TOKYO J. MATH.
VOL. 2, No. 2, 1979

# A Verification for Non-existence of Movable Branch Points of Six Painlevé Transcendents by Formula Manipulations

Shunro WATANABE

*Tsuda College*

## §1. Statement of the problem.

Suppose that the differential equation

$$w''=F(z, w, w') \qquad (\,'\,=d/dz)$$

satisfies that $F$ is rational in $w'$, algebraic in $w$ and analytic in $z$. Among such class of equations, it is well-known that any irreducible equation without movable critical points (i.e., the branch points and essential singular points) must be one of the following six equations, which are called Painlevé equations:

$$w''=6w^2+z\ , \tag{1}$$

$$w''=2w^3+zw+\alpha\ , \tag{2}$$

$$w''=\frac{1}{w}(w')^2-\frac{1}{z}w'+\frac{1}{z}(\alpha w^2+\beta)+\gamma w^3+\frac{\delta}{w}\ , \tag{3}$$

$$w''=\frac{1}{2w}(w')^2+\frac{3}{2}w^3+4zw^2+2(z^2-\alpha)w+\frac{\beta}{w}\ , \tag{4}$$

$$w''=\left(\frac{1}{2w}+\frac{1}{w-1}\right)(w')^2-\frac{1}{z}w'+\frac{(w-1)^2}{z^2}\left(\alpha w+\frac{\beta}{w}\right)+\frac{\gamma w}{z}+\frac{\delta w(w+1)}{w-1}\ , \tag{5}$$

$$w''=\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-z}\right)(w')^2-\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{w-z}\right)w' +\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha+\frac{\beta z}{w^2}+\frac{\gamma(z-1)}{(w-1)^2}+\frac{\delta z(z-1)}{(w-z)^2}\right\}\ . \tag{6}$$

Conversely, the general solutions of these equations, called Painlevé

Received April 28, 1979
Revised June 11, 1979

transcendents, should be free from movable critical points. Painlevé showed the non-existence of movable branch points only for the first equation as follows. In the neighborhood of any arbitrary point $z_0$, the equation (1) is satisfied by the series

$$w=\frac{1}{Z^2}-\frac{1}{10}z_0Z^2-\frac{1}{6}Z^3+hZ^4+\cdots\,, \tag{1a}$$

or

$$w=\frac{1}{Z^2}-\frac{1}{10}zZ^2-\frac{1}{15}Z^3+hZ^4+\cdots\,, \tag{1b}$$

where $Z$ is the abbreviation of $z-z_0$ and $h$ is the second arbitrary parameter. From (1b) we get the series for $w'$, namely,

$$w'=-\frac{2}{Z^3}-\frac{1}{5}zZ-\frac{3}{10}Z^2+4hZ^3+\cdots\,. \tag{1c}$$

On eliminating $Z$ between (1b) and (1c) and writing $w=1/v^2$, it is found that

$$w'=-\frac{2\varepsilon}{v^3}-\frac{\varepsilon}{2}zv-\frac{1}{2}v^2+7\varepsilon hv^3+\cdots\,, \tag{1d}$$

where $\varepsilon=+1$ or $\varepsilon=-1$. If we transform equation (1) by

$$w=\frac{1}{v^2}\,,\quad w'=-\frac{2\varepsilon}{v^3}-\frac{\varepsilon}{2}zv-\frac{1}{2}v^2+\varepsilon uv^3\,, \tag{1e}$$

then the equation (1) becomes the system

$$\begin{cases} v'=\varepsilon+\frac{1}{4}\varepsilon zv^4+\frac{1}{4}v^5-\frac{\varepsilon}{2}uv^6\,, \\ u'=\frac{1}{8}\varepsilon z^2v+\frac{3}{8}zv^2-\varepsilon zuv^3+\frac{1}{4}\varepsilon v^3-\frac{5}{4}uv^4+\frac{3}{2}\varepsilon u^2v^5\,. \end{cases} \tag{1f}$$

From the fact that this system has a unique solution which is analytic in the neighbourhood of $z_0$ and satisfies the initial conditions $u=u_0$, $v=0$ when $z=z_0$, it follows that the general solution $w(z)$ of (1) has a movable pole at $z_0$ and $w(z)$ can have no algebraic branch point at any point $z_0$.

Therefore to prove the non-existence of movable branch points of Painlevé transcendents, it is necessary to calculate the series solutions and the systems corresponding to the equations. However it requires cumbersome calculations which are difficult to fulfil without errors. From this reason it is desirable to calculate these formulas by computer. Fortunately we could use HLISP-REDUCE2 system, a famous system for formula manipulations, and we got following formulas (1a)-(6-2 f) using this system, which must be a verification for hand calculations.

Within my search, [3]-[6] are the only literatures that contain these formulas, and these are identical with our formulas except the arbitrary parameter $h$ of (6-1 a) and (6-1 b) which are not found in those literatures.

## §2. Series solutions obtained by REDUCE 2.

The following formulas (i-j a) and (i-j b) are the series solutions of the Painlevé equation (i). 'j' indicates that the solution (i-j a) and (i-j b) are the $j$-th sub-case. When $i=2, 4$ this classification depends on the sign of the first term. When $i=3, 5, 6$ this classification depends on whether the value of the parameter is zero or not. The solution (i-j b) is obtained from (i-j a) by eliminating $z_0$.

(1a) $$w=\frac{1}{Z^2}-\frac{z_0}{10}Z^2-\frac{1}{6}Z^3+hZ^4+\cdots,$$

(1b) $$w=\frac{1}{Z^2}-\frac{z}{10}Z^2-\frac{1}{15}Z^3+hZ^4+\cdots,$$

(2-1 a) $$w=\frac{1}{Z}-\frac{z_0}{6}Z-\frac{\alpha+1}{4}Z^2+hZ^3+\cdots,$$

(2-1 b) $$w=\frac{1}{Z}-\frac{z}{6}Z-\frac{3\alpha+1}{12}Z^2+hZ^3+\cdots,$$

(2-2 a) $$w=-\frac{1}{Z}+\frac{z_0}{6}Z-\frac{\alpha-1}{4}Z^2+hZ^3+\cdots,$$

(2-2 b) $$w=-\frac{1}{Z}+\frac{z}{6}Z-\frac{3\alpha-1}{12}Z^2+hZ^3+\cdots,$$

(3-1 a) $$w=\frac{1}{gZ}-\frac{\alpha+g}{2z_0g^2}+hZ+\cdots,\quad \gamma=g^2\neq 0,\ z_0\neq 0,$$

(3-1 b) $$w=\frac{1}{gZ}-\frac{\alpha+g}{2zg^2}+\left(h-\frac{\alpha+g}{2z^2g^2}\right)Z+\cdots,\quad \gamma=g^2\neq 0,\ z\neq 0,$$

(3-2 a) $$w=\frac{2z_0}{\alpha Z^2}+h-\frac{h}{z_0}Z+\cdots,\quad \gamma=0,\ z_0\neq 0,$$

(3-2 b) $$w=\frac{2z}{\alpha Z^2}-\frac{2}{\alpha Z}+h-\frac{h}{z}Z+\cdots,\quad \gamma=0,\ z\neq 0,$$

(4-1 a) $$w=\frac{1}{Z}-z_0+\frac{z_0^2+4+2\alpha}{3}Z+hZ^2+\cdots,$$

(4-1 b) $$w=\frac{1}{Z}-z+\frac{z^2-1+2\alpha}{3}Z+\left(h-\frac{2z}{3}\right)Z^2+\cdots,$$

(4-2 a) $$w=-\frac{1}{Z}-z_0+\frac{z_0^2+4+2\alpha}{3}Z+hZ+\cdots,$$

(4-2 b) $$w=-\frac{1}{Z}-z+\frac{z^2+1+2\alpha}{3}Z+\left(h+\frac{2z}{3}\right)Z+\cdots,$$

(5-1 a) $$w=\frac{z_0}{aZ}+h+\cdots,\quad \alpha=\frac{a^2}{2}\neq 0,$$

(5-1 b) $$w=\frac{z}{aZ}+\left(h-\frac{1}{a}\right)+\cdots,\quad \alpha=\frac{a^2}{2}\neq 0,$$

(5-2 a) $$w=\frac{z_0h}{Z^2}+\cdots,\quad \alpha=0,$$

(5-2 b) $$w=\frac{zh}{Z^2}+\cdots,\quad \alpha=0,$$

(6-1 a) $$w=\frac{z_0(z_0-1)}{aZ}+h+\cdots,\quad \alpha=\frac{a^2}{2}\neq 0,$$

(6-1 b) $$w=\frac{z(z-1)}{aZ}+\left(h+\frac{1-2z}{a}\right)+\cdots,\quad \alpha=\frac{a^2}{2}\neq 0,$$

(6-2 a) $$w=\frac{h}{Z^2}+\cdots,\quad \alpha=0,$$

(6-2 b) $$w=\frac{h}{Z^2}+\cdots,\quad \alpha=0.$$

## §3. Systems verified by REDUCE 2.

Using the same method already shown in §1 for the equation (1), we can calculate systems (2-1 f)-(6-2 f) from the series solutions (2-1 b)-(6-2 b). With these systems, immediately we can prove non-existence of movable branch points of the general solutions of the Painlevé equations (2)-(6).

(1f) $$\begin{cases} v'=\varepsilon+\frac{\varepsilon}{4}xv^4+\frac{1}{4}v^5-\frac{\varepsilon}{2}uv^6, \\ u'=\frac{\varepsilon}{8}x^2v+\frac{3}{8}xv^2-\varepsilon xuv^3+\frac{\varepsilon}{4}v^3-\frac{5}{4}uv^4+\frac{3}{2}u^2v^5, \end{cases}$$

where $w=\frac{1}{v^2}$, and $\varepsilon=+1$ or $-1$.

(2-1 f) $$\begin{cases} v'=1+\frac{z}{2}v^2+\frac{2\alpha+1}{2}v^3+uv^4\,, \\ u'=-\frac{2\alpha+1}{4}z-\left(\frac{2\alpha+1}{2}\right)^2 v-zuv-\frac{3(2\alpha+1)}{2}uv^2-2u^2v^3\,, \end{cases}$$

where $w=\frac{1}{v}$.

(2-2 f) $$\begin{cases} v'=-1-\frac{z}{2}v^2-\frac{2\alpha-1}{2}v^3-uv^4\,, \\ u'=\frac{2\alpha-1}{4}z+\left(\frac{2\alpha-1}{2}\right)^2 v+zuv+\frac{3(2\alpha-1)}{2}uv^2+2u^2v^3\,, \end{cases}$$

where $w=\frac{1}{v}$.

(3-1 f) $$\begin{cases} v'=g+\frac{\alpha+g}{gz}v-uv^2\,, \\ u'=\frac{\beta}{z}-\left(\frac{\alpha}{g}+2\right)\frac{u}{z}+(u^2+\delta)v\,, \end{cases}$$

where $w=\frac{1}{v}$, $\gamma=g^2\neq 0$, and $z\neq 0$.

(3-2 f) $$\begin{cases} v'=\varepsilon\sqrt{\frac{\alpha}{2z}}+\frac{v}{2z}-\frac{1}{2}uv^2\,, \\ u'=-\frac{3u}{2z}+\left(\frac{u^2}{2}+\frac{\beta}{z}\right)v+\delta v^3\,, \end{cases}$$

where $w=\frac{1}{v^2}$, $\gamma=0$, and $z\neq 0$.

(4-1 f) $$\begin{cases} v'=1+2zv-2(\alpha-1)v^2-uv^3\,, \\ u'=\{2(\alpha-1)^2-2zu+\beta\}+4(\alpha-1)uv+\frac{3}{2}u^2v^2\,, \end{cases}$$

where $w=\frac{1}{v}$.

(4-2 f) $$\begin{cases} v'=-1-2zv+2(\alpha+1)v^2-uv^3\,, \\ u'=\{2(\alpha+1)^2+2zu+\beta\}-4(\alpha+1)uv+\frac{3}{2}u^2v^2\,, \end{cases}$$

where $w=\dfrac{1}{v}$ .

(5-1 f)
$$\begin{cases} v'=\dfrac{a}{z}-\dfrac{uv}{z} , \\ u'=\left\{-\dfrac{u^2}{2z}+\dfrac{\alpha}{z}+\gamma\right\}+\dfrac{\beta(1-v)^2}{z}+\dfrac{1}{1-v}\left\{\dfrac{(u-a)^2}{z}+\delta z(1+v)\right\} , \end{cases}$$

where $w=\dfrac{1}{v}$ , and $\alpha=\dfrac{a^2}{2}\neq 0$ .

(5-2 f)
$$\begin{cases} v'=\varepsilon u , \\ u'=-2\varepsilon u^2\dfrac{v}{1-v^2}-\dfrac{u}{z}-\dfrac{\varepsilon\beta}{2z^2}v(1-v^2)^2-\dfrac{\varepsilon\gamma}{2z}v-\dfrac{\varepsilon\delta}{2}\dfrac{1+v^2}{1-v^2}v , \end{cases}$$

where $w=\dfrac{1}{v^2}$ , $\alpha=0$ , and $\varepsilon=+1$ or $-1$ .

(6-1 f)
$$\begin{cases} v'=\dfrac{a(1-uv)}{z(z-1)} , \\ u'=-\dfrac{a(uv-1)(z-u)(1+z-zv)}{2z(z-1)(1-zv)}+\dfrac{a(uv-1)^2}{2z(z-1)(1-v)} \\ \qquad +\dfrac{au(u-1+z(v-1))}{2z(z-1)}-\dfrac{uv-1}{1-zv}+\dfrac{\beta(1-v)(1-zv)}{az(1-v)}+\dfrac{\gamma(1-zv)}{az(1-v)} \\ \qquad +\dfrac{\delta(1-v)}{a(1-zv)} , \end{cases}$$

where $w=\dfrac{1}{v}$ , and $\alpha=\dfrac{a^2}{2}\neq 0$ .

(6-2 f)
$$\begin{cases} v'=\varepsilon u , \\ u'=-\varepsilon\left\{\dfrac{1}{1-v^2}+\dfrac{z}{1-zv^2}\right\}u^2v-\left\{\dfrac{1}{z}+\dfrac{1}{z-1}+\dfrac{v^2}{1-zv^2}\right\}u \\ \qquad -\dfrac{\varepsilon v(1-v^2)(1-zv^2)}{2z^2(z-1)^2}\left\{\beta z+\dfrac{\gamma(z-1)}{(1-v)}+\dfrac{\delta z(z-1)}{(1-zv)}\right\} , \end{cases}$$

where $w=\dfrac{1}{v^2}$ , $\alpha=0$ , and $\varepsilon=+1$ or $-1$ .

## References

[1] E. Ince, Ordinary Differential Equations, Chap. XIV, Dover Reprint, New York, 1944.
[2] A. Hearn, REDUCE 2, User's Manual, Univ. of Utah, 1973.
[3] N. Lukashevich, Theory of the fourth Painlevé equation, Differentsial'nye Uravneniya, vol. 3, no. 5 (1967), 771-780.

[4] N. LUKASHEVICH, On the theory of the third Painlevé equation, Differentsial'nye Uravneniya, vol. **3**, no. 11 (1967), 1913–1923.

[5] N. LUKASHEVICH, Solutions of the fifth equation of Painlevé, Differentsial'nye Uravneniya, vol. **4**, no. 8 (1968), 1413–1420.

[6] N. LUKASHEVICH, The theory of Painlevé's equations, Diffrentsial'nye Uravneniya, vol. **6**, no. 3 (1970), 425–430.

*Present Address*:
DEPARTMENT OF MATHEMATICS
Tsuda College
TSUDA-MACHI, KODAIRA, TOKYO 187