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On the Distribution of Zeros of Dirichlet's L-Function on the Line $\sigma=1/2$ (II)

Teluhiko HILANO

Sophia University

§1. Main theorem.

This paper is a continuation of my previous paper [8]. Let χ be a primitive character mod q. We put

$$a=\frac{1}{2}(1-\chi(-1)) , \tag{1.1}$$

$$\begin{aligned} h(s)&=h(s, \chi) \\ &=\left(\frac{\pi}{q}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right) , \end{aligned} \tag{1.2}$$

$$\varepsilon(\chi)=\frac{(-i)^a}{\sqrt{q}}\sum_{m=1}^{q}\chi(m)\exp(2\pi im/q) \tag{1.3}$$

$$f'(s)=h'(s)/h(s) , \tag{1.4}$$

and

$$\begin{aligned} G(s)&=G(s, \chi) \\ &=L(s, \chi)+L'(s, \chi)/(f'(s)+f'(1-s)) . \end{aligned} \tag{1.5}$$

We have proved in [8] the following theorem.

THEOREM 1. *Let $N_G(D)$ be a number of zeros of $G(s)$ in the region*

$$1/2\leqq\sigma\leqq 3 ,$$
$$T\leqq t\leqq T+U .$$

Then, for sufficiently large T and for $U\leqq T/\log(qT/2\pi)$, we have

$$N_0(T+U, \chi)-N_0(T, \chi)\geqq\frac{U}{2\pi}\log\frac{qT}{2\pi}-2N_G(D)+O\left(\frac{U^2}{T}+1\right) .$$

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Using this theorem, we shall give here the detailed proof of the following main theorem.

MAIN THEOREM. *For $\varepsilon>0$, we assume that*

$$\log q \leqq (\log T)^{1-\varepsilon}$$

and put

$$L=\log\frac{qT}{2\pi},$$

$$U=\frac{T}{qL^4}.$$

Then we have

$$N_0(T+U,\chi)-N_0(T,\chi)>\frac{1}{3}(N(T+U,\chi)-N(T,\chi)).$$

§ 2. Preliminary to calculation of $N_G(D)$.

In the following sections, we shall estimate $N_G(D)$. But we shall estimate $N_{G\psi}(D)$ instead of $N_G(D)$, where

$$\psi(s)=\sum_{n\leqq X}\frac{\chi(n)b_n}{n^s}. \tag{2.1}$$

Now we assume the following;

$$X\geqq 1$$
$$\log X \ll L$$
$$|b_n|\leqq 1$$
$$b_1=1.$$

Using Littlewood's theorem, we get

$$\begin{aligned} 2\pi\delta N_G(D) &\leqq \int_T^{T+U}\log\left|(\psi G)\left(\frac{1}{2}-\delta+it\right)\right|dt+O\left(\frac{U}{L}\right) \\ &\leqq \frac{U}{2}\log\left(\frac{1}{U}\int_T^{T+U}\left|(\psi G)\left(\frac{1}{2}-\delta+it\right)\right|^2 dt\right)+O\left(\frac{U}{L}\right). \end{aligned} \tag{2.2}$$

To calculate the above integral, we use an approximate functional equation of $L(s,\chi)$ for a special value of σ.

We put

(2.3) $$g_1(s, \chi)=\sum_{n\leq(q|t|/2\pi)^{1/2}}\frac{\chi(n)}{n^s}$$

(2.4) $$g_2(s, \chi)=\sum_{n\leq(q|t|/2\pi)^{1/2}}\frac{\chi(n)\log n}{n^s}\ .$$

Then we have

THEOREM A. *For* $\sigma=1/2-\delta$ *and* $\log q|t|=O(1/\delta)$, *we have*

(2.5) $$L(s, \chi)=g_1(s, \chi)+\varepsilon(\chi)\frac{h(1-s)}{h(s)}(g_1(1-s, \bar{\chi}))+O((q/|t|)^{1/4})$$

and

(2.6) $$L'(s, \chi)=-g_2(s, \chi)+\varepsilon(\chi)\left\{\left(\frac{d}{ds}\frac{h(1-s)}{h(s)}\right)g_1(1-s, \bar{\chi})+\frac{h(1-s)}{h(s)}g_2(1-s, \bar{\chi})\right\}$$
$$+O((q/|t|)^{1/4}\log q|t|)\ .$$

PROOF. This follows easily from the results in Lavrik [2, 3] or Motohashi [4].

Using this theorem, we can easily get

(2.7) $$G(s)=g_1(s, \chi)+\frac{1}{(f'(s)+f'(1-s))}\left\{-g_2(s, \chi)+\varepsilon(\chi)\frac{h(1-s)}{h(s)}g_2(1-s, \bar{\chi})\right\}$$
$$+O((q/|t|)^{1/4})\ .$$

From Stirling's formula, we have for above σ and $t>0$

(2.8) $$\frac{h(1-s)}{h(s)}=\left(\frac{qt}{2\pi}\right)^{\delta}\exp\left(-\frac{\pi}{2}(a-1/2)i-it\log\frac{qt}{2\pi e}\right)+O(1/t)\ .$$

Let $\theta(t)$ denote the main term of the right-hand side of (2.8). From (2.7) and (2.8) we get

(2.9) $$G\left(\frac{1}{2}-\delta+it\right)=g_1\left(\frac{1}{2}-\delta+it, \chi\right)+\frac{1}{\log\frac{qt}{2\pi}}\left(-g_2\left(\frac{1}{2}-\delta+it, \chi\right)\right.$$
$$\left.+\varepsilon(\chi)\theta(t)g_2\left(\frac{1}{2}+\delta-it, \bar{\chi}\right)\right)+O((q/|t|)^{1/4})$$
$$=H\left(\frac{1}{2}-\delta+it, \chi\right)+H_1\left(\frac{1}{2}-\delta+it, \chi\right)\quad \text{(say!)}\ .$$

We shall estimate

$$\int_T^{T+U}\left|(\psi H)\left(\frac{1}{2}-\delta+it,\chi\right)\right|^2 dt=O(U)\,. \tag{2.10}$$

Using this estimate, we get

$$\int_T^{T+U}\left|(\psi G)\left(\frac{1}{2}-\delta+it\right)\right|^2 dt=\int_T^{T+U}\left|(\psi H)\left(\frac{1}{2}-\delta+it,\chi\right)\right|^2 dt \tag{2.11}$$
$$+O(U^{1/2}q^{1/4}T^{-1/4}(UL+X)^{1/2}+q^{1/2}T^{-1/2}(UL+X))$$

by the same calculation as that of [7]. In order to prove (2.10), we expand the above integral as the sum of 6 terms;

$$\int_T^{T+U}\left|(\psi H)\left(\frac{1}{2}-\delta+it,\chi\right)\right|^2 dt \tag{2.12}$$
$$=I_{11}+I_{22}+I_{33}-2\operatorname{Re} I_{12}-2\operatorname{Re} I_{23}+2\operatorname{Re} I_{13}\,,$$

where

$$I_{11}=\int_T^{T+U}\left|(\psi g_1)\left(\frac{1}{2}-\delta+it,\chi\right)\right|^2 dt\,,$$

$$I_{22}=\int_T^{T+U}\left|(\psi g_2)\left(\frac{1}{2}-\delta+it,\chi\right)\right|^2 \frac{dt}{\log^2\frac{qt}{2\pi}}\,,$$

$$I_{33}=\int_T^{T+U}|\theta(t)|^2\left|(\psi g_2)\left(\frac{1}{2}+\delta-it,\bar{\chi}\right)\right|^2 \frac{dt}{\log^2\frac{qt}{2\pi}}\,,$$

$$I_{12}=\int_T^{T+U}(|\psi|^2 g_1\bar{g}_2)\left(\frac{1}{2}-\delta+it,\chi\right)\frac{dt}{\log\frac{qt}{2\pi}}\,,$$

$$I_{13}=\int_U^{T+U}\overline{\varepsilon(\chi)}\,|\psi|^2\overline{\theta(t)}g_1\left(\frac{1}{2}-\delta+it,\chi\right)\overline{g_2}\left(\frac{1}{2}+\delta+it,\bar{\chi}\right)\frac{dt}{\log\frac{qt}{2\pi}}\,,$$

$$I_{23}=\int_T^{T+U}\overline{\varepsilon(\chi)}\,|\psi|^2\overline{\theta(t)}g_2\left(\frac{1}{2}-\delta+it,\chi\right)\overline{g_2}\left(\frac{1}{2}+\delta-it,\bar{\chi}\right)\frac{dt}{\log^2\frac{qt}{2\pi}}$$

§ 3. Lemmas.

We now list up some lemmas to make calculations a little simpler. Since the most calculations of them are similar to those of Levinson, so we shall not give here their proofs in details.

LEMMA 3.1. *For δ, $0<|\delta|<(c/\log Y)$ and $Y>c'q$, where c and c' are some constants, we have*

$$\sum_{\substack{1\leq j\leq Y\\(j,q)=1}}\frac{1}{j^{1-2\delta}}=\frac{\phi(q)}{q}\Big(\frac{Y^{2\delta}}{2\delta}-\frac{1}{2\delta}\Big)+c_1(q,\delta)+O(\phi(q)Y^{-1+2\delta})$$

$$\sum_{\substack{1\leq j\leq Y\\(j,q)=1}}\frac{\log j}{j^{1-2\delta}}=\frac{\phi(q)}{q}\Big(\frac{Y^{2\delta}}{2\delta}\log Y-\frac{Y^{2\delta}}{(2\delta)^2}+\frac{1}{(2\delta)^2}\Big)+c_2(q,\delta)$$
$$+O(\phi(q)Y^{-1+2\delta}\log Y)$$

$$\sum_{\substack{1\leq j\leq Y\\(j,q)=1}}\frac{\log^2 j}{j^{1-2\delta}}=\frac{\phi(q)}{q}\Big(\frac{Y^{2\delta}}{2\delta}\log^2 Y-2\frac{Y^{2\delta}}{(2\delta)^2}\log Y+2\frac{Y^{2\delta}}{(2\delta)^3}-\frac{2}{(2\delta)^3}\Big)$$
$$+c_3(q,\delta)+O(\phi(q)Y^{-1+2\delta}\log^2 Y)\ ,$$

where $c_k(q,\delta)$'s are some constants depending on only q and δ. Moreover, we can estimate $c_k(q,\delta)$ as

$$c_k(q,\delta)\ll\frac{1}{\delta^{k-1}}(\log\log q)^2 \tag{3.1}$$

for $k=1$, 2, and 3.

PROOF. Combine Lemmas 3.1 and 3.9 in [7].

LEMMA 3.2. *We assume that $m=1$, 2, and 3. Let A be a sufficiently large number and*

$$A^{\delta}=O(1)\ .$$

Then, for r,

$$A\leqq r\leqq B\leqq A+A/\log A\ ,$$

we have

$$\int_A^B\exp\Big(it\log\frac{t}{er}\Big)\Big(\frac{qt}{2\pi}\Big)^{\delta}\frac{dt}{\log^m\frac{qt}{2\pi}}$$
$$=q^{\delta}(2\pi)^{1/2-\delta}r^{1/2+\delta}\exp\Big(-ir+\frac{\pi i}{4}\Big)\Big/\log^m\frac{rq}{2\pi}$$
$$+q^{\delta}E(r)/\log^m qA\ ,$$

where

$$E(r)=O(1)+O\Big(\frac{A}{|A-r|+A^{1/2}}\Big)+O\Big(\frac{B}{|B-r|+B^{1/2}}\Big)\ .$$

For $r<A$ or $r>B$, we can estimate the above integral by

$$q^{s}E(r)/\log^{m} qA\ .$$

(See Lemmas 3.4 *and* 3.5 *in* [7].)

LEMMA 3.3. *Let K be the region in the first quardrant given by*

$$C_1 \leqq (u+\beta_1)(v+\beta_2) \leqq C_2\ ,$$
$$C_3(u+\beta_1) \leqq (v+\beta_2) \leqq C_4(u+\beta_1)$$

where $C_1, C_3>0$ and $\beta_1, \beta_2>0$. Let f be a function on K with continuous partial derivatives. Let $|K|$ denote the area of K and

$$u_M = \max_{(u,v)\in K} u$$
$$v_M = \max_{(u,v)\in K} v$$
$$|f|_M = \max_{(u,v)\in K} |f|$$

and similarly for $|\partial f/\partial u|_M$ and $|\partial f/\partial v|_M$. Then we have

$$\sum_{(m,n)\in K} f(m, n) = \iint_K f(u, v)du\, dv + J$$

and

$$|J| \ll |f|_M(u_M+v_M+1)+(|K|+v_M)\left|\frac{\partial f}{\partial v}\right|_M+|K|\left|\frac{\partial f}{\partial u}\right|_M .$$

(See Lemma 3.7 *in* [7].)

§ 4. Notation and terminology.

For simplicity, we use further notation;

$$\tau=\left(\frac{qT}{2\pi}\right)^{1/2}$$
$$\tau_1=\left(\frac{q(T+U)}{2\pi}\right)^{1/2}$$

k_1, k_2; variables which come from ψ.
j_1, j_2; variables which come from g_1 or g_2.

We put

$$k=(k_1, k_2)$$

and then

$$\begin{aligned} k_l &= kA_l \quad (l=1,\,2) \\ A_M &= \max\,(A_1,\,A_2) \\ k_M &= \max\,(k_1,\,k_2) \\ k_m &= \min\,(k_1,\,k_2) \\ T_1 &= \max\,(T,\,2\pi j_1^2/q,\,2\pi j_2^2/q)\ . \end{aligned}$$

We also assume

$$X \leqq \tau/(qL)\ ,$$

and

$$q \leqq T\ .$$

Let $\sum^*$ denote the summation over relatively prime j's or k's to q.

§ 5. Estimates of I_{11}, I_{22}, I_{33}, and I_{12}.

We can estimate I_{11}, I_{22}, I_{33}, and I_{12} by similar method to that of Levinson with aid of Lemma 3.1. Now we have

PROPOSITION 5.1. *We have*

$$I_{11} = \frac{\phi(q)}{q} UP_0(1,\,1-2\delta)\frac{\tau^{2\delta}}{2\delta} - \Big(\frac{\phi(q)}{q}\,\frac{1}{2\delta} - c_1^*\Big) US_0 + O(R)$$

$$\begin{aligned} I_{22} = {} & \frac{\phi(q)}{q} U\tau^{2\delta}\Big\{\Big(\frac{2}{(2\delta)^3L^2} - \frac{1}{2(2\delta)^2L}\Big)P_0(1,\,1-2\delta) \\ & + \Big(-\frac{1}{(2\delta)^2L^2} + \frac{1}{2(2\delta)L}\Big)P_1(1,\,1-2\delta)\Big\} \\ & + \frac{\phi(q)}{q}\,\frac{U}{L^2}\Big\{-\Big(\frac{2}{(2\delta)^3} - \frac{q}{\phi(q)}c_3^*\Big)S_0 + \Big(\frac{2}{(2\delta)^2} + \frac{q}{\phi(q)}2c_2^*\Big)S_1 \\ & - \Big(\frac{1}{2\delta} - \frac{q}{\phi(q)}c_1^*\Big)S_2\Big\} + O(R) \end{aligned}$$

$$\begin{aligned} I_{33} = {} & \frac{\phi(q)}{q} U\tau^{2\delta}\Big\{\Big(-\frac{2}{(2\delta)^3L^2} - \frac{1}{2(2\delta)^2L}\Big)P_0(1-2\delta,\,1) \\ & + \Big(-\frac{1}{(2\delta)^2L^2} - \frac{1}{2(2\delta)L}\Big)P_1(1-2\delta,\,1)\Big\} \\ & + \frac{\phi(q)}{q}\,\frac{U\tau^{4\delta}}{L^2}\Big\{\Big(\frac{2}{(2\delta)^3} + \frac{q}{\phi(q)}c_6^*\Big)S_0' + \Big(\frac{2}{(2\delta)^2} + \frac{q}{\phi(q)}2c_5^*\Big)S_1' \\ & + \Big(\frac{1}{2\delta} + \frac{q}{\phi(q)}c_4^*\Big)S_2'\Big\} + O(R) \end{aligned}$$

$$2\,\mathrm{Re}\,I_{12}=I_{12}+I_{21}$$
$$=\frac{\phi(q)}{q}U\tau^{2\delta}\Big\{\Big(-\frac{2}{(2\delta)^2L}+\frac{1}{2(2\delta)}\Big)P_0(1,1-2\delta)+\frac{1}{2\delta L}P_1(1,1-2\delta)$$
$$+\frac{\phi(q)}{q}\frac{U}{L}\Big(\frac{2}{(2\delta)^2}+\frac{2q}{\phi(q)}c_2^*\Big)S_0+2\Big(-\frac{1}{2\delta}+\frac{q}{\phi(q)}c_1^*\Big)S_1\Big\}+O(R)\,,$$

where

$$R=\tau XL+U^2L^3/T+\phi(q)XUL^3/\tau\,,$$

and

$$S_0=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k^{1-2\delta}}{(k_1k_2)^{1-2\delta}}\,,$$
$$S_1=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k^{1-2\delta}}{(k_1k_2)^{1-2\delta}}\log(k_1/k)\,,$$
$$S_2=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k^{1-2\delta}}{(k_1k_2)^{1-2\delta}}\log(k_1/k)\log(k_2/k)\,,$$
$$S_0'=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k^{1+2\delta}}{k_1k_2}\,,$$
$$S_1'=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k^{1+2\delta}}{k_1k_2}\log(k_1/k)\,,$$
$$S_2'=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k^{1+2\delta}}{k_1k_2}\log(k_1/k)\log(k_2/k)\,,$$
$$P_0(\alpha,\beta)=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k}{k_M^{\alpha}k_m^{\beta}}\,,$$
$$P_1(\alpha,\beta)=\sum_{1\le k_1,k_2\le X}^{*}\frac{b_{k_1}b_{k_2}k}{k_M^{\alpha}k_m^{\beta}}\log(\tau k_m/k_M)\,,$$

(c_k^* is a constant satisfying an inequality similar to that in Lemma 3.1 *for $k=1, 2,$ and* 3. *For $k=4, 5,$ and* 6 *c_k^* is also a constant satisfying the same inequality as c_{k-3}^* does.)*

§ 6. Estimates of I_{13} and I_{23}.

From the definition, we have

$$(6.1)\qquad I_{13}=\overline{\varepsilon(\chi)}\int_T^{T+U}\sum_{\substack{1\le k_1,k_2\le X\\ 1\le j_1,j_2\le(qt/2\pi)^{1/2}}}\frac{b_{k_1}b_{k_2}\chi(k_1j_1j_2)\overline{\chi}(k_2)\log j_2}{(k_1k_2j_1)^{1/2-\delta}j_2^{1/2+\delta}}$$

$$\times \frac{\left(\frac{qt}{2\pi}\right)^{\delta}}{\log \frac{qt}{2\pi}} \exp\left(\frac{\pi}{2}\left(a-\frac{1}{2}\right)i + it \log \frac{k_2 qt}{2\pi e k_1 j_1 j_2}\right)$$

$$= \overline{\varepsilon(\chi)} \exp\left(\frac{\pi}{2}\left(a-\frac{1}{2}\right)i\right) \sum_{\substack{1\leqq k_1, k_2 \leqq X \\ 1 \leqq j_1, j_2 \leqq \tau_1}} \frac{b_{k_1} b_{k_2} \chi(k_1 j_1 j_2) \overline{\chi}(k_2) \log j_2}{(k_1 k_2 j_1)^{1/2-\delta} j_2^{1/2+\delta}}$$

$$\times \int_T^{T+U} \left(\frac{qt}{2\pi}\right)^{\delta} \frac{\exp\left(it \log \frac{k_2 qt}{2\pi e k_1 j_1 j_2}\right)}{\log \frac{qt}{2\pi}} dt$$

$$= I_{13}^{(0)} + I_{13}^{(R)} ,$$

say, where $I_{13}^{(R)}$ denotes the summation over the error terms when we apply Lemma 3.2 to above inner integrals. This term is estimated by similar method to that of Levinson. Namely

$$|I_{13}^{(R)}| \ll \tau X L^3 .$$

Hence we have

$$I_{13}^{(0)} = \overline{\varepsilon(\chi)} \exp\left(\frac{\pi}{2}(a-1/2)i\right) \sum_{\substack{1\leqq k_1, k_2 \leqq X \\ 1 \leqq j_1, j_2 \leqq \tau_1}}' \frac{b_{k_1} b_{k_2} \chi(k_1 j_1 j_2) \overline{\chi}(k_2) \log j_2}{(k_1 k_2 j_1)^{1/2-\delta} j_2^{1/2+\delta}}$$
$$\times q^{\delta} (2\pi)^{1/2-\delta} \left(\frac{2\pi j_1 j_2 k_1}{q k_2}\right)^{1/2+\delta} \exp\left(-\frac{2\pi i k_1 j_1 j_2}{q k_2} + \frac{\pi i}{4}\right) \Big/ \log \frac{j_1 j_2 k_1}{k_2} , \tag{6.2}$$

where $\sum'$ means the summation over

$$T_1 \leqq 2\pi j_1 j_2 k_1/(q k_2) \leqq T + U . \tag{6.3}$$

This condition is equivalent to

$$\frac{qTk_2}{2\pi k_1} \leqq j_1 j_2 \leqq \frac{q(T+U)k_2}{2\pi k_1} \tag{6.4}$$

and

$$\frac{j_1 k_2}{k_1} \leqq j_2 \leqq \frac{j_1 k_1}{k_2} . \tag{6.5}$$

From (6.5) we may assume that

$$k_2 \leqq k_1 . \tag{6.6}$$

First we consider a sum

(6.7) $$\sum_I = \sum_{j_1}{}' \chi(j_1) \exp\left(-\frac{2\pi i j_1 j_2 k_1}{qk_2}\right).$$

We divide this sum into q sums according to $j_1 \equiv l_1 (\bmod q)$. For each sum, we can apply the same method as that of Levinson.

We define l_2, $0 \leqq l_2 < A_2$, by

$$j_2 A_1 \equiv l_2 \bmod A_2 .$$

If $l_2 \neq 0$, we get

$$\sum_I = O\left(\phi(q)\left(\frac{A_2}{l_2} + \frac{A_2}{A_2 - l_2}\right)\right).$$

Using the partial summation method, we get

$$\sum_{j_1}{}' \frac{j_1^{2\delta}}{\log (j_1 j_2 k_1/k_2)} \exp\left(-\frac{2\pi i j_1 j_2 k_1}{qk_2}\right) = O\left(\phi(q)\left(\frac{A_2}{l_2} + \frac{A_2}{A_2 - l_2}\right)\right),$$

for $l_2 \neq 0$. Since $(A_1, A_2) = 1$, we have

$$\begin{aligned} &\sum_{\substack{j_1, j_2 \\ j_2 A_1 \not\equiv 0 \bmod A_2}}{}' j_1^{2\delta} \log j_2 \chi(j_1 j_2) \exp\left(-\frac{2\pi i j_1 j_2 k_1}{qk_2}\right) \Big/ \log \frac{j_1 j_2 k_1}{k_2} \\ &= O\left(\left(\frac{\tau_1}{A_2} + 1\right)\phi(q) \sum_{l_2=1}^{A_2-1}\left(\frac{A_2}{l_2} + \frac{A_2}{A_2 - l_2}\right)\right) \\ &= O(\phi(q)\tau L) . \end{aligned}$$

Therefore, the contribution to $I_{13}^{(0)}$ from these terms is at most

$$O(\phi(q) q^{-1/2} \tau X L^2) .$$

Hence we have

(6.8) $$I_{13}^{(0)} = I_{13}^{(1)} + O(q^{1/2} \tau X L^2) ,$$

where

$$\begin{aligned} I_{13}^{(1)} = \overline{\varepsilon(\chi)} q^{-1/2} \exp\left(\frac{\pi}{2} a i\right) 2\pi \sum_{\substack{1 \le k_1, k_2 \le X \\ 1 \le j_1, j_2 \le \tau_1 \\ j_2 A_1 \equiv 0 \bmod A_2}}{}' \frac{b_{k_1} b_{k_2} \chi(j_1 j_2 k_1) \overline{\chi}(k_2) \log j_2}{k_1^{-2\delta} k_2 j_1^{-2\delta}} \\ \times \exp\left(-\frac{2\pi i j_1 j_2 k_1}{qk_2}\right) \Big/ \log \frac{j_1 j_2 k_1}{k_2} . \end{aligned}$$

$$\begin{aligned} I_{13}^{(1)} = \overline{\varepsilon(\chi)} q^{-1/2} \exp\left(\frac{\pi}{2} a i\right) 2\pi \sum_{1 \le k_2 \le k_1 \le X}{}^* \frac{b_{k_1} b_{k_2} k_1^{2\delta}}{k_2} \\ \times \sum_{(j_1, j) \in K} j_1^{2\delta} \frac{\log jA_2}{\log j_1 j A_1} \chi(A_1 j_1 j) \exp\left(\frac{-2\pi i j_1 j A_1}{j}\right), \end{aligned}$$

where K is the region given by

$$\frac{qT}{2\pi A_1} \leqq uv \leqq \frac{q(T+U)}{2\pi A_1} \quad \text{and} \quad \frac{u}{A_1} \leqq v \leqq \frac{uA_1}{A_2^2} .$$

Since

$$\frac{1}{\log j_1 j A_1} - \frac{1}{L} = O\left(\frac{U}{TL^2}\right)$$

for $(j_1, j) \in K$, we get

$$I_{13}^{(0)} = \frac{1}{L} I_{13}^{(2)} + O\left(\frac{U}{TL^2} q^{-1/2} \sum_{k_1, k_2} \frac{1}{k_2} \sum_{(j_1, j) \in K} L\right) . \tag{6.9}$$

Because

$$\begin{aligned} \sum_{(j_1, j) \in K} 1 &\ll \sum_j \left(\frac{qU}{2\pi A_1 j} + 1\right) \\ &\ll \frac{qUL}{A_1} + \frac{\tau}{A_2} , \end{aligned}$$

the error term in (6.9) is at most

$$\begin{aligned} &\ll \frac{U}{TL} q^{-1/2} \sum_{k_1, k_2} \left(\frac{qUL}{A_1} + \frac{\tau}{A_2}\right) \frac{1}{k_2} \\ &\ll q^{1/2} \frac{U^2}{T} L^3 + \frac{UX}{T^{1/2}} , \end{aligned}$$

using Lemma 3.6 in [7]. We have

$$\begin{aligned} I_{13}^{(2)} = \overline{\varepsilon(\chi)} q^{-1/2} e^{(\pi/2)ai} 2\pi \sideset{}{^*}\sum_{1 \leqq k_2 \leqq k_1 \leqq X} \frac{b_{k_1} b_{k_2} k_1^{2\delta}}{k_2} \sum_{(j_1, j) \in K} j_1^{2\delta} \log j A_2 \chi(A_1 j_1 j) \\ \times \exp\left(-\frac{2\pi i j_1 j A_1}{q}\right) . \end{aligned} \tag{6.10}$$

Put

$$I_{13}^{(4)}(l_1, l) = \sum_{\substack{j_1 \equiv l_1 \bmod q \\ j \equiv l \bmod q \\ (j_1, j) \in K}} j_1^{2\delta} \log j A_2 . \tag{6.11}$$

Then the above inner most sum in (6.10) is given by

$$I_{13}^{(3)} = \sum_{\substack{0 \leqq l_1 \leqq q-1 \\ 0 \leqq l \leqq q-1}} I_{13}^{(4)}(l_1, l) \chi(A_1 l_1 l) \exp\left(-\frac{2\pi i l_1 l A_1}{q}\right) . \tag{6.12}$$

Now we calculate $I_{13}^{(4)}$; we put

$$j_1 = j_1^* q + l_1 \quad \text{and} \quad j = j^* q + l . \tag{6.13}$$

Then the condition on j_1^* and j^* is that

$$(j_1^*, j^*) \in K(l_1, l) ,$$

where $K(l_1, l)$ is the region defined by

$$\frac{qT}{2\pi A_1} \leqq (qu+l_1)(qv+l) \leqq \frac{q(T+U)}{2\pi A_1} , \quad \frac{qu+l}{A_1} \leqq qv+l \leqq \frac{qu+l_1}{A_2^2} A_1 .$$

Applying Lemma 3.3, we have

$$(6.14) \qquad I_{13}^{(4)}(l_1, l) = \sum_{(j_1^*, j) \in K(l_1, l)} (qj_1^*+l_1)^{2\delta} \log (qj^*+l) A_2$$
$$= \iint_{K(l_1, l)} (qu+l_1)^{2\delta} \log (qv+l) A_2 \, du dv + O(R(l_1, l)) .$$

Using the same notation in Lemma 3.3, we can easily estimate

$$|u_M| \ll \frac{\tau}{q}, \ |v_M| \ll \frac{\tau}{qA_2}, \ |f_M| \ll L, \ \left|\frac{\partial f}{\partial u}\right|_M \ll \frac{qA_1}{\tau A_2}, \ \left|\frac{\partial f}{\partial v}\right|_M \ll \frac{qA_1}{\tau}$$

and

$$\sum_{\substack{0 \leq l_1 < q \\ 0 \leq l < q}} |K(l_1, l)| = \sum_{\substack{0 \leq l_1 < q \\ 0 \leq l < q}} \iint_{(u,v) \in K(l_1, l)} du dv$$
$$= \frac{1}{q^2} \sum_{\substack{0 \leq l_1 < q \\ 0 \leq l < q}} \iint_{(u,v) \in K} du dv$$
$$= |K|$$
$$= \int_{\tau/A_1}^{A_2} (\tau_1^2 - \tau^2) \frac{dv}{A_1 v}$$
$$\ll \frac{qU}{A_1} L .$$

Hence the contribution of $R(l_1, l)$'s to $I_{13}^{(2)}$ is at most

$$(6.15) \qquad \ll q^{-1/2} \sum_{1 \leq k_1, k_2 \leq X} \frac{1}{k_2} \sum{}^* R(l_1, l)$$
$$\ll q^{-1/2} \sum_{1 \leq k_1, k_2 \leq X} \frac{1}{k_2} \sum \left(L\left(\frac{\tau}{q}+1\right) + |K(l_1, l)| \frac{qA_1}{\tau} + \frac{A_2}{A_1} \right)$$
$$\ll q^{-1/2} \tau X L^2 + \frac{q^{1/2} UXL}{\tau} + q^{3/2} X^2 .$$

Since

$$\iint_{K(l_1,l)} (qu+l_1)^{2\delta} \log (qv+l) A_2 du dv = \frac{1}{q^2} \iint_K u^{2\delta} \log v A_2 \, du dv$$
$$= \frac{1}{q^2} F \qquad \text{(say!)}$$

and F is independent of l_1 and l, we get from (6.12)

$$(6.16) \qquad I_{13}^{(3)} = \frac{F}{q^2} \sum_{\substack{0 \leq l_1 < q \\ 0 \leq l < q}}^{*} \chi(A_1 l_1 l) \exp(-2\pi i l_1 l A_1/q) + O\Big(\sum_{\substack{0 \leq l_1 < q \\ 0 \leq l > q}} R(l_1, l) \Big) .$$

Because of $(A_1 l_1, q) = 1$, we have

$$(6.17) \qquad \sum_{\substack{0 \leq l_1 < q \\ 0 \leq l < q}}^{*} \chi(A_1 l_1 l) \exp(-2\pi i l_1 l A_1/q) = \phi(q) \sum_{m=1}^{q}{}^{*} \chi(-m) \exp(2\pi i m/q)$$
$$= \phi(q) q^{1/2} i^{-a} \varepsilon(\chi) .$$

From (6.10)-(6.12) and (6.13)-(6.17), we have

$$I_{13}^{(2)} = 2\pi \frac{\phi(q)}{q^2} \sum_{1 \leq k_2 \leq k_1 \leq X}^{*} \frac{b_{k_1} b_{k_2} k_1^{2\delta}}{k_2} F + O(q^{1/2} \tau X L^2 + q^{1/2} U X L/\tau + q^{3/2} X^2) .$$

On the other hand, we can easily see that

$$F = \frac{1}{8\pi} \tau^{2\delta} q U \Big(-\frac{A_2^{2\delta} L}{\delta A_1^{1+2\delta}} + \frac{2}{\delta A_1} \log\Big(\frac{\tau k_2}{k_1}\Big) + \frac{1}{\delta^2 A_1} - \frac{A_2^{2\delta}}{\delta^2 A_1^{1+2\delta}} \Big) + O\Big(\frac{UL}{T}\Big) .$$

We remark that, for $k_1 = k_2$, the main term of F is zero. From (6.6), we get

$$A_M = A_1 , \quad k_M = k_1 .$$

Using these facts, we finally have

$$(6.18) \qquad 2 \operatorname{Re} I_{13}^{(2)} = \frac{1}{4} \frac{\phi(q)}{q} \tau^{2\delta} U \Big\{ \Big(-\frac{L}{\delta} - \frac{1}{\delta^2} \Big) P_0(1, 1-2\delta) + \frac{1}{\delta^2} P_0(1-2\delta, 1)$$
$$+ \frac{1}{\delta} P_1(1-2\delta, 1) \Big\} + O(q^{1/2} \tau X L^2 + q^{1/2} U X L^2/\tau + q^{3/2} X^2) .$$

Combining (6.1), (6.2), (6.8), (6, 9), and (6.18), we have just estimated I_{13}. We can also estimate I_{23} in similar lines. We have

PROPOSITION 6.1. *We have*

$$2\,\mathrm{Re}\,I_{13}=I_{13}+I_{31}$$

$$=\frac{1}{4}\,\frac{\phi(q)}{q}\tau^{2\delta}U\Big\{\Big(-\frac{1}{\delta}-\frac{1}{\delta^2L}\Big)P_0(1,\,1-2\delta)+\frac{1}{\delta^2L}P_0(1-2\delta,\,1)$$

$$+\frac{2}{\delta L}P_1(1-2\delta,\,1)\Big\}$$

$$+O(q^{1/2}\tau XL^3+q^{1/2}U^2L^3/T+q^{1/2}UXL^2/\tau+q^{3/2}X^2)\ ,$$

$$2\,\mathrm{Re}\,I_{23}=I_{23}+I_{32}$$

$$=\frac{1}{4}\,\frac{\phi(q)}{q}\tau^{2\delta}U\Big\{-\frac{1}{\delta^3L^2}(P_0(1-2\delta,\,1)-P_0(1,\,1-2\delta))$$

$$+\frac{1}{2\delta^2L}(P_0(1-2\delta,\,1)+P_0(1,\,1-2\delta))$$

$$-\frac{1}{\delta^2L^2}(P_1(1-2\delta,\,1)+P_1(1,\,1-2\delta))$$

$$+\frac{1}{\delta L}(P_1(1-2\delta,\,1)-P_1(1,\,1-2\delta))\Big\}$$

$$+O(q^{1/2}\tau XL^3+q^{1/2}U^2L^3/T+q^{1/2}UXL^2/\tau+q^{3/2}X^2)\ .$$

§ 7. Evaluation of the sum of I's.

From Propositions 5.1 and 6.1, we can see that all main terms of I's are different from those of [7] by multiple of $\phi(q)/q$. Hence terms of P's are cancelled as in [7]. Now we have

$$\int_T^{T+U}\Big|\psi H\Big(\frac{1}{\delta}-\delta+it,\,\chi\Big)\Big|^2dt$$

$$=\frac{\phi(q)}{q}U\Big\{S_0\Big[-\Big(\frac{1}{2\delta}-\frac{q}{\phi(q)}c_1^*\Big)-\frac{1}{L^2}\Big(\frac{2}{(2\delta)^3}-\frac{q}{\phi(q)}c_3^*\Big)$$

$$-\frac{1}{L}\Big(\frac{2}{(2\delta)^2}+\frac{2q}{\phi(q)}c_2^*\Big)\Big]$$

$$+S_1\Big[\frac{1}{L^2}\Big(\frac{2}{(2\delta)^2}+\frac{2q}{\phi(q)}c_2^*\Big)-\frac{2}{L}\Big(-\frac{1}{2\delta}+\frac{q}{\phi(q)}c_1^*\Big)\Big]$$

$$+S_2\Big[-\frac{1}{L^2}\Big(\frac{1}{2\delta}-\frac{q}{\phi(q)}c_1^*\Big)\Big]$$

$$+S_0'\frac{\tau^{4\delta}}{L^2}\Big(\frac{2}{(2\delta)^3}+\frac{q}{\phi(q)}c_6^*\Big)$$

$$+S_1'\frac{\tau^{4\delta}}{L^2}\Big(\frac{2}{(2\delta)^2}+\frac{2q}{\phi(q)}c_5^*\Big)$$

$$+S_2'\frac{\tau^{4\delta}}{L^2}\left(\frac{1}{2\delta}+\frac{2q}{\phi(q)}c_4^*\right)\Big\}$$
$$+O(q^{1/2}\tau XL^3+q^{1/2}L^2U^3/T+qUXL^2/\tau+q^{3/2}X^2)\,.$$

§ **8. Calculation of S's.**

First we define some auxiliary functions;

$$F(n,\,w)=\prod_{p|n}\left(1-\frac{1}{p^w}\right)$$
$$F_1(n,\,w)=\prod_{p|n}\left(1+\frac{1}{p^w}\right)$$
$$f_0(c,\,d)=\sum_{1\leq n\leq X/d}{}^{*}\frac{b_{dn}}{n^{1-c\delta}}$$
$$f_1(c,\,d)=\sum_{1\leq n\leq X/d}{}^{*}\frac{b_{dn}\log\dfrac{X}{dn}}{n^{1-c\delta}}\,.$$

Now we can prove

LEMMA 8.1. *We have*

$$S_0=\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1-2\delta)}{d^{1-2\delta}}f_0^2(2,\,d)$$
$$S_1=\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1-2\delta)}{d^{1-2\delta}}f_0(2,\,d)\left(\log\frac{X}{d}f_0(2,\,d)-f_1(2,\,d)\right)$$
$$-\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1-2\delta)}{d^{1-2\delta}}\left(\sum_{p|d}\frac{\log p}{p^{1-2\delta}-1}\right)f_0^2(2,\,d)$$
$$S_2=\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1-2\delta)}{d^{1-2\delta}}\left(\log\frac{X}{d}f_0(2,\,d)-f_1(2,\,d)\right)^2$$
$$-2\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1-2\delta)}{d^{1-2\delta}}\left(\sum_{p|d}\frac{\log p}{p^{1-2\delta}-1}\right)\left(\log\frac{X}{d}f_0(2,\,d)-f_1(2,\,d)\right)f_0(2,\,d)$$
$$+\sum_{1\leq d\leq X}{}^{*}\frac{1}{d^{1-2\delta}}\frac{\partial^2F}{\partial w^2}(d,\,1-2\delta)f_0^2(2,\,d)$$
$$S_0'=\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1+2\delta)}{d^{1-2\delta}}f_0^2(0,\,d)$$
$$S_1'=\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1+2\delta)}{d^{1-2\delta}}f_0(0,\,d)\left(\log\frac{X}{d}f_0(0,\,d)-f_1(0,\,d)\right)$$
$$-\sum_{1\leq d\leq X}{}^{*}\frac{F(d,\,1+2\delta)}{d^{1-2\delta}}\left(\sum_{p|d}\frac{\log p}{p^{1+2\delta}-1}\right)f_0^2(0,\,d)$$

$$S_2'=\sum_{1\leq d\leq X}{}^{*}\frac{F(d,1+2\delta)}{d^{1-2\delta}}\Big(\log\frac{X}{d}f_0(0,d)-f_1(0,d)\Big)^2$$
$$-2\sum_{1\leq d\leq X}{}^{*}\frac{F(d,1+2\delta)}{d^{1-2\delta}}\Big(\sum_{p|d}\frac{\log p}{p^{1+2\delta}-1}\Big)\Big(\log\frac{X}{d}f_0(0,d)-f_1(0,d)\Big)f_0(0,d)$$
$$+\sum_{1\leq d\leq X}{}^{*}\frac{1}{d^{1-2\delta}}\frac{\partial^2 F}{\partial w^2}(d,1+2\delta)f_0^2(0,d)\,.$$

These formulas are proved by the same method of [7]. Now we need exact form of b_n, since we must calculate f_0 and f_1. We put

$$b_n=\frac{\mu(n)}{n^{\delta}}\frac{\log\frac{X}{n}}{\log X}$$

as in [7]. Then we get

$$f_l(c,d)=\frac{\mu(d)}{d^{\delta}}\frac{1}{\log X}\sum_{\substack{1\leq n\leq X/d\\(n,dq)=1}}\frac{\mu(n)\log^{l+1}\frac{X}{nd}}{n^{1-(c-1)\delta}}\,.$$

Now we calculate

$$f^*(Y,l)=\sum_{\substack{1\leq n\leq Y\\(n,dq)=1}}\frac{\mu(n)\log^{l}\frac{Y}{n}}{n^{1-c\delta}}$$

for $l=1$ and 2.

LEMMA 8.2. *For $c=\pm1$ and $\log Y\ll L$, there exists some absolute constant c_1 such that we have*

$$f^*(Y,1)=\frac{1-c\delta\log Y}{F(dq,1-c\delta)}+O\Big(\frac{\log^2 L}{L}+Y^{\sigma_0-1}\log^{c_1}L\Big)\,,$$
$$f^*(Y,2)=\frac{2\log Y-c\delta\log^2 Y}{F(dq,1-c\delta)}+O(\log^3 L+Y^{\sigma_0-1}\log^{c_1}L)\,,$$

where σ_0 is a constant defined by the fact that $\zeta(s)$ is zero-free in the region

$$\sigma\geqq 1-2(1-\sigma_0)\,,\quad |t|\leqq L^2\,.$$

These estimates are independent of q and L.

PROOF. These calculation will be done by a similar method as in [7]. But above estimates are a little different from those of [7]. So we only mention the different part. We may put

$$\sigma_0 = 1 - \frac{c}{\log L},$$

where c is a constant. In [7], the error term of $f^*(Y, 1)$ is of the form

$$O\left(\frac{\log^2 L}{L} + F_1(dq, \sigma_0) Y^{\sigma_0 - 1}\right).$$

Since, for $0 < \alpha < 1/3$, we have

$$\begin{aligned} \log F_1(n, 1-\alpha) &\leqq \sum_{p \mid n} \frac{1}{p^{1-\alpha}} + O(1) \\ &\leqq \sum_{p \leqq (\log n)^{1/(1-\alpha)}} \frac{1}{p^{1-\alpha}} + O\left(\frac{\omega(n)}{\log n} + 1\right) \\ &\leqq (\log n)^{\alpha/(1-\alpha)} (\log \log \log n + O(1)) . \end{aligned}$$

Hence we get

$$F_1(dq, \sigma_0) \ll (\log L)^{c'}$$

for some c'. Q.E.D.

From this lemma we can easily deduce

LEMMA 8.3. *We have*

$$S_0 = \frac{1}{\log^2 X} \sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(n)}{d} \frac{F(d, 1-2\delta)}{F^2(dq, 1-\delta)} \left(1 - \delta \log \frac{X}{d}\right)^2 + O\left(\frac{\log^{c_2} L}{L^2}\right)$$

$$S_1 = \frac{1}{\log^2 X} \sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(d)}{d} \frac{F(d, 1-2\delta)}{F^2(dq, 1-\delta)} \left(\delta \log^2 \frac{X}{d} - \log \frac{X}{d}\right) + O\left(\frac{\log^{c_2} L}{L}\right)$$

$$S_2 = \frac{1}{\log^2 X} \sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(d)}{d} \frac{F(d, 1-2\delta)}{F^2(dq, 1-\delta)} \log^2 \frac{X}{d} + O(\log^{c_2} L)$$

$$S_0' = \frac{1}{\log^2 X} \sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(d)}{d} \frac{F(d, 1+2\delta)}{F^2(dq, 1+\delta)} \left(1 + \delta \log \frac{X}{d}\right)^2 + O\left(\frac{\log^{c_2} L}{L^2}\right)$$

$$S_1' = \frac{-1}{\log^2 X} \sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(d)}{d} \frac{F(d, 1+2\delta)}{F^2(dq, 1+\delta)} \left(\delta \log^2 \frac{X}{d} \log \frac{X}{d}\right) + O\left(\frac{\log^{c_2} L}{L}\right)$$

$$S_2' = \frac{1}{\log^2 X} \sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(d)}{d} \frac{F(d, 1+2\delta)}{F^2(dq, 1+\delta)} \log^2 \frac{X}{d} + O(\log^{c_2} L) .$$

All main terms in the above lemma are linear forms of type

$$\sideset{}{^*}\sum_{1 \leqq d \leqq X} \frac{\mu^2(d)}{d} \frac{F(d, 1+2c\delta)}{F^2(dq, 1+c\delta)} \log^l \frac{X}{d},$$

for $l=0, 1$ and 2, and $c=\pm 1$. Now we cannot apply Lemmas 3.11–3.13 in [7] directly because these terms depend on q as well as X. So we use the same method appearing in the proof of Lemma 8.2. Then we have

LEMMA 8.4. *For $c=\pm 1$, $l=0, 1$, and 2, and $X \leqq Y \leqq 2X$, we have*

$$\sum_{1\leqq d\leqq Y}^{*} \frac{\mu^2(d)}{d} \frac{F(d, 1+2c\delta)}{F^2(dq, 1+c\delta)} \log^l \frac{Y}{d} = \frac{1}{l+1} \frac{\log^{l+1} Y}{F(q, 1)} + O(L^l \log^4 L) .$$

Now we get from Lemmas 8.3 and 8.4.

PROPOSITION 8.5. *We have*

$$S_0 = \frac{1}{F(q, 1)}\left(\frac{1}{\log X} - \delta + \frac{\delta^2 \log X}{3}\right) + O\left(\frac{\log^{c_3} L}{L^2}\right)$$
$$S_1 = \frac{1}{F(q, 1)}\left(-\frac{1}{2} + \frac{\delta \log X}{3}\right) + O\left(\frac{\log^{c_3} L}{L}\right)$$
$$S_2 = \frac{1}{F(q, 1)} \frac{\log X}{3} + O(\log^{c_3} L)$$
$$S'_0 = \frac{1}{F(q, 1)}\left(\frac{1}{\log X} + \delta + \frac{\delta^2 \log X}{3}\right) + O\left(\frac{\log^{c_3} L}{L^2}\right)$$
$$S'_1 = \frac{1}{F(q, 1)}\left(-\frac{1}{2} - \frac{\delta \log X}{3}\right) + O\left(\frac{\log^{c_3} L}{L}\right)$$
$$S'_2 = \frac{1}{F(q, 1)} \frac{\log X}{3} + O(\log^{c_3} L) ,$$

for some constant c_3.

§ 9. Proof of main theorem.

Since

$$F(q, 1) = \frac{\phi(q)}{q} ,$$

we have from §7 and Proposition 8.5

$$(9.1) \quad \frac{1}{U} \int_T^{T+U} \left|\psi H\left(\frac{1}{2} - \delta + it\right)\right|^2 dt$$
$$= \frac{L}{\log X}\left(-\frac{1}{2R} - \frac{1}{2R^2} - \frac{1}{4R^3}\right) + \frac{1}{2}$$

$$+\frac{\log X}{3L}\left(\frac{1}{2}-\frac{R}{2}-\frac{1}{4R}\right)$$
$$+e^{2R}\left(\frac{L}{\log X}\frac{1}{4R^3}+\frac{\log X}{3L}\frac{1}{4R}\right)$$
$$+O\left(\frac{\log^{c_4} L}{L}+q^{1/2}\tau XL^3/U+q^{1/2}UL^3/T+qXL^2/\tau+q^{3/2}X^2/U\right)$$

where

$$R=\delta L\ .$$

Now we put

$$U=\frac{T}{qL^4}\ ,\quad X=\frac{\tau}{q^{5/2}L^8}\ .$$

Then we see that above error term is $O(\log^{c_4} L/L)$. We also assume that, for $\varepsilon>0$,

$$\log q \leqq (\log T)^{1-\varepsilon}\ .$$

Hence we have

$$\frac{\log X}{L}=\frac{1}{2}+O(L^{-\varepsilon})\ .$$

Now the main term of (9.1) becomes of the form

$$F(R)+O(L^{-\varepsilon})\ ,$$

where

(9.2) $$F(R)=e^{2R}\left(\frac{1}{2R^3}+\frac{1}{24R}\right)-\frac{1}{2R^3}-\frac{1}{R^2}-\frac{25}{24R}+\frac{7}{12}-\frac{R}{12}\ .$$

From (2.2), (2.11), (9, 1), and (9.2), we get

$$N_G(D)\leqq\frac{UL}{2\pi}\frac{\log F(R)}{R}+O(UL^{1-\varepsilon})\ .$$

We put $R=1.3$, then we have

$$\log F(R)/R<\frac{1}{3}\ .$$

Hence we have just proved main theorem.

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Present Address:
DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOICHO, CHIYODA-KU, TOKYO 102