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On the Values of the Riemann Zeta Function at Half Integers

Yasushi MATSUOKA

Keio University
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In the study of special values of the Riemann zeta function the following Ramanujan's formulae are well-known: For positive α, β with $\alpha\beta=\pi^2$ and an integer $\nu>1$,

$$\alpha^{\nu}\left\{\frac{\zeta(1-2\nu)}{2}+\sum_{n=1}^{\infty}\sigma_{2\nu-1}(n)e^{-2n\alpha}\right\}=(-\beta)^{\nu}\left\{\frac{\zeta(1-2\nu)}{2}+\sum_{n=1}^{\infty}\sigma_{2\nu-1}(n)e^{-2n\beta}\right\}, \tag{1}$$

$$\begin{aligned}&\left(\frac{1}{4\alpha}\right)^{\nu-1}\left\{\frac{\zeta(2\nu-1)}{2}+\sum_{n=1}^{\infty}\sigma_{1-2\nu}(n)e^{-2n\alpha}\right\}\\&-\left(\frac{-1}{4\beta}\right)^{\nu-1}\left\{\frac{\zeta(2\nu-1)}{2}+\sum_{n=1}^{\infty}\sigma_{1-2\nu}(n)e^{-2n\beta}\right\}\\&+\sum_{k=0}^{[\nu/2]}{}'(-1)^k\pi^{2k}\frac{B_{2k}B_{2\nu-2k}}{(2k)!(2\nu-2k)!}\{(-\alpha)^{\nu-2k}+\beta^{\nu-2k}\}=0,\end{aligned} \tag{2}$$

where $\zeta(s)$ is the Riemann zeta function, $\sigma_a(n)=\sum_{d|n}d^a$, B_n are Bernoulli numbers defined by $\sum_{n=0}^{\infty}(B_n/n!)x^n=x/(e^x-1)$ and the dash in the summation in (2) means that for even ν the last term is $(-1)^{\nu/2}\pi^2(B_\nu/\nu!)^2$ rather than twice the value. These formulae had been found by Ramanujan. Hardy [3] gave two proofs of (1). In 1970 Grosswald [2] proved a more general formula which contains as special cases both (1) and (2).

The values of $\zeta(s)$ at rational points seem to have hardly been studied. In [1] Edwards calculated the approximation $\zeta(1/2)\sim-1.46035496$ and pointed out that Riemann's unpublished papers include a computation of $\xi(1/2)=\log\{-(1/8)\pi^{-1/4}\zeta(1/2)\}$ to several decimal places. Selberg and Chowla [4] obtained a formula for elliptic integrals of the first kind which can be written by the special values of $\zeta(s)$: Let d be a negative integer $\equiv 0$ or $1(\bmod 4)$ such that $d/4$ or d is a square free integer respectively, and let

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$$K=\int_0^{\pi/2}(1-k^2\sin^2 x)^{-1/2}dx \qquad (0<k<1)\,,$$

$$K'=\int_0^{\pi/2}(1-k'^2\sin^2 x)^{-1/2}dx \qquad (k^2+k'^2=1)\,.$$

Then if $iK'/K\in Q(\sqrt{d})$,

$$K=\lambda\left\{\prod_{m=1}^{|d|-1}\zeta\left(\frac{m}{|d|}\right)^{(d/(|d|-m))-(d/m)}\right\}^{w/4h},$$

where λ is an algebraic number, (d/m) is the Kronecker's symbol, h is the class number of $Q(\sqrt{d})$, and

$$w=\begin{cases}6 & \text{if } d=-3\\ 4 & \text{if } d=-4\\ 2 & \text{otherwise.}\end{cases}$$

In this paper we give, by a similar method used in [2], a formula for the values of $\zeta(s)$ at half integers.

THEOREM. *Let ν be an integer greater than* 1. *We put*

$$g_\nu(n)=\sum_{\substack{klm|n\\(k,l,m)\in N^3}} k^{-1/2}l^{2\nu-1}m^{2\nu-3/2}$$

and define, for $x>0$,

$$G_\nu(x)=x^{\nu-1/4}\left\{\sum_{n=1}^{\infty}g_\nu(n)e^{-4\sqrt{\pi n x}}+\frac{(-1)^{\nu-1}(4\nu-2)!\,B_{2\nu}\zeta\left(\frac{1}{2}\right)\zeta\left(2\nu-\frac{1}{2}\right)}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}}\right\}$$
$$+x^{\nu+1/4}\frac{(-1)^{\nu}(4\nu-1)!\,B_{2\nu}\zeta\left(-\frac{1}{2}\right)\zeta\left(2\nu+\frac{1}{2}\right)}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1/2}}\,.$$

Then for any positive α, β with $\alpha\beta=\pi^2$ we have

$$G_\nu(\alpha)=G_\nu(\beta)\,. \tag{3}$$

Equation (3) implies especially

$$G_\nu(2\pi)=G_\nu\left(\frac{\pi}{2}\right),\quad G_\nu(4\pi)=G_\nu\left(\frac{\pi}{4}\right),$$

which leads to the following

COROLLARY.

$$\zeta\left(-\frac{1}{2}\right)\zeta\left(2\nu+\frac{1}{2}\right)=(-1)^{\nu-1}\pi^{2\nu-1}\left\{a_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{4n}}\right.$$
$$+b_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{8n}}+c_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{32n}}$$
$$\left.+d_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{64n}}\right\},$$

$$\zeta\left(\frac{1}{2}\right)\zeta\left(2\nu-\frac{1}{2}\right)=(-1)^{\nu-1}\pi^{2\nu-1}\left\{a'_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{4n}}\right.$$
$$+b'_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{8n}}+c'_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{32n}}$$
$$\left.+d'_\nu\sum_{n=1}^{\infty}g_\nu(n)e^{-\pi\sqrt{64n}}\right\},$$

where $a_\nu, \cdots, d'_\nu$ *are numbers in* $Q(\sqrt[4]{2})$ *defined by*

$$a_\nu=\frac{(2^{\nu-1/4}-2^{-\nu+1/4})2^{4\nu-3/2}}{C_\nu(4\nu-1)!},\quad a'_\nu=\frac{(2^{\nu+1/4}-2^{-\nu-1/4})2^{4\nu-3/2}}{C_\nu(4\nu-2)!},$$
$$b_\nu=-\frac{(2^{2\nu-1/2}-2^{-2\nu+1/2})2^{5\nu-7/4}}{C_\nu(4\nu-1)!},\quad b'_\nu=-\frac{(2^{2\nu+1/2}-2^{-2\nu-1/2})2^{5\nu-7/4}}{C_\nu(4\nu-2)!},$$
$$c_\nu=\frac{(2^{2\nu-1/2}-2^{-2\nu+1/2})2^{7\nu-9/4}}{C_\nu(4\nu-1)!},\quad c'_\nu=\frac{(2^{2\nu+1/2}-2^{-2\nu-1/2})2^{7\nu-9/4}}{C_\nu(4\nu-2)!},$$
$$d_\nu=-\frac{(2^{\nu-1/4}-2^{-\nu-1/4})2^{8\nu-5/2}}{C_\nu(4\nu-1)!},\quad d'_\nu=-\frac{(2^{\nu+1/4}-2^{-\nu-1/4})2^{8\nu-5/2}}{C_\nu(4\nu-2)!},$$

with

$$C_\nu=\frac{B_{2\nu}\{(2^{2\nu-1/2}-2^{-2\nu+1/2})(2^{\nu+1/4}-2^{-\nu-1/4})-(2^{2\nu+1/2}-2^{-2\nu-1/2})(2^{\nu-1/4}-2^{-\nu+1/4})\}}{(2\nu)!}.$$

PROOF OF THEOREM. We put

$$\Phi_\nu(s)=\frac{2\Gamma(2s)\zeta(s)\zeta\left(s+\frac{1}{2}\right)\zeta(s+1-2\nu)\zeta\left(s+\frac{3}{2}-2\nu\right)}{(4\pi)^{2s}}. \tag{4}$$

Then, using the functional equation of the zeta function and Legendre's duplication formula for the gamma function, we have

$$\Phi_\nu\left(2\nu-\frac{1}{2}-s\right)=\Phi_\nu(s). \tag{5}$$

Consider the function

$$f_\nu(t)=\sum_{n=1}^{\infty} g_\nu(n)e^{-4\pi\sqrt{nt}} \qquad (t>0)\ .$$

The series converges absolutely in $t>0$ and uniformly in any interval $\delta \leqq t<\infty$ with $\delta>0$, since

$$(6) \qquad \begin{aligned} |g_\nu(n)| &\leqslant \sum_{klm|n} (lm)^{2\nu-1} \\ &\leqslant n^{4\nu-2}\sum_{klm|n} 1 \leqslant n^{4\nu+1}\ , \end{aligned}$$

so that

$$\sum_{n=1}^{\infty} g_\nu(n)e^{-4\pi\sqrt{nt}} \leqslant \sum_{n=1}^{\infty} n^{4\nu+1}e^{-4\pi\sqrt{\delta n}}<\infty\ .$$

We have

$$\begin{aligned} \int_0^\infty f_\nu(t)t^{s-1}dt &= \int_0^\infty \sum_{N=1}^{\infty} g_\nu(N)e^{-4\pi\sqrt{Nt}}t^{s-1}dt \\ &= \sum_{N=1}^{\infty} g_\nu(N)\int_0^\infty e^{-4\pi\sqrt{Nt}}t^{s-1}dt\ . \end{aligned}$$

The inversion of the order of integration and summation can be justified by the uniform convergence. Substituting $u=4\pi\sqrt{nt}$ in the last integral, we get

$$(7) \qquad \begin{aligned} \int_0^\infty f_\nu(t)t^{s-1}dt &= \sum_{N=1}^{\infty} g_\nu(N)\int_0^\infty e^{-u}\left(\frac{u^2}{(4\pi)^2N}\right)^{s-1}\frac{2u}{(4\pi)^2N}du \\ &= \frac{2\Gamma(2s)}{(4\pi)^{2s}}\sum_{N=1}^{\infty} N^{-s}g_\nu(N)\ . \end{aligned}$$

Taking account of the inequality (6), the last series is absolutely convergent in the half-plane $\mathrm{Re}\, s>4\nu+2$. Thus

$$(8) \qquad \begin{aligned} \sum_{N=1}^{\infty} N^{-s}g_\nu(N) &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k^{-1/2-s}l^{2\nu-1-s}m^{2\nu-3/2-s}n^{-s} \\ &= \zeta\left(s+\frac{1}{2}\right)\zeta(s+1-2\nu)\zeta\left(s+\frac{3}{2}-2\nu\right)\zeta(s) \end{aligned}$$

for $\mathrm{Re}\, s>4\nu+2$ and so for all s (by the theorem of identity). By (4), (7) and (8), we obtain

$$\Phi_\nu(s)=\int_0^\infty f_\nu(t)t^{s-1}dt\ .$$

Using Mellin's inversion formula, we can write

$$f_\nu(t)=\frac{1}{2\pi i}\int_{2\nu+1/4-i\infty}^{2\nu+1/4+i\infty}\Phi_\nu(s)t^{-s}ds\,. \tag{9}$$

To move the line of integration we need the following estimate;

$$\Gamma(\sigma+it)=O(e^{-(\pi/2)|t|}|t|^{\sigma-1/2}\quad (a\leqslant\sigma\leqslant b,\ |t|\geqslant 1)\,,$$
$$\zeta(\sigma+it)=O(|t|^{\tau(\sigma)}\log|t|)\,,$$

where

$$\tau(\sigma)=\begin{cases}\frac{1}{2}-\sigma & (\sigma\leqslant 0)\\ \frac{1}{2} & \left(0\leqslant\sigma\leqslant\frac{1}{2}\right)\\ 1-\sigma & \left(\frac{1}{2}\leqslant\sigma\leqslant 1\right)\\ 0 & (\sigma\geqq 1)\end{cases}\,.$$

(The first inequality follows from the Stirling's formula for $\Gamma(s)$. The second can be found, e.g., in [5].) These inequalities imply

$$\Phi_\nu(\sigma+it)=O(e^{-\pi|t|}|t|^{\Delta})\quad(a\leqslant\sigma\leqslant b,\ |t|\geqslant 1)\,,$$

where a and b are any fixed real numbers, and $\Delta>0$ is a constant independent of t, so that we can shift the line of integration in (9) to any position $(\sigma_0-i\infty,\ \sigma_0+i\infty)$. Taking $\sigma_0=-3/4$, we obtain

$$f_\nu(t)=\sum_{s=2\nu,2\nu-1/2,0,-1/2}\mathrm{Res}(\Phi_\nu(s)t^{-s})+\frac{1}{2\pi i}\int_{-3/4-i\infty}^{-3/4+i\infty}\Phi_\nu(s)t^{-s}ds\,. \tag{10}$$

If we substitute $s=2\nu-(1/2)-S$ and use the functional equation (5), we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{-3/4-i\infty}^{-3/4+i\infty}\Phi_\nu(s)t^{-s}ds&=t^{-2\nu+1/2}\frac{1}{2\pi i}\int_{2\nu+1/4-i\infty}^{2\nu+1/4+i\infty}\Phi_\nu(S)\left(\frac{1}{t}\right)^{-S}dS\\ &=t^{-2\nu+1/2}f_\nu\left(\frac{1}{t}\right)\,.\end{aligned} \tag{11}$$

The residues in the sum are as follows:

$$\operatorname*{Res}_{s=2\nu}(\Phi_\nu(s)t^{-s})=\frac{2\Gamma(4\nu)\zeta(2\nu)\zeta\left(2\nu+\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)t^{-2\nu}}{(4\pi)^{2\nu}}$$

$$=\frac{2\Gamma(4\nu)\zeta(2\nu)\zeta\left(2\nu+\frac{1}{2}\right)\zeta\left(1-\left(-\frac{1}{2}\right)\right)t^{-2\nu}}{(4\pi)^{2\nu}}$$

$$=\frac{(-1)^{\nu}(4\nu-1)!\,B_{2\nu}\zeta\left(-\frac{1}{2}\right)\zeta\left(2\nu+\frac{1}{2}\right)t^{-2\nu}}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}},$$

$$\operatorname*{Res}_{s=2\nu-1/2}(\Phi_\nu(s)t^{-s})=\frac{(-1)^{\nu-1}(4\nu-2)!\,B_{2\nu}\zeta\left(\frac{1}{2}\right)\zeta\left(2\nu-\frac{1}{2}\right)t^{-2\nu+1/2}}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}},$$

$$\operatorname*{Res}_{s=0}(\Phi_\nu(s)t^{-s})=\zeta(0)\zeta\left(\frac{1}{2}\right)\zeta(1-2\nu)\zeta\left(\frac{3}{2}-2\nu\right)$$

$$=\zeta(0)\zeta\left(\frac{1}{2}\right)\zeta(1-2\nu)\zeta\left(1-\left(2\nu-\frac{1}{2}\right)\right)$$

$$=\frac{B_{2\nu}\sqrt{2}\,(-1)^{\nu}\zeta\left(\frac{1}{2}\right)\zeta\left(2\nu-\frac{1}{2}\right)}{4\nu(2\pi)^{2\nu-1/2}}\left(2\nu-\frac{3}{2}\right)\left(2\nu-\frac{1}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{(-1)^{\nu}(4\nu-2)!\,B_{2\nu}\zeta\left(\frac{1}{2}\right)\zeta\left(2\nu-\frac{1}{2}\right)}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}},$$

$$\operatorname*{Res}_{s=-1/2}(\Phi_\nu(s)t^{-s})=\frac{(-1)^{\nu-1}(4\nu-1)!\,B_{2\nu}\zeta\left(-\frac{1}{2}\right)\zeta\left(2\nu+\frac{1}{2}\right)t^{1/2}}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}}.$$

These calculation as well as (10) and (11) imply

$$f_\nu(t)-t^{-2\nu+1/2}f_\nu\left(\frac{1}{t}\right)=(-1)^{\nu}A_\nu t^{-2\nu}+(-1)^{\nu-1}B_\nu t^{-2\nu+1/2}$$
$$+(-1)^{\nu}B_\nu+(-1)^{\nu-1}A_\nu t^{1/2},$$

where

$$A_\nu=\frac{(4\nu-1)!\,B_{2\nu}\zeta\left(-\frac{1}{2}\right)\zeta\left(2\nu+\frac{1}{2}\right)}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}},\quad B_\nu=\frac{(4\nu-2)!\,B_{2\nu}\zeta\left(\frac{1}{2}\right)\zeta\left(2\nu-\frac{1}{2}\right)}{(2\nu)!\,4^{3\nu-1}\pi^{2\nu-1}}.$$

Setting $\pi t=\alpha$ and $\pi/t=\beta$, we obtain

$$\alpha^{\nu-1/4}\left\{f_\nu\left(\frac{\alpha}{\pi}\right)+(-1)^{\nu-1}B_\nu\right\}+\alpha^{\nu+1/4}\frac{(-1)^{\nu}A_\nu}{\sqrt{\pi}}$$
$$=\beta^{\nu-1/4}\left\{f_\nu\left(\frac{\beta}{\pi}\right)+(-1)^{\nu-1}B_\nu\right\}+\beta^{\nu+1/4}\frac{(-1)^{\nu}A_\nu}{\sqrt{\pi}};$$

which leads to the equation (3).

References

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Present Address:
Department of Mathematics
Keio University
Hiyoshi-cho, Kohoku-ku
Yokohama 233