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An Analogue of Paley-Wiener Theorem on Rank 1 Semisimple Lie Groups I

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In the previous paper [9], we have stated some results on Paley-Wiener type theorems on semisimple Lie groups without proof. In this paper we shall give detailed proofs of those theorems.

§ 1. Notation and preliminaries.

Let G be a real reductive Lie group with compact center. We assume that G is in class $\mathscr{H}$ (cf. V. S. Varadarajan [10]). Let K be a maximal compact subgroup of G. Fix a Cartan involution θ on G induced by K. Let P be a parabolic subgroup of G, and $P=MAN$ be the associated Langlands decomposition of P. Then M is a reductive group and is in class $\mathscr{H}$, A is a vector group, which we call the split component of P, and N is the unipotent radical of P. Moreover if P is cuspidal, i.e., $\mathrm{rank}(M)=\mathrm{rank}(K_M)$ $(K_M=K\cap M)$, then there exists a compact Cartan subgroup T of M and $H=TA$ is a Cartan subgroup of G. Now we denote Lie algebras by small German letters and for any real vector space V, we denote by V_c the complex vector space of V and by V^* the dual space of V. Then $\mathfrak{p}=\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$ is the parabolic subalgebra of $\mathfrak{g}$ corresponding to P. In this case, $A=\exp\mathfrak{a}$, $N=\exp\mathfrak{n}$ and P is the normalizer of $\mathfrak{p}$ in G. Let $\mathscr{F}$ be the dual space of $\mathfrak{a}$, i.e., $\mathscr{F}=\mathfrak{a}^*$.

Let $\tau=(\tau_1, \tau_2)$ be a unitary double representation of K on a finite dimensional Hilbert space V. Here we assume that V satisfies the conditions in Harish-Chandra [6] § 8. Then we define the V-valued Schwartz space $\mathscr{C}(G, V)$ and the subspace of τ-spherical functions $\mathscr{C}(G, \tau)$ as usual. Moreover we denote by ${}^0\mathscr{C}(G, \tau)$ the space of τ-spherical cusp forms on G. Next let τ_M be a representation of K_M on V which is the restriction of τ to K_M. Then we can also define $\mathscr{C}(M, V)$, $\mathscr{C}(M, \tau_M)$ and ${}^\circ\mathscr{C}(M, \tau_M)$ respectively.

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Let $\mathscr{E}(G)$ be the set of equivalence classes of irreducible unitary representations of G and $\mathscr{E}^2(G)$ be the subset of $\mathscr{E}(G)$ which consists of equivalence classes of square-integrable representations of G, i.e., the discrete series of G. For other Lie groups we shall define $\mathscr{E}(\ )$ and $\mathscr{E}^2(\ )$ in the same way.

Now we fix a parabolic subgroup $P=MAN$ of G and put $L={}^\circ\mathscr{C}(M, \tau_M)$. Then $\dim L<\infty$. Let $\mathscr{H}_\omega(\omega \in \mathscr{E}^2(M))$ be the smallest closed subspace of L^2-space on M containing all matrix coefficients of ω. Then it is well-known that L is an orthogonal sum of $L(\omega)$ $(\omega \in \mathscr{E}^2(M))$, where $L(\omega)=L \cap (\mathscr{H}_\omega \otimes V)$. Thus, L can be decomposed as

$$L=\bigoplus_{1\leq j\leq m} \bigoplus_{s\in W-W(\omega_j)} L(s\omega_j)\ , \tag{1.1}$$

where $W=W(A)$ is the Weyl group of (G, A) and $W(\omega_j)=\{s\in W; s\omega_j=\omega_j\}$ for $\omega_j \in \mathscr{E}^2(M)$ $(1\leq j\leq m)$. Here we denote an orthonormal basis of $L(\omega_j)$ as follows;

$$\{\phi_i^j; 1\leq i\leq n_j\}\ , \qquad \text{where} \quad n_j=\dim L(\omega_j) \quad (1\leq j\leq m)\ . \tag{1.2}$$

From now on, we shall define a Fourier transform on $\mathscr{C}(G, \tau)$. First of all we shall regard $\mathscr{F}$ as an Euclidean space and define the Schwartz space $\mathscr{C}(\mathscr{F})$ on it as usual. Next for $f\in \mathscr{C}(G, \tau)$ and $\phi_i^j \in {}^\circ\mathscr{C}(M, \tau_M)$ we put

$$\hat{f}(\phi_i^j, \nu)=(c^2\gamma)^{-1}(f, E(P: \phi_i^j: \nu:.)) \quad (\nu \in \mathscr{F}) \tag{1.3}$$

(for notation see Harish-Chandra [6], § 2 and § 11). Then from the results in [5] we obtain that $\hat{f}(\phi_i^j, \nu)$ belongs to $\mathscr{C}(\mathscr{F})$ for fixed $\phi_i^j(1\leq i\leq n_j, 1\leq j\leq m)$. Then we define a Fourier transform $\mathscr{E}_A: \mathscr{C}(G, \tau)\to \mathscr{C}(\mathscr{F})^n(n=\sum_{1\leq j\leq m} n_j)$ as follows;

$$\mathscr{E}_A(f)=(\mathscr{E}_1(f), \mathscr{E}_2(f), \cdots, \mathscr{E}_m(f)) \quad \text{for} \quad f\in \mathscr{C}(G, \tau)\ , \tag{1.4}$$

where $\mathscr{E}_j(f)=(\hat{f}(\phi_1^j, \nu), \hat{f}(\phi_2^j, \nu), \cdots, \hat{f}(\phi_{n_j}^j, \nu))(1\leq j\leq m)$. For simplicity put $e_k=\phi_i^j$, where $k=\sum_{1\leq p\leq j-1} n_p+i(1\leq k\leq n)$.

Let V be an arbitrary element in $\mathscr{C}(\mathscr{F})^n$. Then V can be written as; $V=(V_1, V_2, \cdots, V_m)$ where V_j is an element in $\mathscr{C}(\mathscr{F})^{n_j}$ for $1\leq j\leq m$.

Let $\mathscr{C}(\mathscr{F})^n_*$ be the closed subspace of $\mathscr{C}(\mathscr{F})^n$ consisting of all $V=(V_1, V_2, \cdots, V_m)$ which satisfy the following relations;

$$V_j(s^{-1}\nu)^t={}^\circ\overline{C_{P|P}(s; s^{-1}\nu)}\, V_j(\nu)^t \quad \text{for all } s\in W(\omega_j) \text{ and } \nu\in\mathscr{F}(1\leq j\leq m) \tag{1.5}$$

(for the notation, see T. Kawazoe [9]). Moreover let $\mathscr{H}(\mathscr{F})^n_*$ be the subspace of $\mathscr{C}(\mathscr{F})^n_*$ consisting of V whose each component extends to a holomorphic function on $\mathscr{F}_c$ which is an exponential type and satisfies the following conditions;

if there exists a relation,

$$\sum_{1\leq j\leq m}\sum_{1\leq i\leq n_j}\sum_{1\leq t\leq T_i^j}\sum_{1\leq r\leq M_i^j} A(j, i, t, r)\frac{d^r}{d\nu^r}{}_{|\nu=\nu_i^j(t)}E(P: \phi_i^j: \nu: x)=0 ,$$

(1.6) where $A(j, i, t, r)\in C$ and $\nu_i^j(t)\in\mathscr{F}_c$ for all $1\leqq i\leqq n_j$, $1\leqq j\leqq m$, then

$$\sum_{1\leq j\leq m}\sum_{1\leq i\leq n_j}\sum_{1\leq t\leq T_i^j}\sum_{1\leq r\leq M_i^j} A(j, i, t, r)\frac{d^r}{d\nu^r}{}_{|\nu=\nu_i^j(t)}\alpha_i^j(\nu)=0 ,$$

where $V_j(\nu)=(\alpha_1^j(\nu), \alpha_2^j(\nu), \cdots, \alpha_{n_j}^j(\nu))$ for $1\leqq j\leqq m$.

§ 2. Main results.

Let $P_1, P_2, \cdots, P_r$ be a complete set of cuspidal parabolic subgroups of G, no two of which are associate and $P_i=M_iA_iN_i$ be the corresponding Langlands decomposition of $P_i(1\leqq i\leqq r)$. Now we denote by $\mathscr{C}_i(G, \tau)=\mathscr{C}_{A_i}(G, \tau)(1\leqq i\leqq r)$ the closed subspace of $\mathscr{C}(G, \tau)$ which consists of all f satisfying $f^{(Q)}\sim 0$ for every parabolic subgroup $Q=MAN$ of G such that A is not conjugate to A_i under K. When we apply the preceding argument to P_i, we shall use the notation such that $\mathscr{E}_{A_i}$, $\mathscr{F}_i$ and $n^{(i)}$ instead of $\mathscr{E}_A$, $\mathscr{F}$ and n for P respectively.

THEOREM 1. *If P_i is not G, then the mapping $\mathscr{E}_{A_i}$ is a homeomorphism of $\mathscr{C}_i(G, \tau)$ onto $\mathscr{C}(\mathscr{F}_i)_*^{n(i)}$.*

THEOREM 2. *Assume that the real rank of G is equal to one. Then an element V in $\mathscr{C}(\mathscr{F})_*^n$ belongs to $\mathscr{H}(\mathscr{F})_*^n$ if and only if there exists a function f in $C_c^\infty(G, \tau)$ such that $\mathscr{E}_A(f)=V$, where $P=MAN$ is the minimal parabolic subgroup of G.*

REMARK 1. From the proof of Theorem 1 (see § 3) we can easily prove that

$$\mathscr{E}_{A_i}(f)=0$$

for all f in $\mathscr{C}_{A_j}(G, \tau)(i\neq j, 1\leqq i, j\leqq r)$. Therefore using the decomposition of $\mathscr{C}(G, \tau)$ (see T. Kawazoe [9] (4.5)), we can regard $\mathscr{E}_{A_i}$ as the mapping of $\mathscr{C}(G, \tau)$ onto $\mathscr{C}(\mathscr{F}_i)_*^{n(i)}$. We shall denote this extension by the same notation.

REMARK 2. Noting the proof of the surjection of Theorem 1 (see § 3 (iii)), we can easily obtain the inverse mapping of $\mathscr{E}_{A_k}(1\leqq k\leqq r)$ as follows,

$$\mathscr{E}_{A_k}^{-1}(\mathscr{E}_{A_k}(f))=\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}\int_{\mathscr{F}}\mu(\omega_j,\nu)E(P:\phi_i^j:\nu:x)\hat{f}(\phi_i^j,\nu)d\nu$$

for $f\in\mathscr{C}(G,\tau)$.

§ 3. Proof of Theorem 1.

Assume that P_i is not equal to G. For simplicity we put $P=P_i$ and moreover for the other notations we shall omit the suffix; i.

(i) First of all we shall show that $\mathscr{E}_A(f)$ belongs to $\mathscr{C}(\mathscr{F})_*^n$ for all $f\in\mathscr{C}(G,\tau)$. Now let f be an arbitrary element in $\mathscr{C}(G,\tau)$ and we shall write $\mathscr{E}_A(f)$ as follows;

$$\mathscr{E}_A(f)=(\mathscr{E}_1(f),\ \mathscr{E}_2(f),\ \cdots,\ \mathscr{E}_m(f))\,, \tag{3.1}$$

see notation for (1.4). Then from the definition of $\mathscr{E}_A$ and $\mathscr{E}_j$, it is quite obvious that $\mathscr{E}_j(f)$ belongs to $\mathscr{C}(\mathscr{F})^{n_j}$ for $1\leqq j\leqq m$ and moreover $\mathscr{E}_A(f)$ belongs to $\mathscr{C}(\mathscr{F})^n$. Therefore in order to prove that $\mathscr{E}_A(f)\in\mathscr{C}(\mathscr{F})_*^n$ it is enough to prove that $\mathscr{E}_j(f)$ satisfies the following relation;

$$\mathscr{E}_j(f)(s^{-1}\nu)^t=\overline{{}^\circ C_{P|P}(s;s^{-1}\nu)}\mathscr{E}_j(f)(\nu)^t \text{ for } s\in W(\omega_j) \text{ and } \nu\in\mathscr{F}(1\leqq j\leqq m) \tag{3.2}$$

(for the notation, see (4.2) in [9]).

Here we note that the Eisenstein integral satisfies the relation as follows;

$$E(P:\phi:s^{-1}\nu:x)=E(P:{}^\circ C_{P|P}(s;s^{-1}\nu)\phi:\nu:x) \tag{3.3}$$

for $\phi\in L$, $\nu\in\mathscr{F}$ and $s\in W$ (cf. Harish-Chandra [6] Lemma 17.2) and moreover ${}^\circ C_{P|P}(s;s^{-1}\nu)\phi$ belongs to $L(\omega_j)$ for $\phi\in L(\omega_j)$ and $s\in W(\omega_j)$ $(1\leqq j\leqq m)$. Therefore, using these facts and the definition of the Fourier transform (1.3), we can easily prove that $\mathscr{E}_j(f)$ satisfies (3.2) for $1\leqq j\leqq m$. Thus, we obtain the desired results.

(ii) Next we shall prove that the mapping $\mathscr{E}_A:\mathscr{C}_A(G,\tau)\to\mathscr{C}(\mathscr{F})_*^n$ is injective. Now let f be an element in $\mathscr{C}_A(G,\tau)$ such that $\mathscr{E}_A(f)=0$. Then from the definition of $\mathscr{E}_A(f)$, we have,

$$(\hat{f}(e_1,\nu),\ \hat{f}(e_2,\nu),\ \cdots,\ \hat{f}(e_n,\nu))=0 \tag{3.4}$$

i.e., $\quad (f, E(P:e_i:\nu:.))=0$ for all $1\leqq i\leqq n$ and $\nu\in\mathscr{F}$.

Since $e_i(1\leqq i\leqq n)$ is an orthogonal basis of $\bigoplus_{1\leq j\leq m}L(\omega_j)$, the above relation is valid for all $\psi\in\bigoplus_{1\leq j\leq m}L(\omega_j)$. But, here we note that $\{{}^\circ C_{P|P}(s;s^{-1}\nu)\phi_i^j;1\leqq i\leqq n_j\}$ is an orthogonal basis of $L(s\omega_j)(s\in W)$ and

$$(f, E(P:{}^\circ C_{P|P}(s;s^{-1}\nu)\phi_i^j:\nu:.))=(f, E(P:\phi_i^j:s^{-1}\nu:.))=0 \quad (\nu\in\mathscr{F}) \tag{3.5}$$

by (3.3) and (3.4). Therefore from (1.1) in § 1 we can obtain

$$(3.6)\qquad (f, E(P:\psi:\nu:.))=0 \quad \text{for all} \quad \psi\in L \quad \text{and} \quad \nu\in\mathscr{F} .$$

Then $f^{(P)}\sim 0$ (cf. Harish-Chandra [4] § 20). Then from the fact that f belongs to $\mathscr{C}_A(G, \tau)$, we can easily obtain that $f^{(P')}\sim 0$ for every parabolic subgroup $P'=M'A'N'$ of G such that A' is not conjugate to A. Therefore we have, $f^{(Q)}\sim 0$ for all parabolic subgroups of G. Thus, f must be 0, i.e., the mapping $\mathscr{E}_A$ is injective (cf. Harish-Chandra [4] Lemma 20.1).

(iii) Now we shall prove that the mapping $\mathscr{E}_A: \mathscr{C}_A(G, \tau)\rightarrow\mathscr{C}(\mathscr{F})^n_*$ is surjective. Let V be an arbitrary element in $\mathscr{C}(\mathscr{F})^n_*$. Then V can be written as follows;

$$(3.7)\qquad V=(V_1, V_2, \cdots, V_m)$$

where V_j belongs to $\mathscr{C}(\mathscr{F})^{n_j}$, which denotes $(\alpha^j_1(\nu), \alpha^j_2(\nu), \cdots, \alpha^j_{n_j}(\nu))$ and moreover satisfies the relation (3.2) in (i) with respect to $V_j(1\leqq j\leqq m)$. From now on we shall construct a function f in $\mathscr{C}_A(G, \tau)$ such that $\mathscr{E}_A(f)=V$.

First of all we shall define a function f as follows;

$$(3.8)\qquad f(x)=\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}\hat{\alpha}^j_i(\phi^j_i, x) \qquad (x\in G)$$

(for the notation, see T. Kawazoe [9] (3.2)). Then it is obvious that f belongs to $\mathscr{C}_A(G, \tau)$ (see Harish-Chandra [6] Lemma 26.1). We shall calculate the entry of $\mathscr{E}_A(f)$ which corresponds to ϕ^p_q, i.e., $\hat{f}(\phi^p_q, \nu)$ for $1\leqq q\leqq n_p$, $1\leqq p\leqq m$. Here we shall use the same notations and calculations in Harish-Chandra [6] §§ 26 and 27.

$$\begin{aligned}\hat{f}(\phi^p_q, \nu)&=(c^2\gamma)^{-1}(f, E(P:\phi^p_q:\nu:.))\\&=(c^2\gamma)^{-1}\Big(\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}\hat{\alpha}^j_i(\phi^j_i,.), E(P:\phi^p_q:\nu:.)\Big)\\&=(c^2\gamma)^{-1}\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}(\hat{\alpha}^j_i(\phi^j_i,.))^{(P)}_\nu, \phi^p_q)\\&=\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}\sum_{s\in W}\alpha^j_i(s^{-1}\nu)({}^\circ C_{P|P}(s; s^{-1}\nu)\phi^j_i, \phi^p_q) .\end{aligned}$$

Here we recall that ${}^\circ C_{P|P}(s; s^{-1}\nu)$ is an unitary operator which maps $L(\omega)$ onto $L(s\omega)$, and moreover $L(\omega)$ is orthogonal to $L(s\omega)$ for $s\in W-W(\omega_j)$. Therefore from the decomposition of L (see (1.1) in § 1), we can obtain,

$$\hat{f}(\phi_q^p, \nu) = |W(\omega_P)|^{-1} \sum_{i=1}^{n_p} \sum_{s \in W(\omega_p)} \alpha_i^p(s^{-1}\nu)({}^\circ C_{P|P}(s; s^{-1}\nu)\phi_i^p, \phi_q^p)$$
$$= \alpha_q^p(\nu) ,$$

here we used the relation (3.2). This proves that $\mathscr{E}_A(f) = V$, i.e., the mapping $\mathscr{E}_A$ is surjective.

Therefore from (i), (ii) and (iii) we can prove that the mapping $\mathscr{E}_A$: $\mathscr{C}_A(G, \tau) \to \mathscr{C}(\mathscr{F})^n_*$ is bijective. However from the results of Harish-Chandra [5] we can easily obtain that $\mathscr{E}_A$ is homeomorphism of $\mathscr{C}_A(G, \tau)$ onto $\mathscr{C}(\mathscr{F})^n_*$. This completes the proof of Theorem 1, and moreover Remark 1 in [9].

§4. Proof of Theorem 2.

Let notation be as above and assume that the real rank of G is one.

(i) First of all we shall prove the necessary condition. Let f be in $C_c^\infty(G, \tau)$ and put $V = \mathscr{E}_A(f)$. Then we can prove that V belongs to $\mathscr{C}(\mathscr{F})^n_*$ in the same way in (i) of § 3. Moreover using the fact that $f_\nu^{(P)}$ is in $C_c^\infty(M, \tau_M) \otimes \mathscr{C}(\mathscr{F})$, we can easily obtain that each component of V can be extended to a holomorphic function on $\mathscr{F}_c$ which is an exponential type and satisfies the condition (1.6) in § 1. Thus, we obtain that $\mathscr{E}_A(f)$ belongs to $\mathscr{H}(\mathscr{F})^n_*$.

(ii) Next we shall prove the sufficient condition. Let V be an arbitrary element in $\mathscr{H}(\mathscr{F})^n_*$ and assume that V has the form of (3.7). First of all we shall define the function f by (3.8), i.e.,

$$f(x) = \sum_{j=1}^{m} |W(\omega_j)|^{-1} \sum_{i=1}^{n_j} \int_{\mathscr{F}} \mu(\omega_j, \nu) E(P: \phi_i^j: \nu: x) \alpha_i^j(\nu) d\nu , \tag{4.1}$$

see (3.2) in [9]. It is obvious that f belongs to $\mathscr{C}_A(G, \tau)$.

Now we shall prove that there exists a compactly supported function $F(x)$ which satisfies $\mathscr{E}_A(F) = V$.

At first we shall change the line of the integral in (4.1) as figure 1,

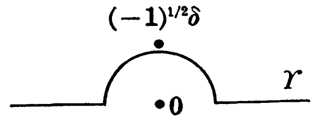

FIGURE 1

where $\delta > 0$ is a sufficiently small real number. This change is valid from the facts that $\alpha_i^j(\nu)$ is an analytic function on $\mathscr{F}_c$ and $\mu(\omega_j, \nu)$ is also analytic on the domain;

$$\{\nu \in \mathscr{F}_c;\ |\mathrm{Im}(\nu)| < \delta\}, \tag{4.2}$$

for a sufficiently small $\delta > 0$ (cf. Harish-Chandra [6] Theorem 25.1).

Next we shall use the following expansion of the Eisenstein integral.

$$e^{\rho(\log(a))} E(P\colon \phi\colon \nu\colon a) = \sum_{s\in W} \Phi(s\nu\colon a) C_{P|P}(s;\nu)\phi(1) \tag{4.3}$$

for $a \in A^+$ and $\nu \in \Gamma'(c) \subset \mathscr{F}_c$, where $A^+ = \exp\mathfrak{a}^+$ ($\mathfrak{a}^+$ is the positive Weyl chamber of $\mathfrak{a}$) and

$$\Phi(\nu\colon a) = \sum_{\lambda\in L} \Gamma_\lambda((-1)^{1/2}\nu - \rho) e^{((-1)^{1/2}\nu - \lambda(\log(a))} \tag{4.4}$$

(for notation see G. Warner [11] p. 289).

Then using the above expansion we have

$$\begin{aligned} f(a) &= e^{-\rho(\log(a))} \sum_{j=1}^{m} |W(\omega_j)|^{-1} \sum_{i=1}^{n_j} \int_\Gamma \mu(\omega_j, \nu) \sum_{s\in W} \Phi(s\nu\colon a) C_{P|P}(s;\nu)\phi_i^j(1)\alpha_i^j(\nu)d\nu \\ &= e^{-\rho(\log(a))} \sum_{j=1}^{m} |W(\omega_j)|^{-1} \sum_{i=1}^{n_j} c^2 \\ &\quad \times \left\{ \int_\Gamma \Phi(\nu\colon a) C_{P|P}(1;\nu)^{*-1} \phi_i^j(1)\alpha_i^j(\nu)d\nu \right. \\ &\quad \left. + \int_{\underline{s}\Gamma} \Phi(\nu\colon a) C_{P|P}(\underline{s};\underline{s}^{-1}\nu)^{*-1} \phi_i^j(1)\alpha_i^j(\underline{s}^{-1}\nu)d\nu \right\}, \end{aligned} \tag{4.5}$$

where $\underline{s}$ is the non-trivial element in W and we used the following relation;

$$\mu(\omega, \nu) C_{P|P}(s;\nu)^* C_{P|P}(s;\nu) = c^2 \quad \text{for } s \in W \text{ and } \nu \in \Gamma \tag{4.6}$$

(cf. Harish-Chandra [6] Lemma 17.1).

Now we note that $C_{P|P}(s; s^{-1}\nu)^{*-1}$ ($s \in W$) and $\Phi(\nu\colon a)$ have no poles on the line $s\Gamma$ and moreover has only finite poles on D, where D is the upper domain of the line $s\Gamma$ (see O. Campoli [2] and G. Warner [11] Chap. 9.1). Therefore using these facts, we can give these poles suffixes as follows. Let $\nu_i^j(t)$ ($1 \leqq t \leqq k_i^j$) be the poles on D of $\Phi(\nu\colon a) C_{P|P}(1;\nu)^{*-1}\phi_i^j(1)$ and let $\nu_i^j(t)$ ($k_i^j + 1 \leqq t \leqq T_i^j$) be the poles on D of $\Phi(\nu\colon a) C_{P|P}(\underline{s};\underline{s}^{-1}\nu)^{*-1}\phi_i^j(1)$. Moreover let $m_i^j(t)$ be the order at $\nu_i^j(t)$ for $1 \leqq t \leqq T_i^j$.

Next we note that $C_{P|P}(s;\nu)$ and $\Phi(\nu\colon a)$ satisfy the following inequalities. Suppose that $\nu + (-1)^{1/2}\eta$ ($\nu, \eta \in \mathscr{F}$) is sufficiently distant from the poles of the following functions. Then for any integer M, there exist constants c_M for which

(4.7) $\|C_{P|P}(s; s^{-1}(\nu+(-1)^{1/2}\eta))^{*-1}\|<c_M(1+|\nu+(-1)^{1/2}\eta|)^M$ for $s \in W$,

(4.8) $\|\Phi'(\nu+(-1)^{1/2}\eta: a)\|<c_M(1+|\nu+(-1)^{1/2}\eta|)^M$

where $\Phi(\nu: a)=\Phi'(\nu: a)e^{(-1)^{1/2}\nu(\log(a))}$ (for these inequalities see O. Campoli [2]). Then since $\alpha_i^j(\nu)(1\leqq i\leqq n_j, 1\leqq j\leqq m)$ is an exponential type, we can change the integral line; $s\Upsilon\to\mathscr{F}+(-1)^{1/2}\eta$, where $s\in W$ as follows;

$$(4.9)\quad f(a)=e^{-\rho(\log(a))}\sum_{j=1}^{m}|W(\omega_j)|^{-1}\sum_{i=1}^{n_j}c^2$$
$$\times\Big\{\sum_{s\in W}\int_{\mathscr{F}}\Phi(\nu+(-1)^{1/2}\eta: a)C_{P|P}(s; s^{-1}(\nu+(-1)^{1/2}\eta))^{*-1}$$
$$\times\phi_i^j(1)\alpha_i^j(s^{-1}(\nu+(-1)^{1/2}\eta))d\nu\Big\}$$
$$+\sum_{t=1}^{k_i^j}\mathrm{Res}_{\nu=\nu_i^j(t)}\,\Phi(\nu: a)C_{P|P}(1; \nu)^{*-1}\phi_i^j(1)\alpha_i^j(\nu)$$
$$+\sum_{t=k_i^j+1}^{T_i^j}\mathrm{Res}_{\nu=\nu_i^j(t)}\Phi(\nu: a)C_{P|P}(\underline{s}; \underline{s}^{-1}\nu)^{*-1}\phi_i^j(1)\alpha_i^j(\underline{s}^{-1}\nu)\ ,$$

where $\eta\in\mathscr{F}^+$ is sufficiently large and satisfies $|\eta|>|\mathrm{Im}\,(\nu_i^j(t))|$ for $1\leqq t\leqq T_i^j$, $1\leqq i\leqq n_j$, $1\leqq j\leqq m$. Let $I_f(a)$ be the integral part of (4.9) and $R_f(a)$ be the residue part of (4.9). Then we can easily prove that for a sufficiently large $a\in A$, $I_f(a)$ must be 0 by the same method in the classical Paley-Wiener theorem on an Euclidean space. (Note that $\alpha_i^j(\nu)$ is an exponential type and (4.7), (4.8).) Thus, for a sufficiently large $a\in A^+$ we have

$$(4.10)\qquad f(a)=R_f(a)\ .$$

Now put $s_i^j(t)=\begin{cases}1 & (1\leqq t\leqq k_i^j)\\ \underline{s}^{-1} & (k_i^j+1\leqq t\leqq T_i^j)\end{cases}$ and let $E_1, E_2, \cdots, E_\gamma$ be a maximal linearly independent subset of

$$(4.11)\quad\Big\{\frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)}E(P: \phi_i^j: s_i^j(t)\nu: x);\ 0\leqq r\leqq m_i^j(t)-1,\ 1\leqq t\leqq T_i^j\ ,$$
$$1\leqq i\leqq n_j \text{ and } 1\leqq j\leqq m\Big\}\ .$$

Therefore we may assume that $E_p(1\leqq p\leqq\gamma)$ can be written as,

$$(4.12)\qquad E_p(x)=\frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu_{i(p)}^{j(p)}(t(p))}E(P: \phi_{i(p)}^{j(p)}: s_{i(p)}^{j(p)}(t(p)\nu): x)$$

for some $1\leqq j(p)\leqq m$, $1\leqq i(p)\leqq n_{j(p)}$, $1\leqq t(p)\leqq T_{i(p)}^{j(p)}$ and $0\leqq r(p)\leqq m_{i(p)}^{j(p)}-1(1\leqq p\leqq\gamma)$. For simplicity put $s_{i(p)}^{j(p)}(t(p))=s(p)$ and $\nu_{i(p)}^{j(p)}(t(p))=\nu(p)(1\leqq p\leqq\gamma)$.

Then there exist $A(j, i, t, r: p) \in C$ for which

$$\frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} E(P: \phi_i^j: s_i^j(t)\nu: x) = \sum_{1\leq p\leq \gamma} A(j, i, t, r: p) E_p(x) \tag{4.13}$$

for all $1\leqq j\leqq m$, $1\leqq i\leqq n_j$, $1\leqq t\leqq T_i^j$ and $0\leqq r\leqq m_i^j(t)-1$. However, since V belongs to $\mathscr{H}(\mathscr{F})_*^n$, each component of V has to satisfy the conditions as follows;

$$\frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} \alpha_i^j(s_i^j(t)\nu) = \sum_{1\leq p\leq \gamma} A(j, i, t, r: p) \frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} \alpha_{i(p)}^{j(p)}(s(p)\nu) \tag{4.14}$$

for all $1\leqq j\leqq m$, $1\leqq i\leqq n_j$, $1\leqq t\leqq T_i^j$ and $0\leqq r\leqq m_i^j(t)-1$.

Here we note that $E_p (1\leqq p\leqq \gamma)$ is a real analytic function on G for all $\nu\in\mathscr{F}_c$. Therefore we can choose compactly supported functions $h_q\in C_c^\infty(G, \tau)(1\leqq q\leqq \gamma)$ which satisfy

$$(h_q, E_p)=\delta_{qp} \quad \text{for all} \quad 1\leqq p, q\leqq \gamma . \tag{4.15}$$

Now we put

$$\underline{G}(x)=f(x)-\sum_{1\leq p\leq \gamma} \frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} \alpha_{i(p)}^{j(p)}(s(p)\nu) h_p(x) \quad (x\in G) . \tag{4.16}$$

Then we have for all $1\leqq j\leqq m$, $1\leqq i\leqq n_j$, $1\leqq t\leqq T_i^j$ and $0\leqq r\leqq m_i^j(t)-1$,

$$\begin{aligned}\frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} \hat{\underline{G}}(\phi_i^j, s_i^j(t)\nu) &= \frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} \hat{f}(\phi_i^j, s_i^j(t)\nu) \\ &- \sum_{1\leq p\leq \gamma} \frac{d^{r(p)}}{d\nu^{r(p)}}\Big|_{\nu=\nu(p)} \alpha_{i(p)}^{j(p)}(s(p)\nu) \frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} \hat{h}_p(\phi_i^j, s_i^j(t)\nu) .\end{aligned} \tag{4.17}$$

By the way, using the relation (4.12) and (4.15), we have

$$\begin{aligned}\frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} \hat{h}_p(\phi_i^j, s_i^j(t)\nu) &= \left(h_p, \frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} E(P: \phi_i^j: s_i^j(t)\nu:.)\right) \\ &= (h_p, \sum_{1\leq q\leq \gamma} A(j, i, t, r: p) E_q) \\ &= A(j, i, t, r: p)\end{aligned} \tag{4.18}$$

for $1\leqq p\leqq \gamma$. Therefore from the relation (4.14) and the fact that $\hat{f}(\phi_i^j, \nu)=\alpha_i^j(\nu)$ (see §3 (iii)),

$$\frac{d^r}{d\nu^r}\Big|_{\nu=\nu_i^j(t)} \hat{\underline{G}}(\phi_i^j, s_i^j(t)\nu)=0 \quad \text{for all } j, i, t \text{ and } r , \tag{4.19}$$

i.e., $\hat{\underline{G}}(\phi_i^j, s_i^j(t)\nu)$ has zero of $m_i^j(t)$-th order at $s_i^j(t)\nu$ for all $1\leqq j\leqq m$,

$1 \leqq i \leqq n_j$, $1 \leqq t \leqq T_i^j$. Now for an arbitrary function g in $\mathscr{C}(G, \tau)$, we put $g' = \mathscr{C}_A^{-1}(\mathscr{C}_A(g))$ and $g^\circ = g - g'$. Then from Theorem 1 it is obvious that g' belongs to $\mathscr{C}_A(G, \tau)$ and g° to ${}^\circ\mathscr{C}(G, \tau)$. Here we apply the preceding argument to $\underline{G}$ instead of f. Thus, we obtain

$$\underline{G}'(a) = I_{\underline{G}'}(a) + R_{\underline{G}'}(a) \quad \text{for} \quad a \in A^+ . \tag{4.20}$$

But, here we note that $\hat{\underline{G}} = \hat{\underline{G}}'$ and (4.19). Then we can easily prove that

$$R_{\underline{G}'}(a) = 0 \quad \text{for} \quad a \in A^+ , \tag{4.21}$$

and moreover for a sufficiently large $a \in A^+$ we have $\underline{G}'(a) = 0$ (see (4.10)). Therefore using the Cartan decomposition; $G = K \cdot CL(A^+) \cdot K$ and the fact that $\underline{G}'$ is a τ-spherical function on G, we can prove that $\underline{G}'$ has a compact support, i.e.,

$$f(x) - \sum_{1 \leqq p \leqq \gamma} C(p) h_p'(x) \in C_c^\infty(G, \tau) \tag{4.22}$$

(note $\hat{f} = \hat{f}'$), where $C(p) = d^{r(p)}/d\nu^{r(p)}|_{\nu=\nu(p)} \alpha_{i(p)}^{j(p)}(s(p)\nu)(1 \leqq p \leqq \gamma)$. Now we put $F(x) = f(x) + \sum_{1 \leqq p \leqq \gamma} C(p) h_p^\circ(x) (x \in G)$. Then it is obvious that

$$\mathscr{C}_A(F) = \mathscr{C}_A(f) = V , \tag{4.23}$$

and moreover $F \in C_c^\infty(G, \tau)$. Here we used (4.22) and

$$\begin{aligned} F(x) &= f(x) + \sum_{1 \leqq p \leqq \gamma} C(p) h_p(x) - \sum_{1 \leqq p \leqq \gamma} C(p) h_p'(x) \\ &= \{f(x) - \sum_{1 \leqq p \leqq \gamma} C(p) h_p'(x)\} + \sum_{1 \leqq p \leqq \gamma} C(p) h_p(x) \quad (x \in G) . \end{aligned} \tag{4.24}$$

Thus F is the desired function on G. This completes the proof of Theorem 2.

REMARK 3. Using Theorem 2 and its proof, we obtained Paley-Wiener type theorem on $\mathscr{C}(G)$ and some relation between an imbedding of the discrete series for G and singularities of $\Phi(\nu\colon a) C_{P|P}(1; \nu)^{*-1} \phi_i^j(1)$ We shall describe these results in a next article.

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