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An Analogue of Paley-Wiener Theorem on Rank 1 Semisimple Lie Groups II

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Introduction

This paper is a continuation of the previous paper [8]. Let G be a connected semisimple Lie group with finite center. We assume that G is not compact and moreover, the real rank of G is one. In [8], we have obtained an analogue of Paley-Wiener theorem on $\mathscr{C}(G, \tau)$ (see Theorem 2 in [8]). However, in that theorem we did not consider the discrete part of $\mathscr{C}(G, \tau)$, i.e., the space of τ-spherical cusp forms on G. Therefore in this paper we shall characterize the discrete part of compactly supported functions on G. Here we note that this characterization depends on $\{E_p; 1\leqq p\leqq\gamma\}$ and does not depend on any choice of $\{h_p; 1\leqq p\leqq\gamma\}$ (see (4.12) and (4.15) in [8]). Next using the proof of Theorem 2 in [8], we shall obtain the relation between a size of a support of a compactly supported function on G and an exponential type of its Fourier transform. To obtain the relation we shall use the same method in the classical Paley-Wiener theorem on an Euclidean space.

In §2 using the results of Harish-Chandra [6], we shall reform the theorem of J. Arthur [1] and obtain some characterization of $\mathscr{C}(G)$. Then applying the above consideration to each K-finite subspace of $\mathscr{C}(G)$, we shall obtain an analogue of Paley-Wiener theorem on $\mathscr{C}(G)$ (see §3).

§1. More precise characterization.

For an arbitrary function g in $\mathscr{C}(G, \tau)$ we shall define g' by $\mathscr{C}_A^{-1}(\mathscr{C}_A(g))$ and g° by $g-g'$. Then from Theorem 1 in [8] we can easily prove that g' belongs to $\mathscr{C}_A(G, \tau)$ and g° to ${}^\circ\mathscr{C}(G, \tau)$. Let notation be as in [8].

LEMMA 1. (i) ${}^\circ\mathscr{C}(G, \tau)$ *is contained in the space which is generated*

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by $\{E_p;\ 1\leqq p\leqq \gamma\}$.

(ii) ${}^\circ\mathscr{C}(G,\tau)$ *is generated by* $\{h_p^\circ;\ 1\leqq p\leqq \gamma\}$.

PROOF. Let F be an arbitrary function in $C_c^\infty(G,\tau)$. Then from Theorem 2 in [8] it is obvious that $\mathscr{C}_A(F)$ belongs to $\mathscr{H}(\mathscr{F})_*^*$ and moreover, there exists a function H in $C_c^\infty(G,\tau)$ such that $\mathscr{C}_A(F)=\mathscr{C}_A(H)$. Thus we obtain,

$$F'=H' . \tag{1.1}$$

On the other hand, from the proof of Theorem 2 in [8] we obtain,

$$\begin{aligned} H^\circ(x)&=\sum_{1\leqq p\leqq \gamma} C(p)h_p^\circ(x) \\ &=\sum_{1\leqq p\leqq \gamma} \frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} \hat{F}(\phi_{i(p)}^{j(p)},\ s(p)\nu)h_p^\circ(x) \quad (x\in G) . \end{aligned} \tag{1.2}$$

However, since $F(x)$ has a compact support, we have,

$$\begin{aligned} &\frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} \hat{F}(\phi_{i(p)}^{j(p)},\ s(p)\nu) \\ &=\left(F,\ \frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} E(P:\ \phi_{i(p)}^{j(p)}:\ s(p)\nu:\ x)\right) \\ &=(F,\ E_p) \quad (1\leqq p\leqq \gamma) . \end{aligned} \tag{1.3}$$

Thus we obtain,

$$H^\circ(x)=\sum_{1\leqq p\leqq \gamma} (F,\ E_p)h_p^\circ(x) \quad (x\in G) . \tag{1.4}$$

By the way, using the relation; (1.1), we can easily prove that $F-H=F^\circ-H^\circ$ and both sides belong to $C_c^\infty(G,\tau)$ and moreover, to ${}^\circ\mathscr{C}(G,\tau)$. Therefore it must be zero, i.e., $F=H$ and $F^\circ=H^\circ$. Thus we obtain,

$$F^\circ(x)=\sum_{1\leqq p\leqq \gamma} (F,\ E_p)h_p^\circ(x) \quad (x\in G) . \tag{1.5}$$

Now we note that $C_c^\infty(G,\tau)$ is dense in $\mathscr{C}(G,\tau)$. Therefore $\{g^\circ;\ g\in C_c^\infty(G,\tau)\}$ must be equal to ${}^\circ\mathscr{C}(G,\tau)$, because $\dim{}^\circ\mathscr{C}(G,\tau)<\infty$ and $g\mapsto g^\circ$ is a continuous projection of $\mathscr{C}(G,\tau)$ onto ${}^\circ\mathscr{C}(G,\tau)$. Then (ii) is quite obvious from (1.2) or (1.5).

Next we shall prove (i). First of all we shall apply the arguments in §1 in [8] to the case of $P=G$. Then $L^G={}^\circ\mathscr{C}(G,\tau)$ can be decomposed as

$$L^G=\bigoplus_{1\leqq j\leqq m'} L^G(\Lambda_j) \tag{1.6}$$

where $\Lambda_j \in \mathscr{E}^2(G)$ and $L^G(\Lambda_j)=L^G \cap (\mathscr{H}_{\Lambda_j} \otimes V)$ for $1 \leqq j \leqq m'$. Moreover, we choose an orthonormal basis of $L^G(\Lambda_j)$ as follows,

$$(1.7) \qquad \{\psi_i^j;\ 1 \leqq i \leqq n'_j\}\ , \quad \text{where} \quad n'_j = \dim L^G(\Lambda_j) \quad \text{for} \quad 1 \leqq j \leqq m'\ .$$

For simplicity we put $e'_k = \psi_i^j$, where $k = \sum_{1 \leqq p \leqq j-1} n'_p + i (1 \leqq k \leqq n' = \sum_{1 \leqq j \leqq m'} n'_j)$. (Note that $n' \leqq \gamma$ by (ii).) Here we may assume that $h_p^\circ (1 \leqq p \leqq \gamma)$ has the following expansion,

$$(1.8) \qquad h_p^\circ = \sum_{1 \leqq k \leqq n'} C_{pk} e'_k \quad C_{pk} \in \boldsymbol{C} \quad \text{for} \quad 1 \leqq p \leqq \gamma \quad \text{and} \quad 1 \leqq k \leqq n'\ .$$

Here we denote by $\underline{M} = (C_{pk})$ the $\gamma \times n'$ matrix whose (p, k)-entry is equal to $C_{pk} (1 \leqq p \leqq \gamma,\ 1 \leqq k \leqq n')$, and in the next lemma we shall prove that $\underline{M}$ does not depend on any choice of $\{h_p;\ 1 \leqq p \leqq \gamma\}$.

Now let F be an arbitrary function in $C_c^\infty(G, \tau)$. Then from (1.5) we have

$$\begin{aligned}(1.9) \qquad (F, e'_k) &= (F^\circ, e'_k) \\ &= \sum_{1 \leqq p \leqq \gamma} (F, E_p)(h_p^\circ, e'_k) \\ &= \Big(F, \sum_{1 \leqq p \leqq \gamma} C_{pk} E_p\Big) \quad (1 \leqq k \leqq n')\ .\end{aligned}$$

Thus, we have,

$$(1.10) \qquad e'_k = \sum_{1 \leqq p \leqq \gamma} C_{pk} E_p \quad (1 \leqq k \leqq n')\ .$$

Therefore (i) is obvious from this relation. Q.E.D.

COROLLARY. *Let F be in $C_c^\infty(G, \tau)$. Then F can be written as*

$$F(x) = F'(x) + \sum_{1 \leqq p \leqq \gamma} (F, E_p) h_p^\circ(x) \quad (x \in G)\ .$$

LEMMA 2. *Let notation be as in the proof of Lemma 1. Then $\underline{M}$ does not depend on any choice of $\{h_p;\ 1 \leqq p \leqq \gamma\}$.*

PROOF. It is enough to prove that $C_{pk} (1 \leqq p \leqq \gamma,\ 1 \leqq k \leqq n')$ which satisfy the relation (1.10) are unique. Suppose there exist constants C'_{pk} for which

$$(1.11) \qquad e'_k = \sum_{1 \leqq p \leqq \gamma} C'_{pk} E_p \quad C'_{pk} \in \boldsymbol{C} \quad \text{for} \quad 1 \leqq k \leqq n'\ .$$

Then from (1.10) we have,

$$(1.12) \qquad \sum_{1 \leqq p \leqq \gamma} (C_{pk} - C'_{pk}) E_p = 0 \quad \text{for} \quad 1 \leqq k \leqq n'\ .$$

However, since $\{E_p;\ 1\leqq p\leqq \gamma\}$ is a maximal linearly independent set (see the definition of $E_p(1\leqq p\leqq \gamma)$ in [8]), we can obtain that $C_{pk}=C'_{pk}$ for all $1\leqq p\leqq \gamma$ and $1\leqq k\leqq n'$. This is the desired relation. Q.E.D.

Now we shall define a more precise Fourier transform on $\mathscr{C}(G, \tau)$ as follows; for $f\in\mathscr{C}(G, \tau)$,

$$F(f)=((f, e'_1), (f, e'_2), \cdots, (f, e'_{n'})) \oplus(\hat{f}(e_1, \nu), \hat{f}(e_2, \nu), \cdots, \hat{f}(e_n, \nu)) \quad \text{for} \quad \nu\in\mathscr{F} . \tag{1.13}$$

Put $\mathscr{E}_0(f)=((f, e'_1), (f, e'_2), \cdots, (f, e'_{n'}))$. Then the mapping $\mathscr{E}_0$ of $\mathscr{C}(G, \tau)$ into $C^{n'}$ coincides with $\mathscr{E}_A$ for the case of $P=G$ in [8] and moreover $F(f)$ can be written as $\mathscr{E}_0(f)\oplus\mathscr{E}_A(f)$. Thus, using Theorem 1 in [8], we can easily obtain the following theorem.

THEOREM 1. *The mapping* F: $\mathscr{C}(G, \tau)\rightarrow C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$ *is a homeomorphism of* $\mathscr{C}(G, \tau)$ *onto* $C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$.

Next we shall define a subspace of $C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$ which becomes the image of compactly supported functions in $\mathscr{C}(G, \tau)$.

Let $a\oplus V$ be an arbitrary element in $C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$. Then we can write a and V as follows.

$$\begin{aligned} a&=(a_1, a_2, \cdots, a_{n'}) \\ &=(a^1_1, a^1_2, \cdots, a^1_{n'_1}, a^2_1, a^2_2, \cdots, a^2_{n'_2}, \cdots, a^{m'}_1, a^{m'}_2, \cdots, a^{m'}_{n'_{m'}}) \\ V&=(v^1_1(\nu), v^1_2(\nu), \cdots, v^1_{n_1}(\nu), v^2_1(\nu), v^2_2(\nu), \cdots, v^2_{n_2}(\nu), \cdots, \\ &\qquad v^m_1(\nu), v^m_2(\nu), v^m_{n_m}(\nu)) \end{aligned} \tag{1.14}$$

where $a^j_i\in C$ and $v^j_i\in\mathscr{C}(\mathscr{F})(1\leqq i\leqq n_j, 1\leqq j\leqq m)$. In this case, we shall use the following notation for simplicity,

$$a=(a^j_i)=(a_k) \quad \text{and} \quad V=(v^j_i) . \tag{1.15}$$

Let $\mathscr{H}$ be the subspace of $C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$ which consists of all $a\oplus V\in C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$ satisfying the following conditions;

$$\begin{aligned} &\text{(i)} \quad V\in\mathscr{H}(\mathscr{F})^n_*. \\ &\text{(ii)} \quad a_k=\sum_{1\leqq p\leqq\gamma} C_{pk}\frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} v^{j(p)}_{i(p)}(s(p)\nu) \quad (1\leqq k\leqq n') . \end{aligned} \tag{1.16}$$

Then we have the following theorem.

THEOREM 2. *Let* f *be a function in* $\mathscr{C}(G, \tau)$. *Then* f *belongs to* $C^\infty_c(G, \tau)$ *if and only if* $F(f)$ *belongs to* $\mathscr{H}$.

PROOF. Let f be in $C_c^\infty(G, \tau)$. Put $F(f)=\mathscr{E}_0(f)\oplus\mathscr{E}_A(f)=((f, e_k'))\oplus(\hat{f}(\phi_i^j, \nu))$. Then from Theorem 2 in [8] we have $\mathscr{E}_A(f)\in\mathscr{H}(\mathscr{F})_*^n$. Moreover, since f has a compact support, we have,

$$(1.17)\qquad (f, e_k')=\sum_{1\leq p\leq\gamma} C_{pk}(f, E_p)\quad \text{(see (1.9))}$$

$$=\sum_{1\leq p\leq\gamma} C_{pk}\frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)}\hat{f}(\phi_{i(p)}^{j(p)}, s(p)\nu)\quad \text{for}\quad 1\leq k\leq n'\ .$$

Therefore $((f, e_k'))$ satisfies the condition (ii) of $\mathscr{H}$. Thus we obtain that $F(f)$ belongs to $\mathscr{H}$.

Next let f be in $\mathscr{C}(G, \tau)$ and $F(f)$ belongs to $\mathscr{H}$. Here we shall write $F(f)$ as $((f, e_k'))\oplus(\hat{f}(\phi_i^j, \nu))$. Since $\mathscr{E}_A(f)$ belongs to $\mathscr{H}(\mathscr{F})_*^n$, there exists a compactly supported function H on G such that $\mathscr{E}_A(H)=\mathscr{E}_A(f)$ (see Theorem 2 in [8]). Moreover, H° can be written as (1.2) where we use f instead of F. Therefore we can easily prove that,

$$(1.18)\qquad (H, e_k')=\sum_{1\leq p\leq\gamma} C_{pk}\frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)}\hat{f}(\phi_{i(p)}^{j(p)}, s(p)\nu)\qquad \text{for}\quad 1\leq k\leq n'\ .$$

Thus from Theorem 1 we have $H=f$ (note that $F(H)=F(f)$, since $F(f)$ belongs to $\mathscr{H}$). In particular, f has a compact support. This completes the proof of Theorem. Q.E.D.

Next we shall obtain the relation between a size of a support of a compactly supported function on G and an exponential type of its Fourier transform.

Let $\mathscr{H}(R)$(R: a positive number) be the subspace of $\mathscr{H}$ which consists of all $a\oplus V=(a_i^j)\oplus(v_i^j(\nu))\in\mathscr{H}$ satisfying the following conditions; for each integer N, there exsist constants C_N for which

$$(1.19)\qquad |v_i^j(\nu+(-1)^{1/2}\eta)|\leq C_N(|\nu+(-1)^{1/2}\eta|)^{-N}e^{R|\eta|}\ \text{for}\ \nu\in\mathscr{F}\ \text{and}$$
$$\eta\in \mathrm{CL}(\mathscr{F}^+)\qquad\qquad (1\leq i\leq n_j,\ 1\leq j\leq m)\ .$$

Moreover let $C_c^\infty(G, \tau: R)$ denote the subspace of $C_c^\infty(G, \tau)$ which consists of all functions in $C_c^\infty(G, \tau)$ such that their supports are contained in a compact set $G_R=\{x\in G;\ \sigma(x)\leq R\}$ (see the definition of σ for V. S. Varadarajan [9]). Then we obtain the following theorem.

THEOREM 3. *f belongs to $C_c^\infty(G, \tau: R)$ if and only if $F(f)$ belongs to $\mathscr{H}(R)$.*

PROOF. Let f be a function in $C_c^\infty(G, \tau: R)$. Then using the same method in the classical Paley-Wiener theorem on an Euclidean space and the definition of the Eisenstein integral, we can easily prove that each component $\hat{f}(\phi_i^j, \nu)$ of $\mathscr{E}_A(f)$ satisfies the above relation (1.19) for $1 \leq i \leq n_j$, $1 \leq j \leq m$. Thus $F(f)$ belongs to $\mathscr{H}(R)$.

Next we assume that $F(f) = \mathscr{E}_0(f) \oplus \mathscr{E}_A(f)$ belongs to $\mathscr{H}(R)$. Then from Theorem 2, f has a compact support. Moreover from Corollary of Lemma 1 f can be written as,

$$f(x) = f'(x) + \sum_{1 \leq p \leq \gamma} (f, E_p) h_p^\circ(x) \tag{1.20}$$
$$= \Big\{ f'(x) - \sum_{1 \leq p \leq \gamma} (f, E_p) h_p'(x) \Big\} + \sum_{1 \leq p \leq \gamma} (f, E_p) h_p(x) \quad (x \in G) .$$

Put $\underline{G}'(x) = f'(x) - \sum_{1 \leq p \leq \gamma} (f, E_p) h_p'(x)$ (this function is same as (4.22) in [8]). Here we note that the support of $h_p (1 \leq p \leq \gamma)$ can be taken sufficiently small (see the construction of $h_p (1 \leq p \leq \gamma)$ in [8]). Therefore we may assume that h_p belongs to $C_c^\infty(G, \tau: R)$ for $1 \leq p \leq \gamma$. Next we note that $\hat{f}(\phi_i^j, \nu) = \hat{f}'(\phi_i^j, \nu) (1 \leq i \leq n_j, 1 \leq j \leq m)$ satisfy the condition (1.19) by the assumption $F(f) \in \mathscr{H}(R)$ and moreover, $\hat{h}_p(\phi_i^j, \nu) = \hat{h}_p'(\phi_i^j, \nu) (1 \leq i \leq n_j, 1 \leq j \leq m)$ satisfy the same condition (1.19) for $1 \leq p \leq \gamma$ (because $h_p \in C_c^\infty(G, \tau: R)$ and we used the necessary condition). Thus we obtain that $\hat{\underline{G}}'$ satisfies the condition (1.19). However we recall that $\underline{G}'(x)$ can be written as,

$$\underline{G}'(x) = \sum_{1 \leq j \leq m} |W(\omega_j)|^{-1} \sum_{1 \leq i \leq n_j} \int_{\mathscr{F}} \mu(\omega_j, \nu) E(P: \phi_i^j: \nu: x) \hat{\underline{G}}'(\phi_i^j, \nu) d\nu . \tag{1.21}$$

Thus, using the proof of Theorem 2 in [8] (in particular (4.7), (4.8) and (4.19)) and the condition (4.19), we can easily prove that $\underline{G}'$ belongs to $C_c^\infty(G, \tau: R)$ by the same method in the classical Paley-Wiener theorem on an Euclidean space. Therefore we obtain that f belongs to $C_c^\infty(G, \tau: R)$. This completes the proof of Theorem 3. Q.E.D.

§ 2. Some results.

In this section we shall describe the results which were obtained in Harish-Chandra [6]. Then using these results, we shall reform the theorem in J. Arthur [1] and obtain some characterization of the Schwartz space $\mathscr{C}(G)$.

First of all we shall obtain a relation between an Eisenstein integral and a matrix coefficient of the principal series for G.

Put $V = C^\infty(K \times K)$. Then for any v_1, v_2 in V we shall define the scalar product (,) as follows;

$$(v_1, v_2) = \int_{K \times K} \overline{v_1(k_1: k_2)} v_2(k_1: k_2) dk_1 dk_2 . \tag{2.1}$$

Then the norm of v in V is defined as,

$$(2.2)\qquad \|v\|^2=\int_{K\times K}|v(k_1\colon k_2)|^2dk_1dk_2$$

and obviously V becames a Hilbert space under this norm. Moreover we shall define a operator $\cdot$, tr and anti-involution * as follows;

$$(2.3)\qquad \begin{aligned} v_1\cdot v_2&=\int_K v_1(k_1\colon k)v_2(k^{-1}\colon k_2)dk\ ,\\ tr(v)&=\int_K v(k\colon k^{-1})dk\ ,\\ v^*(k_1\colon k_2)&=\mathrm{conj}(v(k_2^{-1}\colon k_1^{-1}))\ . \end{aligned}$$

Next we shall define a double representation μ of K on V as follows

$$(2.4)\qquad \begin{aligned} \mu(k)v(k_1\colon k_2)&=v(k_1k\colon k_2)\\ v(k_1\colon k_2)\mu(k)&=v(k_1\colon kk_2) \end{aligned}$$

for all $k\in K$ and $v\in V$. Then it is obvious that μ is a unitary double representation of K on V with respect to the above norm.

Now let F be a finite subset of $\mathscr{E}(K)$ and put $\alpha_F(k)=\sum_{\delta\in F}\alpha_\delta(k)(k\in K)$ where $\alpha_\delta=d(\delta)\,\mathrm{conj}(\chi_\delta)(\chi_\delta$ is the character of the class δ and $d(\delta)=\chi_\delta(1))$. Then we denote by V_F the subspace of V consisting all v in V such that

$$(2.5)\qquad v=\int_K\alpha_F(k)\mu(k)vdk=\int_K\alpha_F(k)v\mu(k)dk\ .$$

Then we can easily prove that V_F is stable under μ and $\dim(V_F)<\infty$. Then let μ_F denote the restriction of μ on V_F.

Now let ω be an element in $\mathscr{E}^2(M)$ and fix it. Let $\underline{\omega}$ be an irreducible representation of M on U_ω whose class belongs to ω. Moreover, let $\mathfrak{H}_\omega$ (resp. $\mathfrak{H}'_\omega$) denote the space of the representation

$$(2.6)\qquad \pi_\omega=\mathrm{Ind}_{K_M}^K(\underline{\omega}\,|\,K_M)\quad (\text{resp. } \pi_{\omega,\nu}=\mathrm{Ind}_{MAN}^G(\underline{\omega}\otimes e^\nu\otimes 1)\ \nu\in\mathscr{F}_c)\ .$$

(cf. Harish-Chandra [6] §4). Then we can easily prove that the mapping: $f\mapsto f|_K$ (the restriction of f on K) is a unitary isomorphism of $\mathfrak{H}'_\omega$ onto $\mathfrak{H}_\omega$. Thus we mayidentify these two spaces under the above mapping. For a fixed finite subset F of $\mathscr{E}(K)$, we put

$$(2.7)\qquad P_F=\int_K\alpha_F(k)\pi_\omega(k)dk\quad\text{and}\quad \mathfrak{H}_\omega^F=P_F(\mathfrak{H}_\omega)\ .$$

Then we define $L={}^\circ\mathscr{C}(M,V_F,\mu_F)$ and $L(\omega)$ as usual and obtain the following lemmas.

LEMMA 3. *For each T in* $\mathrm{End}(\mathfrak{H}_\omega^F)$, *we can associate a Ψ_T in $L(\omega)$ such that the mapping: $T \mapsto d_\omega^{1/2}\Psi_T$ is a linear isometry of* $\mathrm{End}(\mathfrak{H}_\omega^F)$ *with the Hilbert-Schmidt norm onto $L(\omega)$ with L^2-norm, where d_ω is the formal degree of the class ω.*

PROOF. See Harish-Chandra [6] §7. Ψ_T is defined for $T \in \mathrm{End}(\mathfrak{H}_\omega^F)$ as follows;

$$\Psi_T(m)(k_1: k_2) = \mathrm{tr}(\kappa_T(k_2: k_1)\underline{\omega})m)) \quad (m \in M \text{ and } k_1, k_2 \in K)\ , \tag{2.8}$$

where for an orthonormal basis $\{h_i; 1 \leqq i \leqq p\}$(resp. $\{u_j; 1 \leqq j \leqq q\}$) of $\mathfrak{H}_\omega^F$ (resp. U_ω), κ_T is the linear transformation on U_ω given by

$$\kappa_T(k_1: k_2)u = \sum_{1 \leqq i \leqq p} h_i(k_2)((T^* h_i)(k_1), u) \quad \text{for} \quad u \in U_\omega\ . \tag{2.9}$$

Thus Ψ_T can be written as

$$\Psi_T(m)(k_1: k_2) = \sum_{1 \leqq i \leqq p} \sum_{1 \leqq j \leqq q} ((T^* h_i)(k_1), \underline{\omega}(m)u_j)(h_i(k_2), u_j)\ . \tag{2.10}$$

LEMMA 4. *Let notation be as above. Then we have,*

$$E(P: \Psi_T: \nu: x)(1: 1) = \mathrm{tr}(\pi_{\omega,\nu}(x)T) \quad (x \in G)\ . \tag{2.11}$$

PROOF. See Harish-Chandra [6].

Now let τ_1, τ_2 be arbitrary elements in $\mathscr{E}(K)$ and put $F = \{\tau_1, \tau_2\}$. Then we denote by V_{τ_1,τ_2} the subspace of V consisting of all elements v in V_F satisfying;

$$v = \int_K \alpha_{\tau_1}(k)\mu(k)v dk = \int_K \alpha_{\tau_2}(k)v\mu(k)dk\ . \tag{2.11}$$

Here we choose an orthonormal basis of $\mathfrak{H}_\omega^F$ as follows;

$$\{\Phi_{\tau_1,i}, \Phi_{\tau_2,j}; 1 \leqq i \leqq [\tau_1: \omega] \dim \tau_1, 1 \leqq j \leqq [\tau_2: \omega] \dim \tau_2\}\ , \tag{2.12}$$

where $\dim \tau_i (i=1, 2)$ is the dimension of the representation space of τ_i. Thus, using (2.10), we can write Ψ_T as follows.

$$\begin{aligned}\Psi_T(m)(k_1: k_2) = &\sum_{1 \leqq i \leqq d_1} \sum_{1 \leqq r \leqq q} (T^*\Phi_{\tau_1,i}(k_1), \underline{\omega}(m)u_r)(\Phi_{\tau_1,i}(k_2), u_r) \\ &+ \sum_{1 \leqq j \leqq d_2} \sum_{1 \leqq r \leqq q} (T^*\Phi_{\tau_2,j}(k_1), \underline{\omega}(m)u_r)(\Phi_{\tau_2,j}(k_2), u_r)\ ,\end{aligned} \tag{2.13}$$

where $d_1 = [\tau_1: \omega] \dim \tau_1$ and $d_2 = [\tau_2: \omega] \dim \tau_2$. Now we assume that T belongs to $\mathrm{End}(\mathfrak{H}_\omega^{\tau_1}, \mathfrak{H}_\omega^{\tau_2})$. Then we can write Ψ_T as

$$\Psi_T(m)(k_1: k_2) = \sum_{1 \leqq i \leqq d_2} \sum_{1 \leqq r \leqq q} (T^*\Phi_{\tau_2,j}(k_1), \underline{\omega}(m)u_r)(\Phi_{\tau_2,j}(k_2), u_r)\ . \tag{2.14}$$

In this case we can easily prove that Ψ_T is a V_{τ_1,τ_2}-valued function on M. Therefore we obtain the following lemma.

LEMMA 5. *Let notation be as in Lemma 3. If T belongs to* $\mathrm{End}(\mathfrak{H}_\omega^{\tau_1}, \mathfrak{H}_\omega^{\tau_2})$, *then Ψ_T belongs to $L(\omega) \cap (\mathscr{H}_\omega \otimes V_{\tau_1,\tau_2})$.*

Note:

$$L(\omega) \cap (\mathscr{H}_\omega \otimes V_{\tau_1,\tau_2}) = {}^\circ\mathscr{C}(M, V_F, \mu_F) \cap (\mathscr{H}_\omega \otimes V_F) \cap (\mathscr{H}_\omega \otimes V_{\tau_1,\tau_2})$$
$$= {}^\circ\mathscr{C}(M, V_{\tau_1,\tau_2}, \mu_F) \cap (\mathscr{H}_\omega \otimes V_{\tau_1,\tau_2}) .$$

Next we shall reform the results of J. Arthur [1]. Let f be a function in $\mathscr{C}(G)$ (the scalar valued Schwartz space on G). Then we can define a usual Fourier transformation; $\hat{f}(\omega, \nu)$ and $\hat{f}(\Lambda)$ as follows;

$$\text{(2.15)} \qquad \begin{aligned} \hat{f}(\omega, \nu) &= \int_G f(x)\pi^P_{\omega,\nu}(x)dx \quad (\omega \in \mathscr{E}^2(M),\ \nu \in \mathscr{F}) \\ \hat{f}(\Lambda) &= \int_G f(x)\pi_\Lambda(x)dx \quad (\Lambda \in \mathscr{E}^2(G)) , \end{aligned}$$

where $\pi^P_{\omega,\nu} = \mathrm{Ind}^G_{MAN}(\underline{\omega} \otimes e^\nu \otimes 1)$ and π_Λ is the representation of G whose class belongs to Λ. Here we denote by $\mathfrak{H}_\omega$ and $\mathfrak{H}_\Lambda$ the representation spaces of $\pi^P_{\omega,\nu}$ and π_Λ respectively. Then we choose an orthonormol basis of $\mathfrak{H}_\omega$(resp. $\mathfrak{H}_\Lambda$) which transforms under $\pi^P_{\omega,\nu|K}$(resp. $\pi_{\Lambda|K}$) (the restriction of $\pi_{\omega,\nu}$(resp. π_Λ) to K) according to the irreducible representation τ in $\mathscr{E}(K)$ as follows;

$$\text{(2.16)} \qquad \{\Phi_{\tau,i};\ 1 \leqq i \leqq [\tau : \omega] \dim \tau\} \quad (\text{resp. } \{\Phi'_{\tau,i};\ 1 \leqq i \leqq [\Lambda : \tau] \dim \tau\})$$

where $[\tau : \omega] = [\tau_{|M} : \omega]$, $[\Lambda : \tau] = [\Lambda_{|K} : \tau]$. Put $d_\tau = [\tau : \omega] \dim \tau$ and $d'_\tau = [\Lambda : \tau] \dim \tau$.

Now for f in $\mathscr{C}(G)$, we define $V = C^\infty(K \times K)$-valued function $\tilde{f}$ as follows.

$$\text{(2.17)} \qquad \tilde{f}(x)(k_1, k_2) = f(k_1 x k_2) \quad \text{for} \quad k_1, k_2 \in K \quad \text{and} \quad x \in G .$$

Then we can easily prove that the mapping: $f \mapsto \tilde{f}$ is a topological linear isomorphism of $\mathscr{C}(G)$ onto $\mathscr{C}(G, V)$. Here we fix a finite subset $F = \{\tau_1, \tau_2\}$ in $\mathscr{E}(K)$ and put $p_{\tau_i} = \int_K \alpha_{\tau_i}(k)\mu(k)dk\, (i=1, 2)$. Then we define f_{τ_1,τ_2} as follows.

$$\text{(2.18)} \qquad f_{\tau_1,\tau_2}(x) = p_{\tau_1}(\tilde{f}(x))p_{\tau_2} \quad \text{for} \quad f \in \mathscr{C}(G) \quad (x \in G) .$$

Obviously, f_{τ_1,τ_2} belongs to $\mathscr{C}(G, V_{\tau_1,\tau_2}, \mu_F)$ and moreover, the mapping $f \mapsto f_{\tau_1,\tau_2}$ is a topological linear isomorphism of

$$\mathscr{C}(G)_{\tau_1,\tau_2}=\{f\in\mathscr{C}(G);\ \alpha_{\tau_1}*f*\alpha_{\tau_2}=f\}$$

onto $\mathscr{C}(G, V_{\tau_1,\tau_2}, \mu_F)$. Now we apply the arguments in §1 to the pair $(V_{\tau_1,\tau_2}, \mu_{\tau_1,\tau_2})(\mu_{\tau_1,\tau_2}=\mu_F\,|\,V_{\tau_1,\tau_2})$ instead of (V, τ). Then we can obtain the homeomorphism F_{τ_1,τ_2} of $\mathscr{C}(G, V_{\tau_1,\tau_2}, \mu_F)$ onto $C^{n'}\oplus\mathscr{C}(\mathscr{F})^n_*$, where n, n' depend on τ_1, τ_2 (see the definition of the mapping F). Then we have the following lemma.

Lemma 6. *Let notation be as above. Then we can choose an orthonormal basis: $\{\phi_i^j; 1\leqq i\leqq n_j\}$ of $L(\omega_j)$(resp. $\{\psi_i^j; 1\leqq i\leqq n_j'\}$ of $L^G(\Lambda_j)$) satisfying the following relations;*

$$d_{\omega_j}^{1/2}(\Phi_{\tau_1,p}, \hat{f}(\omega_j, \nu)\Phi_{\tau_2,q})=\hat{f}_{\tau_1,\tau_2}(\phi_i^j, \nu), \quad \textit{where} \quad i=d_{\tau_2}(p-1)+q \tag{2.19}$$
$$(1\leqq p\leqq d_{\tau_1},\ 1\leqq q\leqq d_{\tau_2},\ 1\leqq j\leqq m)$$

and

$$d_j^{1/2}(\Phi'_{\tau_1,p}, \hat{f}(\Lambda_j)\Phi'_{\tau_2,q})=(f_{\tau_1,\tau_2}, \psi_i^j), \quad \textit{where} \quad i=d'_{\tau_2}(p-1)+q$$
$$(1\leqq p\leqq d'_{\tau_1},\ 1\leqq q\leqq d'_{\tau_2},\ 1\leqq j\leqq m')$$

for $f\in\mathscr{C}(G)$.

Proof. For each j, p, q, we have,

$$\begin{aligned}(\Phi_{\tau_1,p}, \hat{f}(\omega_j, \nu)\Phi_{\tau_2,q})&=\int_G\overline{f(x)}(\Phi_{\tau_1,p}, \pi^P_{\omega_j,\nu}(x)\Phi_{\tau_2,q})dx\\ &=\int_G\overline{f(x)}\,\mathrm{tr}\,(\pi^P_{\omega_j,\nu}(x)T(\tau_1, \tau_2: j:p, q))dx\end{aligned}$$

where $T(\tau_1, \tau_2: j: p, q)$ is an element in $\mathrm{End}(\mathfrak{H}_\omega^{\tau_1}, \mathfrak{H}_\omega^{\tau_2})$ given by the following conditions;

$$(T(\tau_1, \tau_2: j: p, q)\Phi_{\tau_1,p'}, \Phi_{\tau_2,q'})=\delta_{pp'}\delta_{q,q'} \text{ for } 1\leqq p'\leqq d_{\tau_1} \text{ and } 1\leqq q'\leqq d_{\tau_2}\,. \tag{2.20}$$

Thus using Lemma 4, the above equation can be written as

$$\begin{aligned}&\int_G\overline{f(x)}E(P: \Psi_{T(\tau_1,\tau_2:j:p,q)}: \nu: x)(1: 1)dx\\ &=\int_G\int_{K\times K}\overline{f(x)}(k_1: k_2)E(P: \Psi_{T(\tau_1,\tau_2:j:p,q)}: \nu: x)(k_1: k_2)dk_1dk_2dx\\ &=\int_G(f_{\tau_1,\tau_2}(x), E(P: \Psi_{T(\tau_1,\tau_2:j:p,q)}: \nu: x))_Vdx\\ &=\hat{f}_{\tau_1,\tau_2}(\Psi_{T(\tau_1,\tau_2:j:p,q)}, \nu) \quad \text{(see the definition of } \hat{f} \text{ in [8])}\,.\end{aligned}$$

However from Lemma 5 and its note, we can easily prove that $\{d_{\omega_j}^{1/2}\Psi_{T(\tau_1,\tau_2:j.p,q)};\ 1\leqq p\leqq d_{\tau_1}$ and $1\leqq q\leqq d_{\tau_2}\}$ is an orthonormal basis of $L(\omega_j)$, where $L={}^{\circ}\mathscr{C}(M, V_{\tau_1,\tau_2}, \mu_{\tau_1,\tau_2})(\mu_{\tau_1,\tau_2}=\mu_F|V_{\tau_1,\tau_2})$. Therefore this basis is the desired one.

For the second relation we can choose the desired basis as follows;

$$\{d_{\Lambda_j}^{1/2}(\Phi_{\tau_1,p}, \pi_{\Lambda_j}(x)\Phi_{\tau_2,q})^{\sim}(k_1: k_2);\ 1\leqq p\leqq d_{\tau_1} \text{ and } 1\leqq q\leqq d_{\tau_2}\},$$

(note the orthogonal relation of the matrix coefficients of the discrete series for G). Q.E.D.

Now we shall define a Fourier transformation on $\mathscr{C}(G)$. Put

$$\underline{\mathscr{C}}(\hat{G})=\bigoplus_{\tau_1,\tau_2\in\mathscr{E}(K)}(C^{n'(\tau_1,\tau_2)}\oplus\mathscr{C}(\mathscr{F})_*^{n(\tau_1,\tau_2)}) \tag{2.21}$$

where $n(\tau_1, \tau_2)$ and $n'(\tau_1, \tau_2)$ denote the dependence of n and n' on τ_1, τ_2. Then we denote by $\mathscr{C}(\hat{G})$ the subspace of $\underline{\mathscr{C}}(\hat{G})$ which consists of all $\bigoplus_{\tau_1,\tau_2}(a\oplus V)=\bigoplus_{\tau_1,\tau_2}((a_i^j)\oplus(v_i^j(\nu)))\in\underline{\mathscr{C}}(\hat{G})$ (of couse, j, i depend on τ_1, τ_2) satisfying the following conditions;

(i) for each triplet (p_1, q_1, q_2) of polynomials,

$$\sup_{\substack{\tau_1,\tau_2\in\mathscr{E}(K)\\ 1\leqq j\leqq m'(\tau_1,\tau_2)\\ 1\leqq i\leqq n'_j(\tau_1,\tau_2)}} d_{\Lambda_j}^{1/2}|a_i^j|p_1(|\Lambda_j|)q_1(|\tau_1|)q_2(|\tau_2|)<\infty, \tag{2.22}$$

(ii) for each set (p_1, p_2, q_1, q_2, n) of polynomials p_1, p_2, q_1, q_2 and an integer n,

$$\sup_{\substack{\tau_1,\tau_2\in\mathscr{E}(K)\\ \nu\in\mathscr{F}\\ 1\leqq j\leqq m(\tau_1,\tau_2)\\ 1\leqq i\leqq n_j(\tau_1,\tau_2)}} d_{\omega_j}^{1/2}\left|\left(\frac{d}{d\nu}\right)^n v_i^j(\nu)\right|p_1(|\underline{\omega}_j|)p_2(|\nu|)q_1(|\tau_1|)q_2(|\tau_2|)<\infty$$

(see the definitions of $|\tau_1|$, $|\tau_2|$, $|\Lambda_j|$ and $|\underline{\omega}_j|$ in [1]).

Next we define a Fourier transformation F: $\mathscr{C}(G)\to\mathscr{C}(\hat{G})$ as follows;

$$F(f)=\bigoplus_{\tau_1,\tau_2\in\mathscr{E}(K)}F_{\tau_1,\tau_2}(f_{\tau_1,\tau_2}) \quad \text{for } f\in\mathscr{C}(G). \tag{2.23}$$

Then using Lemma 6, we can reform the results of J. Arthur [1] to the following form.

THEOREM 4. *The mapping F is a homeomorphism of $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$.*

§ 3. An analogue of Paley-Wiener theorem on $\mathscr{C}(G)$.

In this section using the results in the preceding sections, we obtain an analogue of Paley-Wiener theorem on $\mathscr{C}(G)$. First we define $C_c^{\infty}(G: R)$

as usual, i.e., $\{f\in C_c^\infty(G); \mathrm{supp}(f)\subset G_R\}$. Next we define $\mathscr{H}(\hat{G}: R)$ as follows;

$$\mathscr{H}(\hat{G}: R)=\mathscr{C}(\hat{G})\cap\bigoplus_{\tau_1,\tau_2\in\mathscr{E}(K)}\mathscr{H}(\tau_1, \tau_2: R)\ , \tag{2.24}$$

where $\mathscr{H}(\tau_1, \tau_2: R)$ is the space $\mathscr{H}(R)$ in §1 corresponding to the case of $V=V_{\tau_1,\tau_2}$ and $\tau=\mu_{\tau_1,\tau_2}$. Then we obtain the following theorem.

THEOREM 5. *Let notation be as above and f be in $\mathscr{C}(G)$. Then f belongs to $C_c^\infty(G: R)$ if and only if $F(f)$ belongs to $\mathscr{H}(\hat{G}: R)$.*

PROOF. First let f be in $C_c^\infty(G: R)$. Then we can easily prove that the support of f_{τ_1,τ_2} is contained in G_R for all $\tau_1, \tau_2\in\mathscr{E}(K)$. Thus we obtain $f_{\tau_1,\tau_2}\in C_c^\infty(G, \mu_{\tau_1,\tau_2}: R)$ and $F_{\tau_1,\tau_2}(f_{\tau_1,\tau_2})\in\mathscr{H}(\tau_1, \tau_2: R)$ by Theorem 3 for all $\tau_1, \tau_2\in\mathscr{E}(K)$. Therefore $F(f)$ belongs to $\mathscr{H}(\hat{G}: R)$.

Next let $F(f)$ be in $\mathscr{H}(\hat{G}: R)$. Here using the Fourier expansion on $K\times K$, we can obtain,

$$\tilde{f}=\sum_{\tau_1,\tau_2\in\mathscr{E}(K)}f_{\tau_1,\tau_2}\ . \tag{2.25}$$

Then from the assumption $F(f)\in\mathscr{H}(\hat{G}: R)$ we can obtain that $F_{\tau_1,\tau_2}(f_{\tau_1,\tau_2})$ belongs to $\mathscr{H}(\tau_1, \tau_2: R)$ for $\tau_1, \tau_2\in\mathscr{E}(K)$. Therefore using Theorem 3, we have $f_{\tau_1,\tau_2}\in C_c^\infty(G, \mu_{\tau_1,\tau_2}: R)$. Thus, in particular $f_{\tau_1,\tau_2}(1: 1)\in C_c^\infty(G: R)$ and moreover, $f=\tilde{f}(1: 1)\in C_c^\infty(G: R)$. This completes the proof of theorem.

Q.E.D.

NOTE. From the definitions of $\mathscr{C}(\hat{G})$ and $\mathscr{H}(\tau_1, \tau_2: R)$, $\mathscr{H}(\hat{G}: R)$ is the subspace of $\underline{\mathscr{C}}(\hat{G})$ which consists of all $\bigoplus_{\tau_1,\tau_2}((a_k)\bigoplus(v_i^j(\nu))\in\underline{\mathscr{C}}(G)$ satisfying the following conditions; for each $\tau_1, \tau_2\in\mathscr{E}(K)$,

(i) $(v_i^j(\nu))\in\mathscr{H}(\mathscr{F})_*^{n(\tau_1,\tau_2)}$

(ii) $a_k=\sum_{1\le p\le r}C_{pk}(d^{r(p)}/d\nu^{r(p)})_{|\nu=\nu(p)}v_{i(p)}^{j(p)}(s(p)\nu)$ $(1\leqq k\leqq n'(\tau_1, \tau_2))$

(iii) there exist constants C_N for which

$$|v_i^j(\nu+(-1)^{1/2}\eta)|\leqq C_N(|\nu+(-1)^{1/2}\eta|)^{-N}e^{R|\eta|}\quad\text{for } \nu\in\mathscr{F}\text{ and }\eta\in\mathscr{F}^+$$
$$(1\leqq j\leqq m(\tau_1, \tau_2),\ 1\leqq i\leqq n_j(\tau_1, \tau_2))$$

(iv) for each triplet (p_1, q_1, q_2) of polinomials,

$$\sup_{\substack{\tau_1,\tau_2\in\mathscr{E}(K)\\ 1\le k\le n'(\tau_1,\tau_2)}}\left|\sum_{1\le p\le r}d_k^{1/2}C_{pk}\frac{d^{r(p)}}{d\nu^{r(p)}}_{|\nu=\nu(p)}v_{i(p)}^{j(p)}(s(p)\nu)\right|$$
$$\times p_1(|\Lambda_k|)q_1(|\tau_1|)q_2(|\tau_2|)<\infty\ ,$$

where $\Lambda_k=\Lambda_j$, when $e'_k\in L^G(\Lambda_j)$ for $1\leqq k\leqq n'$,

(v) for each set (p_1, p_2, q_1, q_2, n) of polynomials p_1, p_2, q_1, q_2 and an integer n,

$$\sup_{\substack{\tau_1, \tau_2 \in \mathcal{E}(K) \\ \nu \in \mathcal{F} \\ 1 \leqq j \leqq m(\tau_1, \tau_2) \\ 1 \leqq i \leqq n_j(\tau_1, \tau_2)}} d_{\omega_j}^{1/2} \left| \left(\frac{d}{d\nu} \right)^n v_i^j(\nu) \right| p_1(|\underline{\omega}_j|) p_2(|\nu|) q_1(|\tau_1|) q_2(|\tau_2|) < \infty .$$

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