots, x^r, x^{r+1}, \cdots, x^n)$ be a smooth local coordinate system at $p \in S_\mu$ obtained by the above proposition. Let $\boldsymbol{R}^r$, $\boldsymbol{R}^{n-r}$ be r, $n-r$ dimensional cartesian spaces respectively. By the above result, we have the following local product structure of S:

COROLLARY 2.7. *There are neighborhoods V, W of zeros of $\boldsymbol{R}^r$, $\boldsymbol{R}^{n-r}$ respectively such that $(V \times W) \cap S = V \times (W \cap S)$ regarding $V \times W$ as a local coordinate neighborhood at p.*

In the remainder of this section, we shall give another smoothness lemma for the the later use. Let $\boldsymbol{\Gamma}(T_S)$, $\boldsymbol{\Gamma}(T_S')$ be as in §1 and assume $\overline{'S} = S$. A mapping ψ of T into $\boldsymbol{\Gamma}(T_S)$ is said to be *smooth* if $\psi \in \mathscr{M}(T, \boldsymbol{\Gamma}(T_S'))$. Thus, one can define the space $\mathscr{M}(T, \boldsymbol{\Gamma}(T_S))$ with C^∞-topology, where $\overline{'T} = T$ is assumed as in §1. Let Σ be a compact topological space, and $u: \Sigma \to \mathscr{M}(T \times [0, 1], \boldsymbol{\Gamma}(T_S))$ a continuous mapping. We denote the image by $u_{\alpha,t,\lambda}$ for $(\alpha, t, \lambda) \in \Sigma \times T \times [0, 1]$. Solve that equation

$$\frac{\partial}{\partial \lambda} \psi_{\alpha,t,\lambda} = u_{\alpha,t,\lambda} \psi_{\alpha,t,\lambda} \tag{7}$$

with the initial condition $\psi_{\alpha,t,0} =$ identity. Then, we have

LEMMA 2.8. *Notations and assumptions being as above, $\psi_{\alpha,t,\lambda} \in \mathscr{D}(S)$ and $\psi_{\alpha,*,*}$ defines a continuous mapping of Σ into $\mathscr{M}(T \times [0, 1], \mathscr{D}(S))$.*

PROOF. $v_\alpha=(0, \partial/\partial\lambda, u_{\alpha,t,\lambda})$ can be regarded as a tangent vector field on $T\times[0, 1]\times S$. Let ϕ be a C^∞-function on $(-\infty, \infty)$ such that $\phi\equiv 1$ on $[0, 1]$, and $\operatorname{supp}\phi\subset(-\varepsilon, 1+\varepsilon)$. For a sufficiently small ε, $\tilde{v}_\alpha=(0, \phi\partial/\partial\lambda, u_{\alpha,t,\lambda})$ is defined on $T\times[-\varepsilon, 1+\varepsilon]\times S$, and by Corollary 2.4 $\tilde{v}_\alpha$ is a smooth strictly tangent vector field on $T\times[-\varepsilon, 1+\varepsilon]\times S$. Let $\Psi_{\alpha,\lambda}$ be the one parameter subgroup generated by $\tilde{v}_\alpha$. Then, $\Psi_{\alpha,\lambda}\in\mathscr{D}(T\times[-\varepsilon, 1+\varepsilon]\times S)$, and $\Psi_{\alpha,*}$ defines a continuous mapping of Σ into $\mathscr{M}([0, 1], \mathscr{D}(T\times[-\varepsilon, 1+\varepsilon]\times S)$ by the well-known continuity theorem (cf. [7], p. 22 and p. 41). Now set

$$\Psi_{\alpha,\lambda}(t, 0, x)=(t, \lambda, \psi_{\alpha,t,\lambda}(x)) , \qquad \lambda\in[0, 1] . \tag{8}$$

Then, $\psi_{\alpha,t,\lambda}$ is the solution of (7), $\psi_{\alpha,t,\lambda}\in\mathscr{D}(S)$ and by definition $\psi_{\alpha,*,*}\in\mathscr{M}(T\times[0, 1], \mathscr{D}(S))$. It is now obvious that $\psi_{\alpha,*,*}$ defines a continuous mapping of Σ into $\mathscr{M}(T\times[0, 1], \mathscr{D}(S))$.

§ 3. Several properties of expansive sets.

Throughout this section, we assume that S is an expansive subset of N. By Lemma 2.3, S is a disjoint union of $\mathscr{D}(S)_0$-orbits S_λ, $\lambda\in\Lambda$. Let $\Lambda_r=\{\lambda\in\Lambda; \dim S_\lambda\leqq r\}$.

LEMMA 3.1. *Λ_0 is a finite set.*

PROOF. Let S_λ be an orbit with $\dim S_\lambda=0$. Then S_λ is a single point $\{p\}$. Let X_p be an expansive vector field on S at p and $\tilde{X}_p$ an extension of X_p with property (P) in §1. We may assume $\tilde{X}_p\in{}_{\mathrm{loc}}\boldsymbol{\Gamma}_S(T_N)$ without loss of generality. Since p is an isolated zero of X_p, we see that $\bigcap_{\lambda\in\Lambda_0}S_\lambda$ is discrete, hence finite.

Let S_μ be an orbit of $\mathscr{D}(S)_0$ with $\dim S_\mu=r\geqq 1$, and p a point in S_μ. We choose a local coordinate system $(x^1, \cdots, x^r, x^{r+1}, \cdots, x^n)$ on an open neighborhood U of p by the same manner as in Proposition 2.6. Obviously, $(x^1, \cdots, x^r, 0, \cdots, 0)$ gives a local coordinate system of immersed submanifold S_μ. Let X_p be an expansive vector field on S at p, and $\tilde{X}_p$ an extension of X_p with property (P). Let

$$\tilde{X}_p^{(1)}=\sum_{i,j=1}^n a_j^i x^j\partial/\partial x^i$$

be the linear part of $\tilde{X}_p$ at p. Since $\tilde{X}_p^{(1)}$ leaves the tangent space T_pS_μ invariant, we have $a_j^i=0$ for $r+1\leqq i\leqq n$, $1\leqq j\leqq r$. Set

$$\tilde{X}_p=\sum_{i=1}^n h^i(x^1, \cdots, x^n)\partial/\partial x^i$$

by using the above local coordinate system, and let

$$(9)\qquad \tilde{Y}_p = \sum_{i=r+1}^{n} h^i(x^1, \cdots, x^n)\partial/\partial x^i .$$

Since $\tilde{X}_p \in {}_{\mathrm{loc}}\boldsymbol{\Gamma}_S(T_N)$ and $\boldsymbol{\Gamma}_S(T_N)$ is an $\boldsymbol{\Gamma}(1_N)$-module, we have $\tilde{Y}_p \in {}_{\mathrm{loc}}\boldsymbol{\Gamma}_S(T_N)$ by using Proposition 2.6. The linear part of $\tilde{Y}_p$ is given by

$$(10)\qquad \tilde{Y}_p^{(1)} = \sum_{i,j=r+1}^{n} a_j^i x^j \partial/\partial x^i .$$

The eigenvalues of $(a_j^i)_{r+1\leq i,j\leq n}$ are real and positive. Using the same notation as in Corollary 2.7, we see easily that $\tilde{Y}_p|W = \sum_{i=r+1}^{n} h^i(0, \cdots, 0, x^{r+1}, \cdots, x^n)\partial/\partial x^i$ is strictly tangent to $W \cap S$. Hence, regarding $\tilde{Y}_p|W$ as a local vector field $\tilde{Z}_p$ on a neighborhood of p, we have the following:

LEMMA 3.2. *Notations and assumptions being as above, there is a vector field $\tilde{Z}_p$ on a local coordinate neighborhood of p such that*

(i) *$\tilde{Z}_p \in_{\mathrm{loc}} \boldsymbol{\Gamma}_S(T_N)$ and $Z_p(p)=0$,*

(ii) *$\tilde{Z}_p = \sum_{i=r+1}^{n} h^i(x^{r+1}, \cdots, x^n)\partial/\partial x^i$ on a neighborhood of p,*

(iii) *the linear part of $\tilde{Z}_p$ at p with respect to the variables $x^{r+1}, \cdots, x^n$ are real and positive.*

By the above result combined with Corollary 2.4 we have

COROLLARY 3.3. *Let S_μ be an r-dimesional orbit of $\mathscr{D}(S)_0$. Then, the boundary $\overline{S_\mu} - S_\mu$ is a disjoint union of orbits S_λ such that $\dim S_\lambda < r$. In particular, if ${}^r\!S = S$, S is measurable with respect to any smooth volume element on S.*

Under the same notations, the following is also easy, but shows the locally-closedness of each orbit:

COROLLARY 3.4. *For any $p \in S_\mu$, there are neighborhood V, W of zeros of $\boldsymbol{R}^r$, $\boldsymbol{R}^{n-r}$ respectively such that, regarding $V \times W$ as a neighborhood of p by the same manner as in Corollary 2.7, V is a local coordinate neighborhood of S_μ and $W \cap S_\mu = \{0\}$.*

PROOF. We have only to show $W \cap S_\mu = \{0\}$. If there is not such W, then there is a sequence $\{q_m\}$ in $\boldsymbol{R}^{n-r}$ converging to 0 such that $q_m \in S_\mu$. By Corollary 2.7, $V \times \{q_m\}$ is an open subset of the immersed submanifold S_μ. Now, consider the integral curves of $\tilde{Z}_p$ with initial point q_m. This must be contained in S_μ, but this is a contradiction, because $\dim S_\mu = r$.

Now, the goal of the remainder of this section is the following:

PROPOSITION 3.5. *Λ is a finite set.*

PROOF. Since S is compact, we have only to show the locally-finiteness of $\{S_\lambda\}_{\lambda\in\Lambda}$. Assume that there is an s-dimensional orbit S_μ with following property $(*)$:

$(*)$ There is a point $p\in S_\mu$ and infinitely many $\bar{S}_\lambda$, $\lambda\in\Lambda'$ such that $S_\lambda\ni p$.

We may assume that s is the maximum among the dimensions of S_μ with property $(*)$. By Proposition 2.6 and Lemma 3.2, there is a local coordinate system $(x^1, \cdots, x^s, \cdots, x^n)$ such that $\partial/\partial x^i\in {}_{\mathrm{loc}}\Gamma_S(T_N)$ for $1\leqq i\leqq s$, and that there is a local vector field $\tilde{Z}_p$ with properties (i)-(iii) in Lemma 3.2 replacing r by s. By an appropriate change of the variables $x^{s+1}, \cdots, x^n$ in accordance with Sternberg's normalization theorem (cf. Corollary 1.5 [3]), we may assume that $\tilde{Z}_p$ can be written in the form

$$\tilde{Z}_p=\sum_{i=s+1}^{n}\mu_i x^i\partial/\partial x^i+\sum_{i=s+1}^{n}\varphi^i(x^{s+1}, \cdots, x^{i-1})\partial/\partial x^i\ , \tag{11}$$

where $\mu_{s+1}\leqq\mu_{s+2}\cdots\leqq\mu_n$ are the eigenvalues of the linear part of $\tilde{Z}_p$ and $\varphi^i(x^{s+1}, \cdots, x^{i-1})=\sum a^i_\alpha x^\alpha$ are polynomial such that $\alpha_{s+1}\mu_{s+1}+\cdots+\alpha_{i-1}\mu_{i-1}=\mu_i$ and $|\alpha|=\alpha_{s+1}+\cdots+\alpha_{i-1}\geqq 1$. By a suitable change of coordinate $x^i\mapsto\lambda_i x^i$ if necessary, we may assume that the linear part of the second term of (11) has sufficiently small coefficients, say $<\delta$. The second term of (11) is called the nilpotent part of $\tilde{Z}_p$.

Let $f_0(x^{s+1}, \cdots, x^n)=\sum_{i=s+1}^n (x^i)^2$. Since μ_i are posistive and δ is sufficiently small, we see that there is a neighborhood W of 0 of $\boldsymbol{R}^{n-s}$ such that $\tilde{Z}_p f_0>0$ on $W-\{0\}$. Let $\Sigma_p(\varepsilon)$ be an ε-sphere with the center 0 such that $\Sigma_p(\varepsilon)\subset W$. The inequality $\tilde{Z}_p f_0>0$ means that the integral curves of $\tilde{Z}_p$ intersect $\Sigma_p(\varepsilon)$ transversally. Choose a point q_λ in $S_\lambda\cap\Sigma_p(\varepsilon)$ for each $\lambda\in\Lambda'$. Since $S\cap\Sigma_p(\varepsilon)$ is compact, there is an infinite but countable subset Λ'' of Λ' such that $\{q_\lambda; \lambda\in\Lambda''\}$ converges to a point $q\in S\cap\Sigma_p(\varepsilon)$. Let S_σ be the $\mathscr{D}(S)_0$-orbit of q. Considering an expansive vector field X_q on S at q, we see that $\overline{S_\lambda}\ni q$, for infinitely many λ of Λ''. Thus, S_σ has property $(*)$. Since $\overline{S_\sigma}\ni p$, we have $\dim S_\sigma>s$ by Corollary 3.3. This contradicts the maximality of s. Thus, $\{S_\lambda; \lambda\in\Lambda\}$ is locally finite and hence Λ is a finite set.

§ 4. Control of the volume forms near the boundary.

Throughout this section, S means always a compact expansive subset of N such that (i) S has an orientable neighborhood in N, and (ii) the interior $'S$ is nonvoid and connected. We also assume that $\overline{'S}=S$. Let

$\mathscr{V}_S$ be the space of all C^∞-volume forms dV on S such that $\int_S dV=1$. The goal of this section is the following:

PROPOSITION 4.1. *Let dV_0 be an arbitrarily fixed volume form in $\mathscr{V}_S$, and let $h: \sigma^k \to \mathscr{V}_S$ be a continuous mapping. Then, there is a continuous mapping $H: \delta^{k+1} \to \mathscr{D}(S)$ satisfying that $H(0)=$identity and that there is neighborhood W of $\partial S=S-'S$ in S such that $H(q)^*dV_0 \equiv h(q)$ on W for every $q \in \sigma^k$.*

The proof will be given in several lemmas below.

Let D be an $n-r$-dimensional closed disk with the center, 0, and $\widetilde{\mathscr{V}}$ the space of all C^∞-volume forms on D with the C^∞-topology. Let $\widetilde{\mathscr{D}}(D)$ be the group of all C^∞-diffeomorphisms $\tilde{\varphi}$ on D such that $\tilde{\varphi} \equiv$ identity on a neighborhood of ∂D. $\widetilde{\mathscr{D}}(D)$ is a topological group under the C^∞-topology. Let T be a compact subset of a C^∞-manifold N' such that $\overline{'T}=T$. A mapping $\psi: T \to \widetilde{\mathscr{D}}(D)$ is said to be *smooth*, if $\psi \in \mathscr{M}(T, \mathscr{D}(D))$. By $\mathscr{M}(T, \widetilde{\mathscr{D}}(D))$ we denote the space of all smooth mappings of T into $\widetilde{\mathscr{D}}(D)$ with the C^∞-topology. Let $\tilde{Z}_0$ be a C^∞-vector field on D such that $\tilde{Z}_0(0)=0$ and the eigenvalues of the linear part of $\tilde{Z}_0$ at 0 are real and positive. The next lemma plays the role of bricks in the proof of Proposition 4.1.

LEMMA 4.2. *Let $g: T \to \widetilde{\mathscr{V}}$ be an arbitrarily fixed smooth mapping. Suppose $h: \sigma^k \to \mathscr{M}(T, \widetilde{\mathscr{V}})$ is a continuous mapping such that $h(q)(t) \equiv g(t)$ on $\sigma^k \times U_{\partial T}$, where $U_{\partial T}$ is a neighborhood of $\partial T=T-'T$ in T. Then, there is a continuous mapping $H': \delta^{k+1} \to \mathscr{M}(T, \widetilde{\mathscr{D}}(D))$ and a neighborhood $W^{(0)}$ of 0 in D satisfying the following*:

(a) $H'(0)(t)=$identity *for any* $t \in T$,

(b) $H'(d)(t)=$identity *for* $(d, t) \in \delta^{k+1} \times U_{\partial T}$,

(c) $H'(q)(t)^*g(t) \equiv h(q)(t)$ *on* $W^{(0)}$ *for any* $(q, t) \in \sigma^k \times T$.

(If T is a single point, then we put $\partial T=\varnothing$.)

PROOF. For any $dV \in \widetilde{\mathscr{V}}$, there is a smooth local coordinate system $(y^1, \cdots, y^{n-r})$ at 0 in D such that $dV=dy^1 \wedge \cdots \wedge dy^{n-r}$. By using this coordinate system, the Lie derivative $\mathscr{L}_{\tilde{Z}_0} dV$ is given by

$$\mathscr{L}_{\tilde{Z}_0} dV=(\operatorname{div} \tilde{Z}_0)dV=\left(\sum_{i=1}^{n-r} \partial \tilde{Z}_0^i/\partial y^i\right)dV \tag{12}$$

Thus, there is an ε-neighborhood W^ε of 0 such that (i) $\mathscr{L}_{\tilde{Z}_0} dV=\rho dV$, $\rho \geqq a$ (>0) on $\overline{W^\varepsilon}$ and (ii) $\lim_{\theta\to-\infty}(\exp \theta\tilde{Z}_0)(x)=0$ for every $x \in W^\varepsilon$. Set $dV_t=g(t)$ and $dV_{q,t}=h(q)(t)$. We set further $dV_{q,t,\lambda}=(1-\lambda)dV_t+\lambda dV_{q,t}$

for $\lambda \in [0, 1]$. Obviously, $dV_{q,*,*}$ defines a continuous mapping of σ^k into $\mathscr{M}(T\times[0, 1], \widetilde{\mathscr{V}})$. Since $\sigma^k\times T\times[0, 1]$ is compact, we can choose W^ε so that the above properties (i) and (ii) may be fulfilled by every $dV_{q,t,\lambda}$. We set $\mathscr{L}_{\tilde{Z}_0} dV_{q,t,\lambda}=\rho_{q,t,\lambda} dV_{q,t,\lambda}$ on W^ε. Then, $\rho_{q,t,\lambda}\geqq a$ (>0), and by an appropriate use of Lemma 1.1, we see that $\rho_{q,*,*}$ defines a continuous mapping of σ^k into $\mathscr{M}(T\times[0, 1], \Gamma(1_{\overline{W^\varepsilon}}))$.

Now, set $dV_{q,t}-dV_t=h_{q,t,\lambda} dV_{q,t,\lambda}$. Then, by Lemma 1.1 again, we get that $h_{q,*,*}$ defines a continuous mapping of σ^k into $\mathscr{M}(T\times[0, 1], \Gamma(1_D))$. We want to solve the equation

$$\mathscr{L}_{f\tilde{Z}_0} dV_{q,t,\lambda}=h_{q,t,\lambda} dV_{q,t,\lambda} \tag{13}$$

on W^ε. The above equation is equivalent to

$$\tilde{Z}_0 f+\rho_{q,t,\lambda} f=h_{q,t,\lambda} , \tag{14}$$

and hence the continuous solution on W^ε is given by

$$f_{q,t,\lambda}=\int_0^\infty \rho_{q,t,\lambda} e^{-\theta\rho_{q,t,\lambda}}(\exp -\theta\tilde{Z}_0)^* \frac{h_{q,t,\lambda}}{\rho_{q,t,\lambda}} d\theta , \tag{15}$$

because

$$\begin{aligned}\tilde{Z}_0 f &=\tilde{Z}_0\int_0^\infty e^{-\theta}\left(\exp -\frac{\theta}{\rho}\tilde{Z}_0\right)^* \frac{h}{\rho} d\theta=\rho\int_0^\infty e^{-\theta}\frac{\tilde{Z}_0}{\rho}\left(\exp -\frac{\theta}{\rho}\tilde{Z}_0\right)^*\frac{h}{\rho} d\theta \\ &=-\rho\int_0^\infty e^{-\theta}\frac{d}{d\theta}\left(\exp -\theta\frac{\tilde{Z}_0}{\rho}\right)^*\frac{h}{\rho} d\theta \\ &=-\left[e^{-\theta}\left(\exp -\theta\frac{\tilde{Z}_0}{\rho}\right)^*\frac{h}{\rho}\right]_0^\infty -\rho f=h-\rho f .\end{aligned}$$

Since the integration (15) converges uniformly with its all derivatives with respect to $(t, \lambda, x)\in T\times[0, 1]\times\overline{W^\varepsilon}$, we see that $f=f_{q,t,\lambda}$ is an element of $\mathscr{M}(T\times[0, 1], \Gamma(1_{\overline{W^\varepsilon}}))$. Moreover, it is easy to see that $f_{q,*,*}$ defines a continuous mapping of σ^k into $\mathscr{M}(T\times[0, 1], \Gamma(1_{\overline{W^\varepsilon}}))$.

Let ϕ be a C^∞-function on D such that $\operatorname{supp}\phi\subset W^\varepsilon$ and $\phi\equiv 1$ on a neighborhood W^δ of 0 in D. Let $\tilde{u}_{q,t,\lambda}=\phi f_{q,t,\lambda}\tilde{Z}_0$. Then, $\tilde{u}_{q,t,\lambda}$ is a C^∞-vector field on D such that $\tilde{u}_{q,t,\lambda}\equiv 0$ on a neighborhood of ∂D, $u_{q,t,\lambda}(0)=0$ and that

$$\mathscr{L}_{\tilde{u}_{q,t,\lambda}} dV_{q,t,\lambda}=h_{q,t,\lambda} dV_{q,t,\lambda}=dV_{q,t}-dV_t \tag{16}$$

on W^δ. Solve the equation

$$(d/d\lambda)\tilde{\psi}_{q,t,\lambda}=-\tilde{u}_{q,t,\lambda}\tilde{\psi}_{q,t,\lambda} \tag{17}$$

with the initial condition $\tilde{\Psi}_{q,t,0}=$identity. Then, by Lemma 2.8, $\tilde{\Psi}_{q,*,*}$ defines a continuous mapping of σ^k into $\mathscr{M}(T\times[0, 1], \widetilde{\mathscr{D}}(D))$. Since $\sigma^k\times T\times[0, 1]$ is compact and $\tilde{\Psi}_{q,t,\lambda}(0)=0$, there is a neighborhood $W^{(0)}$ of 0 such that $W^{(0)}\subset\tilde{\Psi}_{q,t,\lambda}(W^\delta)$ for any (q, t, λ). Since by (16)

$$(18)\qquad (d/d\lambda)\tilde{\Psi}^*_{q,t,\lambda}dV_{q,t,\lambda}=-\tilde{\Psi}^*_{q,t,\lambda}\mathscr{L}_{u_{q,t,\lambda}}dV_{q,t,\lambda}+\tilde{\Psi}^*_{q,t,\lambda}(dV_{q,t}-dV_t)=0$$

on $W^{(0)}$, we have $\tilde{\Psi}^*_{q,t,\lambda}dV_{q,t,\lambda}\equiv dV_t$ on $W^{(0)}$. Set $H'(\lambda q, t)=\tilde{\Psi}^{-1}_{q,t,\lambda}$ for every $(q, t, \lambda)\in\sigma^k\times T\times[0, 1]$. Then, $H'(0, t)=$identity. If $t\in U_{\partial T}$, then $dV_{q,t}=dV_t$, hence $h_{q,t,\lambda}=0$. Therefore $f_{q,t,\lambda}=0$ and hence $\tilde{\Psi}_{q,t,\lambda}=$identity. It is clear that $dV_{q,t}=dV_{q,t,1}=\tilde{\Psi}^{-1}_{q,t,1}dV_t=H'(q, t)*g(t)$.

Apply the above lemma to the case $T=\{0\}$, and we have

COROLLARY 4.3. *Let $\{p\}$ be a zero-dimensional orbit of $\mathscr{D}(S)_0$. Notations and assumptions being as in Proposition* 4.1, *there is an open neighborhood $W^{(0)}$ of p and a continuous mapping H': $\delta^{k+1}\to\mathscr{D}(S)$ such that $H'(0)=$identity and that $H'(q)*dV_0\equiv h(q)$ on $W^{(0)}$ for any $q\in\sigma^k$, where $dV_0=g(p)$.*

Now, assume that T has an orientable neighborhood in N'. Let $d\mu$ be a C^∞-volume form on T. Let $\widetilde{\mathscr{V}}$ be the space of all C^∞-volume forms on $T\times D$, and $d\mu\wedge dV_t$ be a fixed element of $\widetilde{\mathscr{V}}$, where the volume form dV_t on D may depend on the variable $t\in T$. Let $h: \sigma^k\to\widetilde{\mathscr{V}}$ be a continuous mapping such that $h(q)\equiv d\mu\wedge dV_t$ on $U_{\partial T}\times D$, where $U_{\partial T}$ is a neighborhood of ∂T in T. The following is an immediate conclusion from Lemma 4.2:

COROLLARY 4.4. *Notations and assumptions being as above, there is a continuous mapping H' of δ^{k+1} into the group of diffeomorphisms on $T\times D$ satisfying the following:*

(i) *$H'(0)=$identity*

(ii) *$H'(d)(t, x)=(t, H''(d, t)(x))$ and $H''(d, *)$ defines a continuous mapping of δ^{k+1} into $\mathscr{M}(T, \widetilde{\mathscr{D}}(D))$*

(iii) *$H''(d, t)=$identity if $(d, t)\in\delta^{k+1}\times U_{\partial T}$*

(iv) *$H'(q)^*d\mu\wedge dV_t\equiv h(q)$ on $T\times W^{(0)}$ for any $q\in\sigma^k$.*

Proposition 4.1 will be proved by induction, but before that we need the following:

LEMMA 4.5. *Let S_μ be a $\mathscr{D}(S)_0$-orbit, and ∂S_μ the boundary of S_μ. Let $U_{\partial S_\mu}$ be an open neighborhood of ∂S_μ in $\overline{S_\mu}$. Then, there is a compact connected submanifold $\bar{Q}$ of S_μ such that $\partial\bar{Q}$ is a smooth submanifold contained in $U_{\partial S_\mu}$ and that $\bar{Q}\supset S_\mu-U_{\partial S_\mu}$.*

PROOF. There is a smooth function $f: S_\mu \to \boldsymbol{R}$ such that $f^{-1}((-\infty, c])$ is compact and $\lim_{n\to\infty} f(x_n)=\infty$ for any sequence $\{x_n\}$ converging to a point in ∂S_μ. By a slight change of f if necessary, one may assume that f has countably many critical values. Let c be a sufficiently large number which is not a critical value of f. Then, $f^{-1}(c)\subset U_{\partial S_\mu}$ and a smooth submanifold of S_μ. Take the connected component of $f^{-1}((-\infty, c])$ containing $S_\mu - U_{\partial S_\mu}$.

PROOF OF PROPOSITION 4.1. Notations and assumptions are as in Proposition 4.1. The desired mapping $H: \delta^{k-1}\to \mathscr{D}(S)$ will be obtained by a composition $H(d)=H_l(d)\circ H_{l-1}(d)\circ\cdots\circ H_1(d)$ of continuous mappings $H_i: \delta^{k+1}\to\mathscr{D}(S)$. Let $H': \delta^{k+1}\to\mathscr{D}(S)$ be the mapping obtained in Corollary 4.3. Then, $H'(q)^{-1*}h(q)\equiv dV_0$ on a neighborhood $W^{(0)}$ of p in S. Thus, putting $H(d)=H''(d)\circ H'(d)$ we have only to make $H'': \delta^{k+1}\to\mathscr{D}(S)$ under the assumption that $h(q)\equiv dV_0$ on $W^{(0)}$. Since Λ_0 is finite (Lemma 3.1), we may assume $h(q)\equiv dV_0$ on a neighborhood of $S^{(0)}=\bigcup_{\lambda\in\Lambda_0} S_\lambda$ by the same procedure as above.

Now, we use induction procedure, and assume that $h(q)\equiv dV_0$ on a neighborhood of $S^{(r-1)}=\bigcup_{\lambda\in\Lambda_{r-1}} S_\lambda$. Let S_μ be an r-dimensional orbit. Then, by the assumption, $h(q)\equiv dV_0$ on a neighborhood $U_{\partial S_\mu}$ of ∂S_μ because of Corollary 3.3. By Lemma 4.5, there is a compact submanifold $\bar{Q}$ with smooth boundary $\partial\bar{Q}$ which is contained in $U_{\partial S_\mu}$ and $Q\supset S_\mu - U_{\partial S_\mu}$.

Take a sufficiently small triangulation of $\bar{Q}$ so that every r-dimensional simplex τ may be contained in a local coordinate neighborhood U of N. Let $(x^1, \cdots, x^r, x^{r+1}, \cdots, x^n)$ be a coordinate system on U. By Proposition 2.6, we may assume that $\partial/\partial x^i \in {}_{\mathrm{loc}}\boldsymbol{\Gamma}_S(T_N)$ for $1\leqq i\leqq r$, and hence one can apply Corollary 2.7 and Lemma 3.2. Note that τ is an r-dimensional compact expansive set such that $\overline{'\tau}=\tau$. Therefore, applying Corollary 4.4 successively for the faces of dimension $\leqq r-1$ we obtain that there is a continuous mapping $H^{(1)}: \delta^{k+1}\to\mathscr{D}(S)$ such that $H^{(1)}(0)=$ identity, $H^{(1)}(d)=$identity on $U_{\partial S_\mu}$ for any $d\in\delta^{k+1}$ and $H^{(1)}(q)^* dV_0\equiv h(q)$ for any $q\in\sigma^k$ on a neighborhood $V_{\partial\tau}$ of τ in S. Thus, for the proof, one may assume that $h(q)\equiv dV_0$ on a neighborhood of the $r-1$-dimensional skelton of the triangulated $\bar{Q}$. Apply Corollary 4.4 again to each τ. We have, then, the desired result.

§ 5. Control of the volume forms in the interior.

Throughout this section, notations and assumptions are as in the previous section. The proof of Theorem in § 1 will be given in this section. So recall the statement of Theorem in § 1 first of all.

By Proposition 4.1 we may assume that $h(q)\equiv dV_0$ on a neighborhood $U_{\partial S}$ of ∂S in S, and by Lemma 4.5 there is a connected compact C^∞-submanifold $\bar{Q}$ of $'S$ such that $\partial\bar{Q}\subset U_{\partial S}$ and $\bar{Q}\supset S-U_{\partial S}$. By the bicollaring theorem (cf. p. 23 [4]), there is a neighborhood $V_{\partial\bar{Q}}$ of $\partial\bar{Q}$ such that $V_{\partial\bar{Q}}\subset U_{\partial\bar{S}}$ and $V_{\partial\bar{Q}}$ is diffeomorphic to the direct product $\partial\bar{Q}\times(-\delta, \delta)$. Fix a smooth riemannian metric g' on $\partial\bar{Q}$ and let $d\mu$ be the riemannian volume form on $\partial\bar{Q}$. On $V_{\partial\bar{Q}}=\partial\bar{Q}\times(-\delta, \delta)$, the volume form dV_0 can be written in the form $dV_0=d\mu\wedge f(x', t)dt$. Hence by putting $x^n=\int_0 f(x', t)dt$, one may assume that

$$dV_0=d\mu\wedge dx^n \quad \text{on} \quad \partial\bar{Q}\times(-\delta, \delta)\,. \tag{19}$$

We fix a product riemannian metric $g=g'\times(dx^n)^2$ on $\partial\bar{Q}\times(-\delta, \delta)$. Then, dV_0 is the riemannian volume form with respect to g. We take a suitable extension of g to a neighborhood of $\bar{Q}$ and denote it by the same notation g. Set $dV_q=h(q)$ for $q\in\sigma^k$ and set $dV_{q,\lambda}=(1-\lambda)dV_0+\lambda dV_q$. Then, $dV_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0, 1], \mathscr{V}_S)$. Define a C^∞-function $\rho_{q,\lambda}$ by $dV_{q,\lambda}=\rho_{q,\lambda}dV_0$. Let $g_{q,\lambda}=\rho_{q,\lambda}^{2/n}g$. Then, $g_{q,\lambda}$ is a C^∞-riemannian metric on a neighborhood of $\bar{Q}$ such that $dV_{q,\lambda}$ is the riemannian volume form with respect to $g_{q,\lambda}$. Since $\rho_{q,\lambda}\equiv 1$ on $\partial\bar{Q}\times(-\delta, \delta)$, we have $g_{q,\lambda}\equiv g$ on it. Let $\Delta_{q,\lambda}$ be the Laplacian with respect to $g_{q,\lambda}$, and let $\boldsymbol{\Gamma}(1_{\bar{Q}})$ is the space of all C^∞-functions of $\bar{Q}$ and $\Gamma^k(1_{\bar{Q}})$ the completion of $\boldsymbol{\Gamma}(1_{\bar{Q}})$ by the norm $\|\ \|_k$ given by

$$\|f\|_k^2=\sum_{s=0}^k\int_{\bar{Q}}\langle\nabla^s f, \nabla^s f\rangle dV_0\,.$$

Then, $\{\boldsymbol{\Gamma}(1_{\bar{Q}}), \Gamma^k(1_{\bar{Q}}), k\geqq 0\}$ is an ILB-system and $\Delta_{q,\lambda}$ is a linear mapping of order 2. It is not hard to see that $\Delta_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0, 1], L_2(\boldsymbol{\Gamma}(_{\bar{Q}}), \boldsymbol{\Gamma}(1_{\bar{Q}}))$. Let

$$\boldsymbol{\Gamma}_{\partial\bar{Q}}(1_{\bar{Q}})=\left\{f\in\boldsymbol{\Gamma}(1_{\bar{Q}});\int_{\bar{Q}}f\,dV_0=0,\quad(\partial/\partial x^n)f\equiv 0\quad\text{on}\quad\partial\bar{Q}\right\}\,,$$

and $\Gamma^k_{\partial\bar{Q}}$ its closure in $\Gamma^k(1_{\bar{Q}})$. $\{\boldsymbol{\Gamma}_{\partial\bar{Q}}(1_{\bar{Q}}), \Gamma^k_{\partial\bar{Q}}, k\geqq 0\}$ is also an ILB-system. Let $\boldsymbol{\Gamma}_0=\left\{f\in\boldsymbol{\Gamma}(1_{\bar{Q}});\int_{\bar{Q}}f\,dV_0=0\right\}$, and Γ^k_0 its closure in $\Gamma^k(1_{\bar{Q}})$. For the above $\rho_{q,\lambda}$ we denote by $\rho_{q,\lambda}^{-1}\boldsymbol{\Gamma}_0$ (resp. $\rho_{q,\lambda}^{-1}\Gamma^k_0$) the space $\{\rho_{q,\lambda}^{-1}f; f\in\boldsymbol{\Gamma}_0$ (resp. $\Gamma^k_0)\}$. Since $\rho_{q,\lambda}>0$, we see that $\{\rho_{q,\lambda}^{-1}\boldsymbol{\Gamma}_0, \rho_{q,\lambda}^{-1}\Gamma^k_0, k\geqq 0\}$ is an ILB-system, which is naturally isomorphic to the IBL-system $\{\boldsymbol{\Gamma}_0, \Gamma^k_0, k\geqq 0\}$. Note that $\Delta_{q,\lambda}$ is an isomorphism of $\boldsymbol{\Gamma}_{\partial\bar{Q}}(1_{\bar{Q}})$ onto $\rho_{q,\lambda}^{-1}\boldsymbol{\Gamma}_0$ (cf. [5]), which can be extended to an isomorphism of $\Gamma^{k+2}_{\partial\bar{Q}}$ onto $\rho_{q,\lambda}^{-1}\Gamma^k_0$. The next lemma is an immediate conclusion from Lemma 1.2:

LEMMA 5.1. *Define a C^∞-function $h_{q,\lambda}$ by $dV_q-dV_0=h_{q,\lambda}dV_{q,\lambda}$ ($=h_{q,\lambda}\rho_{q,\lambda}dV_0$). Then, $h_{q,\lambda}\in\rho_{q,\lambda}^{-1}\boldsymbol{\Gamma}_0$ and there exists uniquely $f_{q,\lambda}\in\boldsymbol{\Gamma}_{\partial\bar{Q}}(1_{\bar{Q}})$ such that $\Delta_{q,\lambda}f_{q,\lambda}=h_{q,\lambda}$ on $\bar{Q}$. Moreover, $f_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0,1],\boldsymbol{\Gamma}_{\partial\bar{Q}}(1_{\bar{Q}}))$.*

Let $u_{q,\lambda}=\mathrm{grad}_{q,\lambda}f_{q,\lambda}$ be the gradient vector field of $f_{q,\lambda}$ with respect to the riemannian metric $g_{q,\lambda}$. Since $\mathrm{grad}_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0,1],L_1(\boldsymbol{\Gamma}(1_{\bar{Q}})\boldsymbol{\Gamma}(1_{\bar{Q}}))$, we see that $u_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0,1],\boldsymbol{\Gamma}(T'_{\bar{Q}}))$. Since $h_{q,\lambda}\equiv 0$ on a neighborhood of $\partial\bar{Q}$, there is a positive number δ_1 such that $\mathrm{div}_{q,\lambda}u_{q,\lambda}\equiv 0$ on $\partial\bar{Q}\times[0,\delta_1]$, where $\mathrm{div}_{q,\lambda}$ is the divergent with respect to $g_{q,\lambda}$.

LEMMA 5.2. *Set $u_{q,\lambda}=u'_{q,\lambda}+u^n_{q,\lambda}\partial/\partial x^n$, $u'_{q,\lambda}=\sum_{i=1}^{n-1}u'^i_{q,\lambda}\partial/\partial x^i$. Then,*

$$\int_{\partial\bar{Q}}u^n_{q,\lambda}(x',x^n)d\mu=0\;,\qquad \int_{\partial\bar{Q}}(\partial/\partial x^n)u^n_{q,\lambda}(x',x^n)d\mu=0$$

for $0\leqq x'<\delta_1$, where $0<\delta_1\leqq\delta$ is assumed.

PROOF. Set $\partial Q'=\partial\bar{Q}\times\{x^n\}$, $R=\partial Q\times[0,x^n]$. Since $\mathrm{div}_{q,\lambda}u_{q,\lambda}\equiv 0$ on R, we have

$$0=\int_R \mathrm{div}_{q,\lambda}u_{q,\lambda}d\mu\wedge dx^n=-\int_{\partial\bar{Q}}u^n_{q,\lambda}d\mu+\int_{\partial Q'}u^n_{q,\lambda}d\mu\;.$$

Since $u^n_{q,\lambda}=\partial f_{q,\lambda}/\partial x^n$, we have $\int_{\partial\bar{Q}}u^n_{q,\lambda}d\mu=0$, hence $\int_{\partial Q'}u^n_{q,\lambda}d\mu=0$. On the other hand, since

$$\mathrm{div}_{q,\lambda}u_{q,\lambda}=\mathrm{div}_{\partial\bar{Q}}u'_{q,\lambda}+(\partial/\partial x^n)u^n_{q,\lambda}=0$$

on $\partial Q\times[0,\delta_1)$, where $\mathrm{div}_{\partial\bar{Q}}$ is the divergence on $\partial\bar{Q}$, we have

$$\int_{\partial\bar{Q}}(\partial/\partial x^n)u^n_{q,\lambda}(x',x^n)d\mu=-\int_{\partial\bar{Q}}\mathrm{div}_{\partial Q}u'_{q,\lambda}d\mu=0\;.$$

Let $\phi(x^n)$ be a C^∞-function on $[0,\infty)$ such that $\phi\geqq 0$, $\phi\equiv 1$ on $[0,\delta_1/2]$ and $\phi\equiv 0$ on $[\delta_1,\infty)$. Let Δ' be the Laplacian on $\partial\bar{Q}$ with respect to g'.

LEMMA 5.3. *Regarding $x^n\in[0,\infty)$ as a parameter, there exists uniquely a smooth function $F_{q,\lambda}(x',x^n)$ on $\partial Q\times[0,\infty)$ such that*

$$\begin{cases}\Delta' F_{q,\lambda}(x',x^n)=(\partial/\partial x^n)\phi(x^n)u^n_{q,\lambda}(x',x^n)\\ \int_{\partial\bar{Q}}F_{q,\lambda}(x',x^n)d\mu=0\;.\end{cases}\tag{20}$$

Moreover, $F_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0,1]$, $\boldsymbol{\Gamma}(1_{\partial Q\times[0,2\delta_1]}))$.

PROOF. Since

$$\int_{\partial\bar{Q}}(\partial/\partial x^n)\phi(x^n)u^n_{q,\lambda}(x', x^n)d\mu=\phi'(x^n)\int_{\partial\bar{Q}}u^n_{q,\lambda}d\mu+\phi\int_{\partial\bar{Q}}(\partial/\partial x^n)u^n_{q,\lambda}d\mu=0\ ,$$

the equation (20) can be solved uniquely and by Lemma 1.2 $F_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0, 1], \boldsymbol{\Gamma}(1_{\partial Q\times[0,2\delta_1]})$.

Let $v_{q,\lambda}$ be the gradient vector field of $F_{q,\lambda}$ on ∂Q with respect to g'. Since $v_{q,\lambda}$ contains the parameter x^n, we may regard $v_{q,\lambda}$ as a smooth vector field on $\partial Q\times[0, \infty)$. Note that $v_{q,\lambda}\equiv 0$, if $x^n\geqq\delta_1$. Hence $v_{q,\lambda}$ can be regarded as a stricly tangent vector field on $\bar{Q}$.

Set

$$w_{q,\lambda}=(1-\phi)u_{q,\lambda}+\phi u'_{q,\lambda}+v_{q,\lambda} \tag{21}$$

and $w_{q,\lambda}$ is a strictly tangent vector field on $\bar{Q}$, because

$$w_{q,\lambda}=u'_{q,\lambda}+(1-\phi)u^n_{q,\lambda}\partial/\partial x^n+v_{q,\lambda}$$

and hence has no $\partial/\partial x^n$-component on $\partial Q\times[0, \delta_1/2]$. Note that $w_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0, 1], \boldsymbol{\Gamma}(T_{\bar{Q}}))$.

LEMMA 5.4. *$\mathrm{div}_{q,\lambda}w_{q,\lambda}\equiv h_{q,\lambda}$ on $\bar{Q}$.*

PROOF. If $0\leqq x^n<\delta_1$, then $w_{q,\lambda}=u_{q,\lambda}-\phi u^n_{q,\lambda}\partial/\partial x^n+v_{q,\lambda}$. Thus,

$$\mathrm{div}_{q,\lambda}w_{q,\lambda}=\mathrm{div}_{q,\lambda}u_{q,\lambda}-(\partial/\partial x^n)\phi u^n_{q,\lambda}+\mathrm{div}_{q,\lambda}v_{q,\lambda}\ .$$

Since

$$\mathrm{div}_{q,\lambda}v_{q,\lambda}=\mathrm{div}_{\partial Q}v_{q,\lambda}=\Delta' F_{q,\lambda}=(\partial/\partial x^n)\phi u^n_{q,\lambda}\ ,$$

we have $\mathrm{div}_{q,\lambda}w_{q,\lambda}\equiv\mathrm{div}_{q,\lambda}u_{q,\lambda}\equiv 0$ on $\partial\bar{Q}\times[0, \delta_1)$.

On the complement of $\partial\bar{Q}\times[0, \delta_1)$ in $\bar{Q}$, we have $w_{q,\lambda}\equiv u_{q,\lambda}$. Therefore $\mathrm{div}_{q,\lambda}w_{q,\lambda}\equiv\mathrm{div}_{q,\lambda}u_{q,\lambda}\equiv\Delta_{q,\lambda}f_{q,\lambda}\equiv h_{q,\lambda}$.

Let ψ be a C^∞-function on $[0, \infty)$ such that $\psi\equiv 1$ on $[0, \delta_1/4]$ and $\psi\equiv 0$ on $[\delta_1/2, \infty)$. Set $\tilde{w}_{q,\lambda}=(1-\psi(x^n))w_{q,\lambda}$. Since $\tilde{w}_{q,\lambda}\equiv 0$ on a neighborhood of $\partial\bar{Q}$, we may regard $\tilde{w}_{q,\lambda}$ as an element of $\boldsymbol{\Gamma}_S(T_N)$. Obviously, $\tilde{w}_{q,*}$ defines a continuous mapping of σ^k into $\mathscr{M}([0, 1], \boldsymbol{\Gamma}_S(T_N))$. Note that $h_{q,\lambda}\equiv 0$ on $\partial\bar{Q}\times[0, \delta_1)$ and $\tilde{w}_{q,\lambda}$ has no $\partial/\partial x^n$-component on $\partial Q\times[0, \delta_1/2]$. Hence, we have

LEMMA 5.5. *$\mathrm{div}_{q,\lambda}\tilde{w}_{q,\lambda}=h_{q,\lambda}$ on S, and $\tilde{w}_{q,\lambda}\in\boldsymbol{\Gamma}_S(T_N)$.*

Solve the equation

$$(d/d\lambda)\tilde{\psi}_{q,\lambda}=-\tilde{w}_{q,\lambda}\tilde{\psi}_{q,\lambda} \tag{22}$$

with the initial condition $\tilde{\Psi}_{q,0}$=identity. Then, $\tilde{\Psi}_{q,\lambda}$ is a C^∞-diffeomorphism on S such that $\tilde{\Psi}_{q,\lambda}\equiv$identity on a neighborhood of ∂S. Now, by the same computation as in (18), we see that

$$\tilde{\Psi}^*_{q,\lambda} d V_{q,\lambda} = d V_0 , \qquad q \in \sigma^k,\ \lambda \in [0, 1] .$$

Set $\tilde{H}(\lambda q)=\tilde{\Psi}^{-1}_{q,\lambda}$. Then by Lemma 2.8 $\tilde{H}$ is a continuous mapping of δ^{k+1} into $\mathscr{D}_S(N)$. Thus, restricting $\tilde{H}$ onto S we have a desired mapping. This completes the proof of Theorem in § 1.

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