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Geometry on Complements of Lines in P^2

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Introduction.

Let $\Delta_0, \cdots, \Delta_q$ be projective lines on a complex projective plane $\boldsymbol{P}^2$, where $\Delta_i \neq \Delta_j$ for $i \neq j$. We shall study algebro-geometric properties of the complement $S = \boldsymbol{P}^2 - \bigcup \Delta_j$. For instance, we shall compute the logarithmic geometric genus $\bar{p}_g$, logarithmic irregularity $\bar{q}$, logarithmic m-genus $\bar{P}_m$, logarithmic Kodaira dimension $\bar{\kappa}$, logarithmic Chern numbers $\bar{c}_1^2$, $\bar{c}_2$ of S and establish fundamental relations among them. For the definitions of $\bar{p}_g$, $\bar{q}$, $\bar{P}_m$, $\bar{\kappa}$ we refer the reader to [4] and [5].

THEOREM I. *$\bar{q} = q$ holds. If $q \geqq 2$ and $\bar{p}_g < q-1$, then $S = \boldsymbol{C} \times \Gamma$, $\bar{p}_g = \bar{P}_m = 0$ for any $m \geqq 1$; $\bar{\kappa} = -\infty$, $\bar{c}_1^2 = 3-2q$, $\bar{c}_2 = 1-q$. If $\bar{p}_g = 1$, $q = 2$, then $S = \boldsymbol{C}^{*2}$, $\bar{\kappa} = 0$ and $\bar{c}_1^2 = \bar{c}_2 = 0$. If $\bar{p}_g = q-1 \geqq 2$, then $S = \boldsymbol{C}^* \times \Gamma$, $\bar{g}(\Gamma) \geqq 2$ and $\bar{\kappa}(S) = \bar{\kappa}(\Gamma) = 1$; $\bar{c}_1^2 = \bar{c}_2 = 0$. Finally, if $\bar{p}_g \geqq q$, then $\bar{p}_g \geqq 2q-4$, $\bar{\kappa} = 2$ and $5\bar{c}_2 \geqq 2\bar{c}_1^2$.*

Summarizing the results, we obtain the following

TABLE

Type of Δ	$\bar{\kappa}$	$\bar{q}=q$	$1-\bar{q}+\bar{p}_g$	$\bar{c}_1^2$	$\bar{c}_2$	S
I	$-\infty$	0	1	4	1	$\boldsymbol{C}^2$
		1	0	0	0	$\boldsymbol{C} \times \Gamma$
		$\geqq 2$	$1-\bar{q}$	$3-2q$	$1-q$	$\bar{g}(\Gamma) = q \geqq 1$
II	0	2	0	0	0	$\boldsymbol{C}^* \times \boldsymbol{C}^*$
II$^{1}/_{2}$	1	$\geqq 3$	0	0	0	$\boldsymbol{C}^* \times \Gamma, \bar{g}(\Gamma) = q-1$
III	2	$\geqq 3$	$\geqq 1$ $\geqq \bar{q}-3$	$\geqq 1$	$\geqq \frac{2}{5}\bar{c}_1^2$	

Here, type of Δ is defined as follows:

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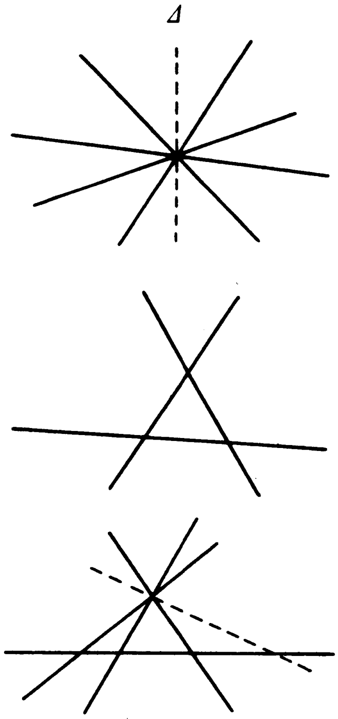
Type

I

II

$\mathrm{II}^1/_2$



$(q \geqq 3)$

III the other case.

FIGURE 1

Moreover, we shall use another kind of type of Δ.

Type

$\Pi_{a,b}$

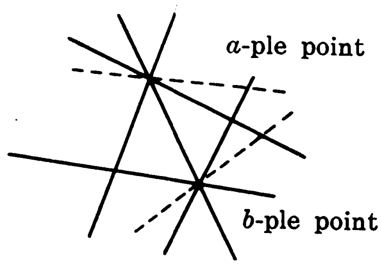


FIGURE 2

Here, $q+2=a+b$.

It is clear that $\mathrm{I}=\Pi_{q+1,1}$, $\mathrm{II}=\Pi_{2,2}$, and $\mathrm{II}^1/_2=\Pi_{q,2}$. S is a product of two curves if and only if Δ is of type $\Pi_{a,b}$.

THEOREM II. *Let $p_1, \cdots, p_s$ be the multiple* ($\geqq 3$) *points of $\Delta=\sum\Delta_j$ and by ν_j denote the multiplicity of Δ at p_j. Define the incomplete linear system on $\boldsymbol{P}^2$:*

$$\Lambda_m = |m(q-2)H|_{B(m)}\ , \quad \textit{where} \quad B(m)=\sum m(\nu_j-2)p_j\ .$$

Then Λ_m is simply generated by Λ_1.

THEOREM III. *Let Δ_j be defined by the homogeneous linear form L_j and regard Δ_q as an infinite line on P^2. Putting $A^2=P^2-\Delta_q$ and letting M be the set of multiple ($\geqq 2$) points of $A^2\cap\Delta$. For any $p\in M$ with the multiplicity $\nu=\nu(p)$, we have lines $\Delta_0, \cdots, \Delta_{\nu-1}$ passing through p. We define homogeneous forms*

$$F_j^{(p)}=\prod_{\substack{0<i<\nu\\ i\neq j}} L_i\cdot\prod_{\nu\leqq k<q} L_k\ .$$

The divisor $(F_j^{(p)})$ defined by $F_j^{(p)}$ belongs to Λ_1. Moreover, $\{(F_j^{(p)});\ p\in M,\ 0<j<\nu(p)\}$ is the base of Λ_1.

EXAMPLE. Consider the following triangle on P^2 and 6 points a, b, c, d, e, f on it. Set $\Lambda_m=|4mH|_{m(a+b+c+d+e+f)}$. Then

$$\dim \Lambda_m=5m^2+3m\ ,$$

and the base of Λ_1 consists of

$$\begin{array}{lll} B+A+D+E\ , & G+A+D+E\ , & A+B+D+G\ , \\ E+B+D+G\ , & B+C+D+E\ , & B+C+G+E\ , \\ G+A+C+E\ , & D+A+B+C\ , & E+A+B+C\ . \end{array}$$

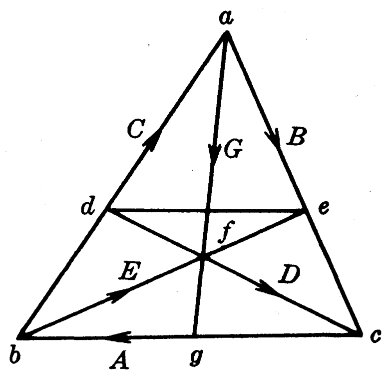


FIGURE 3

Here, $A, B, \cdots$ are divisors defined in the Fig. 3.

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§1. Consider $\Delta=\Delta_0+\cdots+\Delta_q$ as a closed subscheme of P^2 and define sets Σ_e as follows:

$$\Sigma_e=\{p\in P^2;\ \text{mult}_p(\Delta)\geqq e\}\ .$$

Then we have $\Sigma_0 = \boldsymbol{P}^2$, $\Sigma_1 = \Delta$. $\Sigma_2 - \Sigma_3$ consists of double points, say σ points. Set $\Sigma_3 = \{p_1, \cdots, p_s\}$ and let ν_j indicate the multiplicity of Δ at p_j. Performing quadratic transformations successively with centers $p_1, \cdots p_s$, we obtain a complete non-singular surface $\bar{S}$ and a birational morphism $\mu: \bar{S} \to \boldsymbol{P}^2$. Then letting

$E_j = \mu^{-1}(p_j)$, Δ^* be the proper transform of Δ, we have

$$\mu^* \Delta = \Delta^* + \sum \nu_j E_j \ .$$

Since $D := \mu^{-1}(\Delta) = \Delta^* + \sum E_j = (q+1)\mu^* H - \sum(\nu_j - 1)E_j$ where H is the (divisor of) line in $\boldsymbol{P}^2$, and since

$$K(\bar{S}) = K(\boldsymbol{P}^2) + \sum E_j \quad \text{(which indicates a canonical divisor on } \bar{S}) \ ,$$

we obtain

$$K(\bar{S}) + D = (q-2)H - \sum(\nu_j - 2)E_j \ .$$

Here, H stands for $\mu^* H$.

Needless to say, D is a divisor with simple normal crossings. Hence, $\bar{S}$ is a completion of S with smooth boundary D. The completion $\bar{S}$ constructed above is called *the standard completion* of S with smooth boundary D.

Recalling the definitions;

$$\bar{p}_g(S) = \dim T_2(S) \ , \quad \text{where} \quad T_2(S) \quad \text{stands for} \quad H^0(\bar{S}, \mathcal{O}(K(\bar{S}) + D)) \ ,$$
$$\bar{P}_m(S) = \dim H^0(\bar{S}, \mathcal{O}(m(K(\bar{S}) + D))) \ ,$$
$$\bar{q}(S) = \dim T_1(S) \ , \quad \text{where} \quad T_1(S) \quad \text{stands for} \quad H^0(\bar{S}, \Omega^1 \log D) \ ,$$
$$\bar{\kappa}(S) = \kappa(K(\bar{S}) + D, \bar{S}) \ ,$$

we obtain the following

PROPOSITION 1. *$\bar{q}(S) = q$ and*

$$\bar{p}_g(S) = \frac{(q-1)q}{2} - \sum \frac{(\nu_j - 1)(\nu_j - 2)}{2} \ .$$

PROOF. By Theorem 1 ([5]), we have

$$\bar{q}(S) = b_1(S) = q \ .$$

In view of Serre duality, $H^2(K(\bar{S}) + D) = H^0(-D) = 0$ and $H^1(K(\bar{S}) + D) = H^1(-D)$. Since $\bar{S}$ is a regular surface (that is, $q(\bar{S}) = 0$) and D is a connected reduced divisor, we have the following exact sequence:

$$0 \longrightarrow \boldsymbol{C} = H^0(\mathcal{O}_{\bar{S}}) \longrightarrow \boldsymbol{C} = H^0(\mathcal{O}_D) \longrightarrow H^1(\mathcal{O}(-D)) \longrightarrow H^1(\mathcal{O}_{\bar{S}}) = 0 \ .$$

From this, it follows that $H^1(\mathcal{O}(-D))$ (which is abbreviated to $H^1(-D)$) vanishes. From the Riemann-Roch theorem, we derive

$$\bar{p}_g(S)=\frac{D(K(\bar{S})+D)}{2}+1$$
$$=\frac{(q-1)q}{2}-\sum\frac{(\nu_j-1)(\nu_j-2)}{2}\ . \qquad \text{Q.E.D.}$$

On the other hand, it is easy to see that

$$(*) \qquad \frac{q(q+1)}{2}=\sum\frac{\nu_j(\nu_j-1)}{2}+\sigma$$

by counting the number of $\{\Delta_i\cap\Delta_j;\ \text{for}\ i\neq j\}$. Hence, we get another formula of $\bar{p}_g$.

$$\bar{p}_g(S)=\sum^{s}(\nu_j-1)+\sigma-q\ .$$

Fixing Δ_q, we have

$$q=(\sum^{q-1}\Delta_j,\ \Delta_q)=\sum_{p_j\in\Delta_q}(\nu_j-1)+\#((\Sigma_2-\Sigma_3)\cap\Delta_q)\ .$$

Hence,

$$\bar{p}_g(S)=\sum_{p_j\notin\Delta_q}(\nu_j-1)+\#(\Sigma_2-(\Sigma_3\cup\Delta_q))\ .$$

Regarding Δ_q as an infinite line and putting $A^2=P^2-\Delta_q$, $\Delta_j^0=\Delta_j-\Delta_q$, $r_e=\#\{e\text{-ple points of }\sum\Delta_j^0\}$, we can rewrite the formula in Proposition 1 as follows:

$$\bar{p}_g(S)=\sum_{e=2}^{\infty}r_e(e-1)\ .$$

Let l_i be the linear form defining Δ_i^0 in A^2. Then $T_1(S)=\sum Cdl_i/l_i$, $T_2(S)\supset\sum Cdl_i\wedge dl_j/l_il_j$. For $p\in A^2$, we define the vector space $W(p)$ by

$$W(p)=\{\sum Cdl_i\wedge dl_j/l_il_j;\ \Delta_i^0\cap\Delta_j^0\ni p\}\ .$$

Let e be the multiplicity of Δ at p. Then we have the following fact:

i) $W(p)\cap W(p')=0$ for $p\neq p'$,

ii) $\dim W(p)=e-1$.

In order to prove ii), we let $p=$ the origin and $l_j=z+\beta_jw$ where $\beta_1=0$ and $1\leqq j\leqq e$. Put $w=uz$. Then $dl_j/l_j=dz/z+\beta_jdu/(1+\beta_ju)$ and define $\omega_j=dl_1\wedge dl_j/l_1l_j=\beta_jdz\wedge du/z(1+\beta_ju)$. We see that $dl_i\wedge dl_j/l_il_j=\omega_j-\omega_i$. Thus, $W(p)=\sum_{j=2}^{e}C\omega_j$ and it is clear that $\omega_2,\cdots,\omega_e$ are linearly independent. Hence $\dim W(p)=e-1$. Next we shall prove i). Put $p'=(1, 0)$ and choose $\omega \in W(p)\cap W(p')$. Then

$$\omega=\frac{dz}{z}\wedge\sum c_j\frac{d(z+\beta_j w)}{z+\beta_j w}=\frac{d(z+w-1)}{z+w-1}\wedge\sum\delta_j\frac{d(z+w-1+\gamma_j w)}{z+w-1+\gamma_j w}$$

where the c_j and the δ_j are complex number and $z+w-1+\gamma_j w=0$ defines the line passing through p'. Introduce u again by $w=uz$, we have

$$\omega=\frac{dz}{z}\wedge\sum c_j\frac{\beta_j du}{1+\beta_j u}=\psi(z, u)dz\wedge du ,$$

in which $\psi(z, u)$ is a holomorphic function of z at $z=0$. Hence, $c_2=\cdots=c_e=0$ and so $\omega=0$.

By i) and ii), we get

$$\dim\sum Cdl_i\wedge dl_j/l_il_j=\sum\dim W(p)=\sum r_e(e-1)=\dim T_2(S) .$$

From this, follows the

THEOREM 1. $T_2(S)=\sum W(p)=\sum Cdl_i\wedge dl_j/l_il_j$.

Note that this is a special case of Theorem ([1]).

The complete linear system $|m(K(\bar{S})+D)|=|m(q-2)H-\sum m(\nu_j-2)E_j|$ corresponds bijectively to the (incomplete) linear system on $\boldsymbol{P}^2$

$$\Lambda_m=\{C\in|m(q-2)H|;\ \mathrm{mult}_{p_j}(C)\geqq m(\nu_j-2)\} ,$$

which may be written as $|m(q-2)H|_{B(m)}$, $B(m)$ being $\sum m(\nu_j-2)p_j$.

In view of Theorem 1, we get the base of Λ_1 as follows: For a point $p\in\bigcup\Delta_j^0$, we put

$$I(p)=\{i\in[0, q-1];\ \Delta_i^0\ni p\} ,$$

and fix $i_0\in I(p)$. We take $i\in I(p)$ and define effective divisors

$$G_i^{(p)}=\{\textstyle\sum^*\Delta_j;\ j\in[0, q-1]-\{i_0, i\}\} .$$

Then, $\{G_i^{(p)};\ p\in\bigcup\Delta_j^0,\ i\in I(p)-\{i_0\}\}$ is the base of Λ_1.

For this, follows Theorem III in the Introduction.

§2. In general, let $\bar{V}$ be a complete non-singular algebraic variety and D a divisor with normal crossings on $\bar{V}$. Then, $\Omega^1(\log D)$ is a locally free sheaf of rank n. $\Theta(\log D)$ is the dual sheaf of it.

Put $c_i(\bar{V}, D)=c_i(\Theta\log D)\in H^{2i}(\bar{V}, \boldsymbol{Z})$ for any i. The following proposistion is derived from the Hirzebruch formula of Riemann-Roch theorem.

PROPOSITION 2. *$c_n(\bar{V}, D)[\bar{V}]=\chi_V$, where $V=\bar{V}-D$ and χ_V indicates the Euler characteristic of V.*

$c_i(\bar{V}, D)$ may be called the logarithmic Chern classes of $(\bar{V}, D)$. We shall compute Chern numbers $\bar{c}_1^2=c_1(\bar{S}, D)[\bar{S}]$ and $\bar{c}_2=c_2(\bar{S}, D)[\bar{D}]$. By definition,

$$-c_1(\Theta \log D)\sim K(\bar{S})+D=(q-2)H-\sum(\nu_j-2)E_j \ .$$

Hence, $\bar{c}_1^2=(q-2)^2-\sum(\nu_j-2)^2=-5q+3(\sum(\nu_j-1)+\sigma)+4-s-\sigma$. On the other hand, $\bar{c}_2=1-b_1(S)+b_2(S)-b_3(S)$, where $b_j(S)$ denotes the j-th Betti number of S for $j=1, 2, 3$. By the formula: (Proposition 1 [5]) $\bar{q}(S)-q(\bar{S})=b_1(S)-b_1(\bar{S})$, we have

$$b_1(S)=\bar{q}(S)=q \ .$$

PROPOSITION 3. *$b_2(S)=\bar{p}_g(S)$ and $b_3(S)=0$.*

PROOF. Since S is affine and 2-dimensional, it follows that $b_3(S)=0$. By the following lemma, we complete the proof.

LEMMA 1 (Y. Norimatsu). *In general, let $\bar{S}$ be a complete non-singular surface and D a divisor with simple normal crossings, that is, $D=\sum_{j=1}^{s} D_j$ has only normal crossings and each D_j is non-singular. Assume that $S=P^2-\sum C_i$, the C_i being irreducible curves. Then,*

$$b_2(S)=\bar{p}_g(S)+\sum g(D_j)=\bar{p}_g(S)+\sum g(C_i) \ ,$$

where $g(C_i)$ denotes the genus of C_i.

PROOF. From an exact sequence on $\bar{S}$:

$$0\longrightarrow \Omega^1\longrightarrow \Omega^1(\log D)\longrightarrow \bigoplus \mathcal{O}_{D_j}\longrightarrow 0 \ ,$$

follows the exact sequence:

$$\begin{aligned}&0=H^0(\Omega^1)\longrightarrow H^0(\Omega^1\log D)\longrightarrow \bigoplus_{j=1}^{s} H^0(D_j, \mathcal{O})\\ &\xrightarrow{\ \delta\ } H^1(\Omega^1)\longrightarrow H^1(\Omega^1\log D)\longrightarrow \bigoplus_{j=1}^{s} H^1(D_j, \mathcal{O})\\ &\longrightarrow H^2(\Omega^1)=0 \ .\end{aligned}$$

Hence,

$$\begin{aligned}&\dim H^1(\Omega^1\log D)=\sum g(D_j)+\dim(H^1(\Omega^1)/\operatorname{Im}\delta) \ ,\\ &\bar{q}(S)=\dim H^0(\Omega^1\log D)=s-\dim \operatorname{Im}\delta \ .\end{aligned}$$

On the other hand, by the Hodge theory due to Deligne [2],

$$H^2(S, \boldsymbol{C}) \xrightarrow{\sim} H^2(\bar{S}, \mathcal{O}) \oplus H^1(\Omega^1 \log D) \oplus H^0(\bar{S}, \Omega^2 \log D)$$
$$= H^1(\Omega^1 \log D) \oplus H^0(\Omega^2(D)) .$$

Then,

$$b_2(S) = \bar{p}_g(S) + p_g(\bar{S}) + \dim H^1(\Omega^1 \log D)$$
$$= \bar{p}_g(S) + \sum g(D_j) + \dim H^1(\Omega^1) + \bar{q}(S) - s .$$

Since $S = \boldsymbol{P}^2 - \sum C_i$, it follows that

$$\dim H^1(\Omega^1) = s - \bar{q}(S) \quad \text{and} \quad \sum g(D_j) = \sum g(C_i) .$$

Accordingly, we have

$$b_2(S) = \bar{p}_g(S) + \sum g(C_i) = \bar{p}_g(S) + \sum g(D_j) .$$

COROLLARY. *Under the same assumption as above, assume that each C_i is rational. Then $b_2(S) = \bar{p}_g(S)$.*

In other words, any 2-cocycle on S is cohomologous to a logarithmic 2-form. Thus, we have

$$\bar{c}_2 = 1 - q + \bar{p}_g = \frac{(q-1)(q-2)}{2} - \sum \frac{(\nu_j - 1)(\nu_j - 2)}{2} .$$

Thanks to the formula (*), we get

$$\bar{c}_2 = -2q + \sum (\nu_j - 1) + \sigma + 1 .$$

Hence,

$$3\bar{c}_2 - \bar{c}_1^2 = -q - 1 + s + \sigma .$$

If Δ is of type I, then $3\bar{c}_2 - \bar{c}_1^2 = -q - 1 + 1 = -q \leqq 0$.
If Δ is of type II or $\text{II}^1/_2$, then $3\bar{c}_2 - \bar{c}_1^2 = -q - 1 + 1 + q = 0$.
In the other case, $3\bar{c}_2 - \bar{c}_1^2 = -(q+1) + s + \sigma > 0$. Hence,

$$\text{if} \quad \bar{\kappa}(S) = 2 , \qquad \text{then} \quad 3\bar{c}_2 > \bar{c}_1^2 .$$

This is an analog of Miyaoka's inequality in the theory of compact complex surfaces.

Furthermore, by ρ_ν we denote the number of ν-ple points of $\Delta = \bigcup \Delta_j$ $(\nu \geqq 2)$. Then

$$5\bar{c}_2 - 2\bar{c}_1^2 = -3 - \sum (\nu_j - 1) + 2s + \sigma = -3 - \sum_{\nu=2}^{\infty} \rho_\nu (\nu - 3) .$$

We note the following formula (see [3]).

Formula. Let $p_j=\#\{j\text{-gons in the configuration of } \Delta \text{ in } \boldsymbol{P}^2\}$. Then

$$\sum \rho_\nu(\nu-3)+\sum p_j(j-3)+3=0\ .$$

$p_2>0$ if and only if Δ is of type I.

Thus we obtain the following

THEOREM 2. *The notation being as in the above, we have*

$$5\bar{c}_2-2\bar{c}_1^2=-p_2+\sum_{j=4}^{\infty} p_j(j-3)\ .$$

If Δ is of type III, *in other words, if $\bar{q}>0$ and $\bar{c}_1^2>0$, then $\bar{c}_1^2/\bar{c}_2\leqq 5/2$. The equality holds if and only if there are no j-gons ($j\geqq 4$) in the configuration of Δ in $\boldsymbol{P}^2$.*

§ **3.** We shall study $K(\bar{S})+D$ in which $\bar{S}$ is a standard completion of S with smooth boundary D.

THEOREM 3. *If S is not a product of two curves, then $K(\bar{S})+D$ is ample.*

PROOF. It has been proven that S with $\bar{\kappa}(S)\leqq 1$ is the product $C\times\Gamma$ where $C=\boldsymbol{C}$ or $\boldsymbol{C}^*$. Hence, it suffices to show that, under the assumption $\bar{\kappa}(S)=2$, S turns out to be the product of curves, if $K(\bar{S})+D$ is not ample. When $\bar{\kappa}(S)=2$, we have four lines, say, $\Delta_1, \Delta_2, \Delta_3, \Delta_4\subset\Delta$ such that $\Delta_1+\Delta_2+\Delta_3+\Delta_4$ has only normal crossings. We write $\Delta^0=\Delta_1+\Delta_2+\Delta_3+\Delta_4$ and $S^0=\boldsymbol{P}^2-\Delta^0$. By definition, S is an open subset of S^0. By j we denote the open immersion $S\to S^0$. Then μ is the completion of j. Hence, the logarithmic ramification formula applied to $j\colon S\to S^0$ yields

$$K(\bar{S})+D=\mu^*(K(\boldsymbol{P}^2)+\Delta^0)+\bar{R}_j=H+\bar{R}_j\ .$$

Note that $\bar{R}_j\subset D$. Actually, letting $S'=\mu^{-1}(S^0)$ and $D^0=\mu^{-1}(\Delta^0)$, we have a proper birational morphism $j'=\mu|S'\colon S'\to S^0$ and an open immersion $\lambda\colon S\subset S'$, whose completion is the identity $\bar{S}\to\bar{S}$. Then it is easy to see that $\bar{R}_{j'}$ is exceptional for μ and $\bar{R}_\lambda$ the closure of $D-D^0$. Thus, $\bar{R}_j=\bar{R}_{j'}+\bar{R}_\lambda$ is contained in D.

Recalling the Nakai criterion on ample divisors, we shall study the signature of $(K(\bar{S})+D, \Gamma)$ for any irreducible curve Γ on $\bar{S}$.

Case i) $\Gamma\not\subset D$. Then $(K(\bar{S})+D, \Gamma)=(H, \Gamma)+(\bar{R}_j, \Gamma)\geqq\deg\mu(\Gamma)\geqq 1$.

Case ii) $\Gamma=E_j$. Then $(K(\bar{S})+D, E_j)=\nu_j-2\geqq 1$.

Case iii) $\Gamma=\Delta'_a$, that is, Γ is a curve satisfying $\mu(\Gamma)=\Delta_a$ for some line $\Delta_a\subset\Delta$. Then $(K(\bar{S})+D, \Gamma)=\deg\Delta_a+(\bar{R}_{j'}, \Gamma)=1+(\bar{R}_\lambda, \Gamma)+(\bar{R}_{j'}, \Gamma)\geqq 1+(\bar{R}_\lambda, \Gamma)$, because $\bar{R}_{j'}$ is exceptional for $\mu=\bar{j}'$. Note that $\bar{R}_\lambda+D^0=D$.

If $\Gamma\subset\bar{R}_\lambda$, we have a reduced divisor D'' such that $\bar{R}_\lambda=\Gamma+D''$. In this case, $\Delta_a=\mu(\Gamma)\not\subset\Delta^0$.

Changing the indices, if necessary, we write

$$\{p_1, \cdots, p_s\}\cap\Delta_a=\{p_1, \cdots, p_\alpha\} .$$

It is clear that $(E_j, \Gamma)=1$ for $1\leqq j\leqq\alpha$ and that $\Gamma^2=1-\alpha$. If p_ι is the vertex of Δ^0 (in other words, $p_\iota\in\bigcup_{1\leqq i\neq j\leqq 4}(\Delta_i\cap\Delta_j)$), then E_ι is not a component of $\bar{R}_{j'}$. Otherwise, $E_\iota\subset\bar{R}_{j'}$. Define β to be $\#(\{p_1, \cdots, p_\alpha\}\cap$ {the vertices of Δ^0}). We may write

$$\{p_1, \cdots, p_\alpha\}\cap\{\text{the vertices of } \Delta^0\}=\{p_1, \cdots, p_\beta\} .$$

Since $\Delta_a\not\subset\Delta^0$, we see $\beta\leqq 2$. From the argument above, it follows that $E_{\beta+1}+\cdots+E_\alpha\subset\bar{R}_{j'}$. Hence,

$$(\bar{R}_{j'}, \Gamma)\geqq(E_{\beta+1}, \Gamma)+\cdots+(E_\alpha, \Gamma)=\alpha-\beta .$$

Moreover,

$$(\bar{R}_\lambda, \Gamma)=\Gamma^2+(D'', \Gamma)\geqq\Gamma^2=1-\alpha .$$

Hence,

$$(K(\bar{S})+D, \Gamma)\geqq 1+1-\alpha+\alpha-\beta=2-\beta\geqq 0 .$$

Therefore, the assumption $(K(\bar{S})+D, \Gamma)=0$ induces $\beta=2$ and $(D'', \Gamma)=0$. If $\alpha>\beta=2$, it would imply the existence E_3 contained in D''. Hence, $D''\cap\Gamma\supset(E_3, \Gamma)=$one point. This contradicts $D''\cap\Gamma=\varnothing$. Furthermore, $D''\cap\Gamma=\varnothing$ yields the fact that $\mu(D'')\cap\Delta_a=\{p_1, p_2\}$. Therefore, Δ is of type $\Pi_{a,b}$, in which $a=\nu_1$, $b=\nu_2$ and $\nu_1+\nu_2=q+2$. Q.E.D.

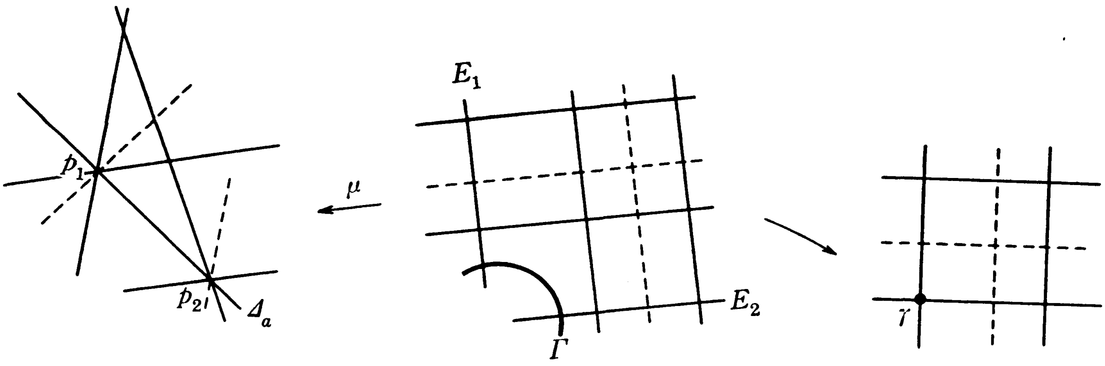


FIGURE 4

Let Γ be an irreducible curve on $\bar{S}$ such that $(K(\bar{S})+D, \Gamma)=0$. Then, as was seen above, $\Gamma^2=-1$ and $\Gamma\cong P^1$. Such a Γ is unique. And Γ is exceptional. Contracting Γ into a non-singular point γ, we obtain a quadratic transformation $\tilde{\lambda}: \bar{S}=Q_{\gamma}(\bar{S}_1)\to\bar{S}_1$. By Fig. 4 we see that $\bar{S}_1=\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $D_1=\tilde{\lambda}(D)=\vartheta_1\times\boldsymbol{P}^1+\boldsymbol{P}^1\times\vartheta_2$, where the ϑ_i are sums of ν_j points on $\boldsymbol{P}^1$. Hence, $S=\bar{S}_1-D_1=(\boldsymbol{P}^1-\vartheta_1)\times(\boldsymbol{P}^1-\vartheta_2)$, a product of two curves. Since $K(\bar{S}_1)+D_1=(K(\boldsymbol{P}^1)+\vartheta_1)\times\boldsymbol{P}^1+\boldsymbol{P}^1\times(K(\boldsymbol{P}^1)+\vartheta_2)$ where $\deg(K(\boldsymbol{P}^1)+\vartheta_i)=-2+\nu_i\geqq 1$, $K(\bar{S}_1)+D_1$ is ample and simply generated. Note that $K(\bar{S})+D=\tilde{\lambda}^*(K(\bar{S}_1)+D_1)$ is not ample, but simply generated.

PROPOSITION 4. *Let $\bar{V}$ be a complete non-singular algebraic surface and D a divisor with normal crossings. Suppose that $K(\bar{V})+D$ is ample. Then any completion $\bar{W}$ of $V=\bar{V}-D$ with smooth boundary Δ dominates $\bar{V}$.*

PROOF. Let φ be a birational map of $\bar{W}$ into $\bar{V}$ which is defined by id: $V\to V$. Assuming that φ is not a morphism, we choose p at which φ is not defined. There is another completion $\bar{Z}$ of V with smooth boundary B such that $\lambda: \bar{Z}\to\bar{W}$ and $g: \bar{Z}\to\bar{V}$ which are defined by id: $V\to V$ are birational morphisms. Then $g=\varphi\cdot\lambda$. Hence $g(\lambda^{-1}(p))$ is a curve. $\lambda^{-1}(p)$ contains an exceptional curve E of the first kind. If $g(E)$ is a point, $\bar{Z}'$ obtained from $\bar{Z}$ by contracting E to a non-singular point is substituted for $\bar{Z}$. Hence, we may assume that $g(E)=C$ is a curve. Since $C\subset D$ and $E\subset B$, we write $D=D'+C$ and $B=B'+E$. Then it follows that $(B', E)\geqq(D', C)$. Next, contracting E to a point z we have a non-singular surface $\bar{Z}_1$ from which $\bar{Z}$ is obtained by a blowing up at z: $\bar{Z}=Q_z(\bar{Z}_1)$. By λ' we denote the birational morphism: $\bar{Z}\to\bar{Z}_1$ and we write $B_1=\lambda'(B)$. Since $\lambda(E)=$ a point, we get a birational morphism $\mu: \bar{Z}_1\to\bar{W}$ which is an extension of the identity: $\bar{Z}_1-B_1=Z=\bar{W}-\Delta$. Since Δ is a divisor with simple normal crossings, B_1 has only simple normal crossings. Hence, $(B', E)=$the multiplicity of $\lambda(B')$ at $z\leqq 2$. Thus, we have $(D', C)\leqq(B', E)\leqq 2$. Finally, by the adjunction formula,

$$(K(\bar{V})+D,\ C)=(K(\bar{V})+C,\ C)+(D',\ C)=-2+(D',\ C)\leqq 0\ .$$

This contradicts the hypothesis that $K(\bar{S})+D$ is ample. Q.E.D.

Combining the above proposition with Theorem 3, we conclude: Let S be a complement of lines in $\boldsymbol{P}^2$ with $\bar{\kappa}(S)=2$. For any completion $\bar{S}'$ with smooth boundary D' of S, we have the birational morphism

$$\varphi: \bar{S}'\longrightarrow\begin{cases}\boldsymbol{P}^1\times\boldsymbol{P}^1 & \text{if } S \text{ is the product of curves}\\ \bar{S} & \text{otherwise}\end{cases}$$

where $(\bar{S}, D)$ is the standard completion of S.

Therefore, let Γ be an arbitrary irreducible curve such that $\varphi_*(\Gamma)\neq 0$. Then, if S is not the product of curves,

$$(K(\bar{S}')+D', \Gamma)=(\varphi^*(K(\bar{S})+D), \Gamma)+(\bar{R}_\varphi, \Gamma)\geqq(K(\bar{S})+D, \varphi_*\Gamma)>0 ,$$

where $\bar{R}_\varphi$ indicates the logarithmic ramification divisor of $\varphi|\bar{S}'$.

PROPOSITION 5. *Assume* $\bar{\kappa}(S)=2$. *Then*

$$\mathrm{Aut}(S)\subset\begin{cases}\mathrm{Aut}(\bar{S}), & \text{if } S \text{ is not a product of curves}\\ \mathrm{Aut}(\boldsymbol{P}^1\times\boldsymbol{P}^1), & \text{if } S \text{ is a product of curves .}\end{cases}$$

This follows from Theorem 3 using a general result in [4]. Namely, it will suffice to apply Proposition 2 in [4].

REMARK. This was proved by I. Wakabayashi by studying the effects of quadratic transformations.

For a surface $S=\boldsymbol{P}^2-\Delta$, we define

$$m(\Delta_i)=\#\{\Delta_j\cap\Delta_i\neq\emptyset,\ j\neq i\}$$

and

$$m(S)=\max\{m(\Delta_i);\quad i\in[0, q]\} .$$

If $m(S)=1$, then Δ is of type I.
If $m(S)=2$, then Δ is of type II.
Then we prove

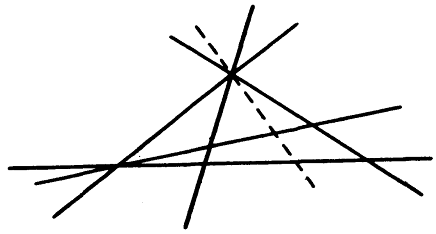
PROPOSITION 6. *If* $\bar{\kappa}(S)=2$, *then* $\bar{p}_g\geqq 2q-4$. *Moreover, if* $\bar{\kappa}(S)=2$ *and* $\bar{p}_g=2q-4$, *then* Δ *is of type* $\Pi_{3,q-1}$.

PROOF. We prove this by induction on q. If $q=3$, $\bar{p}_g=q=3$. Since $m(S)\geqq 3$, there is a line, say Δ_0, such that $m(S)=m(\Delta_0)$. Define S_1 to be $\boldsymbol{P}^2-\Delta_1\cup\cdots\cup\Delta_q$. If $\bar{\kappa}(S_1)=2$, then $\bar{p}_g(S)=\bar{p}_g(S_1)+m(\Delta_0)-1\geqq 2(q-1)-4+2=2q-4$ by the induction hypothesis. If $\bar{\kappa}(S_1)=0$ or 1, then $\Delta_1\cup\cdots\cup\Delta_q$ is of type $\Pi_{2,q-1}$. Since $\bar{\kappa}(S)=2$, $m(S)\geqq q-1$ follows. Thus,

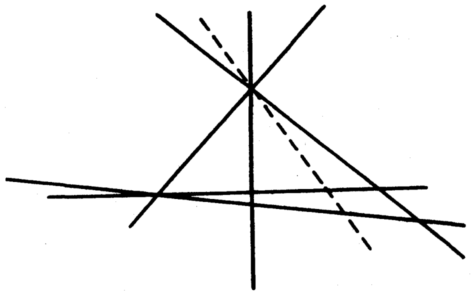
$$\bar{p}_g(S)=\bar{p}_g(S_1)+m(S)-1=q-2+m(S)-1\geqq 2q-4 .$$

If $\bar{\kappa}(S_1)=-\infty$, then $\bar{\kappa}(S_1-\Delta_0)\leqq 1$, which contradicts the hypothesis.

Furthermore, under the assumption $\bar{\kappa}(S)=2$, if $\bar{p}_g=2q-4$, then


,

if $\bar{p}_g=2q-3$, then



If $q\geqq 4$, we have

$$\bar{p}_g\geqq 2q-4=q+q-4\geqq q\ .$$

On the other hand, $\bar{\kappa}(S)=2$, $q=3$ imply that Δ has only normal crossings and so $\bar{p}_g=q=3$.

In particular, $\bar{p}_g=q\geqq 3$ yields $q=3$ or 4. Thus, we complete the proof of Theorem I.

§4. As a generalization of the Kodaira vanishing theorem, we have

LEMMA 2 (Y. Norimatsu). *Let $\bar{V}$ be a complete non-singular algebraic variety and D a divisor with simple normal crossings. For any ample divisor A on $\bar{V}$, we have*

(i) $H^p(\bar{V}, \Omega^q(\log D)\otimes\mathcal{O}(A))=0$ *for* $p+q>n=\dim\bar{V}$,

(ii) $H^p(\bar{V}, \Omega^q(\log D)\otimes\mathcal{O}(-A))=0$ *for* $p+q<n=\dim\bar{V}$.

PROOF. Using an exact sequence by Deligne [2], we could derive this from the Kodaira vanishing theorem on $\bar{V}$ as well as on each component of D. Q.E.D.

We get the following formula: For a surface $S=P^2-\bigcup\Delta_j$ with $\bar{\kappa}(S)=2$,

$$\begin{aligned}\bar{P}_m(S)&=\frac{1}{2}\bar{c}_1^2m^2-\frac{\bar{c}_1c_1}{2}m+1\\&=\frac{1}{2}((q-2)^2-\sum(\nu_j-2)^2)m^2+\frac{1}{2}(3(q-2)-\sum(\nu_j-2))m+1\\&=\frac{1}{2}m(q-2)(m(q-2)+3)-\sum\frac{(m(\nu_j-2)+1)(m(\nu_j-2))}{2}+1\ .\end{aligned}$$

Here $\bar{c}_1=c_1((\Omega^1\log D)^{\vee})=c_1(\Theta(\log D))$ and $c_1=c_1(\Theta_{\bar{S}})$, $\bar{S}$ being the standard completion of S with smooth boundary D. Actually, by the Riemann

Roch theorem, we have

$$\chi(m(K(\bar{S})+D))=\frac{(m(q-2)+1)(m(q-2)+2)}{2}-\sum\binom{m(\nu_j-2)+1}{2}.$$

Since $(m-1)(K(\bar{S})+D)$ is ample if S is not a product of curves, we have

$$H^1(m(K(\bar{S})+D))=H^2(m(K(\bar{S})+D))=0$$

by Lemma 2. Hence, in this case, we get the formula. When Δ is of $\Pi_{a,b}$, we have to compute $\bar{P}_m(S)$ by the product formula, that is,

$$\begin{aligned}\bar{P}_m(S)&=\bar{P}_m(\boldsymbol{P}^1-a \text{ points})\cdot\bar{P}_m(\boldsymbol{P}^1-b \text{ points})\\&=(m(a-2)+1)(m(b-2)+1)\\&=\frac{(m(a+b-4)+1)(m(a+b-4)+2)}{2}-\frac{(m(a-2))(m(a-2)-1)}{2}\\&\quad-\frac{(m(b-2))(m(b-2)-1)}{2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\bar{P}_m(S)&=\frac{m\bar{c}_1(m\bar{c}_1-c_1)}{2}+1=\frac{\bar{c}_1^2}{2}m^2-\frac{\bar{c}_1c_1}{2}m+1\\&=\frac{(d+1)(d+2)}{2}-\sum\binom{n_i+1}{2},\end{aligned}$$

where $d=m(q-2)$, $n_i=m(\nu_i-2)$.

If $K(\bar{S})+D$ is ample, then $m(K(\bar{S})+D)+\delta\mu^*H$ is ample too for any $m\geqq 1$ and $\delta\geqq 0$. Recalling that

$$|m(K(\bar{S})+D)+\delta\mu^*H|=|(m(q-2)+\delta)H-\sum m(\nu_j-2)E_j|$$

corresponds to $|(m(q-2)+\delta)H|_{B(m)}$ we get the following

THEOREM 4. *If* $\bar{\kappa}(S)=2$, *we have*

$$\dim|(m(q-2)+\delta)H|_{B(m)}=\frac{d(d+3)}{2}-\sum\frac{(n_j+1)n_j}{2},$$

where $d=m(q-2)+\delta$ *and* $n_j=m(\nu_j-2)$.

PROOF. It suffices to show that $H^1(m(K(\bar{S})+D)+\delta H)=0$. For $m=1$, this is obvious. Assume that $K(\bar{S})+D$ is ample. Then, since $(m-1)(K(\bar{S})+D)+\delta H$ is ample for $m\geqq 2$, $H^1(m(K(\bar{S})+D)+\delta H)$ vanishes by Lemma 2. If S is a product of two curves, the same formula as above is derived by a simple computation.

§ 5. We shall prove the following

Theorem 5. *When $\bar{\kappa}(S)\geqq 0$, the linear system $|K(\bar{S})+D|$ is simply generated.*

Here we use the following terminology. Let D be a divisor on V. $|D|$ is *simply generated* if for any m the natural homomorphism

$$\otimes^m H^0(V, \mathcal{O}(D)) \longrightarrow H^0(V, \mathcal{O}(mD))$$

is surjective.

Subspace $\boldsymbol{L}\subset H^0(V, \mathcal{O}(D))$ defines a linear system and $\boldsymbol{L}_m$'s$\subset H^0(V, \mathcal{O}(mD))$ define linear systems $\Lambda_m\subset|mD|$. If $\otimes^m \boldsymbol{L}_1\rightarrow \boldsymbol{L}_m$ is surjective, then we say that Λ_m is simply generated by Λ_1.

Proof of Theorem 5. If S is a product of two curves, the assertion is easily checked. Hence, we assume that $K(\bar{S})+D$ is ample. Put $A_m=H^0(m(K(\bar{S})+D))$ and the assertion that

$$A_{m-1}\otimes A_1 \longrightarrow A_m \text{ (the linearization of the product)}$$

is surjective is denoted by S_m. We show the S_m by the induction on q. If $q=3$, then $\bar{S}=\boldsymbol{P}^2$ and $D\sim 3H$. Hence $A_m=H^0(\boldsymbol{P}^2, \mathcal{O}(mH))$ and so S_m is obvious. Assume $q\geqq 4$ and take a component Δ_q of Δ such that $S'=\boldsymbol{P}^2-\Delta_0\cup\cdots\cup\Delta_{q-1}$ is still of hyperbolic type, i.e., $\bar{\kappa}(S')=2$. And denote by $\bar{S}'$ and D_1' the standard completion of S' and its smooth boundary. Furthermore, we may assume that $K(\bar{S}')+D_1'$ is ample. Set $L=$the proper image of Δ_q in $\bar{S}$. First, we prove the vanishing property.

Proposition 6.

$$H^1(mK(\bar{S})+\alpha D+(m-\alpha)D')=0$$

for $m\geqq 1$, $\alpha\geqq 0$ and $m\geqq\alpha$, where D' is defined to be by $D=D'+L$.

Proof. Case 1: $\alpha\geqq 2$. First, we note that $K(\bar{S})+D'$ is pseudo-ample. In fact, let $\Delta_q\cap\{p_1, \cdots, p_s\}=\{p_1, \cdots, p_\varepsilon\}$. If there exists $i\in[1, \varepsilon]$ such that $\nu_i\geqq 4$, then $\bar{S}$ itself is the standard completion of S' with smooth boundary D'. Hence, $K(\bar{S})+D'$ is ample. If $\varepsilon\geqq 1$ and $\nu_1=\cdots=\nu_\varepsilon=3$, then we have a proper birational morphism $\lambda\colon \bar{S}\rightarrow\bar{S}'$ which is a completion of $S\subset S'$. $\lambda|S$ has no logarithmic ramification divisor. Hence $K(\bar{S})+D'=\lambda^*(K(\bar{S}')+\bar{D}_1')$, which is pseudo-ample. Finally, if $\varepsilon=0$, then $\bar{S}$ is regarded as a standard completion of S'. Hence $K(\bar{S})+D'$ is ample. Therefore, $(m-1)K(\bar{S})+(\alpha-1)D+(m-\alpha)D'=(\alpha-1)(K(\bar{S})+D)+(m-\alpha)(K(\bar{S})+D')$ is ample. By Lemma 2, we obtain

$$H^1(mK(\bar{S})+\alpha D+(m-\alpha)D')=0 .$$

Case 2: $\alpha=0$ or 1. We have to show that both $H^1(m(K(\bar{S})+D'))$ and $H^1(mK(\bar{S})+D+(m-1)D')$ vanish. Note that D and D' are reduced and connected curves and that by induction hypothesis $Bs|(m-1)(K(\bar{S})+D')|$ is void and $|(m-1)(K(\bar{S})+D')|$ has an irreducible member. Hence, both $|D'+(m-1)(K(\bar{S})+D')|$ and $|D+(m-1)(K(\bar{S})+D')|$ have reduced connected members, say F. By Serre duality we have $H^1(K(\bar{S})+F)=H^1(-F)$, which vanishes from the fact that $\bar{S}$ is a regular surface.

We use the following

LEMMA 3. *Let X and Y be divisors on a complete algebraic variety V and Z an effective divisor on V. We assume that* i) $H^1(X-Z)=0$, ii) $H^0(X|Z)\otimes H^0(Y)\to H^0((X+Y)|Z)$ *is surjective,* iii) *the natural map $H^0(X-Z)\otimes H^0(Y)\to H^0(X+Y-Z)$ is surjective. Then*

$$H^0(X)\otimes H^0(Y)\longrightarrow H^0(X+Y)$$

is surjective, too.

PROOF. This follows from the diagram below:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X-Z)\otimes H^0(Y) & \longrightarrow & H^0(X)\otimes H^0(Y) & \longrightarrow & H^0(X|Z)\otimes H^0(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0((X+Y)-Z) & \longrightarrow & H^0(X+Y) & \longrightarrow & H^0((X+Y)|Z) & & \\
 & & \downarrow & & \vdots\downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 . & &
\end{array}$$

N.B. ii) is replaced by the following $\text{ii})_a$ and $\text{ii})_b$

$\text{ii})_a$ $H^0(Y)\to H^0(Y|Z)$ is surjective,

$\text{ii})_b$ $H^0(X|Z)\otimes H^0(Y|Z)\to H^0((X+Y)|Z)$ is surjective.

Put $X=(m-1)K(\bar{S})+\alpha D+(m-1-\alpha)D'$, $Y=K(\bar{S})+D$ and $Z=L$. Then $X-Z=(m-1)K(\bar{S})+(\alpha-1)D+(m-\alpha)D'$. Hence, by Proposition 6, i) is verified. Next, we can verify $\text{ii})_a$ by making use of the exact sequence:

$$H^0(Y)\longrightarrow H^0(Y|Z)\longrightarrow H^1(Y-Z)=H^1(K(\bar{S})+D')=0 .$$

Since $Z=\boldsymbol{P}^1$ and $\deg(Y|Z)=(K(\bar{S})+D, L)>0$ and $\deg(X|Z)=((m-1)K(\bar{S})+$

$\alpha D+(m-1-\alpha)D', L)\geqq 0$, we check ii)$_b$. Thus ii) is satisfied.

Next, put $A_{\alpha,m-\alpha}=H^0(mK(\bar{S})+\alpha D+(m-\alpha)D')$ and let $S_{\alpha,m-\alpha}$ denote the assertion that

$$A_{\alpha-1,m-\alpha}\otimes A_{1,0}\longrightarrow A_{\alpha,m-\alpha}$$

is surjective, where $\alpha\geqq 1$.

In view of Lemma 3, we obtain the implication

$$S_{\alpha,m-\alpha}\Longrightarrow S_{\alpha+1,m-\alpha-1}\,.$$

$S_{m,0}$ is the desired assertion.

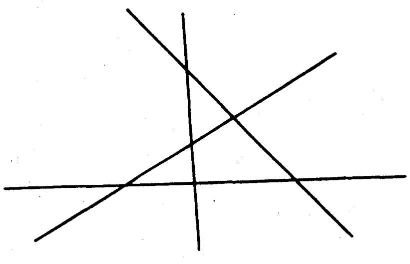
We have to prove $S_{1,m-1}$, in other words, the surjectivity of

$$A_{0,m-1}\otimes A_{1,0}\longrightarrow A_{1,m-1}\,.$$

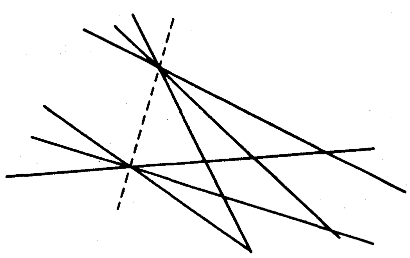
Setting $X=K(\bar{S})+D$, $Z=L$ and $Y=(m-1)(K(\bar{S})+D')$, we use the lemma. Actually, $H^1(X-Z)=H^1(K(\bar{S})+D')=H^1(-D')=0$ and $H^0(X-Z)=A_{0,1}$. By $H^0(mK(\bar{S})+mD')=H^0(mK(\bar{S}')+mD')$, $K(\bar{S})+D'$ is simply generated. Thus the condition iii) of Lemma 3 is checked. Therefore, by Lemma 3 we establish the surjectivity. Q.E.D.

Theorem II follows from Theorem 5 immediately.

§ **6.** For our surface S, S is measure-hyperbolic, if and only if S is of hyperbolic type, that is, $\bar{\kappa}(S)=2$. Next we shall discuss when S is hyperbolic. In order to state the result we introduce the type $\Pi'_{a,b}$ of Δ as follows;



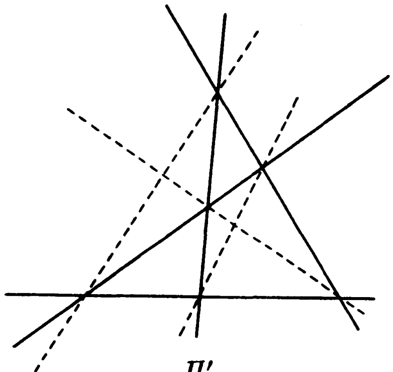
$\Pi'_{2,2}$



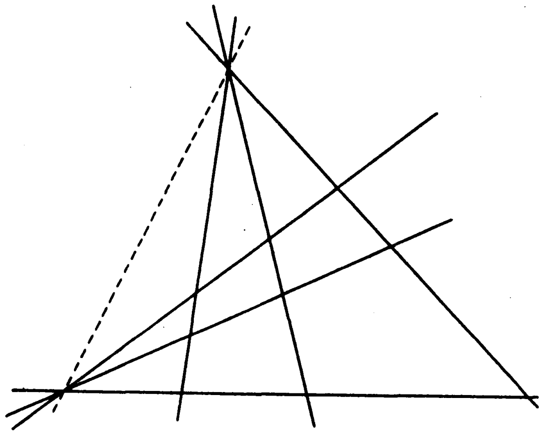
$\Pi'_{a,b}$

Note that $\Pi'_{a,b}$ is obtained from $\Pi_{a+1,b+1}$ by removing the line connecting 2 multiple points. $\Pi'_{2,2}$ represents Δ consisting of four lines in general position.

THEOREM 6. *Let Γ be a curve on S with $\bar{\kappa}(S)=2$ such that $\bar{\kappa}(\Gamma)=0$. Then it is one of the following lines dotted in the following figures, where $a+b\geqq 5$.*



$\Pi'_{2,2}$



$\Pi'_{a,b}$

PROOF. It suffices to prove for Δ of type $\Pi'_{2,2}$. The S is isomorphic to Spec $\boldsymbol{C}[x, y, x^{-1}, y^{-1}, (x+y+1)^{-1}]$. As Γ lies over $\boldsymbol{C}^* \times \boldsymbol{C}^*$, Γ turns out to be a factor of $\boldsymbol{C}^* \times \boldsymbol{C}^*$, namely, $\Gamma \times \boldsymbol{C}^* \simeq \boldsymbol{C}^* \times \boldsymbol{C}^*$ by a corrolary to Theorem 4 [5]. Hence, there are new variables u, v such that $x = au^\alpha v^\beta$, $y = bu^\gamma v^\delta$ where $\alpha\delta - \beta\gamma = \pm 1$ and $a, b \neq 0$ and such that Γ is defined by $u = 1$. S is defined by

$$au^\alpha v^\beta + bu^\gamma v^\delta \neq 1$$

on $\boldsymbol{C}^* \times \boldsymbol{C}^*$. Recalling that $\Gamma \subset S$, we have

$$av^\beta + bv^\delta \neq 1 \qquad \text{whenever} \quad v \neq 0 \quad \text{or} \quad \infty .$$

Thus,

$$av^\beta + bv^\delta = 1 + cv^\varepsilon$$

for some $c \neq 0$ and $\varepsilon \in \boldsymbol{Z}$. Then we have the following three cases.

Cace 1: $\beta = 0$ and $\delta \neq 0$. Then $a = 1$, from $\alpha\delta - \beta\gamma = \pm 1$ it follows that $\alpha = \pm 1$. Hence, $u - 1 = x^{\pm 1} - 1$. This implies that Γ is defined by $x = 1$.

Case 2: $\delta = 0$ and $\beta \neq 0$. Then $b = 1$ and by the same argument as above we conclude that Γ is defined by $y = 1$.

Case 3: $\beta = \delta$, $a + b = 0$, $\varepsilon = 0$, and $c = -1$. Then $\beta = \delta = \pm 1$ from $\alpha\delta - \beta\gamma = \pm 1$. In view of the inverse transformation $u^{\pm 1} = (x/a)^\delta (y/b)^{-\beta}$, we get $y = -x$ from $u = 1$.

Accordingly, Γ is defined by $x = 1$ or $y = 1$ or $x = -y$. Q.E.D.

It is very probable that S is hyperbolic if and only if $\bar{\kappa}(S) = 2$ and the type of Δ is not $\Pi'_{,b}$.

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