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On π-uniform Vector Bundles

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In this paper, we define a notion of "π-uniform vector bundle" over a P^1-bundle $\pi: V \rightarrow W$, where V, W are algebraic varieties. First, generalizing a result of E. Sato ([5], Proposition 3), we give a necessary and sufficient condition in order that a vector bundle over a P^1-bundle is π-uniform (Lemma). By virtue of the Lemma, we give a cohomological condition in order that a vector bundle over the trivial ruled surface $P^1 \times P^1$ is decomposable (Theorem 1). Also we generalize a result of S. Shatz in [6] (Corollary, p. 106) (Theorem 2).

In [2], Schwarzenberger defined the notion of 'uniform vector bundle' on a projective space P^n. Our 'π-uniform vector bundle' is an analogue of his, and is suitable for our situation of P^1-bundle $\pi: V \rightarrow W$. In his paper on uniform vector bundles [5], E. Sato developed some methods for treating such bundles. This paper is inspired by [5].

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§1. A criterion for π-uniform vector bundles.

Let k be an algebraically closed field of arbitrary characteristic and $\pi: V \rightarrow W$ a P^1-bundle, where V, W are algebraic varieties over k. By a vector bundle E on V, we mean a locally free $\mathcal{O}_V$-sheaf module of finite rank, where $\mathcal{O}_V$ is the structure sheaf of V. We use the following notation; $h^i(V, E) := \dim_k H^i(V, E)$.

DEFINITION 1. We say that a vector bundle E on V is π-uniform, if the restriction $E|_{\pi^{-1}(p)}$ of E to $\pi^{-1}(p)$ is mutually isomorphic for any point p of W.

First the following proposition is an immediate consequence of Definition 1.

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PROPOSITION 1. (1) *Any line bundle on V is π-uniform.*

(2) *A direct sum of line bundles is π-uniform.*

(3) *The dual of π-uniform vector bundle is π-uniform.*

(4) *If E is a π-uniform vector bundle on V, then so is $E\otimes L$ for any line bundle L on V.*

Now, the following is a key lemma.

LEMMA. *Let E be a vector bundle of rank r on V. Then E is π-uniform if and only if one of the following conditions* (A), (B) *holds.*

(A) *E is an extension of π-uniform vector bundles,*

$$0\longrightarrow E_1\longrightarrow E\longrightarrow E_2\longrightarrow 0\ ,$$

where E_1, E_2 satisfy $h^1(\boldsymbol{P}^1, (E_2|_{\pi^{-1}(p)})^*\otimes(E_1|_{\pi^{-1}(p)}))=0$ for any point p of W. Here $(E_2|_{\pi^{-1}(p)})^*$ is the dual of $E_2|_{\pi^{-1}(p)}$.

(B) *$E\cong\pi^*(F)\otimes L$, where F is a vector bundle on W, and L is a line bundle on V.*

The proof of Lemma is essentially due to E. Sato ([5], where Sato treats the case of $\pi\colon \mathrm{Proj}\,(\mathcal{O}_{P^n}\oplus\mathcal{O}_{P^n}(1))\rightarrow \boldsymbol{P}^n$; the general case can be handled similarly).

REMARK 1. If $W\cong\boldsymbol{P}^1$, then a π-uniform vector bundle which satisfies the condition (B) is a direct sum of line bundles.

REMARK 2. If W is an affine variety and $V\cong\boldsymbol{P}^1_W$, then a π-uniform vector bundle of rank 2 which satisfies the condition (A) on V is decomposable.

§2. π-uniform vector bundles on rational ruled surfaces.

Let $\pi\colon \boldsymbol{F}_n=\mathrm{Proj}\,(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(n))\rightarrow\boldsymbol{P}^1$ be a rational ruled surface over k, where n is a non-negative integer. We summarize some well-known facts on $\boldsymbol{F}_n$ from M. Maruyama in ([3], Chapter IV, 3). There is a minimal section M on $\boldsymbol{F}_n$ with $(M, M)=-n$. Let F be a fibre of $\boldsymbol{F}_n$. Every divisor D on $\boldsymbol{F}_n$ is linearly equivalent to $aM+bN$, where $a=(D, N)$ and $b=(D, M)+an$. Also we use the notions of decomposable vector bundle and of simple vector bundle in the usual sense (cf. [3]). Note that every simple vector bundle is indecomposable.

2.1. Let $E(a, b)$ be the set of vector bundles which are obtained by the following extension;

$$0 \longrightarrow \mathcal{O}_{F_n} \longrightarrow E \longrightarrow \mathcal{O}_{F_n}(aM+bN) \longrightarrow 0 .$$

Note that, by Proposition 1, if a vector bundle E of rank 2 is decomposable, then E is π-uniform. We show here the existence of elements of $E(a, b)$, which are π-uniform but indecomposable. More precisely, let $U(a, b)$ be the subset of π-uniform vector bundles in $E(a, b)$. By Proposition 1, $\mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(aM+bN)$ is contained in $U(a, b)$ and we put $N(a, b) = U(a, b) - \{\mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(aM+bN)\}$. By virtue of the Oda's lemma [4], then we have;

PROPOSITION 2. (1) *The case* $n>0$;

(i) *If* $-an+b-2 \geqq 0$ *and* $b \leqq 0$, *then* $E(a, b) = U(a, b)$ *and every element of* $N(a, b)$ *is indecomposable and not simple.*

(ii) *If* $a \leqq -1$ *and* $b \geqq 1$, *then* $E(a, b) = U(a, b)$ *and every element of* $N(a, b)$ *is simple.*

(2) *The case* $n=0$: *If* $a \leqq -1$ *and* $b \geqq 2$, *then* $E(a, b) = U(a, b)$ *and every element of* $N(a, b)$ *is simple.*

COROLLARY. *There are indecomposable and* π*-uniform vector bundles on* $F_n (n \geqq 0)$.

EXAMPLE. The tangent bundle T_{F_n} of F_n has the following exact sequence;

$$0 \longrightarrow \mathcal{O}_{F_n}(2M+nN) \longrightarrow T_{F_n} \longrightarrow \mathcal{O}_{F_n}(2N) \longrightarrow 0 .$$

By Lemma, we see that T_{F_n} is π-uniform. When $n=0$, $T_{F_0} = \mathcal{O}_{F_0}(2M) \oplus \mathcal{O}_{F_0}(2N)$. But when $n>0$, the above sequence does not split. Therefore by Proposition 2, we see that T_{F_1} is simple, and $T_{F_n} (n \geqq 2)$ is indecomposable and not simple.

2.2. Here, we give a cohomological criterion in order that a vector bundle of rank 2 on $P^1 \times P^1$ is decomposable. Let E be a vector bundle of rank r on F_n, and let $E(\ell)$ denote $E \otimes \mathcal{O}_{F_n}(\ell M + \ell(n+1)N)$, where ℓ is an integer.

PROPOSITION 3. *If* $h^1(F_n, E(\ell)) = 0$ *for any integer* ℓ, *then* E *is* π*-uniform.*

PROOF. For simplicity, we give the proof for $r=2$. We consider the following exact sequence;

$$0 \longrightarrow \mathcal{O}_{F_n}(-N) \longrightarrow \mathcal{O}_{F_n} \longrightarrow \mathcal{O}_N \longrightarrow 0 .$$

Tensoring with $E(\ell M + (\ell n + \ell + 1)N)$, we have

$$0 \longrightarrow E(\ell) \longrightarrow E(\ell M+(\ell n+\ell+1)N) \longrightarrow E(\ell M)|_N \longrightarrow 0 .$$

From the assumption, we see that

$$(1) \qquad 0 \longrightarrow H^0(F_n, E(\ell)) \longrightarrow H^0(F_n, E(\ell M+(\ell n+\ell+1)N)) \\ \longrightarrow H^0(P^1, E(\ell M)|_N) \longrightarrow 0 .$$

Now we suppose that E is not π-uniform. Then we may assume that for two fibres N_1, N_2 of π, $E|_{N_i} \cong \mathcal{O}_{P^1}(a_1^i) \oplus \mathcal{O}_{P^1}(a_2^i)$ with $a_1^i \geqq a_2^i (i=1, 2)$ and $a_1^1 \neq a_1^2$. Therefore, it is clear that $h^0(P^1, E(\ell M)|_{N_1}) \neq h^0(P^1, E(\ell M)|_{N_2})$. But this contradicts to the exact sequence (1). q.e.d.

THEOREM 1. *Let E be a vector bundle of rank 2 on F_0. If $h^1(F_0, E(\ell))=0$ for any integer ℓ, then E is decomposable.*

PROOF. We put $c_1(E)=aM+bN$ and $c_2(E)=c$. By virtue of the Riemann-Roch Theorem,

$$\chi(E(\ell)) = h^0(F_0, E(\ell)) - h^1(F_0, E(\ell)) + h^2(F_0, E(\ell)) \\ = 2\ell^2 + (a+b+4)\ell + (ab+a+b+2-c) .$$

From the assumption, we have $\chi(E(\ell)) \geqq 0$ for any integer ℓ, and $(a+b+4)^2 - 8(ab+a+b-c+2) = (a-b)^2 + 2(4c-2ab) \leqq 0$. Therefore we get $c_1^2(E) - 4c_2(E) = 2ab - 4c \geqq 0$. From Proposition 3, E is π-uniform, and we may put $E|_N \cong \mathcal{O}_{P^1}(a_1) \oplus \mathcal{O}_{P^1}(a_2)$ with $a_1 \geqq a_2$ and $a_1 + a_2 = a$. By Lemma and Remark 1, it is enough to prove the case of the condition (A) of Lemma. We may assume that E is obtained by the extension of line bundles L_1, L_2 on F_0;

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

where $L_i = \mathcal{O}_{F_0}(a_i M + b_i N)$. Also we may assume $h^1(P^1, (L_{2|N})^* \otimes (L_{1|N}))$ $h^0(P^1, \mathcal{O}_{P^1}(a-2a_1-2))=0$ (i.e., $a-2a_1 \leqq 1$). Now using the above, we show $h^1(F_0, L_2^* \otimes L_1) = 0 \cdots (2)$. By Künneth formula we have $h^1(F_0, L_2^* \otimes L_1) = h^0(P^1, \mathcal{O}_{P^1}(\alpha)) \cdot h^1(P^1, \mathcal{O}_{P^1}(\beta)) + h^1(P^1, \mathcal{O}_{P^1}(\alpha)) \cdot h^0(P^1, \mathcal{O}_{P^1}(\beta))$, where $\alpha = 2a_1 - a$, and $\beta = 2b_1 - b$. If $\alpha = -1$, then $h^0(P^1, \mathcal{O}_{P^1}(\alpha)) = h^1(P^1, \mathcal{O}_{P^1}(\alpha)) = 0$, and we have (2). If $\alpha = 0$, then E is an inverse image of vector bundle of rank 2 on P^1, and (2) holds. Finally if $\alpha > 0$, then we have $\beta \geqq 0$ since $2ab - 4c = \alpha\beta \geqq 0$. By $\alpha > 0$, $\beta \geqq 0$, we have $h^1(P^1, \mathcal{O}_{P^1}(\alpha)) = h^1(P^1, \mathcal{O}_{P^1}(\beta)) = 0$, and we have (2). But (2) implies that E is decomposable. q.e.d.

REMARK 3. By virtue of Maruyama's Theorem ([3], Theorem 4.6), the inequality $c_1^2(E) - 4c_2(E) \geqq 0$ in the proof of Theorem 1 implies that

E is not simple. By the way, from Proposition 2, there are many indecomposable, not simple and π-uniform vector bundles on $F_n (n \geqq 1)$. Hence it is impossible to generalize the above result to general rational ruled surfaces.

§3. The direct image under a finite flat covering map of ruled surfaces.

In [6], Shatz showed; let $\pi: X \to C$ be a ruled surface, $\phi: D \to C$ a finite flat covering of degree s. If θ: $Y \to X$ is the induced covering of X by base extension, and if L is any line bundle on Y, then the direct image $\theta_* L$ is a π-uniform vector bundle of rank s on X. Moreover, under the same hypothesis as in the above result, the conclusion is still valid when L is replaced by a π_1-uniform vector bundle V on Y whose restriction to the fibre of Y has the special form $\mathcal{O}_{P^1}(a)^{\oplus r}$, where π_1 is a morphism π_1: $Y \to D$ (cf. [6] Proposition 7 and its Corollary). But, using Lemma, these results can be generalized to arbitrary π_1-uniform vector bundles of rank 2 on Y.

THEOREM 2. *If E is a π_1-uniform vector bundle of rank 2 on Y, then the direct image $\theta_* E$ is a π-uniform vector bundle on X.*

PROOF. We have only to prove the case (A) in Lemma, because the case of (B) is contained in Shatz's result. We consider the extension; $0 \to L_1 \to E \to L_2 \to 0$. Since θ is a finite flat covering, the direct image of the above extension is a short exact sequence;

$$0 \longrightarrow \theta_* L_1 \longrightarrow \theta_* E \longrightarrow \theta_* L_2 \longrightarrow 0 .$$

Shatz proved that for any line bundle L on Y, the restriction of the direct image $\theta_* L$ to the fibres of π has the form $L_{|\pi^{-1}(d)}{}^{\oplus s}$ for any d of D (cf. [6] Proposition 7). Hence, we have $\theta_* L_{i|\pi^{-1}(c)} \cong L_{i|\pi_1^{-1}(d)}{}^{\oplus s}$, where $c = \phi(d)$, and there is an isomorphism $H^1(\boldsymbol{P}^1, (\theta_* L_{2|\pi^{-1}(c)})^* \otimes (\theta_* L_{1|\pi^{-1}(c)})) \xrightarrow{\sim} \bigoplus H^1(\boldsymbol{P}^1, (L_{2|\pi_1^{-1}(d)})^* \otimes (L_{1|\pi_1^{-1}(d)}))$. By Lemma, $h^1(\boldsymbol{P}^1, (L_{2|\pi_1^{-1}(d)})^* \otimes (L_{1|\pi_1^{-1}(d)})) = 0$, and therefore we have $h^1(\boldsymbol{P}^1, (\theta_* L_{2|\pi^{-1}(c)})^* \otimes (\theta_* L_{1|\pi^{-1}(c)})) = 0$. This completes the proof.

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