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On Lattice Isomorphisms of $C(X)^+$

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Introduction.

This paper originates from a question raised by Professor J. J. Schäffer during the meeting on Banach spaces at Kent State University in August, 1977. The question is this: Let X and Y be compact Hausdorff spaces, let $C(X)^+$(resp. $C(Y)^+$) be the lattice of non-negative continuous functions on X(resp. Y), and let T be a lattice isomorphism of $C(X)^+$ onto $C(Y)^+$; does T preserve the strict inequality $<$? Here, for f and g in $C(X)^+$, $f<g$ means that $f(x)<g(x)$ for each x in X. By Kaplansky's theorem [7], if $C(X)^+$ and $C(Y)^+$ are lattice isomorphic, then X and Y are homeomorphic, and so we may assume that $X=Y$. It turns out that the answer to Schäffer's question depends on the space X, and the rather unexpected result is: Each lattice isomorphism of $C(X)^+$ onto itself preserves the strict inequality if and only if X is not the Stone-Čech compactification of a non-compact, σ-compact, locally compact Hausdorff space. If X satisfies this condition, we say that the space X has property (S). (The reason for our choice of the letter "S" should, by now, be clear.) Professor E. Hewitt then started to ask us questions concerning the case where X and Y are not assumed to be compact. Then we can no longer assume that $X=Y$, and the answer to Schäffer's question (as generalized by Hewitt) depends on the topological properties of X and Y. Property (S), suitably generalized, again plays the central role. The purpose of the present paper is to present the answers to the questions of Schäffer and Hewitt, to investigate related questions, and to establish further properties of spaces with (S).

The paper is organized as follows: Section 1 contains characterizations of those compact Hausdorff spaces X such that each lattice isomorphism $C(X)^+\to C(X)^+$ preserves the strict inequality. The proofs are relatively simple and transparent.

Section 2 contains generalizations of the results of Section 1 to non-

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compact spaces. The lack of compactness introduces various complications which unfortunately render the section somewhat opaque. (The reader who is not interested in the non-compact case can proceed from Section 1 directly to Section 4.) More specifically, we first assume that, for Tychonoff spaces X and Y, there exists a lattice isomorphism $C(X)^+ \to C(Y)^+$ that does not preserve the strict inequality, and from this assumption we deduce certain topological properties of X and Y. These properties are subsequently shown to be sufficient for the existence of non-strict lattice isomorphisms. For the convenience of organization, property (S) is defined purely topologically, and it is shown that each lattice isomorphism of $C(X)^+$ onto itself is strict if and only if X has property (S). One indispensable tool in our analysis is the isomorphism σ of Boolean algebra of regular open subsets of X onto that of Y that corresponds to a lattice isomorphism $C(X)^+ \to C(Y)^+$. The construction of σ and the derivation of a few of the properties of σ that are needed in Section 2 are carried out in the beginning of the section. Additional properties of σ are established in Section 3. For instance, it is shown that σ can be described by a continuous one-to-one map of X into the real-compactification υY of Y. From this it is a simple matter to recover Shirota's theorem [10]: If X and Y are realcompact and if $C(X)^+$ and $C(Y)^+$ are lattice isomorphic, then X and Y are homeomorphic. This is a generalization of the theorem of Kaplansky mentioned above.

In Section 4, we consider the class of topological spaces with property (S). The product of a family of spaces with property (S) has again property (S). The notion of weakly sequential spaces is introduced, and it is proved that a weakly sequential space has property (S) and that the product of a family of first-countable spaces is weakly sequential. Although a quotient of a space with property (S) need not have property (S), a quotient of weakly sequential space is again weakly sequential. Finally, we give necessary and sufficient conditions for Y to satisfy the following: $X \times Y$ has property (S) for each Tychonoff space X.

We are grateful to Professor J. J. Schäffer for raising the original question and for the subsequent correspondence on the subject, and to Professor E. Hewitt for encouraging us to consider the non-compact case. We also acknowledge with gratitude Professors W. W. Comfort, E. van Douwen, and E. Michael for enlightening us with topological information concerning property (S) and for giving us permission to incorporate some of their remarks into the present paper.

§ 1. Lattice isomorphisms (the compact case).

Let X and Y be compact Hausdorff spaces, and let $C(X)^+$ and $C(Y)^+$ denote the lattices of all non-negative continuous real-valued functions on X and Y respectively. A map $T: C(X)^+ \to C(Y)^+$ is called a *lattice isomorphism* if it is one-to-one, onto, and, for f and g in $C(X)^+$, $Tf \leqq Tg$ if and only if $f \leqq g$. The last condition is equivalent to $T(f \wedge g) = T(f) \wedge T(g)$ (or $T(f \vee g) = T(f) \vee T(g)$) for arbitrary f and g in $C(X)^+$. Clearly the inverse of a lattice isomorphism is also a lattice isomorphism. A lattice isomorphism $T: C(X)^+ \to C(Y)^+$ is said to be *strict* if $Tf < Tg$ whenever $f < g$, where $f < g$ means that $f(x) < g(x)$ for each x in X. For a real-valued function f on X, the *support* of f (denoted by $\operatorname{supp} f$) is defined by $\{x: f(x) \neq 0\}^-$.

LEMMA 1.1. *Let T be a lattice isomorphism $C(X)^+ \to C(Y)^+$, where X and Y are compact Hausdorff spaces. Then for each y in Y, there is a unique point $\rho(y)$ in X such that $Tf(y) = 0$ whenever $f \in C(X)^+$ and $\rho(y) \notin \operatorname{supp} f$. Moreover, the map $y \mapsto \rho(y)$ is continuous.*

PROOF. Fix a point y in Y, and let $\mathscr{U}$ be the family of all open subsets U of X such that $Tf(y) = 0$ whenever $f \in C(X)^+$ and $\operatorname{supp} f \subset U$. Then $\mathscr{U}$ is closed under finite unions. In fact, suppose that $U_1, U_2 \in \mathscr{U}$ and $\operatorname{supp} f \subset U_1 \cup U_2$ for some f in $C(X)^+$. Then there are f_1, f_2 in $C(X)^+$ such that $\operatorname{supp} f_i \subset U_i (i=1, 2)$ and $f = f_1 \vee f_2$. By the definition of $\mathscr{U}$, $Tf_i(y) = 0$ for $i = 1, 2$, whence $Tf(y) = T(f_1 \vee f_2)(y) = (Tf_1 \vee Tf_2)(y) = 0$. This shows that $U_1 \cup U_2 \in \mathscr{U}$. Let $V = \bigcup \mathscr{U}$. If $\operatorname{supp} f \subset V$, then $\operatorname{supp} f \subset U_1 \cup \cdots \cup U_n$ where $U_i \in \mathscr{U} (i = 1, \cdots, n)$. Since $U_1 \cup \cdots \cup U_n \in \mathscr{U}$, we have $Tf(y) = 0$. Hence $V \in \mathscr{U}$. If $V = X$, then $Tf(y) = 0$ for all f in $C(X)^+$ in contradiction to the fact that T is onto. Hence $V \neq X$. Suppose that $X \sim V$ contains two distinct points x_1 and x_2. Then there are disjoint open neighborhoods W_1 and W_2 of x_1 and x_2 respectively. Since $W_1 \notin \mathscr{U}$, there is a g_1 in $C(X)^+$ such that $\operatorname{supp} g_1 \subset W_1$ and $Tg_1(y) > 0$. Similarly, there is a g_2 in $C(X)^+$ such that $\operatorname{supp} g_2 \subset W_2$ and $Tg_2(y) > 0$. Since $g_1 \wedge g_2 = 0$, $0 = T(g_1 \wedge g_2)(y) = (Tg_1 \wedge Tg_2)(y) > 0$. This contradiction shows that $X \sim V$ is a singleton. Let $\rho(y)$ be the unique element in $X \sim V$, then $\rho(y)$ satisfies the condition of the lemma. Suppose that a point x in X also satisfies the condition. Then $X \sim \{x\} \in \mathscr{U}$, and hence $X \sim \{x\} \subset V = X \sim \{\rho(y)\}$. Therefore $x = \rho(y)$. To see the continuity of ρ, let W be an open neighborhood of $\rho(y)$. Then there exists an h in $C(X)^+$ such that $\operatorname{supp} h \subset W$ and $Th(y) > 0$. Let $U = \{z: z \in Y \text{ and } Th(z) > 0\}$. Then U is a neighborhood of y and $\rho[U] \subset W$.

We call the map $\rho: Y \to X$ in Lemma 1.1 the *map associated* with the lattice isomorphism $T: C(X)^+ \to C(Y)^+$.

LEMMA 1.2. *Let T and ρ be as in Lemma 1.1. Then ρ is a homeomorphism, and $\rho^{-1}: X \to Y$ is the map associated with $T^{-1}: C(Y)^+ \to C(X)^+$.*

PROOF. Let $\tilde{\rho}$ be the map associated with T^{-1}. Suppose that $\rho\tilde{\rho}(x) \notin \operatorname{supp} f$ for some x in X and f in $C(X)^+$. Then there is an open neighborhood V of $\tilde{\rho}(x)$ in Y such that $\rho[V] \cap \operatorname{supp} f = \varnothing$. By the definition of ρ, we then have $Tf \mid V \equiv 0$ and, consequently, $\tilde{\rho}(x) \notin \operatorname{supp}(Tf)$. Since $\tilde{\rho}$ is associated with T^{-1}, $0 = T^{-1}T(f)(x) = f(x)$. It follows that $\rho\tilde{\rho}$ is associated with the identity: $C(X)^+ \to C(X)^+$, and consequently $\rho\tilde{\rho} = \mathrm{id}$. Reversing the rôles of T and T^{-1}, we see also that $\tilde{\rho}\rho = \mathrm{id}$. Hence ρ is a homeomorphism and $\tilde{\rho} = \rho^{-1}$.

It follows from Lemma 1.2 that, if X and Y are compact Hausdorff spaces such that $C(X)^+$ and $C(Y)^+$ are isomorphic as lattices, then X and Y are homeomorphic. This was proved by Kaplansky [7] and was subsequently generalized by Shirota [10] (cf. Remark 3.8(*b*) below). Since our analysis of T would yield Kaplansky's theorem automatically, we began this section with two compact Hausdorff spaces X and Y. In the rest of the section, however, we only consider lattice isomorphisms of $C(X)^+$ onto itself.

LEMMA 1.3. *Let T and ρ be as in Lemma 1.1 with $X = Y$. If f and g are elements of $C(X)^+$ such that $f(x) \leqq g(x)$ for each x in an open subset U of X, then $Tf(y) \leqq Tg(y)$ for each y in $\rho^{-1}[U]$. In particular, $f \mid U = g \mid U$ implies $Tf \mid \rho^{-1}[U] = Tg \mid \rho^{-1}[U]$.*

PROOF. Let h be a member of $C(X)^+$ such that $\operatorname{supp} h \subset \rho^{-1}[U]$. Then, for each x in $X \sim U$, $\rho^{-1}(x) \notin \operatorname{supp} h$ and hence $T^{-1}h(x) = 0$. It follows that $f \wedge T^{-1}h \leqq g \wedge T^{-1}h$, whence $h \wedge Tf \leqq h \wedge Tg$. Therefore $Tf \leqq Tg$ on $\rho^{-1}[U]$.

Let U be an open subset of X. Then by $C^*(U)^+$ we shall denote the lattice of all bounded continuous non-negative functions on U.

LEMMA 1.4. *Let T and ρ be as in Lemma 1.1 with $X = Y$, let U be an open subset of X, and let $V = \rho^{-1}[U]$. Then there is a lattice isomorphism $T_U: C^*(U)^+ \to C^*(V)^+$ such that, for each f in $C(X)^+$, $T_U(f \mid U) = Tf \mid V$.*

PROOF. Given $g \in C^*(U)^+$, we define $T_U(g)$ as follows: Let $x \in U$ and let W be an open neighborhood of x such that $W^- \subset U$. Then $g \mid W^-$ can be extended to an element h in $C(X)^+$. Define $T_U(g)(\rho^{-1}(x)) = T(h)(\rho^{-1}(x))$.

Then by Lemma 1.3, $T_U(g)(\rho^{-1}(x))$ is well-defined. Since $T_U(g)$ and $T(h)$ agree on $\rho^{-1}[W]$, $T_U(g)$ is continuous at $\rho^{-1}(x)$. From the definition it is clear that $T_U(f|U)=Tf|V$ for each f in $C(X)^+$. Also in view of Lemma 1.3, T_U preserves the order relation $\leqq$. Consequently, $T_U(g)\in C^*(V)^+$ whenever $g\in C^*(U)^+$. Finally by applying the same construction to T^{-1} and V, we see immediately that $(T^{-1})_V T_U=\mathrm{id}$. Similarly $T_U(T^{-1})_V=\mathrm{id}$. Therefore T_U is a lattice isomorphism.

The next theorem is the main theorem of this section. This theorem will be generalized in the next section with a considerably more complicated proof.

THEOREM 1.5. *Let X be a compact Hausdorff space. Then the following conditions are equivalent*:

(1) *Each lattice isomorphism of $C(X)^+$ onto itself is strict.*

(2) *X is not the Stone-Čech compactification of a non-compact, locally compact, σ-compact Hausdorff space.*

(3) *X is not the Stone-Čech compactification of a non-pseudo-compact Tychonoff space.*

PROOF. (1)$\Rightarrow$(2). Suppose that $X=\beta U$ where U is a non-compact, locally compact, σ-compact subspace of X. Then U is a proper dense open F_σ-set in X. Therefore, there is a ϕ in $C(X)^+$ such that $U=\{x: \phi(x)>0\}$. We define a map $T_0: C^*(U)^+\to C^*(U)^+$ as follows:

$$T_0(f)(x)=\begin{cases} f(x) & \text{if } f(x)\geqq 1 \\ (f(x))^{\phi(x)} & \text{if } f(x)\leqq 1 \end{cases} \quad (f\in C^*(U)^+,\ x\in U)\ .$$

The inverse T_0^{-1} is given by a similar formula:

$$T_0^{-1}(g)(x)=\begin{cases} g(x) & \text{if } g(x)\geqq 1 \\ (g(x))^{1/\phi(x)} & \text{if } g(x)\leqq 1 \end{cases} \quad (g\in C^*(U)^+,\ x\in U)\ .$$

Hence T_0 is a lattice isomorphism. The restriction map $r: C(X)^+\to C^*(U)^+$ is a lattice isomorphism. Hence we can define a lattice isomorphism $T: C(X)^+\to C(X)^+$ by $T=r^{-1}T_0 r$. Clearly, $T(1)=1$ and $T(1/2)(x)=(1/2)^{\phi(x)}$ for x in X. Since $\{x: \phi(x)=0\}$ is not empty, $T(1/2)<T(1)$ is false. Thus, if X does not satisfy condition (2), then there is a non-strict lattice isomorphism of $C(X)^+$ onto itself.

(2)$\Rightarrow$(3). Suppose that Y is a non-pseudo-compact Tychonoff space and $X=\beta Y$. Then there is a bounded continuous real-valued function f on Y such that $f(y)>0$ for each y in Y and $\inf\{f(y): y\in Y\}=0$. Let $\bar{f}$ be the continuous extension of f to X, and let $U=\{x: \bar{f}(x)>0\}$. Then

U is a proper dense open F_σ-set in X, and $X=\beta U$ since $Y\subset U\subset \beta Y=X$. Hence X does not satisfy condition (2).

(3)$\Rightarrow$(1). Assume that X does not satisfy condition (1). Then there is a lattice isomorphism $T: C(X)^+\rightarrow C(X)^+$ which is not strict. Let f, g be members of $C(X)^+$ such that $f<g$ but $Tf<Tg$ is false. Let $V=\{x: Tf(x)<Tg(x)\}$. Then V is a proper open F_σ-subset of X. Let $\rho: X\rightarrow X$ be the map associated with T, and let $U=\rho[V]$. Then, since ρ is a homeomorphism, U is a proper open F_σ-subset of X. We claim that V and, hence, U are dense in X. For, otherwise, there would be a non-void open set W such that $W\cap V=\emptyset$. Then $Tf|W=Tg|W$, and therefore $f|\rho[W]=g|\rho[W]$ by Lemma 1.3 (applied to T^{-1}). This contradicts the assumption that $f<g$. We now show that $X=\beta U$. Let $h\in C^*(U)^+$. Then there is a positive number $\varepsilon>0$ such that $f|U\leqq\varepsilon h+f|U\leqq g|U$. If $\varepsilon h+f|U$ can be extended to a continuous function k on X, then $\varepsilon^{-1}(k-f)$ is a continuous extension of h to X. Hence for the purpose of extending h continuously to X, we may assume that $f\leqq h\leqq g$ on U. Then by Lemma 1.4, $T_U(h)\in C^*(V)^+$ and $Tf\leqq T_U(h)\leqq Tg$ on V. Consequently, if we extend $T_U(h)$ to a function l on X in such a way that l agrees with Tf and Tg outside V, then l is continuous and $l\in C^*(X)^+$. Let $\bar{h}=T^{-1}(l)$. Then by Lemma 1.4, $T_U(h)=l|V=T(\bar{h})|V=T_U(\bar{h}|U)$. Since T_U is one-to-one, we have $h=\bar{h}|U$. Hence h admits a continuous extension to X. This shows that $X=\beta U$, and, since U is clearly not pseudo-compact, X does not satisfy condition (3).

In §2, condition (2) of Theorem 1.5 is generalized to an arbitrary Tychonoff space and is called *property* (S).

§2. Lattice isomorphisms (the general case).

In this section we generalize the results of §1 to Tychonoff (i.e., completely regular and T_1) spaces. Although the spirit of the proof for the general case remains the same as that of the compact case, the detail becomes far more complicated.

One of the reasons for the complication is that, for an arbitrary Tychonoff space X, we must distinguish $C(X)^+$ (the lattice of all continuous non-negative functions on X) from $C^*(X)^+$ (the lattice of all bounded continuous non-negative functions on X). For Tychonoff spaces X and Y, a *lattice isomorphism* $T: C(X)^+\rightarrow C(Y)^+$ or $T: C^*(X)^+\rightarrow C^*(Y)^+$ is defined in the same way as in the compact case (§1). As before, a lattice isomorphism is said to be *strict* if it preserves the strict inequality $<$.

In the general case, it is not always possible to construct the map associated with a lattice isomorphism. The following lemma provides a substitute in the form of an isomorphism of Boolean algebras of regular open sets. An open subset U of a topological space X is called *regular* if $U=U^{-0}$. Here A^0 denotes the interior of the set A. We note that if A is closed then A^0 is regular. Let $\mathscr{R}(X)$ denote the family of all regular open subsets of X. It is well-known that $\mathscr{R}(X)$ is a complete Boolean algebra [1, p. 216]. The Boolean operations are given as follows: For $\mathscr{A}\subset\mathscr{R}(X)$, $\vee\mathscr{A}=(\cup\mathscr{A})^{-0}$ and $\wedge\mathscr{A}=(\cap\mathscr{A})^{-0}$; and for $U\in\mathscr{R}(X)$, $U'=X\sim U^-$. If $U, V\in\mathscr{R}(X)$, then $U\cap V\in\mathscr{R}(x)$ and, therefore, $U\wedge V=U\cap V$. The partial ordering of $\mathscr{R}(X)$ is given by the inclusion: $U\leqq V$ if and only if $U\subset V$.

Let X be a Tychonoff space. For an f in $C(X)^+$, let $Z(f)=\{x: f(x)=0\}$. A subset of X of the form $Z(f)$ is called a *zero-set* in X. For a regular open subset U of X, let $I_U=\{f: f\in C(X)^+ \text{ and } U\subset Z(f)\}$. A similar object can be defined for $C^*(X)^+$, namely $I_U\cap C^*(X)^+$, which we continue to denote by I_U. We note that, for $U, V\in\mathscr{R}(X)$, $V\subset U$ if and only if $I_U\subset I_V$. We also note that $f\in I_{U'}$ if and only if $\{x: f(x)>0\}\subset U$.

THEOREM 2.1. *Let X and Y be Tychonoff spaces and let $T: C(X)^+\to C(Y)^+$ (or $T: C^*(X)^+\to C^*(Y)^+$) be a lattice isomorphism. Then there is a corresponding isomorphism $\sigma: \mathscr{R}(X)\to\mathscr{R}(Y)$ of Boolean algebras such that $T[I_U]=I_{\sigma(U)}$ for each U in $\mathscr{R}(X)$. The inverse isomorphism $\sigma^{-1}: \mathscr{R}(Y)\to\mathscr{R}(X)$ corresponds to T^{-1}.*

PROOF. We give a proof only for $T: C(X)^+\to C(Y)^+$. The proof of the other case is essentially identical.

For U in $\mathscr{R}(X)$, let $\sigma(U)=(\cap\{Z(Tf): f\in I_U\})^0$. Then clearly $T[I_U]\subset I_{\sigma(U)}$ and $\sigma(V)\subset\sigma(U)$ whenever $V\subset U$. If $f\in I_U$ and $g\in I_{U'}$, then $T(f)\wedge T(g)=T(f\wedge g)=T(0)=0$, or equivalently $Z(Tf)\cup Z(Tg)=Y$. It follows that $(\cap\{Z(Tf): f\in I_U\})\cup(\cap\{Z(Tg): g\in I_{U'}\})=Y$ and hence $\sigma(U)\cup(\cap\{Z(Tg): g\in I_{U'}\})=Y$. Therefore $\sigma(U')\supset(Y\sim\sigma(U))^0=Y\sim\sigma(U)^-=\sigma(U)'$. Suppose that $f\notin I_U$. Then there is a g in $I_{U'}$ such that $0\neq g\leqq f$. Then $Tg\in I_{\sigma(U')}\subset I_{\sigma(U)'}$ and $0\neq Tg\leqq Tf$. Therefore $Tf\notin I_{\sigma(U)}$. This shows that $T[I_U]=I_{\sigma(U)}$.

Since $T^{-1}: C(Y)^+\to C(X)^+$ is a lattice isomorphism, it follows from what is already proved that there exists an order preserving map $\bar{\sigma}: \mathscr{R}(Y)\to\mathscr{R}(X)$ such that $T^{-1}[I_V]=I_{\bar{\sigma}(V)}$ for each $V\in\mathscr{R}(Y)$. Then for such V, $I_V=TT^{-1}[I_V]=T[I_{\bar{\sigma}(V)}]=I_{\sigma\bar{\sigma}(V)}$. Hence $V=\sigma\bar{\sigma}(V)$ for each V in $\mathscr{R}(Y)$, i.e., $\sigma\bar{\sigma}=\text{id}$. Similarly $\bar{\sigma}\sigma=\text{id}$. Consequently σ is an isomorphism of partially ordered sets and, hence, of Boolean algebras, and $\bar{\sigma}=\sigma^{-1}$. This completes the proof.

REMARK 2.2. The isomorphism σ constructed in Theorem 2.1 necessarily preserves all the Boolean operations. In particular, if $U, V \in \mathscr{R}(X)$, then $\sigma(U')=\sigma(U)'$ and $\sigma(U\cap V)=\sigma(U\wedge V)=\sigma(U)\wedge\sigma(V)=\sigma(U)\cap\sigma(V)$. Let $\mathscr{A}$ be a subfamily of $\mathscr{R}(X)$ such that $\cup\mathscr{A}$ is dense in X. Then $\vee\mathscr{A}=X$, and therefore $Y=\sigma(X)=\vee\{\sigma(U): U\in\mathscr{A}\}$, i.e., $\cup\{\sigma(U): U\in\mathscr{A}\}$ is dense in Y.

In the following lemma, the notation is that of Theorem 2.1.

LEMMA 2.3. *Let f and g be members of $C(X)^+$ (or $C^*(X)^+$) such that $f(x)\leqq g(x)$ for each point x in U, where $U\in\mathscr{R}(X)$. Then $Tf(y)\leqq Tg(y)$ for each y in $\sigma(U)$.*

PROOF. Again the proofs for two cases are identical. Let h be an arbitrary function in $C^*(X)^+$ such that $\{y: h(y)>0\}\subset\sigma(U)$. Then $h\in I_{\sigma(U)'}$, and therefore $T^{-1}(h)\in I_{U'}$ by Theorem 2.1. By hypothesis, $f\wedge T^{-1}(h)\leqq g\wedge T^{-1}(h)$. It follows that $(Tf)\wedge h=T(f\wedge T^{-1}(h))\leqq T(g\wedge T^{-1}(h))=(Tg)\wedge h$. The conclusion now follows.

A subset U of a Tychonoff space X is said to be *C^*-imbedded* in X if each bounded continuous real-valued function on U admits a continuous real-valued extension to X. Clearly this is the case if and only if the restriction map $C^*(X)^+\to C^*(U)^+$ is onto.

LEMMA 2.4. *Let U be a C^*-embedded subspace of a Tychonoff space X. If $f\in C(U)^+$ and if $f\leqq g$ on U for some g in $C(X)^+$, then f can be extended to a function in $C(X)^+$.*

PROOF. Let $h=f/(g+1)$; then $h\in C^*(U)^+$. Let $\bar{h}$ be a continuous non-negative extension of h to X. Then $(g+1)\bar{h}$ extends f and is a member of $C(X)^+$.

A subset U of a Tychonoff space X is called a *cozero-set* if $X\sim U$ is a zero-set. A Tychonoff space X is said to have property (S) if there is no dense proper cozero-set that is C^*-embedded in X. If X is compact Hausdorff, then property (S) is equivalent to condition (2) (hence, to each of conditions (1), (2), and (3)) of Theorem 1.5. Corollary 2.6 below generalizes Theorem 1.5 to Tychonoff spaces. The next two theorems are the main results of this section.

THEOREM 2.5. *Let X and Y be Tychonoff spaces. If there exists a lattice isomorphism $T: C(X)^+\to C(Y)^+$ which is not strict, then:*

(i) X does not have property (S); and

(ii) Y does not have property (S).

THEOREM 2.5*. *Let X and Y be Tychonoff spaces. If there exists a lattice isomorphism $T: C^*(X)^+\to C^*(Y)^+$ which is not strict, then:*

(i) *Either X does not have property (S) or X is not pseudo-compact; and*

(ii) *Y does not have property (S).*

PROOF. We shall give a proof for Theorem 2.5*. Modifications for the proof of Theorem 2.5, if necessary, will be given in parentheses.

We break the proof into four steps.

I. We first prove that, by modifying the given T, we obtain a lattice isomorphism $\tilde{T}: C^*(X)^+ \to C^*(Y)^+$ such that $\tilde{T}1(y)=0$ for some y in Y. By hypothesis there exist elements f and g of $C^*(X)^+$ such that $f<g$ and such that $Tf<Tg$ is false. We may and do assume that $f=0$, because the map $h \mapsto T(f+h)-T(f)$ is a lattice isomorphism $C^*(X)^+ \to C^*(Y)^+$, which maps $g-f$ to $Tg-Tf$. Let M be a positive number such that $g<M$, and let $\theta(x)=(\log M-\log g(x))/\log 2$. Define $T_1: C^*(X)^+ \to C^*(X)^+$ by

$$T_1(h)(x)=\begin{cases}\dfrac{1}{2}Mh(x) & \text{if} \quad h(x)\geqq 2\,,\\ M\Big(\dfrac{1}{2}h(x)\Big)^{\theta(x)} & \text{if} \quad h(x)\leqq 2\,.\end{cases}$$

Then it is easy to check that T_1 is a lattice isomorphism such that $T_1(1)=g$. (For the case of Theorem 2.5, define $T_1: C(X)^+ \to C(X)^+$ by $T_1(h)=gh$ for each h in $C(X)^+$.) Hence $\tilde{T}=TT_1$ is a lattice isomorphism such that $\tilde{T}(1)=T(g)$, and therefore $\tilde{T}(1)(y)=0$ for some y in Y.

II. By step I, we may assume that the isomorphism T of the theorem satisfies $T1(y)=0$ for some y in Y. Let $\sigma: \mathscr{R}(X) \to \mathscr{R}(Y)$ be the isomorphism of Boolean algebras that corresponds to T, and let N denote the set of positive integers. For each n in N, let $W_n=\{y: T1(y)\geqq n^{-1}\}^0$ and $V_n=\sigma^{-1}(W_n)$, and let $W=\cup\{W_n: n\in N\}=Y\sim Z(T1)$ and $V=\cup\{V_n: n\in N\}$. Since $W_n\subset W_{n+1}$ and $W_n'\neq\varnothing$ for each n, it follows that $V_n\subset V_{n+1}$ and $V_n'\neq\varnothing$. The interior of $Z(T1)$ is empty. For, otherwise, there would be a non-void regular open set U such that $U\subset Z(T1)$. Then, by Lemma 2.3, $1=0$ on $\sigma^{-1}(U)$, which is absurd. Therefore W is dense in Y, and consequently V is dense in X by Remark 2.2.

For each n, by forming the composite of $T1$ and a suitable continuous function on the real line, we obtain a continuous function $\psi_n: Y\to[0, 1]$ such that

$$\psi_n(y)=1 \quad \text{if} \quad T1(y)\geqq n^{-1}\,, \quad \text{and}$$
$$\psi_n(y)=0 \quad \text{if} \quad T1(y)\leqq (n+1)^{-1}\,.$$

Let $g_n=\psi_n\cdot T1$. Then, for $m\geqq n$, $g_m|W_n=T1|W_n$. Also $0\leqq g_n\leqq T1$ and $g_n\in I_{W'_{n+1}}$ for each n. Let $\phi_n=T^{-1}g_n$. Then, from Theorem 2.1 and Lemma

2.3, we see that:

(a) $\phi_m | V_n \equiv 1$ for $m \geqq n$;

(b) $0 \leqq \phi_n \leqq 1$ for each n; and

(c) $\phi_n \in I_{V'_{n+1}}$, i.e., $\{x: \phi_n(x) > 0\} \subset V_{n+1}$ for each n.

Finally, let $\phi = \sum \{2^{-n}\phi_n: n \in N\}$. By ($b$), ϕ is well-defined and is continuous. It follows from (a) and (c) that

(d) $V = \{x: \phi(x) > 0\}$.

If $x \in V'_m$, then $\phi(x) \leqq \sum \{2^{-n}: n \geqq m\} = 2 \cdot 2^{-m}$ from (c). Since, as noted above, $V'_m \neq \phi$ for each m, we also have

(e) $\inf \{\phi(x): x \in X\} = 0$.

III. Now we distinguish two cases according to $V = X$ or $V \neq X$.

Case 1. $V = X$. In this case, the existence of the function ϕ satisfying (d) and (e) shows that X is not pseudo-compact. (We show in step IV that Case 1 cannot occur for Theorem 2.5.)

Case 2. $V \neq X$. In this case, V is a proper dense cozero-set. We shall prove that V is C^*-embedded in X. Let f be an arbitrary element of $C^*(V)^+$ such that $0 \leqq f \leqq 1$. For each n in N, let f_n be the element of $C^*(X)^+$ defined as follows:

$$f_n | V = f \cdot \phi_n | V \quad \text{and} \quad f_n \equiv 0 \quad \text{on} \quad X \sim V .$$

Then, from (a), $f_m | V_n = f_n | V_n = f | V_n$ for $m \geqq n$, and, from (b),

$$0 \leqq f_n \leqq 1 .$$

Therefore, by Lemma 2.3, $Tf_m | W_n = Tf_n | W_n$ for $m \geqq n$. Furthermore, since $0 \leqq Tf_n \leqq T1$, $Tf_n \equiv 0$ on $Z(T1) = Y \sim W$. It follows that $h(y) = \lim_n Tf_n(y)$ exists for all y in Y. Clearly, $h | W_n = Tf_n | W_n$ and $0 \leqq h \leqq T1$. Consequently $h \in C^*(Y)^+$. (In case of Theorem 2.5, $h \in C(Y)^+$.) Let $\bar{f} = T^{-1}(h)$. Then, by Lemma 2.3 again, $\bar{f} | V_n = f_n | V_n = f | V_n$ for each n, i.e., $\bar{f} | V = f$. This proves that V is C^*-embedded in X, and, therefore, X does not have property (S).

IV. Finally we show that the proper dense cozero-set W is C^*-embedded in Y. (We shall also rule out Case 1 of part III for Theorem 2.5.) This will complete the proof.

Let h be an arbitrary member of $C(W)^+$. For each n, define a function h_n on Y as follows:

$$h_n | W = h \cdot \psi_n | W \quad \text{and} \quad h_n \equiv 0 \quad \text{on} \quad Y \sim W .$$

From the properties of $\{\psi_n\}$ (see part II), it follows that h_n is continuous and $h_m | W_n = h_n | W_n = h | W_n$ for $m \geqq n$.

(Consider the case of Theorem 2.5. Since $T^{-1}(h_m)|V_n = T^{-1}(h_n)|V_n$ for $m \geqq n$, $f(x) = \lim_n T^{-1}(h_n)(x)$ exists for each x in V and $f|V_n = T^{-1}(h_n)|V_n$ for each n. Hence f is continuous on V. Assume now that $V = X$. Then $f \in C(X)^+$, and $Tf|W_n = h_n|W_n = h|W_n$ for each n, i.e., $Tf|W = h$. This shows that each function in $C(W)^+$ can be extended to a continuous function on Y. This is absurd, because the function $y \mapsto (T1(y))^{-1}$ on W cannot be extended continuously to Y. This proves that Case 1 cannot occur for Theorem 2.5.)

Now assume that the h satisfies $h \leqq 1$. Then clearly $0 \leqq h_n \leqq 1$ for each n. As in the last paragraph, $f(x) = \lim_n T^{-1}(h_n)(x)$ exists for each x in V, f is continuous on V and $0 \leqq f \leqq T^{-1}(1)|V$. Since $f \in C^*(V)^+$ and since either $V = X$ or V is C^*-embedded in X, there exists an $\bar{f}$ in $C^*(X)^+$ such that $\bar{f}|V = f$. (For the case of Theorem 2.5, we only know that $f \in C(V)^+$. However, since $f \leqq T^{-1}(1)$ on V, there exists an $\bar{f}$ in $C(X)^+$ such that $\bar{f}|V = f$ by Lemma 2.4.) Then, as in the last paragraph, we can check that $T\bar{f}$ extends h. Hence W is C^*-embedded in Y.

COROLLARY 2.6. *Let X be a Tychonoff space. Then the following conditions are equivalent*:

(1) *X has property (S).*

(2) *There is no proper, dense, open F_σ-subset of X which is C^*-embedded.*

(3) *Each lattice isomorphism*: *$C(X)^+ \to C(X)^+$ is strict.*

(4) *Each lattice isomorphism*: *$C^*(X)^+ \to C^*(X)^+$ is strict.*

PROOF. The implications (1)$\Rightarrow$(3) and (1)$\Rightarrow$(4) follow from Theorem 2.5. The implication (2)$\Rightarrow$(1) is obvious.

(1)$\Rightarrow$(2): Assume that U is a proper, dense, open F_σ in X which is C^*-embedded in X. Write $U = \bigcup \{F_n : n \in N\}$, where each F_n is a closed subset of X. Let $x_0 \in X \sim U$, and, for each n, let f_n be a continuous function: $X \to [0, 1]$ such that $f_n(x_0) = 0$ and $f_n|F_n \equiv 1$. Let $f = \sum \{2^{-n} f_n : n \in N\}$; then f is continuous and $\varnothing \neq Z(f) \subset X \sim U$. Therefore the cozero-set $X \sim Z(f)$ is dense, proper and is C^*-embedded in X. Thus the negation of (2) implies the negation of (1), i.e., (1)$\Rightarrow$(2).

Finally assume that X does not have property (S). Then there is a ϕ in $C(X)^+$ such that $U = \{x : \phi(x) > 0\}$ is a dense proper subset of X which is C^*-embedded in X. We define a map $T_0 : C(U)^+ \to C(U)^+$ as in the proof of Theorem 1.5:

$$T_0(g)(x) = \begin{cases} g(x) & \text{if } g(x) \geqq 1 \\ (g(x))^{\phi(x)} & \text{if } g(x) \leqq 1 \end{cases} \quad (g \in C(U)^+, x \in U) .$$

The inverse T_0^{-1} is given by a similar formula:

$$T_0^{-1}(g)(x)=\begin{cases} g(x) & \text{if } g(x)\geqq 1 \\ (g(x))^{1/\phi(x)} & \text{if } g(x)\leqq 1 \end{cases} \quad (g\in C(U)^+,\ x\in U)\ .$$

Therefore T_0 is a lattice isomorphism and the restriction of T_0 to $C^*(U)^+$ is a lattice isomorphism: $C^*(U)^+\to C^*(U)^+$. We note that $T_0(g)\leqq g\vee 1$ and $T_0^{-1}(g)\leqq g\vee 1$ for each g in $C(U)^+$. Now let $f\in C(X)^+$. Then $T_0(f|U)\leqq f\vee 1$ and $T_0^{-1}(f|U)\leqq f\vee 1$ on U. Hence, by Lemma 2.4, $T_0(f|U)$ and $T_0^{-1}(f|U)$ extend continuously to X. Let $T(f)$ and $\tilde{T}(f)$ denote the extensions respectively. Then clearly $T\tilde{T}=\mathrm{id}$ and $\tilde{T}T=\mathrm{id}$. Since both T and $\tilde{T}$ preserve the partial ordering of $C(X)^+$, T is a lattice isomorphism. Furthermore, T induces a lattice isomorphism $C^*(X)^+\to C^*(X)^+$. From the definition of T, it is clear that $T(1/2)(x)=(1/2)^{\phi(x)}$ and $T(1)=1$. Since $Z(\phi)\neq\varnothing$, $T(1/2)<T(1)$ does not hold, and therefore, T is not strict. Hence both (3) and (4) fail. This proves that (3)$\Rightarrow$(1) and (4)$\Rightarrow$(1).

COROLLARY 2.7. *Let X be a Tychonoff space. Then the following two conditions are equivalent*:

(1) *For each Tychonoff space Y, if T is a lattice isomorphism*: $C^*(X)^+\to C^*(Y)^+$, *then T is necessarily strict.*

(2) *The space X is pseudo-compact and has property* (*S*).

PROOF. The implication (2)$\Rightarrow$(1) follows from Theorem 2.5. Suppose that X does not satisfy (2). Then either X is non-pseudo-compact or it fails to have property (S). In the latter case, there is a non-strict lattice isomorphism $C^*(X)^+\to C^*(X)^+$ by Corollary 2.6. So assume that X is not pseudo-compact. Then there is an element f of $C^*(X)^+$ such that $f>0$ and $\inf\{f(x): x\in X\}=0$. The natural map: $C^*(X)^+\to C^*(\beta X)^+$ is obviously a lattice isomorphism, but it is not strict because the extension $\bar{f}$ of f fails to satisfy $\bar{f}>0$. In either case, therefore, X does not satisfy (1).

The proofs of the following two corollaries are similar to the one above.

COROLLARY 2.8. *Let X be a Tychonoff space. Then the following two conditions are equivalent*:

(1) *For each Tychonoff space Y, if T is a lattice isomorphism*: $C(X)^+\to C(Y)^+$, *then T is necessarily strict.*

(2) *The space X has property* (*S*).

COROLLARY 2.9. *Let Y be a Tychonoff space. Then the following conditions are equivalent*:

(1) *For each Tychonoff space X, if T is a lattice isomorphism*: $C(X)^+ \to C(Y)^+$, *then T is necessarily strict.*

(2) *For each Tychonoff space X, if T is a lattice isomorphism*: $C^*(X)^+ \to C^*(Y)^+$, *then T is necessarily strict.*

(3) *Then space Y has property* (S).

To conclude this section, we discuss briefly some lattices of continuous functions other than those of non-negative functions. For a topological space X, let $C(X)$ (resp. $C^*(X)$) be the lattice of all continuous (resp. bounded continuous) real-valued functions on X, and let $C(X)^{++} = \{f: f \in C(X)^+, f > 0\}$ and $C^*(X)^{++} = \{f: f \in C^*(X)^+, f > 0\}$. The notions of lattice isomorphisms and strict lattice isomorphisms can easily be extended for these new lattices. The following theorem, which was communicated to us by Professor Schäffer, makes it clear that we have not lost generality by considering exclusively the lattices of non-negative functions.

THEOREM 2.10. *For topological spaces X and Y, the following statements are equivalent*:

(1) *Each lattice isomorphism*: $C(X) \to C(Y)$ *is strict.*

(2) *Each lattice isomorphism*: $C(X)^+ \to C(Y)^+$ *is strict.*

(3) *Each lattice isomorphism*: $C(X)^{++} \to C(Y)^{++}$ *is strict.*

Also the following statements are equivalent:

(1)* *Each lattice isomorphism*: $C^*(X) \to C^*(Y)$ *is strict.*

(2)* *Each lattice isomorphism*: $C^*(X)^+ \to C^*(Y)^+$ *is strict.*

(3)* *Each lattice isomorphism*: $C^*(X)^{++} \to C^*(Y)^{++}$ *is strict.*

PROOF. A continuous order isomorphism of $(-\infty, \infty)$ and $(0, \infty)$(e.g. $x \mapsto e^x$) induces an order isomorphism of $C(X)$ and $C(X)^{++}$ which preserves the strict ordering $<$. Therefore (1) and (3) are equivalent. Assume (1) and let $T: C(X)^+ \to C(Y)^+$ be a lattice isomorphism. Extend T to $C(X)$ by the formula: $\tilde{T}(f) = T(f^+) - T(f^-)(f \in C(X))$, where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. It is easy to see that $\tilde{T}$ is order preserving. Since $T(f^+) \wedge T(f^-) = T(f^+ \wedge f^-) = 0$, $\tilde{T}(f)^+ = T(f^+)$ and $\tilde{T}(f)^- = T(f^-)$. If we denote by $\tilde{T}^{-1}$ a similar extension of T^{-1} to $C(Y)$, then $\tilde{T}^{-1}\tilde{T}(f) = T^{-1}(\tilde{T}(f)^+) - T^{-1}(\tilde{T}(f)^-) = T^{-1}(T(f^+)) - T^{-1}(T(f^-)) = f^+ - f^- = f$. Hence $\tilde{T}^{-1}\tilde{T} = \mathrm{id}$, and similarly $\tilde{T}\tilde{T}^{-1} = \mathrm{id}$. Hence $\tilde{T}$ is a lattice isomorphism, and it is strict by the assumption. Therefore T is strict. Finally assume (2), and let $T: C(X) \to C(Y)$ be a lattice isomorphism. If $f \in C(X)$, then $h \mapsto T(f+h) - T(f)$ is a lattice isomorphism: $C(X)^+ \to C(Y)^+$. Hence $T(f+h) > T(f)$ for each h such that $h > 0$. Therefore T is strict. The proof of the *-version is identical.

§3. The isomorphism σ.

In Section 2, the isomorphism σ of Boolean algebras of regular open sets was introduced as a substitute for the associated map ρ, which played an important rôle for the compact case. For the purpose of proving the main Theorem 2.5, it was not necessary to delve into the nature of σ. In the present section we shall examine the map σ more closely. For the moment, let X and Y be compact Hausdorff spaces and let $T: C(X)^+ \to C(Y)^+$ be a lattice isomorphism. Then the connection between the associated map $\rho: Y \to X$ (Lemma 1.1) and $\sigma: \mathscr{R}(X) \to \mathscr{R}(Y)$ that corresponds to T (Theorem 2.1) can easily be described: If $U \in \mathscr{R}(X)$ and $f \in I_U$, then $U \cap \operatorname{supp} f = \varnothing$. Hence, if $y \in \rho^{-1}[U]$, then $\rho(y) \notin \operatorname{supp} f$ and, hence, $Tf(y)=0$ by the definition of the associated map. In other words, $T[I_U] \subset I_{\rho^{-1}[U]}$. Since ρ^{-1} is associated with T^{-1}, we also have $T^{-1}[I_{\rho^{-1}[U]}] \subset I_U$. It follows that $T[I_U] = I_{\rho^{-1}[U]}$, i.e.,$\sigma(U) = \rho^{-1}[U]$. One of the results of this section is that the map σ can be described in terms of a map $X \to \beta Y$ in case X and Y are only assumed to be Tychonoff spaces. Throughout the section, unless otherwise stated, $\sigma: \mathscr{R}(X) \to \mathscr{R}(Y)$ is the isomorphism of Boolean algebras that corresponds to a lattice isomorphism $T: C(X)^+ \to C(Y)^+$ or $T: C^*(X)^+ \to C^*(Y)^+$, where X and Y are Tychonoff spaces.

Two subsets A and B of a topological space are said to be *completely separated* in the space if there is a continuous function f on the space into $[0, 1]$ such that $f|A \equiv 0$ and $f|B \equiv 1$. The following simple lemma is quite useful throughout this section.

LEMMA 3.1. *Regular open subsets U and V of X are completely separated in X if and only if $\sigma(U)$ and $\sigma(V)$ are completely separated in Y.*

PROOF. Suppose that U and V are completely separated. Then there is a continuous function $f: X \to [0, 1]$ such that $f|U \equiv 0$ and $f|V \equiv 1$. Let $g = f \cdot T^{-1}1$; then $g|U \equiv 0$ and $g|V = T^{-1}1|V$. Hence by Lemma 2.3, $Tg|\sigma(U) \equiv 0$ and $Tg|\sigma(V) \equiv 1$. Clearly this implies that $\sigma(U)$ and $\sigma(V)$ are completely separated. Since σ^{-1} corresponds to the lattice isomorphism T^{-1}, the converse follows from what is already proved.

Let X be a dense subspace of a space Z. Then there is a natural isomorphism $\alpha: \mathscr{R}(Z) \to \mathscr{R}(X)$. In fact α and α^{-1} are given by

$$\alpha(U) = U \cap X \qquad (U \in \mathscr{R}(Z))$$
$$\alpha^{-1}(V) = \operatorname{int}_Z cl_Z V \qquad (V \in \mathscr{R}(X)) .$$

Here $cl_Z A$ and $\operatorname{int}_Z A$ denote respectively the closure of A in Z and the interior of A in Z.

LEMMA 3.2. *For each x in X, the intersection $\cap\{cl_{\beta Y}\sigma(U): x \in U \in \mathscr{R}(X)\}$ is a singleton in βY.*

PROOF. Let $A = \cap\{cl_{\beta Y}\sigma(U): x \in U \in \mathscr{R}(X)\}$. By the compactness of βY, the set A is non-void. Suppose that $y_1 \in A$, $y_2 \in A$, and $y_1 \neq y_2$. Then there are open neighborhoods U_1 and U_2 of y_1 and y_2 respectively such that $U_i \in \mathscr{R}(\beta Y)(i=1, 2)$ and U_1 and U_2 are completely separated. By the above remark, there are V_1 and V_2 in $\mathscr{R}(X)$ such that $\sigma(V_1) = U_1 \cap Y$ and $\sigma(V_2) = U_2 \cap Y$. By Lemma 3.1, V_1 and V_2 are completely separated. In particular $V_1^- \cap V_2^- = \varnothing$. Hence either $x \notin V_1^-$ or $x \notin V_2^-$. We may, therefore, assume that $x \notin V_1^-$, i.e., $x \in V_1'$. Hence $y_1 \in A \subset cl_{\beta Y}\sigma(V_1') = cl_{\beta Y}\sigma(V_1') = cl_{\beta Y}(U_1' \cap Y) = cl_{\beta Y} U_1' = \beta Y \sim U_1$. This contradicts the fact that $y_1 \in U_1$. Therefore A is a singleton.

For each x in X, let $\tau(x)$ denote the unique element in $\cap\{cl_{\beta Y}\sigma(U): x \in U \in \mathscr{R}(X)\}$. We say that the map $\tau: X \to \beta Y$ is *induced* by the lattice isomorphism $T: C(X)^+ \to C(Y)^+$ or $C^*(X)^+ \to C^*(Y)^+$. The next theorem summarizes the properties of the map τ.

THEOREM 3.3. *Let $\tau: X \to \beta Y$ be as above. Then:*

(i) *The map τ is continuous and one-to-one;*

(ii) *For each regular open subset U of X, $cl_{\beta Y}\sigma(U) = cl_{\beta Y}\tau[U]$. Moreover, τ is the unique continuous mapping: $X \to \beta Y$ that satisfies this relation.*

(iii) *For each regular open subset U of X,*

$$\sigma(U) = (\mathrm{int}_{\beta Y}\, cl_{\beta Y}\tau[U]) \cap Y .$$

PROOF. (i) Let $x \in X$, and let V be an open neighborhood of $\tau(x)$ in βY. Then by the compactness of βY, there exists a regular open neighborhood U of x such that $cl_{\beta Y}\sigma(U) \subset V$. Then $\tau[U] \subset V$, and consequently τ is continuous. Next suppose that x_1 and x_2 are two distinct points of X. Then there are completely separated regular open neighborhoods U_1 and U_2 of x_1 and x_2 respectively. Then by Lemma 3.1, $\sigma(U_1)$ and $\sigma(U_2)$ are completely separated in Y and, hence, in βY. Therefore $(cl_{\beta Y}\sigma(U_1)) \cap (cl_{\beta Y}\sigma(U_2)) = \varnothing$, and this implies that $\tau(x_1) \neq \tau(x_2)$.

(ii) Let $U \in \mathscr{R}(X)$. From the definition of τ, $\tau[U] \subset cl_{\beta Y}\sigma(U)$ and, therefore, $cl_{\beta Y}\tau[U] \subset cl_{\beta Y}\sigma(U)$. Suppose now that $y \in \sigma(U)$ and V is a regular open neighborhood of y in βY. Then there is a member W of $\mathscr{R}(X)$ such that $\sigma(W) = V \cap Y$. Since $\sigma(U \cap W) = \sigma(U) \cap \sigma(W) \neq \varnothing$, $U \cap W \neq \varnothing$. Let $x \in U \cap W$; then $\tau(x) \in cl_{\beta Y}(\sigma(U) \cap \sigma(W)) \subset cl_{\beta Y}\sigma(W) \subset cl_{\beta Y} V$. Hence $\tau[U] \cap cl_{\beta Y} V \neq \varnothing$, and therefore $y \in cl_{\beta Y}\tau[U]$. This proves that $\sigma(U) \subset cl_{\beta Y}\tau[U]$. The second statement of (ii) is obvious.

(iii) This follows from (ii) and the formula: $\sigma(U)=Y\cap \operatorname{int}_{\beta Y} cl_{\beta Y}\sigma(U)$ (cf. the remark preceding Lemma 3.2).

EXAMPLE 3.5.

(*a*) Let $\phi: Y\to X$ be a homeomorphism. Then ϕ induces the lattice isomorphisms $T_\phi: C(X)^+\to C(Y)^+$ and $T_\phi: C^*(X)^+\to C^*(Y)^+$, where $T_\phi(f)=f\circ\phi(f\in C(X)^+$ or $f\in C^*(X)^+)$. The corresponding σ is given by $\sigma(U)=\phi^{-1}[U](U\in\mathscr{R}(X))$. Hence the map τ induced by T_ϕ is $i\phi^{-1}$, where $i: Y\to\beta Y$ is the inclusion map.

(*b*) Suppose that $Y\subset X\subset\beta Y$. Then the restriction defines a lattice isomorphism $T: C^*(X)^+\to C^*(Y)^+$. It can be verified easily that the corresponding $\sigma: \mathscr{R}(X)\to\mathscr{R}(Y)$ is given by $\sigma(U)=U\cap Y(U\in\mathscr{R}(X))$ and that the induced map τ is simply the inclusion: $X\to\beta Y$. The map $Y\to\beta X$ induced by $T^{-1}: C^*(Y)^+\to C^*(X^+)$ is the composite of two inclusion maps: $Y\to X$ and $X\to\beta X$.

So far, the results are valid for lattice isomorphisms $T: C^*(X)^+\to C^*(Y)^+$ as well as for lattice isomorphisms $T: C(X)^+\to C(Y)^+$. The next theorem, however, gives a more precise information on the ranges of only those maps τ that are induced by lattice isomorphisms $C(X)^+\to C(Y)^+$. The crucial difference lies in the following lemma.

LEMMA 3.6. *Let $T: C(X)^+\to C(Y)^+$ be a lattice isomorphism, and let $\sigma: \mathscr{R}(X)\to\mathscr{R}(Y)$ be the Boolean algebra isomorphism that corresponds to T. If $\{U_n: n\in N\}$ is a sequence in $\mathscr{R}(X)$ such that $\cap\{U_n: n\in N\}\neq\varnothing$, then $\cap\{\sigma(U_n): n\in N\}\neq\varnothing$.*

PROOF. Without loss of generality, we can assume that U_n' and U_{n+1} are completely separated for each n. Then by Lemma 3.1, $\sigma(U_n)'$ and $\sigma(U_{n+1})$ are completely separated. In particular $\sigma(U_{n+1})^-\subset\sigma(U_n)$. Assume that $\cap\{\sigma(U_n): n\in N\}=\cap\{\sigma(U_n)^-: n\in N\}=\varnothing$. For each n, let ψ_n be a continuous function on Y into $[0, 1]$ such that $\psi_n\equiv 0$ on $\sigma(U_n)'$ and $\psi_n\equiv 1$ on $\sigma(U_{n+1})$. Then $\operatorname{supp}\psi_n=\{y: \psi_n(y)>0\}^-\subset\sigma(U_n)^-$, and the family $\{\operatorname{supp}\psi_n: n\in N\}$ is locally finite. Therefore the function $\phi=\sum\{\psi_n\cdot T(n): n\in N\}$ is well-defined and $\phi\in C^+(Y)$. Since $\phi\geqq T(n)$ on $\sigma(U_{n+1})$, $T^{-1}\phi\geqq n$ on U_{n+1} by Lemma 2.3. Therefore $\cap\{U_n: n\in N\}=\varnothing$, contrary to the hypothesis. Hence $\cap\{\sigma(U_n): n\in N\}\neq\varnothing$.

Let X be a Tychonoff space. Then each f in $C(X)^+$ can be extended to a continuous function $\bar{f}: \beta X\to[0, \infty]$, where $[0, \infty]$ is given the order-topology under which it is compact Hausdorff. Let $R_f=\{x: \bar{f}(x)<\infty\}$, and let $\upsilon X=\bigcap\{R_f: f\in C(X)^+\}$. Then $X\subset\upsilon X\subset\beta X$, and υX is called the *real-compactification* of X. If $X=\upsilon X$, then X is called *real-compact*. The space υX itself is real-compact, i.e., $\upsilon\upsilon X=\upsilon X$. A Tychonoff space which is Lindeöf is real-compact, and a topological space is real-compact if and only if it is homeomorphic to a closed subspace of the product of a family of copies of the real line (see, for instance, [4, p. 272]).

THEOREM 3.7. *Let $T: C(X)^+ \to C(Y)^+$ be a lattice isomorphism, and let $\tau: X \to \beta Y$ be the map induced by T. Then the image of τ is contained in υY.*

PROOF. Suppose that, for some $x_0 \in X$, $y_0 = \tau(x_0) \notin \upsilon Y$. Then there exists a continuous map $f: \beta Y \to [0, \infty]$ such that $f(y_0) = \infty$ and $f(y) < \infty$ for each y in Y. For each positive integer n, let $V_n = \{y: f(y) \geqq n\}$. Then clearly $(\cap \{V_n: n \in N\}) \cap Y = \emptyset$. Since V_n is a neighborhood of $y_0 = \tau(x_0)$, there exists a regular open neighborhood U_n of x_0 such that $\tau[U_n] \subset V_n$. Then by Theorem 2.3, $\sigma(U_n) \subset cl_{\beta Y}\tau[U_n] \subset V_n$. Consequently $\cap \{\sigma(U_n): n \in N\} \subset (\cap \{V_n: n \in N\}) \cap Y = \emptyset$, whereas $x_0 \in \cap \{U_n: n \in N\}$. This contradicts Lemma 3.6.

REMARK 3.8.

(*a*) Both Lemma 3.6 and Theorem 3.7 are false for lattice isomorphisms $C^*(X)^+ \to C^*(Y)^+$. Let X be a Tychonoff space which is not pseudo-compact. Then $\upsilon X \neq \beta X$. The restriction map $T: C^*(\beta X)^+ \to C^*(X)^+$ is a lattice isomorphism, and the induced map τ is the identity map: $\beta X \to \beta X$ (Example 3.5(b)). Therefore Theorem 3.7 and, hence, Lemma 3.6, are no longer valid for lattice isomorphisms $C^*(X)^+ \to C^*(Y)^+$.

(*b*) Let $T: C(X)^+ \to C(Y)^+$ be a lattice isomorphism. Then, because of Theorem 3.7, we may regard the induced map as a map $X \to \upsilon Y$ rather than a map $X \to \beta Y$. Now assume that both X and Y are real-compact. Then the lattice isomorphisms T and T^{-1} induce $\tau: X \to Y$ and $\bar{\tau}: Y \to X$ respectively, and, by using Theorem 3.3(ii), it is straight forward to prove that $\tau\bar{\tau} = \mathrm{id}$ and $\bar{\tau}\tau = \mathrm{id}$. In particular, X and Y are homeomorphic. This is Shirota's theorem [10], which generalizes a Kaplansky's theorem for compact X and Y [7].

The next theorem explains why we need not consider lattice isomorphisms of the type $C^*(X)^+ \to C(Y)^+$.

THEOREM 3.9. *Let X and Y be Tychonoff spaces. If $C^*(X)^+$ and $C(Y)^+$ are lattice isomorphic, then Y is pseudo-compact, and consequently $C(Y)^+ = C^*(Y)^+$.*

PROOF. If $C^*(X)^+$ and $C(Y)^+$ are lattice isomorphic, then $C(\beta X)^+$ and $C(\upsilon Y)^+$ are lattice isomorphic. Since both βX and υY are real-compact, they are homeomorphic by Remark 3.8(*b*). Hence υY is compact. This implies that $\upsilon Y = \beta Y$ or, equivalently, that Y is pseudo-compact.

§ 4. Spaces with property (*S*).

In this section we discuss a number of conditions for a Tychonoff space to have property (*S*). We also consider the permanence of property (*S*). The most notable fact is that property (*S*) is preserved under arbitrary products. In contrast, neither a subspace nor a continuous image of a space with property (*S*) has the property in general. Finally, there is a wide class of spaces Y such that $X\times Y$ has property (*S*) for an arbitrary Tychonoff space X.

LEMMA 4.1. *Let U be a cozero-set in a Tychonoff space X. If there is a sequence in U that converges to a point outside U, then U is not C^*-embedded in X.*

PROOF. Let ϕ be a continuous function on X into $[0, 1]$ such that $U=\{x: \phi(x)>0\}$, and let $\{x_n: n\in N\}$ be a sequence in U which converges to a point x_0 in $X\sim U$. We may assume that $\phi(x_i)\neq\phi(x_j)$ if $i\neq j$. Let $t_n=\phi(x_n)$, and let g be a bounded continuous real-valued function on $(0, 1]$ such that $g(t_n)=(-1)^n$. Then the function $g\circ\phi$ cannot be continuously extended to X. Hence U is not C^*-embedded in X.

It follows immediately from the lemma that each first countable Tychonoff space has property (*S*). We shall see that this fact can be considerably generalized. Let A be a subset of the product $X\times Y$. Then, for each x in X, the *section* A_x of A at x is the set $\{y: (x, y)\in A\}$.

LEMMA 4.2. *Let U be a cozero-subset of the product $X\times Y$ of Tychonoff spaces X and Y. If U is C^*-embedded in $X\times Y$, then for each x in X, the section U_x is C^*-embedded in Y. If, furthermore, U is dense in $X\times Y$, then the closure U_x^- is open in Y.*

PROOF. Let ϕ be a continuous function on $X\times Y$ into $[0, 1]$ such that $U=\{(x, y): \phi(x, y)>0\}$. Fix a point x_0 in X, and for each (x, y) in U, let $\psi(x, y)=|(\phi(x_0, y)/\phi(x, y))|\wedge 1$. Then ψ is a continuous function on U such that $\psi(x_0, y)=1$ whenever $y\in U_{x_0}$ and such that $\psi\equiv 0$ on $U\cap[X\times(Y\sim U_{x_0})]$. Let f be a bounded real-valued continuous function on U_{x_0}. Then define a function $\tilde{f}$ on U as follows:

$$\tilde{f}(x, y)=\begin{cases}\psi(x, y)f(y) & \text{if } (x, y)\in U\cap(X\times U_{x_0}) ,\\ 0 & \text{if } (x, y)\in U\cap[X\times(Y\sim U_{x_0})] .\end{cases}$$

Then $\tilde{f}$ is bounded and continuous on U, and $\tilde{f}(x_0, y)=f(y)$ for each y in U_{x_0}. Since U is C^*-embedded in $X\times Y$, $\tilde{f}$ can be extended to a continuous function $\bar{f}$ on $X\times Y$, and the function $y\mapsto\bar{f}(x_0, y)$ is a continuous extension of f to Y. Thus U_{x_0} is C^*-embedded in Y.

Assume next that U is dense, and let $\bar{\psi}$ be a continuous extension of ψ to $X\times Y$. Since $U\cap[X\times(Y\sim U_{x_0}^-)]$ is dense in $X\times(Y\sim U_{x_0}^-)$ and since $\psi\equiv 0$ on $U\cap[X\times(Y\sim U_{x_0})]$, $\psi(x, y)=0$ whenever $y\notin U_{x_0}^-$. By the continuity of $\bar{\psi}$, $\bar{\psi}(x_0, y)=1$ for each y in $U_{x_0}^-$. It follows that $y\mapsto\bar{\psi}(x_0, y)$ is the characteristic function of the set $U_{x_0}^-$. Hence $U_{x_0}^-$ is open.

THEOREM 4.3. *Let* $\{X_\gamma: \gamma\in\Gamma\}$ *be a family of Tychonoff spaces with property* (*S*). *Then the product* $\times\{X_\gamma: \gamma\in\Gamma\}$ *has property* (*S*).

PROOF. Let $X=\times\{X_\gamma: \gamma\in\Gamma\}$, and assume that X does not have property (S). Then there is a dense proper cozero-subset U of X which is C^*-embedded. Since U is an F_σ, $U=\bigcup\{C_n: n\in N\}$, where $\{C_n: n\in N\}$ is a sequence of closed subsets of X. Fix a point v in $X\sim U$. For each n, there exists a closed neighborhood F_n of v which is disjoint from C_n. We may assume that F_n can be expressed as $F_n=\times\{F_{n,\gamma}: \gamma\in\Gamma\}$, where $F_{n,\gamma}=X_\gamma$ for all but a finite number of γ's and each $F_{n,\gamma}$ is a zero-set in X_γ. Then F_n is a zero-set in X. Let $F=\bigcap\{F_n: n\in N\}$. Then $F\cap U=\varnothing$, $v\in F$, and $F=\times\{F_\gamma: \gamma\in\Gamma\}$ where $F_\gamma=\bigcap\{F_{n,\gamma}: n\in N\}$. Obviously F and each F_γ are zero-sets in X and X_γ respectively. Since $U\subset X\sim F\subset X$, $X\sim F$ is C^*-embedded in X. Also $X\sim F$ is a dense proper cozero-set in X. Hence we may and do assume that $U=X\sim F$. Fix a γ_0 in Γ. Then $X_{\gamma_0}\sim F_{\gamma_0}$ is a certain section of U, where we regard X as $(\times\{X_\gamma: \gamma\neq\gamma_0\})\times X_{\gamma_0}$. Hence by Lemma 4.2, $X_{\gamma_0}\sim F_{\gamma_0}$ is C^*-embedded in X_{γ_0}. Since X_{γ_0} has property (S), the cozero-set $X_{\gamma_0}\sim F_{\gamma_0}$ cannot be dense in X_{γ_0}. Hence the interior of F_{γ_0} in X_{γ_0} is non-void for each γ_0 in Γ.

Now, since U is dense in X, the interior of F in X must be empty. It follows that $\{\gamma: F_\gamma\neq X_\gamma\}$ cannot be finite. Let $\{\gamma_n: n\in N\}$ be a sequence in Γ such that $F_{\gamma_n}\neq X_{\gamma_n}$ for each n and such that $\gamma_n\neq\gamma_m$ if $n\neq m$. For each n, choose an a_n in $X_{\gamma_n}\sim F_{\gamma_n}$. Define a sequence $\{x_n\}$ in X as follows:

$$x_n(\gamma)=\begin{cases}a_n & \text{if } \gamma=\gamma_n\\ v(\gamma) & \text{if } \gamma\neq\gamma_n\ .\end{cases}$$

Then $x_n\in X\sim F=U$ and the sequence $\{x_n\}$ converges to v, which is outside U. In view of Lemma 4.1, the set U cannot be C^*-embedded in X. This contradiction establishes the theorem.

A subset A of a topological space X is called *sequentially closed* if A contains the limits of each convergent sequence in A. If each sequentially closed subset of X is closed, then X is said to be *sequential*. A topological space X is said to be *weakly sequential* if each sequentially closed open subset of X is closed. Equivalently, a topological space X is weakly sequential if and only if, whenever U is a non-closed open subset of X, there is a sequence in U that converges to a point outside U. The following theorem is clear from Lemma 4.1 and the definition of property (S).

THEOREM 4.4. *A weakly sequential Tychonoff space has property* (S).

The proof of the following theorem is straight forward and is omitted.

THEOREM 4.5. *A quotient of a weakly sequential space is weakly sequential.*

The following theorem in the present generality was observed by Professor E. Michael. The proof is also due to him.

THEOREM 4.6. *Let* $\{X_\gamma: \gamma \in \Gamma\}$ *be a family of first countable topological spaces. Then the product* $X = \times\{X_\gamma: \gamma \in \Gamma\}$ *has the following property: If* U *is an open subset of* X *and if* $x \in U^-$, *then there is a sequence in* U *that converges to* x. *In particular,* X *is weakly sequential.*

PROOF. Let Σ be the subspace of X consisting of all points y such that $\{\gamma: x(\gamma) \neq y(\gamma)\}$ is countable. Then by Noble [9], Σ is a Fréchet space, i.e., whenever $A \subset \Sigma$ and $y \in \bar{A} \cap \Sigma$, there is a sequence in A that converges to y. Since Σ is dense in X, $x \in U^- \cap \Sigma = (U \cap \Sigma)^- \cap \Sigma$. Hence there is a sequence in U (in fact, in $U \cap \Sigma$) which converges to x.

By combining Theorems 4.4, 4.5, and 4.6, we obtain the following corollary.

COROLLARY 4.7. *Let* $\{X_\gamma: \gamma \in \Gamma\}$ *be a family of first countable Tychonoff spaces. Then the product* $X = \times\{X_\gamma: \gamma \in \Gamma\}$ *and each quotient space of* X *that is Tychonoff has property* (S).

REMARK 4.8.

(*a*) Professor E. Michael remarked that the conclusions of Theorem 4.6 and Corollary 4.7 are valid when first countable spaces are replaced by "bi-sequential spaces" or "W-spaces." See Michael [8] for the definition and the properties of bi-sequential spaces, and see Gruenhage [6] for the definition and the properties of W-spaces.

(*b*) It follows from Corollary 4.7 that $[0, 1]^A$ has property (S) for an arbitrary index set A. Hence each Tychonoff space can be embedded in a compact Hausdorff space with property (S). This shows, in particular, that property (S) is not hereditary.

(c) Let X be the product of a family of first countable compact Hausdorff spaces. Then, by Corollary 4.7, each continuous image of X has property (S). Suppose that Y is a Tychonoff space such that βY is a continuous image of X. Then, βY has property (S), and hence Y is pseudo-compact by Theorem 1.5. This fact generalizes a theorem of Engelking and Pełczyński [5], who prove this result under the assumption that X be a Cantor cube, that is, $X=\{0, 1\}^A$ for some index set A.

The next theorem shows, in part, that for $X\times Y$ to have property (S) it is not necessary that both spaces X and Y have property (S).

THEOREM 4.9. *Let Y be a Tychonoff space with property (S). Then the following conditions are equivalent*:

(1) *The product $X\times Y$ has property (S) for each Tychonoff space X.*

(2) *The product $(\beta N)\times Y$ has property (S), where N is the space of positive integers with the discrete topology.*

(3) *Each non-void open and closed subset of Y contains a G_δ-set which is not open.*

PROOF. The implication (1)$\Rightarrow$(2) is trivial. Now assume condition (2). Let Y_0 be an open and closed subset of Y. Then $(\beta N)\times Y_0$ is an open and closed subset of $(\beta N)\times Y$, and, therefore, it has property (S). Hence in order to prove (3), it is sufficient to prove that, if each G_δ-set in Y is open, then $(\beta N)\times Y$ does not have property (S). In fact, we show that, if each G_δ-set in Y is open, then $N\times Y$ is C^*-embedded in $(\beta N)\times Y$. (Note that $N\times Y$ is a cozero-set in $(\beta N)\times Y$.) Let f be a continuous bounded real-valued function on $N\times Y$. We can extend f to a real-valued function $\bar{f}$ on $(\beta N)\times Y$ in such a way that, for each fixed y in Y, the function $x\mapsto\bar{f}(x, y)$ on βN is the continuous extension of the function $n\mapsto f(n, y)$ on N. We show that $\bar{f}$ is in fact continuous on $(\beta N)\times Y$. Let $(x_0, y_0)\in(\beta N)\times Y$, and let $\varepsilon>0$. For each n in N, let $U_n=\{y: |f(n, y)-f(n, y_0)|<\varepsilon/2\}$. Then by assumption, $V=\bigcap\{U_n: n\in N\}$ is an open neighborhood of y_0. By the definition of $\bar{f}$, $|\bar{f}(x, y)-\bar{f}(x, y_0)|\leqq\varepsilon/2$ for all x in βN and y in V. Let $W=\{x: |\bar{f}(x, y_0)-\bar{f}(x_0, y_0)|<\varepsilon/2\}$; then W is an open neighborhood of x_0. If $(x, y)\in W\times V$, then $|\bar{f}(x, y)-\bar{f}(x_0, y_0)|\leqq |\bar{f}(x, y)-\bar{f}(x, y_0)|+|\bar{f}(x, y_0)-\bar{f}(x_0, y_0)|<\varepsilon/2+\varepsilon/2=\varepsilon$. Thus $\bar{f}$ is continuous on $(\beta N)\times Y$.

Finally assume condition (3), and assume that the product $X\times Y$ does not have property (S) for some Tychonoff space X. Then there is a proper dense cozero-set U in $X\times Y$ which is C^*-embedded. As in the proof of Theorem 4.3, we may assume that $U=X\times Y\sim(A\times B)$, where A (resp. B) is a non-void zero-set in X (resp. Y). The cozero-set $Y\sim B$ in

Y is a section of U and, therefore, is C^*-embedded in Y by Lemma 4.2. Since Y has property (S), this implies that $(Y\sim B)^-=Y\sim B^0\neq Y$, i.e., $B^0\neq\emptyset$. Because U is dense, it then follows that $A^0=\emptyset$. By Lemma 4.2, $Y\sim B^0$ and, hence, B^0 are open and closed. Consequently $U\cap(X\times B^0)=(X\sim A)\times B^0$ is C^*-embedded in $X\times B^0$. By condition (3), there is a sequence $\{W_n: n\in N\}$ of open subsets of B^0 such that $W_n\supset W_{n+1}$ for each n and such that the intersection $C=\bigcap\{W_n: n\in N\}$ is not open. Let $y_0\in C\sim C^0$, and, for each n, let g_n be a continuous function on B^0 into $[0, 1]$ such that $g_n(y_0)=1$ and $g_n\equiv 0$ on $B^0\sim W_n$. Let $g=\Sigma\{2^{-n}g_n: n\in N\}$. Then g is a continuous function on B^0 such that $0\leqq g\leqq 1$, $g(y_0)=1$ and each neighborhood of y_0 contains a point y with $g(y)<1$. Let f be a continuous non-negative function on X such that $A=Z(f)$, and define a continuous function h on $(X\sim A)\times B^0$ by

$$h(x, y)=(g(y))^{1/f(x)}(x\in X\sim A,\ y\in B^0)\ .$$

Since $(X\sim A)\times B^0$ is C^*-embedded in $X\times B^0$, there is a continuous extension $\bar{h}$ of h on $X\times B^0$. Since $h(x, y_0)=1$ for each x in $X\sim A$ and since $X\sim A$ is dense in X, $\bar{h}(x_0, y_0)=1$. On the other hand, an arbitrary neighborhood of y_0 contains a point y such that $g(y)<1$ and, therefore, $\bar{h}(x_0, y)=0$. This contradicts the continuity of $\bar{h}$.

EXAMPLES AND REMARKS 4.10.

(*a*) By Theorem 4.9, the product $X\times[0, 1]$ has property (S) for each Tychonoff space X. Since $X\times[0, 1]\to X$ is a continuous open map, it follows that property (S) is not preserved by continuous open maps in general.

(*b*) Let Ω be the first uncountable ordinal number, and let X be the space of all ordinal numbers α such that $0\leqq\alpha\leqq\Omega$ with the order topology. Then X has property (S). For, if not, then there would be a proper open dense F_σ-set U in X such that $X=\beta U$. If $\Omega\notin U$, then the F_σ-set U cannot be dense in X. Hence $\Omega\in U$. This implies that there is an ordinal number α such that $\alpha\notin U$ and $0\leqq\alpha<\Omega$. Since α has a countable base of neighborhoods, there is a sequence in U that converges to α. Then by Lemma 4.1, U cannot be C^*-embedded in X. Therefore X has property (S). Let $Y=\{\alpha: 0\leqq\alpha<\Omega\}$, then $X=\beta Y$. Therefore the Stone-Čech compactification of a locally compact normal space *can* have property (S).

(*c*) The space βN, of course, lacks property (S). However $\beta N\sim N$ has property (S). For, according to [11, Corollary 3.27], no proper F_σ-set in $\beta N\sim N$ can be dense. Professor Comfort pointed out a generalization: Let X be a locally compact and real-compact Hausdorff space. Then

$\beta X \sim X$ has property (S). This follows from [2, Theorem 15.18(*b*)].

(*d*) We owe the following remarks to Professor van Douwen: If Y is non-pseudo-compact Tychonoff space, then βY contains a copy of βN. (In fact, let f be a continuous function on Y into $[0, \infty]$ such that $\sup\{f(y): y \in Y\} = \infty$. Then there is a countable subset A of Y such that $f[A]$ is a closed discrete subset of $[0, \infty)$. The closure $\bar{A}$ of A in βY is homeomorphic to βN.) Since βN contains a non-normal subspace (see e.g. [4; Example 3.6.19]), βY is not hereditarily normal. Furthemore, since $|\beta N| = 2^c$, $|\beta Y| \geqq 2^c$. In view of Theorem 1.5, we can conclude that a compact Hausdorff space has property (S) if either (i) X is hereditarily normal, or (ii) $|X| < 2^c$. Condition (ii) was also communicated to us by Professor Comfort.

(*e*) In a forthcoming paper [3], Professor van Douwen proves the following theorem: If $\{X_\gamma: \gamma \in \Gamma\}$ is a family of compact Hausdorff spaces such that X_γ is perfect for at least two γ's or such that X_γ has at least two points for an infinite number of γ's, then each subspace Z of $\times \{X_\gamma: \gamma \in \Gamma\}$ such that $\beta Z = \times \{X_\gamma: \gamma \in \Gamma\}$ must be pseudo-compact. By Theorem 1.5, the product space $\times \{X_\gamma: \gamma \in \Gamma\}$ has property (S). Thus, for instance, $\beta R \times \beta R$ has property (S) although βR does not have the property. Here, R denotes the real line.

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