TOKYO J. MATH.
VOL. 1, NO. 2, 1978

On Graded Rings, II (Z^n-graded rings)

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Introduction

Let k be a field, S an affine semigroup, i.e., a finitely generated additive submonoid of N^n, and $k[S]$ the semigroup ring of S over k. Then S is called normal if the ring $k[S]$ is integrally closed. (This condition does not depend on the field k. See Proposition 1, [10].) In [10] Hochster proved that $k[S]$ is a Macaulay ring if S is a normal semigroup and deduced from this fact that, if G is a torus over k and if G acts on a finite-dimensional vector space V over k rationally, then the ring A^G of invariants under the induced action of G on the symmetric algebra A of V is a Macaulay ring. (His proof of the above fact on semigroup rings depends on a certain result concerning the shellability of real polytopes.) Further in [18] Stanley studied the Hilbert functions of the algebra $k[S]$ and gave a criterion of $k[S]$ to be a Gorenstein ring in case S is a normal semigroup. It seems to be interesting to ask when the ring $k[S]$ is Macaulay (resp. Gorenstein) in case S is not necessarily normal.

The main purpose of our paper is to give a purely ring-theoretic proof of the Hochster's result on normal semigroups and, applying our way of proof further to arbitrary affine semigroups S, to find a criterion of the ring $k[S]$ to be Macaulay (resp. Gorenstein) in terms of S. Note that this was achieved by the authors and Suzuki [5] in case S is a simplicial monoid.

For this purpose we will develope a certain theory of graded rings and modules. Let H be a finitely generated free abelian group. By definition, an H-graded ring is a commutative Noetherian ring R together with a family $\{R_h\}_{h\in H}$ of subgroups such that $R=\bigoplus_{h\in H}R_h$ and $R_hR_g\subset R_{h+g}$ for all $h, g\in H$. Similarly an H-graded R-module is an R-module M for which there is given a family $\{M_h\}_{h\in H}$ of subgroups so that $M=\bigoplus_{h\in H}M_h$ and $R_hM_g\subset M_{h+g}$ for all $h, g\in H$. A homomorphism

Received February 14, 1978

$f: M \to N$ of H-graded R-modules is an R-linear map such that $f(M_h) \subset N_h$ for all $h \in H$. We denote by $M_H(R)$ the category consisting of all the H-graded R-modules and their homomorphisms. Of course our typical examples of graded rings are semigroup rings $k[S]$. In fact, if we denote by T^s the image of $s \in S$ in $k[S]$, then $k[S]$ is naturally an H-graded ring for $H = Z^n$——$k[S]_h = kT^h$ if $h \in S$ and $k[S]_h = (0)$ if $h \notin S$.

In [4] the authors investigated Z-graded rings and, even though the main subject of the present paper is to study semigroup rings, it is at the same time one of our purposes to generalize the results of [4] to H-graded rings with H an arbitrary finitely generated free abelian group.

In Chapter 1 the local conditions of H-graded rings are studied. It is proved in Section 1 that every simple H-graded ring is an (abelian) group ring over a field. In Section 2 it is showed that various properties of H-graded rings—being Macaulay rings, Gorenstein rings, or regular rings—are determined by their graded local data only. In Section 3 the structure of minimal injective resolutions of H-graded R-modules in $M_H(R)$ is determined. We construct in Section 4 a complex associated with a given H-graded R-module, which we call the Cousin complex. We will classify the H-graded R-modules by the behavior of their Cousin complexes.

In Chapter 2 we assume that R is an H-graded ring defined over a field. We give in Section 1 a duality of Noetherian H-graded R-modules and Artinian H-graded R-modules. We will define the canonical module of R in Section 2 and give a condition of R to be a Gorenstein ring in terms of the canonical module.

Thus some of the results of [4] are generalized in these two chapters.

In Chapter 3 R is assumed to be an affine semigroup ring over a field. Section 1 is devoted to give some properties of R which we shall need later. In Section 2 we will compute the local cohomology modules of R and give an explicit description of the canonical module. From them we deduce the Hochster's result on normal semigroup rings. These prepare to establish a criterion of the ring R to be Macaulay (resp. Gorenstein) for more general semigroup rings R, which we will find in Section 3. We will give in Section 2, as an application of the above results, a necessary and sufficient condition of the Rees algebra $\bigoplus_{i \geq 0} \mathfrak{a}^i$ to be a Gorenstein ring, where $\mathfrak{a}$ is the ideal $(X_1, X_2, \cdots, X_n)^d (d > 0)$ of the polynomial ring $k[X_1, X_2, \cdots, X_n]$ over a filed k.

Throughout this paper we denote by H a finitely generated free abelian group and by $R = \bigoplus_{h \in H} R_h$ a Noetherian H-graded ring.

CHAPTER I. LOCAL CONDITIONS OF H-GRADED RINGS AND MODULES.

Let $R=\bigoplus_{h\in H} R_h$ be a Noetherian H-graded ring with H a finitely generated free abelian group.

§ 1. The structure of simple H-graded rings.*

We begin with the definition of simple H-graded rings.

DEFINITION 1.1.1. We say that R is a simple H-graded ring if every non-zero homogeneous element of R is a unit. (This is equivalent to the condition that R has no proper H-graded ideals except (0).)

For example, let k be a field and G a subgroup of H. We denote by $k[G]$ the group ring of G over k and by T^g the image of $g\in G$ in $k[G]$. Then $k[G]$ can be regarded as an H-graded ring whose grading is given by

$$k[G]_h=\begin{cases} kT^h(h\in G) \\ (0)(h\notin G)\ . \end{cases}$$

Note that $k[G]$ is a regular *UFD*. (In fact, let $e_1, e_2, \cdots, e_n$ be a free basis of G and put $T_i=T^{e_i}$ for $1\leqq i\leqq n$. Then $T_1, T_2, \cdots, T_n$ are algebraically independent over the field k and $k[G]=k[T_1, T_2, \cdots, T_n, T_1^{-1}, T_2^{-1}, \cdots, T_n^{-1}]$.) Obviously $k[G]$ is a simple H-graded ring. The aim of this section is to show that the converse of this fact is also true, i.e., if R is a simple H-graded ring, then it is isomorphic to a group ring $k[G]$ for some field k and some subgroup G of H.

From now on, we put $k=R_0$ and $G=\{h\in H/R_h$ contains a unit of $R\}$. (Thus $G=\{h\in H/R_h\neq(0)\}$ if R is a simple H-graded ring.) Of course G is a subgroup of H. First we note

LEMMA 1.1.2. *Let $g\in G$ and u a unit of R contained in R_g. Then $u^{-1}\in R_{-g}$ and $R_g=ku$.*

COROLLARY 1.1.3. *Suppose that R is a simple H-graded ring. Then k is a field and $[R_g\colon k]=1$ for all $g\in G$.*

In order to state the main theorem we need some definitions. Let M be an H-graded R-module and $h\in H$. We denote by $M(h)$ the H-graded R-module which coincides with M as the underlying R-module and whose grading is given by $M(h)_g=M_{h+g}(g\in H)$. An H-graded R-module F is called free if it is isomorphic to a direct sum of H-graded

* In this section we do not need the Noetherian assumption on R.

R-modules of the form $R(h)$ $(h \in H)$. (This is equivalent to the condition that F has an R-free basis consisting of homogeneous elements.)

The next theorem is the main result of this section.

THEOREM 1.1.4. *The following conditions are equivalent.*

(1) *R is a simple H-graded ring.*

(2) *k is a field and $R \cong k[G]$ as H-graded k-algebras.*

(3) *Every H-graded R-module is free.*

PROOF. (1)⟹(2) Certainly k is a field. Now let $U(R)$ denote the group of all the units of R. Then there exists a homomorphism $f\colon G \to U(R)$ of groups such that $f(g) \in R_g$ for all $g \in G$. (In fact, let $e_1, e_2, \cdots, e_n$ be a free basis of the free abelian group G and T_i a unit of R contained in R_{e_i}. If we define $f(e_i) = T_i$ for every $1 \leqq i \leqq n$, then f has the required property. See 1.1.2.) By virtue of the universal property of the group ring $k[G]$, the map f can be extended to a homomorphism $\bar{f}\colon k[G] \to R$ of k-algebras. Obviously $\bar{f}$ is a desired isomorphism, as $[R_g\colon k] = 1$ for all $g \in G$ (cf. 1.1.3).

(2)⟹(1) This is trivial.

(1)⟹(3) Let $\{h_i\}_{i \in I}$ be a system of representatives of $H \bmod G$ and M an H-graded R-module. We put $M_i = \bigoplus_{g \in G} M_{g+h_i}$ and regard it as a G-graded R-module. (Of course we put $[M_i]_g = M_{g+h_i}$ for all $g \in G$. Note that R can be regarded as a G-graded ring, since $R_h = (0)$ if $h \notin G$.) Then, since $M = \bigoplus_{i \in I} M_i$ as R-modules, in order to show that M has an R-free basis consisting of homogeneous elements, we may assume $H = G$ without loss of generality. In this situation it is easy to prove that every k-basis of M_0 is an R-free basis of M.

(3)⟹(1) Let $\mathfrak{a}$ be an H-graded ideal of R $(\mathfrak{a} \neq R)$. Then, as $R/\mathfrak{a}$ is a non-zero free R-module, we have $(0)\colon R/\mathfrak{a} = (0)$. Thus $\mathfrak{a} = (0)$, and hence R is a simple H-graded ring. This completes the proof.

We will close this section with some definitions and a certain remark, which we shall need later. As above, we put $k = R_0$ and $G = \{h \in H / R_h$ contains a unit of $R\}$.

DEFINITION 1.1.5. Let $\mathfrak{m}$ be an H-graded ideal of R $(\mathfrak{m} \neq R)$. We say that $\mathfrak{m}$ is H-maximal if $R/\mathfrak{m}$ is a simple H-graded ring.

Of course every H-maximal ideal is a prime ideal of R, as every simple H-graded ring is an integral domain (cf. 1.1.4). Note that H-maximal ideals are not necessarily maximal ideals of R.

DEFINITION 1.1.6. We say that R is H-local if it has a unique H-maximal ideal.

Now let $R_G=\bigoplus_{g\in G} R_g$ and consider it an H-graded ring whose grading is given by

$$[R_G]_h=\begin{cases}R_h(h\in G)\\(0)(h\notin G)\ .\end{cases}$$

Note that R_G is a simple H-graded ring if k is a field. (In fact, let $g\in G$ and u a unit of R contained in R_g. Then, by 1.1.2, we have $R_g=ku$. Hence every non-zero element x of R_g has the form $cu(0\neq c\in k)$ and so it is a unit of R. Of course $x^{-1}=c^{-1}u^{-1}\in R_{-g}$ and hence x^{-1} is contained in R_G. Thus every non-zero homogeneous element of R_G is a unit of R_G.) Moreover we have

PROPOSITION 1.1.7. *Suppose that k is a field and put $\mathfrak{m}=\bigoplus_{h\notin G} R_h$. Then $(R, \mathfrak{m})$ is an H-local ring and $R/\mathfrak{m}\cong R_G$ as H-graded k-algebras.*

PROOF. First we will prove that $\mathfrak{m}$ is actually an ideal of R. Let $x\in R_h$ and $y\in R_g$ where $h, g\in H$. If $xy\notin\mathfrak{m}$, then we have $h+g\in G$. Therefore xy is a unit of R, as is mentioned above. Hence y is a unit of R and so $g\in G$. Thus $xy\in\mathfrak{m}$ if $g\notin G$ and we have the assertion. Of course $\mathfrak{m}$ is a unique H-maximal ideal of R, because every homogeneous element of R not contained in $\mathfrak{m}$ is a unit of R. It is clear that $R/\mathfrak{m}$ is isomorphic to R_G as H-graded k-algebras.

§ 2. A relation between $\mu_i(\mathfrak{p}, M)$ and $\mu_i(\mathfrak{p}^*, M)$.

Let $\mathfrak{p}$ be a prime ideal of R. We denote by S the set of all the homogeneous elements of R not contained in $\mathfrak{p}$. Then $S^{-1}R$ (resp. $S^{-1}M$ for an H-graded R-module M) is again an H-graded ring (resp. an H-graded $S^{-1}R$-module) (cf. n°9, Sect. 2, Chap. 2, [2]). $S^{-1}R$ (resp. $S^{-1}M$) is called the homogeneous localization of R (resp. M) at $\mathfrak{p}$ and will be denoted by $R_{(\mathfrak{p})}$ (resp. $M_{(\mathfrak{p})}$). Let $\mathfrak{p}^*$ denote the largest H-graded ideal of R contained in $\mathfrak{p}$. Then $(R_{(\mathfrak{p})}, \mathfrak{p}^*R_{(\mathfrak{p})})$ is an H-local ring and $\mathfrak{p}^*$ is a prime ideal of R. Notice that $M_{(\mathfrak{p})}\neq(0)$ if and only if $M_\mathfrak{p}\neq(0)$, where $M_\mathfrak{p}$ denotes the usual localization of M at $\mathfrak{p}$.

Let $V_H(M)$ be the set of H-graded prime ideals $\mathfrak{p}$ of R such that $M_{(\mathfrak{p})}\neq(0)$. Recall that $\mathrm{Ass}_R M\subset V_H(M)$ (cf. n°1, Sect. 1, Chap. 4, [2]).

PROPOSITION 1.2.1. *Let M be an H-graded R-module and $\mathfrak{p}\in V_H(M)$. Let $n=\dim M_\mathfrak{p}$. Then there exists a chain $\mathfrak{p}=\mathfrak{p}_0\supsetneqq\mathfrak{p}_1\supsetneqq\cdots\supsetneqq\mathfrak{p}_n$ in $V_H(M)$.*

This is proved by induction on n.

PROPOSITION 1.2.2. *Let M be an H-graded R-module and $\mathfrak{p}$ a prime* *ideal of R. If $M_{\mathfrak{p}}\neq(0)$, then $\mathfrak{p}^*\in V_H(M)$ and* $\dim M_{\mathfrak{p}}=\dim M_{\mathfrak{p}^*}+\dim R_{\mathfrak{p}}/\mathfrak{p}^*R_{\mathfrak{p}}$.

PROOF. Since $M_{\mathfrak{p}}=[M_{(\mathfrak{p}^*)}]_{\mathfrak{p}R_{(\mathfrak{p}^*)}}$, we have $M_{(\mathfrak{p}^*)}\neq(0)$. Now consider the second assertion. After homogeneous localization at $\mathfrak{p}$ we may assume that $(R, \mathfrak{p}^*)$ is an H-local ring. We put $d=\dim R_{\mathfrak{p}}/\mathfrak{p}^*R_{\mathfrak{p}}$, $n=\dim M_{\mathfrak{p}}$ and $n^*=\dim M_{\mathfrak{p}^*}$. Obviously $n\geqq n^*+d$ and we will show that $n\leqq n^*+d$ by induction on n^*.

Case (1). R is an integral domain and $M=R$. If $n^*=0$, $\mathfrak{p}^*=(0)$ and of course $n=d$. Suppose that $n^*>0$ and assume that the assertion holds for $\mathfrak{p}$ with $\dim R_{\mathfrak{p}^*}<n^*$. Let a be a non-zero homogeneous element of R contained in $\mathfrak{p}^*$. As $\dim R_{\mathfrak{p}}/aR_{\mathfrak{p}}=n-1$, there exists a chain $\mathfrak{p}=\mathfrak{p}_0\supsetneqq\mathfrak{p}_1\supsetneqq\cdots\supsetneqq\mathfrak{p}_{n-1}=\mathfrak{q}$ of prime ideals of R with $\mathfrak{q}\ni a$. Since $\mathfrak{q}$ is a minimal prime divisor of a, $\mathfrak{q}\in\mathrm{Ass}_R R/aR$ and hence is H-graded. As $\dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}=n-1<n$ and as $\dim R_{\mathfrak{p}^*}/\mathfrak{q}R_{\mathfrak{q}^*}<n^*$, we have by the assumption on n^* that $\dim R_{\mathfrak{p}^*}/\mathfrak{q}R_{\mathfrak{p}^*}=\dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}-\dim R_{\mathfrak{p}}/\mathfrak{p}^*R_{\mathfrak{p}}$. Thus $n^*>(n-1)-d$. Hence we have the assertion.

Case (2). General case. If $n^*=0$, every prime ideal $\mathfrak{q}$ of R with $M_{\mathfrak{q}}\neq(0)$ contains $\mathfrak{p}^*$ since $\mathrm{Ass}_R M=\{\mathfrak{p}^*\}$. Suppose that $n^*>0$ and assume that the assertion holds for $\mathfrak{p}$ with $\dim M_{\mathfrak{p}^*}<n^*$. As $n=\dim M_{\mathfrak{p}}$ there is a chain $\mathfrak{p}=\mathfrak{p}_0\supsetneqq\mathfrak{p}_1\supsetneqq\cdots\supsetneqq\mathfrak{p}_n$ of prime ideals of R such that $M_{\mathfrak{p}_n}\neq(0)$. If we put $\mathfrak{q}=\mathfrak{p}_d$, $\dim M_{\mathfrak{q}}=n-d$ and $\dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}=d$. If $\mathfrak{q}\subset\mathfrak{p}^*$, $n-d\leqq n^*$ and the result follows. Suppose that $\mathfrak{q}\not\subset\mathfrak{p}^*$. Then $\mathfrak{q}$ is not graded as $(R, \mathfrak{p}^*)$ is an H-local ring. Of course $\mathfrak{q}^*\in V_H(M)$ and $\mathfrak{q}^*\subset\mathfrak{p}^*$. We put $e=\dim R_{\mathfrak{q}}/\mathfrak{q}^*R_{\mathfrak{q}}$ (Note that $e>0$ since $\mathfrak{q}$ is not H-graded.) and $f=\dim R_{\mathfrak{p}^*}/\mathfrak{q}^*R_{\mathfrak{p}^*}$. Applying Case (1) to $R/\mathfrak{q}^*$, we see (*) $\dim R_{\mathfrak{p}}/\mathfrak{q}^*R_{\mathfrak{p}}=f+d$ and we have $f\geqq e>0$ since $\dim R_{\mathfrak{p}}/\mathfrak{q}^*R_{\mathfrak{p}}\geqq d+e$. Thus $\dim M_{\mathfrak{q}^*}<n^*$. For, if $\dim M_{\mathfrak{q}^*}=n^*$, $\mathfrak{q}^*=\mathfrak{p}^*$ and therefore $\dim R_{\mathfrak{p}}/\mathfrak{q}^*R_{\mathfrak{p}}=d$. By (*) this implies that $f=0$—this is a contradiction. Now we have by the assumption on n^* that $n-d=\dim M_{\mathfrak{q}^*}+e$. As $f+\dim M_{\mathfrak{q}^*}\leqq n^*$ we see that

$$\begin{aligned} n-d &\leqq (n^*-f)+e \\ &= n^*+(e-f) \\ &\leqq n^* . \end{aligned}$$

Thus we have verified the assertion.

Let A be an arbitrary Noetherian ring and M an A-module. For every prime ideal $\mathfrak{p}$ of A and for every integer $i\geqq 0$, we put

$$\mu_i(\mathfrak{p}, M)=\dim_{k(\mathfrak{p})}\mathrm{Ext}^i_{A_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$$

(Here $k(\mathfrak{p})$ denotes the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.) and call it the i-th Bass number of M at $\mathfrak{p}$. Bass [1] showed that, if $0\to M\to E^0\to\cdots\to E^i\to\cdots$ is a minimal injective resolution of M, $\mu_i(\mathfrak{p}, M)$ is equal to the number of the injective envelopes $E_A(A/\mathfrak{p})$ of $A/\mathfrak{p}$ which appear in E^i as direct summands.

Let M, N be H-graded R-modules and $h \in H$. We denote by $\underline{\mathrm{Hom}}_R(M, N)_h$ the abelian group of all the homomorphisms from M into $N(h)$. We put $\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{h \in H} \underline{\mathrm{Hom}}_R(M, N)_h$ and consider it as an H-graded R-module with $\{\underline{\mathrm{Hom}}_R(M, N)_h\}_{h \in H}$ as its grading. The derived functor of $\underline{\mathrm{Hom}}_R(\cdot,\cdot)$ will be denoted by $\underline{\mathrm{Ext}}^i_R(\cdot,\cdot)$. Since R is Noetherian, $\underline{\mathrm{Ext}}^i_R(M, N) = \mathrm{Ext}^i_R(M, N)$ as the underlying R-module if M is finitely generated as an R-module.

THEOREM 1.2.3. *Let M be an H-graded R-module and $\mathfrak{p}$ a prime ideal of R. Let $d = \dim R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}$. Then $0 \leqq d \leqq \mathrm{rank}\, H$. Moreover $\mu_i(\mathfrak{p}, M) = 0$ for $0 \leqq i < d$, and $\mu_i(\mathfrak{p}, M) = \mu_{i-d}(\mathfrak{p}^*, M)$ for $d \leqq i$.*

PROOF. After homogeneous localization at $\mathfrak{p}$ we may assume that $(R, \mathfrak{p}^*)$ is an H-local ring. Since $R/\mathfrak{p}^*$ is a simple H-graded ring, we have by 1.1.4 that $d \leqq \mathrm{rank}\, G$ (Here $G = G(R/\mathfrak{p}^*)$.). Thus we have the first assertion. For the second, let $\{f_1, f_2, \cdots, f_d\}$ be an $R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}$-regular sequence such that $\mathfrak{p}R_\mathfrak{p} = \mathfrak{p}^*R_\mathfrak{p} + (f_1, f_2, \cdots, f_d)$. (Recall that $R/\mathfrak{p}^*$ is a regular ring. See 1.1.4.) We put $\mathfrak{q}_s = \mathfrak{p}^*R_\mathfrak{p} + (f_1, f_2, \cdots, f_s)$ for $0 \leqq s \leqq d$. Then we have an exact sequence

$$(E_s)\quad 0 \longrightarrow R_\mathfrak{p}/\mathfrak{q}_s \xrightarrow{f_{s+1}} R_\mathfrak{p}/\mathfrak{q}_s \longrightarrow R_\mathfrak{p}/\mathfrak{q}_{s+1} \longrightarrow 0$$

of $R_\mathfrak{p}$-modules for $0 \leqq s < d$. Applying $\mathrm{Ext}^i_{R_\mathfrak{p}}(\cdot, M_\mathfrak{p})$ to (E_s) we have a long exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_{s+1}, M_\mathfrak{p}) \longrightarrow \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_s, M_\mathfrak{p}) \xrightarrow{f_{s+1}} \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_s, M_\mathfrak{p}) \longrightarrow \cdots \\ \longrightarrow \mathrm{Ext}^1_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_{s+1}, M_\mathfrak{p}) \longrightarrow \cdots\end{aligned}$$

of $R_\mathfrak{p}$-modules. Since $\mathrm{Ext}^i_R(R/\mathfrak{p}^*, M)$ is the underlying module of $\underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M)$ and since $R/\mathfrak{p}^*$ is a simple H-graded ring, we have by 1.1.4 that $\mathrm{Ext}^i_R(R/\mathfrak{p}^*, M)$ is a free $R/\mathfrak{p}^*$-module. As $\mathfrak{q}_0 = \mathfrak{p}^*R_\mathfrak{p}$ we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_0, M_\mathfrak{p}) \xrightarrow{f_1} \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_0, M_\mathfrak{p}) \longrightarrow \mathrm{Ext}^{i+1}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_1, M_\mathfrak{p}) \longrightarrow 0$$

fore very integer $i \geqq 0$ and $\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_1, M_\mathfrak{p}) = (0)$. (Recall that f_1 is $R_\mathfrak{p}/\mathfrak{q}_0$-regular.) Of course $\mathrm{Ext}^{i+1}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_1, M_\mathfrak{p}) \cong \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_0, M_\mathfrak{p})/f_1 \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_0, M_\mathfrak{p})$ is again a free $R_\mathfrak{p}/\mathfrak{q}_1$-module as $\mathfrak{q}_1 = \mathfrak{q}_0 + f_1 R_\mathfrak{p}$.

Now it follows by induction on s that $\mathrm{Ext}^{i+d}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \cong \mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}, M_\mathfrak{p})/(f_1, f_2, \cdots, f_d)\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}, M_\mathfrak{p})$ for every $i \geqq 0$ and

$\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})=(0)$ for $0\leqq i<d$. Thus we have

$$\begin{aligned}\mu_{i+d}(\mathfrak{p}, M)&=\dim_{k(\mathfrak{p})}\mathrm{Ext}^{i+d}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})\\&=\mathrm{rank}_{R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}}\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}, M_\mathfrak{p})\\&=\mathrm{rank}_{R/\mathfrak{p}^*}\mathrm{Ext}^i_R(R/\mathfrak{p}^*, M)\\&=\dim_{k(\mathfrak{p}^*)}\mathrm{Ext}^i_{R_{\mathfrak{p}^*}}(k(\mathfrak{p}^*), M_{\mathfrak{p}^*})\\&=\mu_i(\mathfrak{p}^*, M)\end{aligned}$$

for every $i\geqq 0$ and $\mu_i(\mathfrak{p}, M)=0$ for $0\leqq i<d$.

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and M a Macaulay A-module of $\dim_A M=n$. We put $r_A(M)=\dim_k\mathrm{Ext}^n_A(k, M)$ $(=\mu_n(\mathfrak{m}, M))$, and call it the type of M. Various properties of the invariant $r_A(M)$ are discussed by [9]. M is called a Gorenstein A-module if $\dim A=n$ and M has injective dimension equal to n. The concept of Gorenstein modules was given by Sharp [16] in which we will find useful characterizations of Gorenstein modules. If A is not necessarily a local ring, Gorenstein modules are defined by their local data.

COROLLARY 1.2.4. *Let M be a finitely generated H-graded R-module and $\mathfrak{p}$ a prime ideal of R such that $M_\mathfrak{p}\neq(0)$. Let $d=\dim R_\mathfrak{p}/\mathfrak{p}^*R_\mathfrak{p}$. Then* depth $M_\mathfrak{p}=$ depth $M_{\mathfrak{p}^*}+d$. *Moreover $M_\mathfrak{p}$ is a Macaulay* (resp. *Gorenstein*) *$R_\mathfrak{p}$-module if and only if $M_{\mathfrak{p}^*}$ is a Macaulay* (resp. *Gorenstein*) *$R_{\mathfrak{p}^*}$-module. In this case $r_{R_\mathfrak{p}}(M_\mathfrak{p})=r_{R_{\mathfrak{p}^*}}(M_{\mathfrak{p}^*})$.*

This follows immediately from 1.2.3 (cf. [1] and (3.11), [16]).

PROPOSITION 1.2.5. *Let $\mathfrak{p}$ be a prime ideal of R. Then the following conditions are equivalent.*

(1) *$R_\mathfrak{p}$ is a regular local ring.*

(2) *$R_{\mathfrak{p}^*}$ is a regular local ring.*

PROOF. We have only to show (2)$\Rightarrow$(1). After homogeneous localization at $\mathfrak{p}$ we may assume that $(R, \mathfrak{p}^*)$ is an H-local ring. Note that R is an integral domain, since $R_{\mathfrak{p}^*}$ is an integral domain and since the functor $R_{\mathfrak{p}^*}\otimes_R\cdot$ is faithfully flat on the category $M_H(R)$. We put $d=\dim R_{\mathfrak{p}^*}$ and prove by induction on d. If $d=0$, R is a simple H-graded ring and is certainly a regular ring (cf. 1.1.4). Now suppose that $d>0$ and assume that the assertion holds for $d-1$. Let a be a homogeneous element of $\mathfrak{p}^*$ not contained in $\mathfrak{p}^{*2}$. Then $\dim R_\mathfrak{p}/aR_\mathfrak{p}=d-1$ and $R_{\mathfrak{p}^*}/aR_{\mathfrak{p}^*}$ is a regular local ring. As $(\mathfrak{p}/aR)^*=\mathfrak{p}^*/aR$ in R/aR, we have by the assumption on d that $R_\mathfrak{p}/aR_\mathfrak{p}$ is a regular local ring. This shows that $R_\mathfrak{p}$ is a regular local ring, since a is $R_\mathfrak{p}$-regular.

§ 3. Minimal injective resolutions.

Let M be an H-graded R-module. We donote by $\underline{E}_R(M)$ the injective envelope of M in $M_H(R)$. First we note

LEMMA 1.3.1. *Let E be an H-graded R-module. Then the following conditions are equivalent.*

(1) *E is an injective object of $M_H(R)$.*

(2) *$\underline{\mathrm{Ext}}^1_R(R/\mathfrak{a}, E)=(0)$ for every H-graded ideal $\mathfrak{a}$ of R.*

(3) *$\underline{\mathrm{Ext}}^i_R(\cdot, E)=(0)$ for every integer $i>0$.*

The proof is similar to that of non-graded case and so we omit it (cf. Theorem 3.2, [3]). Of course (2) is equivalent to the following condition: Let $\mathfrak{a}$ be an H-graded ideal of R and $h\in H$. Then every homomorphism from $\mathfrak{a}(h)$ into E can be extended over $R(h)$.

COROLLARY 1.3.2. *Suppose that R is a simple H-graded ring. Then every H-graded R-module is an injective object of $M_H(R)$.*

THEOREM 1.3.3. (1) $\mathrm{Ass}_R\underline{E}_R(M)=\mathrm{Ass}_R M$ *for every H-graded R-module M.*

(2) *Let E be an H-graded R-module. Then E is an indecomposable injective object of $M_H(R)$ if and only if $E\cong[\underline{E}_R(R/\mathfrak{p})](h)$ for some H-graded prime ideal $\mathfrak{p}$ of R and for some $h\in H$. In this case* $\mathrm{Ass}_R E=\{\mathfrak{p}\}$ *and $\mathfrak{p}$ is uniquely determined for E.*

(3) *Every injective object E of $M_H(R)$ can be decomposed into a direct sum of indecomposable injective objects of $M_H(R)$. This decomposition is uniquely determined by E up to isomorphisms.*

The proof is similar to that of non-graded case (cf. [14]).

THEOREM 1.3.4. *Let M be an H-graded R-module and let*

$$0\longrightarrow M\longrightarrow \underline{E}^0_R(M)\longrightarrow\cdots\longrightarrow \underline{E}^i_R(M)\xrightarrow{d_i}\underline{E}^{i+1}_R(M)\longrightarrow\cdots$$

denote a minimal injective resolution of M in $M_H(R)$. Then, for every H-graded prime ideal $\mathfrak{p}$ of R and for every integer $i\geqq 0$, $\mu_i(\mathfrak{p}, M)$ is equal to the number of the H-graded R-modules of the form $[\underline{E}_R(R/\mathfrak{p})](h)$ $(h\in H)$ which appear in $\underline{E}^i_R(M)$ as direct summands.

PROOF. Since $\underline{E}^i_R(M)=\underline{E}_R(B^i)$ where $B^i=\mathrm{Ker}\, d^i$, it suffices to prove in case $i=0$. Moreover, after homogeneous localization at $\mathfrak{p}$, we may assume that $(R, \mathfrak{p})$ is an H-local ring. Now let us express $\underline{E}_R(M)=\bigoplus_{\mathfrak{q}\in V_H(R), h\in H} m(\mathfrak{q}, h)[\underline{E}_R(R/\mathfrak{q})](h)$ where $m(\mathfrak{q}, h)$ denotes the multiplicity of

$[\underline{E}_R(R/\mathfrak{q})](h)$. Then, recalling that $\underline{\mathrm{Hom}}_R(R/\mathfrak{a}, \underline{E}_R(N)) = \underline{E}_{R/\mathfrak{a}}(\underline{\mathrm{Hom}}_R(R/\mathfrak{a}, N))$ for every H-graded ideal $\mathfrak{a}$ of R and for every H-graded R-module N (cf. [1]), we have by 1.3.2

$$\begin{aligned}\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, M) &= \underline{\mathrm{Hom}}_R(R/\mathfrak{p}, \underline{E}_R(M)) \\ &= \bigoplus_{\mathfrak{q}\in V_H(R), h\in H} m(\mathfrak{q}, h)\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, [\underline{E}_R(R/\mathfrak{q})](h)) \\ &= \bigoplus_{\mathfrak{q}\in V_H(R), h\in H} m(\mathfrak{q}, h)\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, [R/\mathfrak{q}](h)) \\ &= \bigoplus_{h\in H} m(\mathfrak{p}, h)[R/\mathfrak{p}](h)\ .\end{aligned}$$

Thus we have verified the assertion that $\mu_0(\mathfrak{p}, M) = \sum_{h\in H} m(\mathfrak{p}, h)$.

COROLLARY 1.3.5. *Let $\mathfrak{m}$ be a maximal ideal of R and assume that $\mathfrak{m}$ is an H-graded ideal. Then $\underline{E}_R(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$ as the underlying R-module.*

For an H-graded R-module M, let $\underline{\mathrm{id}}_R M$ (resp. $\mathrm{id}_R M$) denote the injective dimension of M in $M_H(R)$ (resp. as the underlying R-module).

THEOREM 1.3.6. *Let $n = \mathrm{rank}\, H$. Then $\mathrm{id}_R M \leqq \underline{\mathrm{id}}_R M + n$ for every H-graded R-module M.*

PROOF. It suffices to prove in case M is an injective object of $M_H(R)$. Let $\mathfrak{p}$ be a prime ideal of R such that $M_\mathfrak{p} \neq (0)$. We put $d = \dim M_\mathfrak{p} - \dim M_{\mathfrak{p}^*}$. If $\mathfrak{p}$ is H-graded, then $\mu_i(\mathfrak{p}, M) = 0$ for every $i > 0$ since $\underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}, M) = (0)$. Now suppose that $\mathfrak{p}$ is not H-graded. Then, since $\mu_{i+d}(\mathfrak{p}, M) = \mu_i(\mathfrak{p}^*, M)$ by 1.2.3, we have $\mu_{i+d}(\mathfrak{p}, M) = 0$ for every $i > 0$ by virtue of the result in case $\mathfrak{p}$ is an H-graded ideal. Thus $\mu_i(\mathfrak{p}, M) = 0$ for every $i > n$ and for every prime ideal $\mathfrak{p}$ of R, since $d = d(\mathfrak{p}) \leqq n$ for every $\mathfrak{p}$ (cf. 1.2.3).

§ 4. Cousin complexes and local cohomology modules.

The Cousin complexes were given by Hartshorne [7] in terms of algebraic geometry, and in this section we will reconstruct them in terms of algebra—namely of H-graded R-modules. The method is due to Sharp [15] and so, though he considered no sort of grading, we may refer the detail to [15].

We put $U^i_H(M) = \{\mathfrak{p} \in V_H(M) / \dim M_\mathfrak{p} \geqq i\}$ for every H-graded R-module M and for every integer $i \geqq 0$.

LEMMA 1.4.1. *Let U and U' be subsets of $V_H(R)$ such that $U' \subset U$ and suppose that every element of U/U' is minimal in U. Let M be an H-graded R-module and assume that $V_H(M) \subset U$. Then*

$$\begin{aligned} \varphi\colon M &\longrightarrow \bigoplus_{\mathfrak{p}\in U/U'} M_{(\mathfrak{p})} \\ x &\longmapsto \{x/1\} \end{aligned}$$

is a well-defined homomorphism of H-graded R-modules and $V_H(\text{Coker}\,\varphi) \subset U'$.

Construction of $\underline{C}_R(M)$. Let M be an H-graded R-module. We put $M^{-2}=(0)$, $M^{-1}=M$, and $d^{-2}=0$. Let $i\geqq 0$ be an integer and assume that there is a complex

$$M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^{0} \longrightarrow \cdots \longrightarrow M^{i-2} \xrightarrow{d^{i-2}} M^{i-1}$$

of H-graded R-modules such that $V_H(\text{Coker}\ d^{i-2}) \subset U_H^i(M)$. Of course this assumption is satisfied for $i=0$. We put $M^i=\bigoplus_{\mathfrak{p}\in U_H^i(M)/U_H^{i+1}(M)}[\text{Coker}\ d^{i-2}]_{(\mathfrak{p})}$ and we define $d^{i-1}=\varphi\circ\varepsilon$ where $\varepsilon\colon M^{i-1}\to\text{Coker}\ d^{i-2}$ is the canonical epimorphism and φ: $\text{Coker}\ d^{i-2}\to M^i$ denotes the homomorphism induced by 1.4.1. Then $d^{i-2}\circ d^{i-1}=0$, and $V_H(\text{Coker}\ d^{i-1})\subset U_H^{i+1}(M)$ by 1.4.1. Thus inductively we obtain a complex $\underline{C}_R(M)$

$$0 \longrightarrow M=M^{-1} \xrightarrow{d^{-1}} M^{0} \longrightarrow \cdots \longrightarrow M^{i} \xrightarrow{d^{i}} M^{i+1} \longrightarrow \cdots$$

of H-graded R-modules which we call the Cousin complex of M.

By virtue of 1.2.4 the proof of the following theorem is the same as that of non-graded case (cf. [16]).

THEOREM 1.4.2. *Let M be a non-zero finitely generated H-graded R-module. Then M is a Macaulay* (resp. *Gorenstein*) *R-module if and only if $\underline{C}_R(M)$ is exact* (resp. *$\underline{C}_R(M)$ provides a minimal injective resolution of M in $M_H(R)$*).

Suppose that $(R, \mathfrak{m})$ is an H-local ring. For every integer $i\geqq 0$ and for every H-graded R-module M we put

$$\underline{H}_{\mathfrak{m}}^i(M)=\varinjlim_t \underline{\text{Ext}}_R^i(R/\mathfrak{m}^t, M)$$

and call it the i-th local cohomology module of M (cf. [6]). $\underline{H}_{\mathfrak{m}}^0(\cdot)$ is a left exact functor and $\{\underline{H}_{\mathfrak{m}}^i(\cdot)\}_{i\geqq 0}$ are its derived functors.

THEOREM 1.4.3. *Suppose that $(R, \mathfrak{m})$ is an H-local ring. Let M be a Macaulay H-graded R-module with $\dim M_{\mathfrak{m}}=n$. Then*

(1) $\underline{\text{Ext}}_R^n(N, M)\cong\underline{\text{Hom}}_R(N, \underline{H}_{\mathfrak{m}}^n(M))$ *for every finitely generated H-graded R-module N such that $V_H(N)\subset\{\mathfrak{m}\}$.*

(2) $M^n\cong\underline{H}_{\mathfrak{m}}^n(M)$.

(3) *M is a Gorenstein R-module if and only if $\underline{H}^n_{\mathfrak{m}}(M)$ is an injective object of $M_H(R)$.*

The proof of 1.4.3 is similar to that of Z-graded case and we omit it (cf. Theorem 1.3.4, [4]).

CHAPTER II. THE CANONICAL MODULES OF H-GRADED RINGS DEFINED OVER A FIELD.

Let k be a field.

DEFINITION. A Noetherian H-graded ring R is said to be defined over k if $R_0=k$ and if $k\cong R/\mathfrak{m}$, where $\mathfrak{m}$ denotes the unique H-maximal ideal of R (cf. 1.1.7). R is a finitely generated k-algebra in this case. In fact, if we express $\mathfrak{m}=(x_1, x_2, \cdots, x_n)$ with x_i homogeneous, then we have that $R=k[x_1, x_2, \cdots, x_n]$. Suppose that $H=Z^n$ and assume that $R_h=(0)$ for all $h\in Z^n/N^n$. Then a Noetherian H-graded ring R is defined over k if $R_0=k$. A homomorphism of H-graded ringsdefined over k is, by definition, a k-algebra map which preserves degrees.

In this chapter we assume that R is a Noetherian H-graded ring defined over k.

§ 1. A duality of Noetherian H-graded R-modules and Artinian H-graded R-modules.

We denote $k=R/\mathfrak{m}$ by $\underline{k}$ if we regard it as an H-graded R-module. Note that $k=R_0$ is an H-graded subring of R. Let M be an H-graded R-module. We define $M^*=\underline{\mathrm{Hom}}_k(M, \underline{k})$ and call it the graded k-dual of M. M^* is actually an H-graded R-module with $\{\mathrm{Hom}_k(M_{-h}, k)\}_{h\in H}$ as its grading. $M=M^{**}$ if and only if $[M_h : k]$ is finite for all $h\in H$. $[\cdot]^*$: $M_H(R)\to M_H(R)$ is a contravariant exact functor.

THEOREM 2.1.1. $R^*=\underline{E}_R(\underline{k})$.

PROOF. R^* is an indecomposable object of $M_H(R)$, since $R=R^{**}$. Moreover R^* contains $\underline{k}=\underline{k}^*$ as an H-graded R-submodule. Thus it suffies to show that R^* is an injective object of $M_H(R)$. Let $\mathfrak{a}$ be an H-graded ideal of R and $h\in H$. For every homomorphism $f: \mathfrak{a}(h)\to R^*$, we can find $g: R=R^{**}\to[R(h)]^*$ so that $i^*\circ g=f^*$ where $i: \mathfrak{a}(h)\to R(h)$ denotes the inclusion map. Therefore $g^*\circ i=f$ and hence, by 1.3.1, we have the assertion.

COROLLARY 2.1.2. *R^* is the injective envelope of $\underline{k}$ as the under-*

lying R-module.

PROOF. See 1.3.5.

COROLLARY 2.1.3. *Let M be an H-graded R-module. Then the following conditions are equivalent.*

(1) *M is an Artinian R-module.*

(2) *There is an exact sequence $0 \to M \to \bigoplus_{i=1}^{r} R^*(h_i)$ of H-graded R-modules.*

The proof is similar to that of non-graded case (cf. [14] and 2.1.2).

Let $N_H(R)$ (resp. $A_H(R)$) denote the full subcategory of $M_H(R)$ consisting of all the Noetherian (resp. Artinian) H-graded R-modules. By virtue of 2.1.3 we have

THEOREM 2.1.4. *$[\cdot]^*: N_H(R) \to A_H(R)$ establishes an equivalence of categories.*

§ 2. **The canonical modules.**

Let $n = \dim R_{\mathfrak{m}}$.

DEFINITION. $K_R = (\underline{\boldsymbol{H}}_{\mathfrak{m}}^n(R))^*$.

We call K_R the canonical module of R. As $\underline{\boldsymbol{H}}_{\mathfrak{m}}^n(R)$ is an Artinian R-module, K_R is a finitely generated H-graded R-module (cf. 2.1.4).

THEOREM 2.2.2. *The following conditions are equivalent.*

(1) *R is a Macaulay ring.*

(2) *$\underline{\mathrm{Ext}}_R^i(M, K_R) \cong (\underline{\boldsymbol{H}}_{\mathfrak{m}}^{n-i}(M))^*$ for every finitely generated H-graded R-module M and for every integer i.*

The proof is similar to the $\boldsymbol{Z}$-graded case (cf. 2.1.6, [4]).

COROLLARY 2.2.3. *Suppose that R is a Macaulay ring. Then K_R is a Gorenstein R-module with $r_{R_{\mathfrak{m}}}((K_R)_{\mathfrak{m}}) = 1$. In particular R is a Gorenstein ring if and only if $K_R = R(h)$ for some $h \in H$.*

PROOF. $\underline{\mathrm{Ext}}_R^i(R/\mathfrak{m}, K_R) \cong (\underline{\boldsymbol{H}}_{\mathfrak{m}}^{n-i}(R/\mathfrak{m}))^*$ for all i. Hence $\mu_i(\mathfrak{m}, K_R) = \delta_{in}$ for every $i \geq 0$. This proves the first assertion (cf. (3.11) Theorem, [16] and 1.2.4). Suppose that R is a Gorenstein ring. Then $\underline{\boldsymbol{H}}_{\mathfrak{m}}^n(R) = R^*(-h)$ for some $h \in H$ by 1.4.3, since $R^* = \underline{E}_R(\underline{k})$ by 2.1.1. Therefore $K_R = (R^*(-h))^* = R(h)$.

PROPOSITION 2.2.4. *Let P be a Macaulay H-graded ring defined over k and let $f: P \to R$ denote a finite homomorphism of H-graded rings* *defined over k. Let $d=\dim P_{\mathfrak{n}}-\dim R_{\mathfrak{m}}$, where $\mathfrak{n}$ denotes the H-maximal ideal of P. Then $K_R=\underline{\mathrm{Ext}}_P^d(R, K_P)$.*

PROOF. Since f is finite, $\underline{H}_{\mathfrak{n}}^j(R)=\underline{H}_{\mathfrak{m}}^j(R)$ for all j. Thus by 2.2.2 we have $K_R=(\underline{H}_{\mathfrak{m}}^n(R))^*=(\underline{H}_{\mathfrak{n}}^n(R))^*=\underline{\mathrm{Ext}}_P^d(R, K_P)$.

COROLLARY 2.2.5. *Suppose that R is a Macaulay ring and let a be a homogeneous non-zero divisor of R. Then a is regular on K_R and $K_{R/aR}\cong[K_R/aK_R](\deg a)$.*

The proof is the same as that of $\boldsymbol{Z}$-graded case (cf. 2.2.10, [4]).

COROLLARY 2.2.6. *Suppose that $R=k[X_1, X_2, \cdots, X_n]$ is a polynomial ring and assume that each X_i is a homogeneous element of R. Then $K_R=R(-e)$ where $e=\sum_{i=1}^n \deg X_i$.*

PROOF. Let $K_R=R(h)$ $(h\in H)$ (cf. 2.2.3). Then, as $K_{R/(X_1,X_2,\cdots,X_n)}\cong[K_R/(X_1, X_2, \cdots, X_n)K_R](e)$ by 2.2.5, we have $k\cong k(e+h)$. (Note that $k=R/(X_1, X_2, \cdots, X_n)$.) Thus $h=-e$.

REMARK 2.2.7. Suppose that $H=\boldsymbol{Z}^n$. For every $h=(h_1, h_2, \cdots, h_n)\in H$ we put $|h|=\sum_{i=1}^n h_i$. Let M be an H-graded R-module. We define $R_d=\bigoplus_{|h|=d} R_h$ (resp. $M_d=\bigoplus_{|h|=d} M_h$) for every integer d. Then clearly $\{R_d\}_{d\in \boldsymbol{Z}}$ (resp. $\{M_d\}_{d\in \boldsymbol{Z}}$) is a $\boldsymbol{Z}$-grading of R (resp. M) and we denote this $\boldsymbol{Z}$-graded ring R (resp. this $\boldsymbol{Z}$-graded R-module M) by $R^\#$ (resp. $M^\#$). Notice that, in case $R^\#$ is defined over k, $[K_R]^\#$ coincides with $K_{R^\#}$ (cf. Chap. 2, [4]).

CHAPTER III. AFFINE SEMIGROUP RINGS.

§ 1. Some general properties of semigroup rings.

In this chapter, we put $H=\boldsymbol{Z}^n$ and we use H and $\boldsymbol{Z}^n$ interchangeably. Let R be a Noetherian H-graded ring. We call R a semigroup ring over a field k if $R_0=k$, R is an integral domain and if $[R_h: k]\leqq 1$ for every $h\in H$. In this case, if we put $S=\{h\in H; R_h\neq 0\}$, S is closed under addition. That is, S is an additive subsemigroup of H.

PROPOSITION 3.1.1. *If R is a semigroup ring over k, R is isomorphic to an H-graded subring of $k[H]=k[X_1, X_1^{-1}, \cdots, X_n, X_n^{-1}]$ as H-graded rings.*

PROOF. Let Q be the localization of R by all non-zero homogeneous elements of R. Then Q is an H-simple ring and by 1.1.1, $Q \cong k[L]$ where L is the additive subgroup of H generated by S. So, Q is isomorphic to a H-graded subalgebra of $k[H]$ as H-graded algebras and so is R.

NOTATION. In this situation, we write $R=k[S]$. By 3.1.1, we can consider R as the subring of $k[H]$ generated by $\{X^s; s \in S\}$. (As usual, we write $X^s = X_1^{s_1} \cdots X_n^{s_n}$ for $s=(s_1, \cdots, s_n)$.) We always write L the additive subgroup of H generated by S.

DEFINITION 3.1.2. (1) Let A be a subset of L. A is an S-ideal if $A+S=\{a+s; a \in A, s \in S\} \subset A$. A is a prime S-ideal if $A \not\supseteq S$, A is an S-ideal and if S/A is closed under addition. (By S/A we mean the set of elements of S which are not in A. We define $S-A=\{s-a; s \in S, a \in A\}$.) A is an inverse S-ideal if $A-S \subset A$. It is clear that A is an inverse S-ideal if and only if $-A=\{-a; a \in A\}$ is an S-ideal.

(2) Let A be an S-ideal or an inverse S-ideal. We write $k[A]$ the k-vector space spanned by the set $\{X^a; a \in A\}$. If A is an S-ideal, $k[A]$ is naturally an R-submodule of $Q=k[L]$. If A is an inverse S-ideal, we define the R-module structure of $k[A]$ by

$$X^s.\ X^a = \begin{cases} X^{s+a} & (s+a \in A) \\ 0 & (s+a \notin A) \end{cases}$$

for $s \in S$ and $a \in A$. Then it is easy to see that $(k[A])^* = k[-A]$ if A is an S-ideal or an inverse S-ideal (cf. Chap. 2, § 1 for the definition of $(\ )^*$). Note that if A, B are S-ideals (resp. inverse S-ideals) with $A \subset B$, then $k[A]$ is a submodule (resp. quotient module) of $k[B]$ and $k[B/A]$ is a quotient module (resp. a submodule) of $k[B]$.

PROPOSITION 3.1.3. (1) *If P is a prime S-ideal, $\mathfrak{p}=k[P]$ is an H-graded prime ideal of R. (We consider the empty set as a prime S-ideal and we put $k[\varnothing]=(0)$.) Conversely, every H-graded prime ideal of R is of the form $k[P]$ for some prime S-ideal P.*

(2) *If $\mathfrak{p}=k[P]$, $R_{(\mathfrak{p})}=k[U_P]$ and $\underline{E}_R(R/\mathfrak{p}) \cong k[-U_P]$, where U_P is an S-ideal defined by $U_P = S-(S/P)=\{s-s'; s \in S, s' \in S/P\}$.*

PROOF. (1) If P is a prime S-ideal, $R/k[P] \cong k[S/P]$ is an integral domain. Conversely, if $\mathfrak{p}$ is an H-graded prime ideal of R, $R/\mathfrak{p}=k[S']$ for some additive subsemigroup S' of S. If we put $P=S/S'$, P is a prime S-ideal and $\mathfrak{p}=k[P]$.

(2) By the definition of $R_{(\mathfrak{p})}$, $R_{(\mathfrak{p})}=k[U_P]$ is easy to see. As $k[-U_P] \cong (R_{(\mathfrak{p})})^*$, $k[-U_P]$ is an H-injective $R_{(\mathfrak{p})}$-module (see the proof of 2.1.1) and

hence an H-injective R-module. Furthermore, we have $-U_P \cap S = S/P$ by the lemma below. So $R/\mathfrak{p}$ is an H-graded submodule of $k[-U_P]$. As $k[-U_P]$ is clearly indecomposable since $k[U_P]$ is a fractional ideal of R, we have $\underline{E}_R(R/\mathfrak{p}) \cong k[-U_P]$.

LEMMA 3.1.4. $-U_P \cap S = S/P$.

PROOF. If $a \in -U_P \cap S$ and if $a \in P$, $a+b \in S/P$ for some $b \in S$. Since P is an S-ideal, $a+b \in P$. Contradiction! This shows $-U_P \cap S \subset S/P$ and the converse implication is clear.

The following example plays an es sential role in next section.

EXAMPLE 3.1.5. Let L be a subgroup of $H=\boldsymbol{Z}^n$ and $S=L \cap \boldsymbol{N}^n$. For a subset I of $\{1, 2, \cdots, n\}$, we put

$$\boldsymbol{Z}^{(I)} = \{(x_1, \cdots, x_n) \in H;\ x_j = 0 \quad \text{for} \quad j \notin I\}$$

and

$$P_I = \{(x_1, \cdots, x_n) \in S \mid x_i > 0 \quad \text{for some} \quad i \in I\}\ .$$

Then $\{P_I \mid I \subset \{1, \cdots, n\}\}$ is the set of all prime S-ideals (if $I=\phi$, we put $P_\phi = (0)$). Hence, there are at most 2^n prime S-ideals. (It may happen that $P_I = P_J$ for some $I \neq J$.) In particular, if $L=H$ and $R=k[X_1, \cdots, X_n]$, there are exactly 2^n H-graded prime ideals of R. If $\mathfrak{p}=k[P_I]=(X_i)_{i \in I}$, we sometimes write $k[X_i^{-1}, X_j, X_j^{-1} | i \in I, j \notin I]$ for $\underline{E}_R(R/\mathfrak{p})$.

PROOF. If $\tilde{R}=k[X_1, \cdots, X_n]$, an H-graded prime of $\tilde{R}$ is obviously of the form $\mathfrak{p}_I$ for some $I \subset \{1, \cdots, n\}$, since H-homogeneous elements of R are monomials. As R is direct summand of $\tilde{R}$ as an R-module, we have $\mathfrak{a}\cdot\tilde{R} \cap R = \mathfrak{a}$ for every ideal $\mathfrak{a}$ of R. Let $\mathfrak{p}$ be an H-graded prime ideal of R. Then $\mathfrak{p}\cdot\tilde{R}_{(\mathfrak{p})}$ is an H-graded prime ideal of $\tilde{R}_{(\mathfrak{p})}$ and not equal to $\tilde{R}_{(\mathfrak{p})}$ itself. If $\mathfrak{p}$ is a H-graded prime ideal of $\tilde{R}_{(\mathfrak{p})}$ which contains $\mathfrak{p}\cdot\tilde{R}_{(\mathfrak{p})}$, $\mathfrak{p}=\tilde{\mathfrak{p}} \cap R$. So every H-graded prime ideal of R is the contraction of some H-graded prime ideal of $\tilde{R}$, which concludes the proof.

NOTATION 3.1.6. If R_1 and R_2 are H-graded rings and if M_1 (resp. M_2) is an H-graded R_1- (resp. R_2-) module, we define

$$R_1 \# R_2 = \bigoplus_{h \in H} [(R_1)_h \otimes_k (R_2)_h] \quad \text{and} \quad M_1 \# M_2 = \bigoplus_{h \in H} [(M_1)_h \otimes_k (M_2)_h]\ .$$

$M_1 \# M_2$ is an H-graded $R_1 \# R_2$-module by putting $(M_1 \# M_2)_h = (M_1)_h \otimes_k (M_2)_h$. The functors $M_1 \#.$ and $.\# M_2$ are clearly exact functors from the category of H-graded R_2- (resp. R_1-) modules to the category of H-graded $R_1 \# R_2$-modules.

If $R_i=k[S_i]$ for some semigroup $S_i\subset H$ $(i=1, 2)$, then $R_1\# R_2=k[S_1\cap S_2]$. If $R=k[S]$ and if $A_i(i=1, 2)$ is an S-ideal (resp. an inverse S-ideal), then $k[A_1]\# k[A_2]=k[A_1\cap A_2]$ is an H-graded R-module and $A_1\cap A_2$ is an S-ideal (resp. an inverse S-ideal).

REMARK 3.1.7. In the same situation as in 3.1.6, we shall always consider $R_1\otimes_k R_2$ (resp. $M_1\otimes_k M_2$) as an H-graded ring (resp. H-graded $R_1\otimes_k R_2$-module) by the grading $(R_1\otimes_k R_2)_h=\bigoplus_{h_1+h_2=h}[(R_1)_{h_1}\otimes_k (R_2)_{h_2}]$ (resp. $(M_1\otimes_k M_2)_h=\bigoplus_{h_1+h_2=h}[(M_1)_{h_1}\otimes_k (M_2)_{h_2}]$). For example, $k[X_1, \cdots, X_n]=k[X_1]\otimes_k\cdots\otimes_k k[X_n]$ as H-graded rings, if we put $\deg X_1=(1, 0, \cdots, 0),\cdots$, $\deg X_n=(0, \cdots, 0, 1)$.

§ **2. Local cohomology groups of normal affine semigroup rings.**

In this section, let S be a subsemigroup of N^n such that $R=k[S]$ is normal. (In such a case, we say that S is a normal affine semigroup.) Note that R is an H-graded ring defined over k in the sense of Chapter 2. Then we know that S is isomorphic to a semigroup of the form $L\cap N^n$ where L is a subgroup of Z^n (cf. [10], § 2). So we may work in the situation of 3.1.5. But let us repeat the definitions.

NOTATION 3.2.1. Let L be a subgroup of $H=Z^n$ and let $S=L\cap N^n$. We assume that S generates the group L. We put $R=k[S]$ and $\tilde{R}=k[N^n]=k[X_1, \cdots, X_n]$. Then R is a normal ring and $\dim R=\operatorname{rank} L$. As usual, we put $\mathfrak{m}=R\cap(X_1, \cdots, X_n)\tilde{R}$. $\mathfrak{m}$ is the unique H-maximal ideal of R and $R/\mathfrak{m}=\underline{k}$.

DEFINITION 3.2.2. We define the complex $\tilde{C}_i^{\cdot}(i=1, 2, \cdots, n)$ by $\tilde{C}_i^0=k[X_i, X_i^{-1}]$, $\tilde{C}_i^1=k[X_i, X_i^{-1}]/k[X_i]$, $\tilde{C}_i^p=0$ for $p\neq 0, 1$ and $d: \tilde{C}_i^0\to\tilde{C}_i^1$ being the canonical surjection (the H-graded module structure of $k[X_i]$ or $k[X_i, X_i^{-1}]$ is the same as in 3.1.7). We sometimes write $X_i^{-1}\cdot k[X_i^{-1}]$ for $k[X_i, X_i^{-1}]/k[X_i]$. We define
$\tilde{C}^{\cdot}=\tilde{C}_1^{\cdot}\otimes_k\tilde{C}_2^{\cdot}\otimes_k\cdots\otimes_k\tilde{C}_n^{\cdot}$, the tensor product of complexes of H-graded modules. Thus the p-th component of $\tilde{C}^{\cdot}$ is given by $\tilde{C}^p=\bigoplus_{\#(I)=p}\tilde{C}_I$, where I is a subset of $\{1, 2, \cdots, n\}$ and $\tilde{C}_I=\{\bigotimes_{j\notin I}k[X_j, X_j^{-1}]\}\otimes_k\{\bigotimes_{i\in I}X_i^{-1}k[X_i^{-1}]\}$. In particular, $\tilde{C}^0=k[H]$ and $\tilde{C}^n=X_1^{-1}\cdots X_n^{-1}\cdot k[X_1^{-1}, \cdots, X_n^{-1}]$.

PROPOSITION 3.2.3. (1) *The complex $\tilde{C}^{\cdot}$ is a resolution of $\tilde{R}$ by H-graded $\tilde{R}$-modules.*

(2) *The complex $\tilde{C}^{\cdot}$ is the Cousin complex of $\tilde{R}$ as an H-graded $\tilde{R}$-module.*

(3) *The complex $\tilde{C}^{\cdot}$ is the minimal injective resolution of $\tilde{R}$ in*

the category of H-graded $\tilde{R}$-modules.

PROOF. (1) As $H^0(\tilde{C}_i^{\cdot})=k[X_i]$ and $H^q(\tilde{C}_i^{\cdot})=0$ for $q\neq 0$, $H^0(\tilde{C}^{\cdot})=\bigotimes_{i=1}^{n} k[X_i]=\tilde{R}$ and $H^q(\tilde{C}_i^{\cdot})=0$ for $q\neq 0$ by Künneth formula ([13], Chap. V, (10.1)).

(2) To prove this, we compare degree q parts of $\tilde{C}^{\cdot}$ and $\underline{C}_{\tilde{R}}^{\cdot}(\tilde{R})$. For $q=0$, both sides are $k[H]$. So we may proceed by induction on q. If $\mathfrak{p}$ is a H-graded prime ideal of height q in $\tilde{R}$, $(\tilde{C}^q_{(\mathfrak{p})})$ is the cokernel of the map $(\tilde{C}^{q-2})_{(\mathfrak{p})}\to(\tilde{C}^{q-1})_{(\mathfrak{p})}$ since $(\tilde{C}^r)_{(\mathfrak{p})}=0$ for $r>q$. As Cousin complex is compatible with localization, we have the desired result by induction hypothesis.

(3) This is a direct consequence of (2) and 1.4.2.

DEFINITION 3.2.4. If M is a H-graded $\tilde{R}$-module, we put $M_L=\bigoplus_{h\in L} M_h$. It is clear that $(\tilde{R})_L=R$, M_L is an R-module and that the functor $(.)_L$ is an exact functor from the category of H-graded $\tilde{R}$-modules to the category of H-graded R-modules.

DEFINITION 3.2.5. We put $C^{\cdot}=(\tilde{C}^{\cdot})_L$. By 3.2.3 and the exactness of $(.)_L$, $C^{\cdot}$ is a resolution of R by H-graded R-modules. We write $C_I=(\tilde{C}_I)_L$ (cf. 3.2.2). We have $C^p=\bigoplus_{\#(I)=p} C_I$. We want to compute the local cohomology groups of R using the resolution $C^{\cdot}$ of R. For this purpose, we recall some facts concerning local cohomology groups.

PROPOSITION 3.2.6. (1) *For every q, $\underline{H}^q_{\mathfrak{m}}(R)$ is an Artinian R-module.*

(2) *If $R=\bar{R}/\mathfrak{a}$ for some Gorenstein H-graded ring $\bar{R}$ and H-graded ideal $\mathfrak{a}$ of $\bar{R}$, then there is an isomorphism of H-graded R-modules*

$$(\underline{H}^q_{\mathfrak{m}}(R))^*\cong \mathrm{Ext}^{d-q}_{\bar{R}}(R,\bar{R})(h)$$

for some $h\in H$ ($d=\dim\bar{R}$).

(3) *If $\mathfrak{p}$ is an H-graded prime ideal of R and if $R_{(\mathfrak{p})}$ is a Macaulay ring, $[(\underline{H}^q_{\mathfrak{m}}(R))^*]_{(\mathfrak{p})}=0$ for $q<\dim R$.*

PROOF. (1) is well known. (2) follows from 2.2.2 and 2.2.3. (3) is an easy consequence of (2).

LEMMA 3.2.7. *Let M be an H-graded R-module.*

(1) *If M satisfies the condition;*

(*) *For every element $x\in M$, $x\neq 0$, $\mathrm{Ann}_R(x)$ is an $\mathfrak{m}$-primary ideal, then $\underline{H}^0_{\mathfrak{m}}(M)=M$ and $\underline{H}^q_{\mathfrak{m}}(M)=0$ for $q\neq 0$.*

(2) *If M is a direct sum of modules which satisfy the condition;*

(**) *There exists a homogeneous element $f\in\mathfrak{m}$ such that the* *multiplication map f_M of f on M is bijective, then $\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(M)=0$ for every integer q.*

PROOF. See [4], 2.2.1 and 2.2.2.

Now, let us compute the local cohomology groups of R.

LEMMA 3.2.8. *For a subset I of $\{1, 2, \cdots, n\}$, we put $I'=\{1, 2, \cdots, n\}/I$.*

(1) *If $\boldsymbol{Z}^{(I')}\cap L\cap N^n=\{0\}$, then C_I satisfies the condition (*) of 3.2.7.*

(*We have defined $\boldsymbol{Z}^{(I)}$ in* 3.1.5.)

(2) *If $\boldsymbol{Z}^{(I')}\cap L\cap N^n\supsetneqq\{0\}$, then C_I satisfies the condition (**) of 3.2.7.*

PROOF. (1) Let us take $X^c\in C_I$, $c=(c_1, \cdots, c_n)$ and $X^a\in\mathfrak{m}$, $a=(a_1, \cdots, a_n)$. Then by the assumption of (1), $a_i>0$ for some $i\in I$. So there is an integer m such that $ma_i+c_i>0$. By the definition of C_I, $(X^a)^m.X^c=0$. This shows that C_I satisfies the condition (*).

(2) Let us take $a=(a_1, \cdots, a_n)\in\boldsymbol{Z}^{(I')}\cap L\cap N^n$, $a\neq 0$. Then $X^a\in\mathfrak{m}$ and $a_i=0$ for every $i\in I$. This shows that the multiplication map of X^a on C_I is bijection.

COROLLARY 3.2.9. *$\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R)\cong H^q(\underline{\boldsymbol{H}}^0_{\mathfrak{m}}(C^\cdot))\cong H^q('C^\cdot)$, where $'C^\cdot$ is the subcomplex of $C^\cdot$ which is the direct sum of all C_I's that satisfy the condition $\boldsymbol{Z}^{(I')}\cap L\cap N^n=\{0\}$.*

PROOF. This is an immediate consequence of 3.2.5, 3.2.8, and 3.2.7.

LEMMA 3.2.10. *If $h=(h_1, \cdots, h_n)\in L$ and if $h_i\geqq 0$ for some i, $1\leqq i\leqq n$, then $(\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R))_h=0$ for every $q\geqq 0$.*

PROOF. Assume that $(\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R))_h\neq 0$ for some q. For simplicity, let us assume $h_1, \cdots, h_t$ are non-negative and $h_{t+1}, \cdots, h_n$ are negative. Then by the definition of the complex $C^\cdot$, $(\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R))_{h'}\neq 0$ for every $h'=(h'_1, \cdots, h'_n)$, such that $h'_1, \cdots, h'_t$ are non-negative and $h'_{t+1}, \cdots, h'_n$ are negative. This contradicts the fact that $\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R)$ is an Artinian R-module.

The following theorem was proved by M. Hochster and R. Stanley. But we need to put a proof for this theorem to prove Theorem 3.3.3.

THEOREM 3.2.11. *We put $S_+=\{(s_1, \cdots, s_n)\in S|s_i>0$ for every i, $1\leqq i\leqq n\}$ and we assume that $S_+\neq\phi$. Also, we put $L_-=-S_+=\{-a|a\in S_+\}$. Then*

$$\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R)=\begin{cases}k[L_-] & (q=\dim R)\\ 0 & (\textit{otherwise})\ .\end{cases}$$

COROLLARY 3.2.12. (1) (M. Hochster, [10]) *$R=k[S]$ is a Macaulay ring.*

(2) (R. Stanley, [18]) $K_R \cong k[S_+]$.

PROOF. 3.2.12 is a direct consequence of 3.2.11 (cf. Chapter 2, § 2 for the definition of K_R). We shall prove 3.2.11. Let $h, h' \in L_-$. Then by 3.2.9 and the definition of $C^{\cdot}$, $[(\underline{H}^q_{\mathfrak{m}}(R))_h : k]=[(\underline{H}^q_{\mathfrak{m}}(R))_{h'} : k]$ for every q and the multiplication of $X^s (s \in S)$ induces a bijection from $(\underline{H}^q_{\mathfrak{m}}(R))_h$ to $(\underline{H}^q_{\mathfrak{m}}(R))_{h+s}$ if $h+s \in L_-$. So $\underline{H}^q_{\mathfrak{m}}(R)$ must be a direct sum of some copies of $k[L_-]$. But as $(k[L_-])^* = k[S_+]$ is a torsion-free R-module and as $R_{(\mathfrak{p})}$ is a Macaulay ring at least for $\mathfrak{p}=(0)$, $\underline{H}^q_{\mathfrak{m}}(R)=0$ for $q<\dim R$ by 3.2.6, (3). As for $q=d=\dim R$, we know that $(\underline{H}^d_{\mathfrak{m}}(R))^* = K_R$ is an R-module of rank 1. So $\underline{H}^d_{\mathfrak{m}}(R)=k[L_-]$.

EXAMPLE 3.2.13. We put $R_0=k[T_1, \cdots, T_m]$ and $\mathfrak{a}=(T_1, \cdots, T_m)^d$. We want to determine the cases when the Rees algebra $\bigoplus_{i\geq 0} \mathfrak{a}^i$ is a Gorenstein ring. It is easy to see that this Rees algebra is isomorphic to the semigroup ring $R=k[S]$, where $S=L \cap N^{m+2}$ and

$$L=\{(a_1, \cdots, a_m, b, c) \in Z^{m+2} \mid c=\sum_{i=1}^{m} a_i - db\} .$$

Then the generators of the S-ideal S_+ are

$$\{(1, \cdots, 1, dp, m-dp) \mid p \geq 1, m-1 \geq dp\}$$

and

$$\{(a_1, \cdots, a_m, b_0 d, 1) \mid \sum_{i=1}^{m} a_i = b_0 d+1\} ,$$

where b_0 is the smallest integer satisfying $b_0 \geq 1$ and $b_0 d+1 \geq m$. So, S_+ requires $\binom{b_0 d}{m-1}+[(m-1)/d]'$ generators (where $[(m-1)/d]'$ is the largest integer which is smaller than $(m-1)/d$). In particular, $R=k[S]$ is a Gorenstein ring if and only if $m=1$ or $d=m-1$.

DEFINITION 3.2.14. Let L be a subgroup of Z^n. We put

$$\Delta_L=\{I \subset \bar{n} \mid L \cap Z^{(I)} \cap N^n=\{0\}\} \quad (\text{we put } \bar{n}=\{1, 2, \cdots, n\}) .$$

It is clear by definition that if $I \in \Delta_L$ and if $J \subset I$, then $J \in \Delta_L$. Thus Δ_L is a simplicial complex and we have

$$\dim |\Delta_L| + \operatorname{rank} L+1=n . \quad (\text{See [17], Chapter 3, § 1.})$$

EXAMPLE 3.2.15. (i) If $\operatorname{rank} L=n$, then $\Delta_L=\{\phi\}$.

(ii) If $L=\{(a, \cdots, a) \mid a \in Z\}$, $\Delta_L=2^{\bar{n}}/\{\bar{n}\}$. Thus $|\Delta_L|$ is homotopic to the $(n-2)$-dimensional sphere.

(iii) If L is as in 3.2.13, then $I \in \Delta_L$ is either a subset of $\{1, \cdots, m\}$ or a subset of $\{m+1, m+2\}$. Thus $|\Delta_L|$ is homotopic to two points.

PROPOSITION 3.2.16. *If* $h \in L_-$, $(\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R))_h = \tilde{H}_{n-q-1}(\Delta_L; k)$. (*If* $\Delta_L = \{\phi\}$, $\tilde{H}_q(\phi; k) = 0$ *for* $q \geqq 0$ *and* $\tilde{H}_{-1}(\phi; k) = k$.)

PROOF. Recall that $\underline{\boldsymbol{H}}^q_{\mathfrak{m}}(R) = H^q('C^{\cdot})$ where $'C^{\cdot} = \underline{\boldsymbol{H}}^0_{\mathfrak{m}}(C^{\cdot})$. In 3.2.9, we have seen that $('C^{\cdot})_h$ has the basis $\{I \subset \bar{n} \mid \boldsymbol{Z}^{(I')} \cap L \cap \boldsymbol{N}^n = \{0\}\} = \{I \subset \bar{n} \mid I' \in \Delta_L\}$. It is easy to verify that the complex $('C^{\cdot})_h$ is isomorphic to the chain complex associated to the simplicial complex Δ_L.

COROLLARY 3.2.17.

$$\tilde{H}_q(\Delta_L; k) \cong \begin{cases} k & (q = n-1-\operatorname{rank} L) \\ 0 & (\text{otherwise}) . \end{cases}$$

PROOF. This is clear from 3.2.16 and 3.2.11, since $\dim R = \operatorname{rank} L$.

§ 3. General affine semigroup rings.

In this section let S be a finitely generated subsemigroup of $\boldsymbol{N}^n$ and let L be the subgroup of $\boldsymbol{Z}^n$ generated by S. We define $\bar{S}$ the "normalization" of S. That is,

$$\bar{S} = \{x \in L \mid nx \in S \text{ for some positive integer } n\} .$$

Then it is known that $\bar{S}$ is a normal semigroup and that $k[\bar{S}]$ is the normalization of $k[S]$ (cf. [10], § 1). As in § 2, we assume that $\bar{S} = L \cap \boldsymbol{N}^n$. We need further notations.

NOTATION 3.3.1. $R = k[S]$ and $\mathfrak{m} = k[S/\{0\}]$ (in the notation of 3.1.2.

$$F_i = \{(s_1, \cdots, s_n) \in S \mid s_i = 0\} , \quad \bar{F}_i = \{(s_1, \cdots, s_n) \in \bar{S} \mid s_i = 0\} \quad (1 \leqq i \leqq n) .$$

L_i (resp. $\bar{L}_i$) is the group generated by F_i (resp. $\bar{F}_i$). Note that L_i is a subgroup of finite index of $\bar{L}_i$. We assume that $L_i \neq L_j$ for $i \neq j$ (cf. 3.3.2).

$$S_i = S + L_i = S - F_i , \quad C_i = L/S_i \ (1 \leqq i \leqq n)$$

$$S' = \bigcap_{i=1}^n S_i , \quad C = \bigcap_{i=1}^n C_i .$$

LEMMA 3.3.2. (1) *For every* i, $1 \leqq i \leqq n$, $\operatorname{rank} L_i = \operatorname{rank} L - 1$.

(2) *We may assume that* $\bar{L}_i \neq \bar{L}_j$ *if* $i \neq j$.

PROOF. (1) If we put $\bar{P}_i = \bar{S}/\bar{F}_i$, $\bar{P}_i$ is a prime $\bar{S}$-ideal and, by 3.1.5, there is no prime $\bar{S}$-ideal between $\bar{P}_i$ and ϕ. So $ht(k[\bar{P}_i]) = 1$ and rank L_i = rank $\bar{L}_i = \dim(\bar{R}/k[\bar{P}_i]) = \dim \bar{R} - 1$ = rank $L - 1$.

(2) If $\bar{L}_i = \bar{L}_j$ for some $i \neq j$, we may construct a semigroup isomorphic to S in $\boldsymbol{Z}^{n-1}$ by deleting bases e_i, e_j and adding a new base which is a linear combination of e_i and e_j.

The main theorem of this chapter is the following

THEOREM 3.3.3. (1) *$R = k[S]$ is a Macaulay ring if and only if $S = S'$.*

(2) *$\underline{\boldsymbol{H}}^d_{\mathfrak{m}}(R) = k[C]$ as H-graded R-modules ($d = \dim R$ = rank L).*

(3) *R is a Gorenstein ring if and only if there exists an element $c \in L$ such that $c + S = -C$.*

We prove this theorem in several steps. First, we construct a complex.

DEFINITION 3.3.4. For every i, $1 \leqq i \leqq n$, we define the complex $D_i^{\cdot}$ as follows.

$D_i^0 = k[L]$, $D_i^1 = k[L]/k[S_i] = k[C_i]$ and $d\colon D_i^0 \to D_i^1$ is the canonical surjection map. We define the complex $D^{\cdot}$ by

$$D^{\cdot} = D_1^{\cdot} \# D_2^{\cdot} \# \cdots \# D_n^{\cdot}$$

the Segre product of the complexes of H-graded R-modules (cf. 3.1.6). Thus

$$D^p = \bigoplus_{\#(I)=p} \{ \mathop{\#}_{i \in I} k[C_i] \} = \bigoplus_{\#(I)=p} D_I\ , \quad \text{where } D_I = k[\bigcap_{i \in I} C_i]\ .$$

The complex $D^{\cdot}$ is a resolution of $k[S']$ in the category of H-graded R-modules (cf. the proof of 3.2.3). If $\bar{S} = S$, the complex $D^{\cdot}$ coincides with the complex $C^{\cdot}$ of 3.2.5.

LEMMA 3.3.5. *For every i, $1 \leqq i \leqq n$, there is an integer N_i such that the i-th coefficient of any element in C_i is smaller than N_i. (If $(x_1, \cdots, x_i, \cdots, x_n) \in C_i$, then $x_i < N_i$.)*

PROOF. Take an element $s = (s_1, \cdots, s_n) \in S$, $s \notin L_i$. Then the group L' generated by s and L_i has the same rank as L by 3.3.2. If we take the elements $a_1 = 0, a_2, \cdots, a_t$ from S ($t = [L\colon L']$) such that $L = \bigcup_{j=1}^t (a_j + L')$ and if we write $a_j = (a_{j1}, \cdots, a_{jt})$, it is clear that we can take $N_i = \max(a_{1i}, \cdots, a_{ti})$.

LEMMA 3.3.6. *Let us take $I \subset \bar{n}$ and put $I' = \bar{n}/I$ as usual. Then*

(1) *If $I' \in \Delta_L$* (cf. 3.2.14 *for the definition of Δ_L*), *D_I satisfies the condition* (*) *of* 3.2.7.

(2) *If* $I' \notin \Delta_L$, D_I *satisfies the condition* (**) *of* 3.2.7.

PROOF. The proof is similar to that of 3.2.8 if we use 3.3.5.

COROLLARY 3.3.7. $\underline{H}^q_{\mathfrak{m}}(k[S']) = H^q('D^\cdot)$ *where* $'D^\cdot$ *is the subcomplex of* $D^\cdot$ *which is the direct sum of all* D_I*'s such that* $I' \in \Delta_L$.

LEMMA 3.3.8. $(\underline{H}^q_{\mathfrak{m}}(k[S']))_h = 0$ *for every* $q \geqq 0$ *if* $h \notin C$.

PROOF. As $S' \subset \bar{S}$, $k[S']$ is a finitely generated R-module. So, $\underline{H}^q_{\mathfrak{m}}(k[S'])$ is an Artinian R-module. Then the proof is similar to that of 3.2.10.

(Conclusion of the proof of 3.3.3.) We have seen that $(\underline{H}^q_{\mathfrak{m}}(k[S']))_h = 0$ unless $h \in C$. If $h \in C$, it is easy to see that $(H^q('D^\cdot))_h \cong \tilde{H}_{n-q-1}(\Delta_L; k)$ (cf. 3.2.16). Then, by 3.2.17, we have

$$\underline{H}^q_{\mathfrak{m}}(k[S']) = \begin{cases} k[C] & (q = \dim R) \\ 0 & (\text{otherwise}). \end{cases}$$

This proves that $k[S']$ is a Macaulay ring and that $\underline{H}^d_{\mathfrak{m}}(k[S']) = k[C]$ ($d = \dim R$). Next, we put $M = k]S']/k[S]$. Then we know that $\mathrm{Ass}_R(M)$ consists of H-graded prime ideals of R. ([2], Chap. 4, § 3, Proposition 1.) But by 3.3.2, H-graded prime ideal of height 1 of R is of the form $\mathfrak{p}_i = k[S/F_i]$ for some i, $1 \leqq i \leqq n$. But it is clear that $R_{(\mathfrak{p}_i)} = (k[S'])_{(\mathfrak{p}_i)} = k[S_i]$. So $M_{(\mathfrak{p}_i)} = 0$ for every i. Thus $\dim M \leqq d-2$. If we write the long exact sequence of local cohomolgy groups associated to the short exact sequence

$$0 \longrightarrow R \longrightarrow k[S'] \longrightarrow M \longrightarrow 0$$

noting that $\underline{H}^q_{\mathfrak{m}}(M) = 0$ for $q \geqq d-1$, we can see that R is a Macaulay ring if and only if $M = 0$ and that $\underline{H}^d_{\mathfrak{m}}(R) = k[C]$. Thus we have proved (1) and (2). As for (3), we can prove that $k[-C]$ is a Macaulay R-module by the same process as the proof of 3.3.3. So (3) results from 2.2.3.

Before closing this paper, let us calculate some examples. It will be of some interest to compare the proof of 3.3.9 with that in [11].

EXAMPLE 3.3.9. ([11], Example 2.2.) Let a, b, t be indeterminates and let $R = k[a^2, a^3, b, ab, a^4t, a^2bt, b^2t] \subset k[a, b, t]$. If we consider the subsemigroup S of N^4 generated by (2, 0, 0, 2), (3, 0, 0, 3), (0, 1, 0, 2), (1, 1, 0, 3), (4, 0, 1, 0), (2, 1, 1, 0), and (0, 2, 1, 0), $k[S]$ is isomorphic to R and $\bar{S} = L \cap N^4$, where

$$L = \{(a, b, t, u) \in \boldsymbol{Z}^4 \mid a + 2b = 4t + u\} .$$

Note that $\bar{S}$ is generated by (1, 0, 0, 1), (0, 1, 0, 2), (4, 0, 1, 0), (2, 1, 1, 0), and (0, 2, 1, 0) and that $S=\{(a, b, t, u) \in \bar{S} \mid u \neq 1\}$. Let us determine L_i, S_i, and $C_i (i=1, \cdots, 4)$ for this S. We can see easily,

L_1 is generated by (0, 1, 0, 2) and (0, 2, 1, 0)
L_2 is generated by (1, 0, 0, 1) and (0, 0, 1, −4)
L_3 is generated by (1, 0, 0, 1) and (0, 1, 0, 2)
L_4 is generated by (2, 1, 1, 0) and (0, 2, 1, 0)

$$S_1=\{(a, b, t, u) \in L \mid a \geqq 0\} \qquad C_1=\{(a, b, t, u) \in L \mid a<0\}$$
$$S_2=\{(a, b, t, u) \in L \mid b \geqq 0\} \qquad C_2=\{(a, b, t, u) \in L \mid b<0\}$$
$$S_3=\{(a, b, t, u) \in L \mid t \geqq 0\} \qquad C_3=\{(a, b, t, u) \in L \mid t<0\}$$
$$S_4=\{a, b, t, u) \in L \mid u \geqq 0, u \neq 1\} \qquad C_4=\{(a, b, t, u) \in L \mid u<0 \text{ or } u=1\}.$$

As $S=\{(a, b, t, u) \in \bar{S} \mid u \neq 1\}$, we have $S=\bigcap_{i=1}^4 S_i$ and $C=(-1, -1, -1, 1)-S$.

Thus, by 3.3.3, $R=k[S]$ is a Gorenstein ring.

EXAMPLE 3.3.10. Let $R=k[a^4, a^3b, ab^3, b^4] \subset k[a, b]$ and $R=R_0[a^4t, b^4t] \subset R_0[t]$. Then we can prove that R is a Macaulay ring by the same process as in 3.3.9. In this case, the generators of K_R are a^3bt, a^2b^2t and ab^3t. (In this case, R_0 is not a Macaulay ring. But R_0 is a Buchsbaum ring and (a^4, b^4) is a parameter system for R_0. In general, if R_0 is a 2-dimensional local Buchsbaum domain and if (x, y) is a parameter system for R_0, it is proved in [19] that $R_0[xt, yt]$ is a Macaulay ring.

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