TOKYO J. MATH.
VOL. 2, No. 2, 1979

On Unimodal Linear Transformations and Chaos II

Shunji ITO, Shigeru TANAKA and Hitoshi NAKADA

Tsuda College and Keio University

Introduction

In part II we consider the general unimodal linear transformations, that is, a family of maps from [0, 1] into itself which take the extremum at c for some $c\in(0,1)$ and are linear on each intervals $[0,c]$ and $[c,1]$. It is not difficult to show that, except for some trivial exceptions, the consideration of the general unimodal linear transformations defined above can be reduced to that of the special class $\{f_{a,b};\ b>1,\ ab>1,\ a+b\geqq ab\}$ defined in the following way:

$$f_{a,b}(x)=\begin{cases} ax+\dfrac{a+b-ab}{b} & \text{for} \quad 0\leqq x\leqq 1-\dfrac{1}{b} \\ -b(x-1) & \text{for} \quad 1-\dfrac{1}{b}\leqq x\leqq 1\ . \end{cases}$$

In the cases which will be discussed below there will appear phenomena called "window" and "islands", which did not occur in the case $a=b$ of part I. Let us explain these cases, dividing the case $b=4$ into several classes according to the behavior of the corresponding $f_{a,b}$.

1) The case of $0<a<1/4$ (that is, the case of $ab<1$).

In this case, there exists a unique periodic orbit with period 2 and all points except the fixed point approach this periodic orbit. So this class is a stable class, and we omit this class from further consideration.

2) The case of $a=1/4$ (that is, the case of $ab=1$).

Let $A_0=[0, 3/4]$ and $A_1=[13/16, 1]$, then we have $f_{a,b}A_0=A_1$, $f_{a,b}A_1=A_0$, and $f^4_{a,b}|_{A_i}$ is the identity map on $A_i (i=0, 1)$ and every orbit starting from $(3/4, 13/16)-\{4/5\}$ enters into $A_0\cup A_1$. So, this class is also stable.

3) The case of $1/4<a\leqq 4/15$ (that is, the case of $ab>1$, $(a+b-ab)/b\geqq b/(b+1)$).

There exist a natural number m and intervals $A_0, A_1, \cdots, A_{2^m-1}$ such that $f_{a,b}A_i=A_{i+1}$ for $0\leqq i\leqq 2^m-2$ and $f_{a,b}A_{2^m-1}=A_0$, and every orbit starting from $[0,1]-\bigcup_{i=0}^{2m-1}A_i$ (except the fixed point of $f^{2m}_{a,b}$) enters into

Received May 10, 1979

$\bigcup_{i=0}^{2^m-1} A_i$. In this case, $f_{a,b}$ has an invariant measure (absolutely continuous with respect to the Lebesgue measure) whose support is equal to $\bigcup_{i=0}^{2^m-1} A_i$, and, with respect to this measure, $f_{a,b}$ is ergodic but not weakly mixing. But $f_{a,b}^{2^m}|_{A_i}$ is weak Bernoulli. And $f_{a,b}$ has period $2^m \times$ odd ($\neq 1$) as the maximal period (in the sense of Šarkovskii [8]). We denote by D_0 the domain of parameters (a, b) with above properties. (See Figure 1.)

4) The case of $4/15 < a \leqq 1/3$ (that is, the case of $b/(b+1) > (a+b-ab)/b \geqq 1 - 1/b$).

In the case $a = 1/3$, $f_{a,b}$ has period 3 as the maximal period. The interval $4/15 < a < 1/3$ can be divided into sub-intervals $a_m \leqq a < a_{m-1}$, in which $f_{a,b}$ has period $2m+1$ as the maximal period, for $m \geqq 2$. For a in

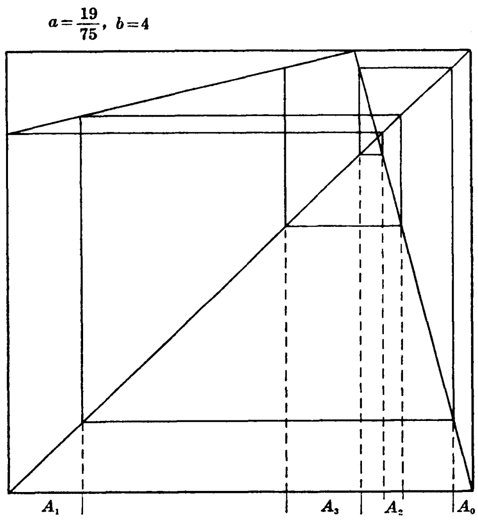

FIGURE 1

each of these intervals, $f_{a,b}$ has an invariant measure (absolutely continuous with respect to the Lebesgue measure) whose support is equal to $[0, 1]$, and with respect to this measure, $f_{a,b}$ is weak Bernoulli. We denote by D_1 the domain of parameters with these properties.

These cases mentioned above are essentially the same as those of part I ($a = b$); that is, case 3) (resp. case 4)) corresponds to the case $1 < a \leqq \sqrt{2}$ (resp. $\sqrt{2} < a \leqq (\sqrt{5}+1)/2$) of part I. But as we mention in the following, phenomena quite different from those for the case $a = b$ will appear in general.

5) The case of $1/3 < a \leqq 1/2$ (that is, the case of $a^2 b \leqq 1$, $(a+b-ab)/b < 1 - 1/b$).

In this case, there exists a stable periodic orbit with period 3 and almost all orbits approach this periodic orbit, and so $f_{a,b}$ does not have an absolutely continuous invariant measure. We call this case "window".

The topological entropy of $f_{a,b}$ is equal to $\log(\sqrt{5}+1)/2$ in this case. We denote this domain of parameters by $D_2^{(1)}$. (The case $a=1/2$ is a little bit different, but essentially the same as mentioned above.) (See Figure 2.)

6) The case of $1/2<a\leqq(1+\sqrt{257})/32$ (that is, the case of $a^2b>1$, $a+b\geqq a^2b^2$, $(a+b-ab)/b<1-1/b$).

In this case there exist sub-intervals J_0, J_1, J_2 of [0, 1] which satis-

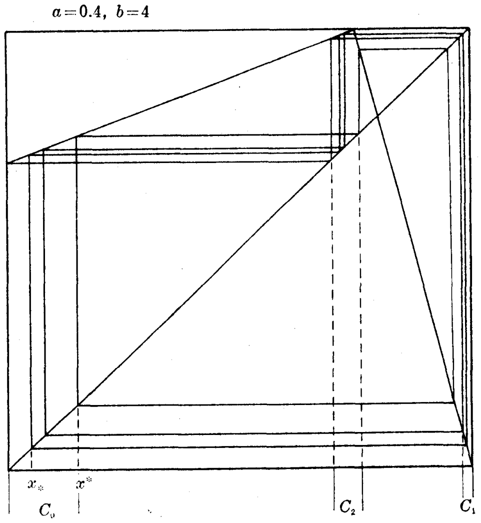

FIGURE 2

fy that $f_{a,b}J_i=J_{i+1}$ for $i=0, 1$, $f_{a,b}J_2=J_0$ and almost all orbits starting from $[0, 1]-\bigcup_{i=0}^{2} J_i$ enter into $\bigcup_{i=0}^{2} J_i$. And $f_{a,b}$ has an absolutely continuous invariant measure whose support is equal to $\bigcup_{i=0}^{2} J_i$. With respect to this measure, $f_{a,b}$ is ergodic but not weakly mixing. In this sense these intervals J_i behave like islands of stability. So, we will call this case "islands". On the other hand, in $[0, 1]-\bigcup_{i=0}^{2} J_i$ there exists an uncountable subset B of Lebesgue measure 0, invariant under $f_{a,b}$, on which $f_{a,b}$ behaves chaotically. In this case the topological entropy of $f_{a,b}$ is also equal to $\log(\sqrt{5}+1)/2$. We denote this case by $D_2^{(2)}$. (See Figure 3.)

7) The case of $(1+\sqrt{257})/32<a<4/3$ (that is, the case of $a+b<a^2b^2$, $(a+b-ab)/b<1-1/b$).

In this case truly chaotic phenomenon appears, that is, $f_{a,b}$ has period 3, and has an absolutely continuous invariant measure with its support [0, 1] and with respect to this measure, $f_{a,b}$ is weak Bernoulli.

The Table 1 summarizes these phenomena mentioned above.

As we have indicated in the remarks above we see that these unimodal linear transformations (though they represent quite simple models)

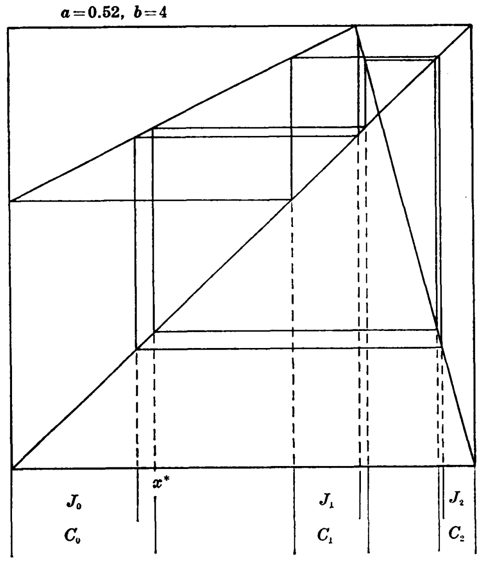

FIGURE 3

Table 1

	maximal period	topological entropy	support of $h_{a,b}(x)$ (cf. [5], [9])	ergodicity w. r. t. $h_{a,b}(x)dx$
$D_0^{(m)}$	$2^m \times$ odd		$A_0 \cup A_1 \cup \cdots \cup A_{2^m-1}$	ergodic but not weakly mixing
∂D_0	6	$\log \sqrt{2}$	[0,1]	ergodic but not weakly mixing
$D_1^{(2m+1)}$	$2m+1$		[0,1]	weak Bernoulli
∂D_1	3	$\log \frac{1+\sqrt{5}}{2}$	[0,1]	weak Bernoulli
$\mathring{D}_k^{(1)}$	3	$\log \gamma_k$	there exists no a.c.	invariant measure
$\partial D_k^{(1)}$	3	$\log \gamma_k$	$J_0 \cup J_1 \cup \cdots \cup J_k$	not ergodic
$\mathring{D}_k^{(2)}$	3	$\log \gamma_k$	$J_0 \cup J_1 \cup \cdots \cup J_k$ or $\bigcup_{i=0}^{k} (J_{i,1} \cup J_{i,2})$	ergodic but not weakly mixing
$\partial D_k^{(2)}$	3	$\log \gamma_k$	$J_0 \cup J_1 \cup \cdots \cup J_k$	ergodic but not weakly mixing
D_k^*	3		[0,1]	weak Bernoulli
D^*			[0,1]	weak Bernoulli

show much complicated behavior. (cf. [6], [7].)

Finally, we explain the organization of this paper. In § 1, we will divide the domain of parameters into several subdomains for the sake of subsequent discussions. In § 2, we will treat the cases of "window" and "islands", which are the characteristic features of the cases in discussion. In § 3, we will give the explicit form of the density function of an absolutely continuous invariant measure of $f_{a,b}$ (cf. [3]), and investigate the ergodicity of $f_{a,b}$ with respect to this measure.

§ 1. Definitions and fundamental properties.

In part II, we consider the transformation $f_{a,b}$ on [0, 1] defined by

$$(1) \qquad f_{a,b}(x)=\begin{cases} ax+\dfrac{a+b-ab}{b} & \text{for} \quad 0\leqq x\leqq 1-\dfrac{1}{b} \\ -b(x-1) & \text{for} \quad 1-\dfrac{1}{b}\leqq x\leqq 1 , \end{cases}$$

for a pair of parameters (a, b) which satisfies $b>1$, $ab>1$, and $a+b\geqq ab$. We notice that $b/(b+1)$ is a fixed point of $f_{a,b}$ for any (a, b).

Let us define the fundamental partition $\{I_0, I_1\}$ of $f_{a,b}$ in the same manner as in part I, that is, let $I_0=[0, 1-1/b]$ and $I_1=(1-1/b, 1]$ in the case when, for some natural number n, $f^n_{a,b}(0)=0$, $f^i_{a,b}(0)\neq 0$ for $1\leqq i\leqq n-1$ and the number

$$(2) \qquad k=\#\left\{i;\ 0\leqq i\leqq n-2,\ f^i_{a,b}(0)>1-\frac{1}{b}\right\}$$

is odd, and let $I_0=[0, 1-1/b)$ and $I_1=[1-1/b, 1]$ otherwise.

The reason why we define the fundamental partition in two different ways is, as in part I, that we can prove the following Theorem 1.1 by using this $\{I_0, I_1\}$, and that this distinction is convenient for representation of $f_{a,b}$ by a symbolic dynamical system. But to consider measure theoretical problems, the difference of the fundamental partitions in the two cases are not essential.

Let us represent $f_{a,b}$ by a symbolic dynamical system. Let us define the space Ω, the shift operator σ on Ω and the order relation in Ω as in part I. Let $\pi_{a,b}$ be a map from [0, 1] into Ω defined by

$$(3) \qquad \pi_{a,b}(x)(n)=j\ , \qquad \text{if} \quad f^n_{a,b}(x)\in I_j \quad (j=0 \text{ or } 1)\ .$$

Let $Y_{a,b}=\pi_{a,b}[0, 1]$ and let $X_{a,b}$ be the closure of $Y_{a,b}$. Then we can prove the following theorem in the same way as in the proof of Theorem 3.1 of part I.

THEOREM 1.1. *We can characterize $X_{a,b}$ as follows:*

$$X_{a,b}=\{\omega\in\Omega;\ \sigma^n\omega\geqq\omega^0_{a,b}\quad \textit{for every}\quad n\geqq 0\}\ , \tag{4}$$

where we denote by $\omega^x_{a,b}$ the image of x under $\pi_{a,b}$.

Now we divide the domain $D=\{(a, b);\ b>1,\ ab>1,\ a+b\geqq ab\}$ into subdomains depending on the behavior of $f_{a,b}$. Let

$$D_0=\left\{(a, b)\in D;\ \frac{a+b-ab}{b}\geqq\frac{b}{b+1}\right\}\ , \tag{5}$$

$$D_1=\left\{(a, b)\in D;\ \frac{b}{b+1}>\frac{a+b-ab}{b}\geqq 1-\frac{1}{b}\right\}\ . \tag{6}$$

In $D_0\cup D_1$ we have

$$\omega^0_{a,b}(0)=0,\ \omega^0_{a,b}(1)=1\ , \tag{7}$$

that is, $f_{a,b}(0)\in I_1$. For $k\geqq 2$ let

$$D_k=\{(a, b)\in D;\ a<1,\ 1+a^{-1}+\cdots+a^{-(k-1)}<b\leqq 1+a^{-1}+\cdots+a^{-k}\}\ . \tag{8}$$

The relation $1+a^{-1}+\cdots+a^{-(k-1)}<b\leqq 1+a^{-1}+\cdots+a^{-k}$ is equivalent to

$$f^i_{a,b}(0)\in I_0\quad\text{for}\quad 1\leqq i\leqq k-1\ ,\qquad f^k_{a,b}(0)\in I_1\ . \tag{9}$$

We divide D_k into three subdomains as follows:

$$D^{(1)}_k=\{(a, b)\in D_k;\ a^kb\leqq 1\}\ , \tag{10}$$

$$D^{(2)}_k=\{(a, b)\in D_k;\ a^kb>1,\ a+b\geqq a^kb^2\}\ , \tag{11}$$

$$D^*_k=D_k-(D^{(1)}_k\cup D^{(2)}_k)\ . \tag{12}$$

And finally, let

$$D^*=\left\{(a, b)\in D;\ a>1,\ \frac{a+b-ab}{b}<\frac{b}{b+1}\right\}\ . \tag{13}$$

(See Figure 4.)

In the remainder of this section, we sub-divide D_0 and D_1 further, and investigate the behavior of $f_{a,b}$ in detail. The results for these domains D_0 and D_1 are essentially the same as those for the case $1<a\leqq(1+\sqrt{5})/2$ of part I. So, with each result, we mention the corresponding result of part I and omit the proof. First of all we notice that $f_{a,b}$ has no periodic point of odd period (except the fixed point

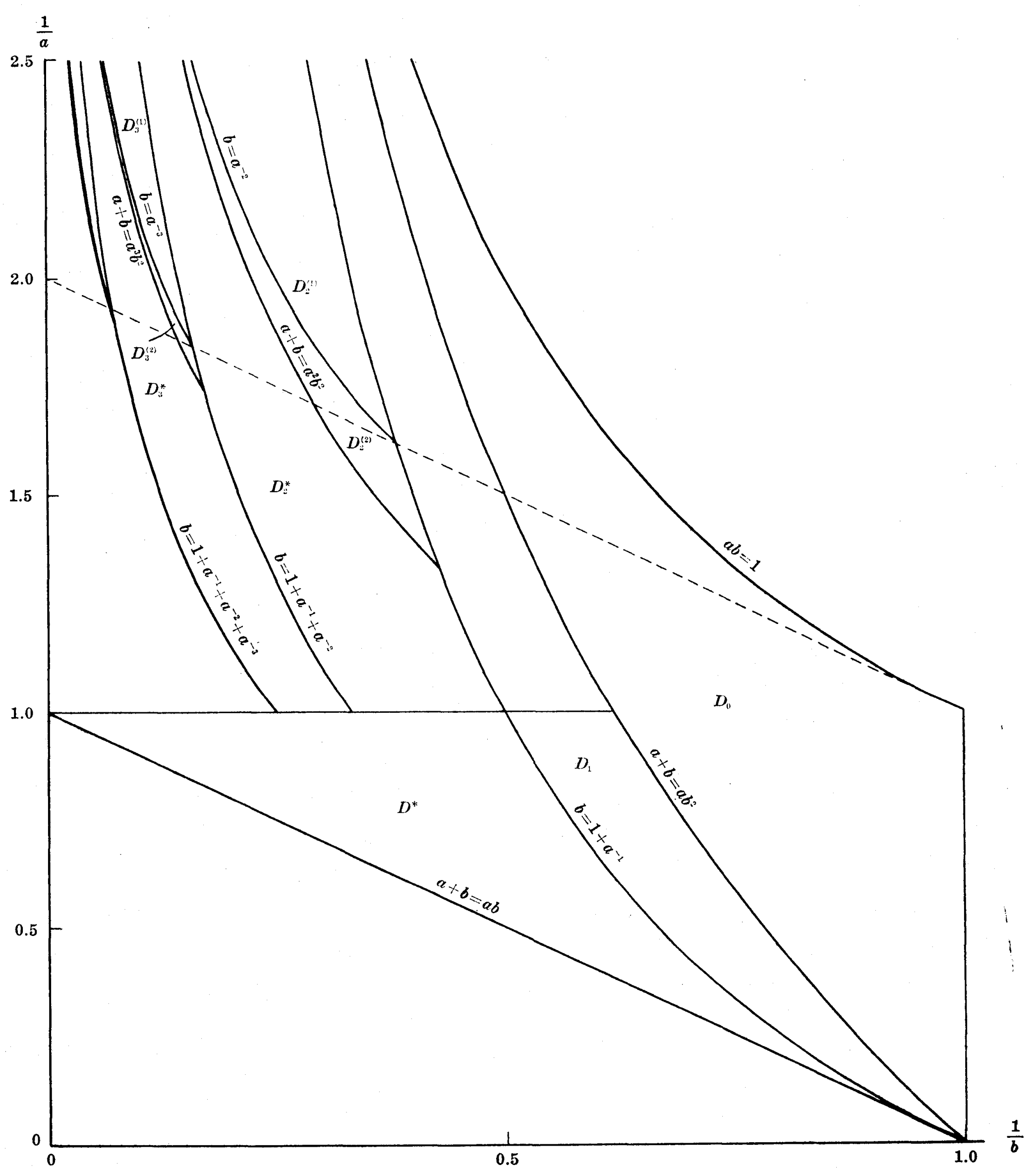

FIGURE 4

$b/(b+1)$) in the case D_0, which follows from the relation

$$f_{a,b}\left[0,\ \frac{b}{b+1}\right]=\left[\frac{b}{b+1},\ 1\right],\quad f_{a,b}\left[\frac{b}{b+1},\ 1\right]=\left[0,\ \frac{b}{b+1}\right]. \tag{14}$$

LEMMA 1.1 (*Lemmas* 2.1 *and* 2.2 *of part* I). *Let* $(a, b)\in D_0$ *and let* $A_0=[f_{a,b}(0), 1]$ *and* $A_1=[0, f^2_{a,b}(0)]$. *Then*

$$f_{a,b}A_0=A_1,\ f_{a,b}A_1=A_0\ , \tag{15}$$

and $f_{a,b}^2|_{A_j}$ $(j=0$ *or* $1)$ *is linearly conjugate to* $f_{b^2,ab}$, *that is, there exists a linear isomorphism* φ *from* A_j *onto* $[0, 1]$ *such that* $\varphi \circ f_{a,b}^2 \circ \varphi^{-1} = f_{b^2,ab}$.

Let us define the numbers $p(m)$ for $m \geqq 1$ inductively as follows:

$$(16)\qquad \begin{cases} p(1)=1\ , \\ p(m)=\begin{cases} 2p(m-1) & \text{if } m \text{ is even} \\ 2p(m-1)-1 & \text{if } m \text{ is odd}\ . \end{cases} \end{cases}$$

For $m \geqq 1$ let

$$(17)\qquad D_0^{(m)} = \{(a, b) \in D_0;\ a^{p(m)} b^{p(m+1)} \leqq a+b < a^{p(m+1)} b^{p(m+2)}\}\ .$$

Then we have

THEOREM 1.2. (*Theorem* 2.3 *of part* I. *Also see* (63).)

(i) *If* $(a, b) \in D_0^{(m)}$, *then* $f_{a,b}$ *has no periodic point with period* $2^k \times odd$ *for* $0 \leqq k < m$.

(ii) $(a, b) \in D_0^{(m)}$ *implies* $(b^2, ab) \in D_0^{(m-1)}$ *for* $m \geqq 2$ *and* $(a, b) \in D_0^{(1)}$ *implies* $(b^2, ab) \in D^*$.

We note the following facts concerning the location of $D_0^{(m)}$ in D_0. First of all, the curve $a+b=ab^2$ (which is a part of the boundary of D_0, and equivalent to $(a+b-ab)/b=b/(b+1)$) does not intersect the curves $ab=1$ and $b=1$. The curve $a+b=a^{p(m)}b^{p(m+1)}$ intersects the curve $ab=1$ at $(\rho_{1,m}^{-1}, \rho_{1,m})$ and meets the line $b=1$ at $(\rho_{2,m}, 1)$, where $\rho_{1,m}(\rho_{2,m})$ is the maximal root of the equation $b^{p(m+1)-p(m)+1}-b^2-1=0$ $(a^{p(m)}-a-1=0$, respectively). We also notice that $\rho_{1,m}$ and $\rho_{2,m}$ are decreasing to 1 as $m \to \infty$.

For $m \geqq 1$, let

$$(18)\qquad D_1^{(2m+1)} = \{(a, b) \in D_1;\ ab^{2m}-b^{2m-1}-ab^{2m-2}-1 \geqq 0\ ,\ ab^{2m-2}-b^{2m-3}-ab^{2m-4}-1<0\}\ .$$

Then we have

THEOREM 1.3 (*Theorem* 2.2 *of part* I). *If* $(a, b) \in D_1^{(2m+1)}$, *then the maximal period* (*in the sense of Šarkovskii*) *of* $f_{a,b}$ *is* $2m+1$.

§ 2. The case of "window" and "islands".

In this section, we show that the fundamental partition is not a generator of $f_{a,b}$ if and only if $(a, b) \in \bigcup_{k=2}^{\infty} D_k^{(1)}$, and show that $D_k^{(1)}$ is the case of "window" and $D_k^{(2)}$ is the case of "islands".

Let $(a, b) \in D_k$ for some k and let

$$x_0=1-\frac{1}{b}-\frac{1}{ab}-\cdots-\frac{1}{a^{k-1}b}\ ; \tag{19}$$

then we can easily show that $x_0\geqq 0$, $f^i_{a,b}(x_0)\in I_0$ for $0\leqq i\leqq k-2$ and $f^{k-1}_{a,b}(x_0)=1-1/b$. In the case $(a,b)\in D_k^{(1)}$, we have

$$f^k_{a,b}(0)\in I_1\ ,\qquad f^{k+1}_{a,b}(0)\in I_0\ \text{ and }\ f^{k+1}_{a,b}(0)\leqq x_0\ . \tag{20}$$

On the other hand in the case $(a,b)\in D_k^{(2)}$, we have

$$\begin{cases} f^k_{a,b}(0)\in I_1\ , & f^{k+1}_{a,b}(0)\in I_0\ , \qquad x_0<f^{k+1}_{a,b}(0)<f_{a,b}(x_0)\ , \\ f^i_{a,b}(0)\in I_0 \text{ for } k+2\leqq i\leqq 2k-1\ , & f^{2k}_{a,b}(0)\in I_1 \text{ and } f^{2k+1}_{a,b}(0)\geqq f^k_{a,b}(0)\ . \end{cases} \tag{21}$$

THEOREM 2.1. *The fundamental partition of $f_{a,b}$ is a generator of $f_{a,b}$ if and only if $(a,b)\notin\bigcup_{k=2}^{\infty}D_k^{(1)}$.*

PROOF. Let $(a,b)\in D_k^{(1)}$ for some k, then from (20) we obtain that $f^{k+1}_{a,b}[0,x_0]\subset[0,x_0]$ and that any $x\in[0,x_0)$ has the same symbolic representation $\pi_{a,b}(x)=\dot{0}0\cdots0\dot{1}$ with period $k+1$. So $\{I_0,I_1\}$ is not a generator. Let $(a,b)\in D_0\cup D_1$. If $\pi_{a,b}(x)=\pi_{a,b}(x')$ for some $x\neq x'$, then we can show that $|f^{2i}_{a,b}(x)-f^{2i}_{a,b}(x')|\geqq(ab)^i|x-x'|$ for every $i\geqq 0$, which contradicts the inequality $ab>1$. And so $\{I_0,I_1\}$ is a generator in these cases. Next let $(a,b)\in D_k^{(2)}\cup D_k^*$ for some $k\geqq 2$. If $\pi_{a,b}(x)=\pi_{a,b}(x')$ for some $x\neq x'$, then we can show as above that $|f^{(k+1)i}_{a,b}(x)-f^{(k+1)i}_{a,b}(x')|\geqq(a^kb)^i|x-x'|$ for every $i\geqq 0$, which contradicts the inequality $a^kb>1$. So $\{I_0,I_1\}$ is a generator. In the case of D^*, it is clear that $\{I_0,I_1\}$ is a generator.

Now let us investigate the case $D_k^{(1)}\cup D_k^{(2)}$ more precisely. In the remainder of this section we assume that $(a,b)\in D_k^{(1)}\cup D_k^{(2)}$. Let

$$\begin{cases} x^*=\dfrac{a^{k-1}b^2-a^{k-1}b-a^{k-2}b-\cdots-a^2b-ab-b}{a^{k-1}b^2-1}\ , \\ x_*=\dfrac{a^kb-a^k-a^{k-1}-\cdots-a^2-a}{a^kb+1}\ . \end{cases} \tag{22}$$

We can easily show that $x^*>x_0>x_*$ and that x^* and x_* are periodic points of $f_{a,b}$ with period $k+1$ with the following symbolic representations:

$$\pi_{a,b}(x^*)=\dot{0}0\cdots01\dot{1}\ ,\qquad \pi_{a,b}(x_*)=\dot{0}0\cdots00\dot{1}\ . \tag{23}$$

LEMMA 2.1. *Let $C_0=[0,x^*]$, then $f^i_{a,b}C_0$ $(0\leqq i\leqq k)$ are disjoint and $f^{k+1}_{a,b}C_0=C_0$.*

PROOF. From (9) we obtain

$$f_{a,b}^k(0)=\frac{a^k+a^{k-1}+\cdots+a^2+a+b-a^kb}{b}, \tag{24}$$

and by the definition of x^* we obtain

$$f_{a,b}^k(x^*)=1-\frac{a^{k-1}b-a^{k-1}-a^{k-2}-\cdots-a^2-a-1}{a^{k-1}b^2-1}. \tag{25}$$

And so we obtain

$$f_{a,b}^k(0)-f_{a,b}^k(x^*)=\frac{a^{k-2}(b-1-a^{-1}-\cdots-a^{-(k-1)})(a+b-a^kb^2)}{b(a^{k-1}b^2-1)}\geqq 0. \tag{26}$$

If we notice that $f_{a,b}^{k-1}C_0\ni 1-1/b$, then we can show that $f_{a,b}^kC_0=[f_{a,b}^k(x^*), 1]$, which completes the proof.

Let α, β be a pair of real numbers which satisfy $\alpha>1, \beta>0$ and $1/\alpha+1/\beta\leqq 1$. We denote by $g_{\alpha,\beta}$ the map from $[0, 1]$ into itself defined by

$$g_{\alpha,\beta}(x)=\begin{cases}\alpha x & \text{for } 0\leqq x\leqq\dfrac{1}{\alpha}\\ -\beta x+\dfrac{\alpha+\beta}{\alpha} & \text{for } \dfrac{1}{\alpha}\leqq x\leqq 1.\end{cases} \tag{27}$$

Then we have

LEMMA 2.2. (i) *If $\beta<1$, then any orbit of $g_{\alpha,\beta}$ approaches the fixed point $(\alpha+\beta)/\alpha(\beta+1)$ of $g_{\alpha,\beta}$.*

(ii) *If $\beta=1$, then every point of $[1/\alpha, 1]-\{(\alpha+\beta)/\alpha(\beta+1)\}$ is periodic point with period 2 and, for any $x\in(0, 1/\alpha)$, $g_{\alpha,\beta}^n(x)\in[1/\alpha, 1]$ for some n.*

(iii) *If $\beta>1$, then $g_{\alpha,\beta}|_{[(\alpha+\beta-\alpha\beta)/\alpha,1]}$ is linearly conjugate to $f_{\alpha,\beta}$ and, for any $x\in(0, (\alpha+\beta-\alpha\beta)/\alpha)$, $g_{\alpha,\beta}^n(x)\in[(\alpha+\beta-\alpha\beta)/\alpha, 1]$ for some n.*

PROOF. All assertions are clear from the definition of $g_{\alpha,\beta}$.

LEMMA 2.3. *$f_{a,b}^{k+1}|_{C_0}$ is linearly conjugate to $g_{a^{k-1}b^2,a^kb}$.*

PROOF. It is clear if we notice that $f_{a,b}^{k-1}C_0\ni 1-1/b$.

LEMMA 2.4. *Denote by λ the Lebesgue measure on $[0, 1]$. Then we have $\lambda(\bigcup_{n=0}^{\infty} f_{a,b}^{-n}C_0)=1$.*

PROOF. Let

$$C_1=f_{a,b}^{-1}C_0, \qquad C_2=f_{a,b}^{-1}C_1, \qquad C_j=f_{a,b}^{-1}C_{j-1}\cap I_0 \quad \text{for } 3\leqq j\leqq k. \tag{28}$$

We can easily show that these sets are disjoint and

$$(29)\quad \begin{cases} \lambda(C_0)=\dfrac{b(a^{k-1}b-a^{k-1}-a^{k-2}-\cdots-a-1)}{a^{k-1}b^2-1}\,, \\ \lambda(C_1)=\dfrac{1}{b}\lambda(C_0)\,, \qquad \lambda(C_2)=\dfrac{a+b}{ab}\lambda(C_1)\,, \\ \lambda(C_j)=\dfrac{1}{a^{j-2}}\lambda(C_2) \quad \text{for} \quad 3\leqq j\leqq k\,. \end{cases}$$

Let us define intervals $C(a_0, a_1, \cdots, a_n)$ for $n\geqq 0$ and for sequences $(a_0, a_1, \cdots, a_n)$ of 0 and 1 inductively as follows:

$$(30)\quad \begin{aligned} &C(a_0)=f_{a,b}^{-1}\left(\bigcup_{j=2}^{k} C_j\right)\cap I_{a_0}\,, \\ &C(a_0, a_1, \cdots, a_n)=f_{a,b}^{-1}C(a_0, a_1, \cdots, a_{n-1})\cap I_{a_n}\,. \end{aligned}$$

Then we have

$$(31)\quad \bigcup_{n=0}^{\infty} f_{a,b}^{-n}C_0=\bigcup_{n=0}^{\infty}\ \bigcup_{(a_1,a_2,\cdots,a_n)\in\Omega_n^*} C(1, a_1, a_2, \cdots, a_n)\cup\left(\bigcup_{j=0}^{k} C_j\right)$$

where Ω_n^* is the set of all sequences $(a_1, a_2, \cdots, a_n)$ such that each a_i is equal to 0 or 1 and that no more than k 0's appear consecutively. Moreover the sets appearing in the union of the right-hand side of (31) are disjoint. For each $(a_1, a_2, \cdots, a_n)\in\Omega_n^*$,

$$(32)\quad \lambda(C(1, a_1, a_2, \cdots, a_n))=a^{-n(0)}b^{-n(1)-1}\lambda\left(\bigcup_{j=2}^{k} C_j\right)\,,$$

where $n(1)=\sum_{i=1}^{n} a_i$ and $n(0)=n-n(1)$. So it follows that

$$(33)\quad \begin{aligned} &\lambda\left(\bigcup_{n=0}^{\infty}\ \bigcup_{(a_1,a_2,\cdots,a_n)\in\Omega_n^*} C(1, a_1, a_2, \cdots, a_n)\right) \\ &=\sum_{m=1}^{\infty}\ \sum_{\substack{m_0,m_1,\cdots,m_{k-1}\geqq 0\\ m_0+m_1+\cdots+m_{k-1}=m}} \frac{m!}{m_0!\,m_1!\cdots m_{k-1}!}a^{-m_1-2m_2-\cdots-(k-1)m_{k-1}}b^{-m}\lambda\left(\bigcup_{j=2}^{k} C_j\right) \\ &=\sum_{m=1}^{\infty}(1+a^{-1}+a^{-2}+\cdots+a^{-(k-1)})^m b^{-m}\lambda\left(\bigcup_{j=2}^{k} C_j\right)\,. \end{aligned}$$

Using (29) and (30) we obtain

$$(34)\quad \begin{aligned} \lambda\left(\bigcup_{n=0}^{\infty} f_{a,b}^{-n}C_0\right)&=\lambda(C_0)+\lambda(C_1)+\lambda\left(\bigcup_{j=2}^{k} C_j\right)\frac{1}{1-(1+a^{-1}+a^{-2}+\cdots+a^{-(k-1)})b^{-1}} \\ &=1\,. \end{aligned}$$

THEOREM 2.2. *In the case of* $D_k^{(1)}$, *almost all points of* $[0, 1]$ *are* *asymptotically periodic. Especially, in the case $a^k b<1$, almost all (with respect to the Lebesgue measure) orbits approach the periodic orbit starting from x_*.*

PROOF. This theorem follows from Lemmas 2.1, 2.2 ((i) and (ii)), 2.3 and 2.4.

THEOREM 2.3. *In the case of $D_k^{(2)}$, let $J_j=[f_{a,b}^j(0), f_{a,b}^{k+j+1}(0)]$ for $0\leqq j\leqq k-1$ and $J_k=[f_{a,b}^k(0), 1]$. Then we have*

(i) *$J_j\subset f_{a,b}^j C_0$ for $0\leqq j\leqq k$, and so J_j's are disjoint.*

(ii) *$f_{a,b}J_j=J_{j+1}$ for $0\leqq j\leqq k-1$ and $f_{a,b}J_k=J_0$.*

(iii) *$f_{a,b}^{k+1}|_{J_j}$ is linearly conjugate to $f_{a^{k-1}b^2,a^k b}$.*

(iv) *For almost all $x\in[0, 1]-\bigcup_{j=0}^k J_j$, $f_{a,b}^n(x)\in\bigcup_{j=0}^k J_j$ for some n.*

PROOF. (i)~(iii) follow from Lemmas 2.1, 2.2 ((iii)), 2.3 and 2.4. To prove (iv) it is sufficient to show that, for all $x\in(f_{a,b}^{k+1}(0), x^*)$, $f_{a,b}^n(x)\in J_0$ for some n. But this is easy to see if we notice that $|f_{a,b}^{k+1}(x)-x^*|=a^{k-1}b^2|x-x^*|$ and $a^k b>1$.

Next, we give a proposition concerning $(a^{k-1}b^2, a^k b)$.

PROPOSITION 2.1. *Let $(a, b)\in D_k^{(2)}$. If $a+b<a^{2k}b^3$, then $(a^{k-1}b^2, a^k b)\in D_1\cup D^*$. On the other hand, if $a+b\geqq a^{2k}b^3$, then $(a^{k-1}b^2, a^k b)\in D_0^{(1)}$.*

PROOF. By definitions of D_1, D^* and $D_0^{(1)}$, we can easily show that $(a^{k-1}b^2, a^k b)\in D_1\cup D^*$ if and only if $a+b<a^{2k}b^3$ and that $(a^{k-1}b^2, a^k b)\in D_0^{(1)}$ if and only if $a^{2k}b^3\leqq a+b<a^{4k-1}b^6$. But it is clear that $a+b<a^{4k-1}b^6$ follows from $(a, b)\in D_k^{(2)}$, so we have Proposition 2.1.

REMARK. It is evident that $f_{a,b}$ has a periodic point with period 3 in the case $D-(D_0\cup D_1)$. So, Theorem 2.2 shows that $D_k^{(1)}$ is the case of "window" and Theorem 2.3 shows that $D_k^{(2)}$ is the case of "islands".

Finally, we will give a result concerning the topological entropy in the case $D_k^{(1)}\cup D_k^{(2)}$. Let γ_k be the maximal root of the equation $\gamma^k-\gamma^{k-1}-\cdots-\gamma-1=0$. We can easily show that $1<\gamma_k<2$ and γ_k increases to 2 as $k\to\infty$.

THEOREM 2.4 (cf. [2]). *The topological entropy of $f_{a,b}$ is equal to $\log\gamma_k$ for the case of $D_k^{(1)}\cup D_k^{(2)}$.*

PROOF. Denote by $h_{\mathrm{top}}(f_{a,b})$ the topological entropy of $f_{a,b}$ and denote by $N_{a,b}^{(n)}$ the number of $f_{a,b}$-admissible words of length n, that is,

$$N_{a,b}^{(n)}=\#\{(a_0, a_1, \cdots, a_{n-1});\ \pi_{a,b}(x)(i)=a_i \text{ for } 0\leqq i\leqq n-1, \text{ for some } x\}\ . \tag{35}$$

It is well known that $h_{\text{top}}(f_{a,b})=\lim_{n\to\infty}(1/n)\log N_{a,b}^{(n)}$. We can easily show that

$$\begin{aligned}&\pi_{a,b}(0)(i)=0 \quad \text{for} \quad 0\leqq i\leqq k-1\,, \qquad \pi_{a,b}(0)(k)=1\\ &\text{and } \pi_{\gamma_k,\gamma_k}(0)=\dot{0}0\cdots 01\dot{1}=\pi_{a,b}(x^*)\,.\end{aligned} \tag{36}$$

And therefore $X_{a,b}\supseteqq X_{\gamma_k,\gamma_k}$, which implies $h_{\text{top}}(f_{a,b})\geqq h_{\text{top}}(f_{\gamma_k,\gamma_k})=\log\gamma_k$. But it is easy to see by virtue of Lemma 2.1 that

$$X_{a,b}-X_{\gamma_k,\gamma_k}=\{\omega\in X_{a,b};\ \sigma^n\omega=\pi_{a,b}(x) \text{ for some } n \text{ and some } x\in C_0\} \tag{37}$$

and $\pi_{a,b}(x)=00\cdots 0{*}100\cdots 0{*}1\cdots$ for every $x\in C_0$. So we get

$$N_{a\ b}^{(n)}\leqq\sum_{m=0}^{n}N_{\gamma_k\ \gamma_k}^{(n-m)}2^{[m/(k+1)]+1}\leqq C\gamma_k^n\,. \tag{38}$$

The last inequality follows from the inequality $N_{\gamma_k,\gamma_k}^{(n)}\leqq C'\gamma_k^n$, which has been shown in § 4 of part I. So we obtain $h_{\text{top}}(f_{a,b})\leqq\log\gamma_k$, which completes the proof.

§ 3. $f_{a,b}$-expansion and the density of invariant measure.

In this section we consider the case when the fundamental partition is a generator, that is, the case $D-(\bigcup_{k=2}^{\infty}D_k^{(1)})$.

Let us define $N_0(x, n)$ and $N_1(x, n)$ for $x\in[0, 1]$ and $n\geqq 0$ by

$$N_j(x, n)=\begin{cases}1 & \text{if} \quad n=0\\ \#\ \{i;\ 0\leqq i\leqq n-1,\ \omega_{a,b}^x(i)=j\} & \text{if} \quad n\geqq 1\,.\end{cases} \tag{39}$$

Then we have

Lemma 3.1 ($f_{a,b}$-expansion). *If $(a, b)\in D-(\bigcup_{k=2}^{\infty}D_k^{(1)})$, then we have the so-called $f_{a,b}$-expansion for $x\in[0, 1]$ as follows*

$$x=1-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{1}{a}\right)^{N_0(x,n)}\left(-\frac{1}{b}\right)^{N_1(x,n)}, \tag{40}$$

where the sum in the right-hand side converges absolutely.

Proof. Let us define $\varepsilon(j)$ and $\delta(j)$ for $j=0$ or 1 by

$$\varepsilon(j)=\begin{cases}\dfrac{1}{a} & \text{for } j=0\\[2mm] -\dfrac{1}{b} & \text{for } j=1\,,\end{cases} \tag{41}$$

(42)
$$\delta(j)=\begin{cases}1 & \text{for } j=0\\ 0 & \text{for } j=1\,.\end{cases}$$

Then it follows from (1) that

(43)
$$x=\varepsilon(\omega^x_{a,b}(0))f_{a,b}(x)+1-\frac{a+b}{ab}\delta(\omega^x_{a,b}(0))\,.$$

By using (43) successively we obtain, for any natural number N,

(44)
$$x=\sum_{n=0}^{N-1}\left(1-\frac{a+b}{ab}\,\delta(\omega^x_{a,b}(n))\right)\prod_{i=0}^{n-1}\varepsilon(\omega^x_{a,b}(i))+\prod_{i=0}^{N-1}\varepsilon(\omega^x_{a,b}(i))f^{N+1}_{a,b}(x)\,.$$

It is easy to see that

$$-\frac{a+b}{ab}\delta(\omega^x_{a,b}(n))\prod_{i=0}^{n-1}\varepsilon(\omega^x_{a,b}(i))+\prod_{i=0}^{n}\varepsilon(\omega^x_{a,b}(i))$$
$$=-\frac{1}{b}\prod_{i=0}^{n-1}\varepsilon(\omega^x_{a,b}(i))=-\frac{1}{b}\left(\frac{1}{a}\right)^{N_0(x,n)}\left(-\frac{1}{b}\right)^{N_1(x,n)},$$

and so we get (40) by letting N go to infinity in (44). The absolute convergence is proved as follows. In the case $D^{(2)}_k\cup D^*_k$, $\pi_{a,b}(x)$ has no consecutive 0's of length longer than k for any $x\in[0,1]$; so by using the inequality $a^kb>1$ we obtain the absolute convergence. We can show this in the same manner in the case $D_0\cup D_1\cup D^*$.

Define a function $h_{a,b}(x)$ on $[0,1]$ by

(45)
$$h_{a,b}(x)=\sum_{n=0}^{\infty}\left(\frac{1}{a}\right)^{N_0(0,n)}\left(-\frac{1}{b}\right)^{N_1(0,n)}I_{[f^n_{a,b}(0),1]}(x)\,.$$

By the absolute convergence of (40), we see that $h_{a,b}(x)$ is a function of bounded variation. Now let us prove that $h_{a,b}$ is the density of an invariant measure for $f_{a,b}$.

LEMMA 3.2. *For any Borel set $A\subset[0,1]$, we have*

(46)
$$\int_A h_{a,b}(x)dx=\int_{f^{-1}_{a,b}A}h_{a,b}(x)dx\,.$$

PROOF. It is enough to show that

(47)
$$h_{a,b}(x)=\frac{1}{a}\,h_{a,b}\left(\frac{1}{a}x-\frac{a+b-ab}{ab}\right)I_{[(a+b-ab)/ab,1]}(x)+\frac{1}{b}\,h_{a,b}\left(-\frac{1}{b}x+1\right).$$

We can show (47) in the same manner as for the proof of Theorem 2.1 in part I.

To prove $h_{a,b}(x)\geqq 0$, we prepare several lemmas as follows:

LEMMA 3.3 (*Li-Yorke* [5]). *Let an integrable function* $h(x)$ *on* $[0, 1]$ *satisfy* (46). *Denote by* $P(N, Z)$ *the set of* $x\in[0, 1]$ *which satisfies* $h(x)>0(<0, =0$, *respectively*). *Then we have that*

$$f_{a,b}P=P \text{ a.e. } \textit{and } f_{a,b}N=N \text{ a.e.,} \tag{48}$$

where a.e. *means almost everywhere with respect to the Lebesgue measure.*

PROOF. To simplify the notation, we write f for $f_{a,b}$ in this proof. From the assumption we have

$$\begin{aligned}\int_P h(x)dx&=\int_{f^{-1}P} h(x)dx\\ &=\int_{f^{-1}P\cap P} h(x)dx+\int_{f^{-1}P\cap N} h(x)dx+\int_{f^{-1}P\cap Z} h(x)dx\\ &\leqq\int_{f^{-1}P\cap P} h(x)dx\leqq\int_P h(x)dx\ .\end{aligned} \tag{49}$$

So we obtain that

$$f^{-1}P\supset P \text{ a.e. and } f^{-1}P\cap N=\varnothing \text{ a.e.,} \tag{50}$$

which imply that

$$fP\subset P\subset f^{-1}(fP)\subset f^{-1}P\ . \tag{51}$$

From (46), (50) and (51) it is easy to see that

$$\begin{aligned}0&=\int_{f^{-1}(fP)-fP} h(x)dx\\ &=\int_{f^{-1}(fP)-P} h(x)dx+\int_{P-fP} h(x)dx\\ &=\int_{P-fP} h(x)dx\ ,\end{aligned} \tag{52}$$

so we obtain that $fP=P$ a.e. The assertion $fN=N$ a.e. can be proved in the same manner.

LEMMA 3.4. *Let* $h(x)$ *satisfy the same assumption as in Lemma* 3.4 *and let a Borel set* $B\subset[0, 1]$ *satisfy, for some* n_0,

$$f^n_{a,b}B\cap B=\varnothing \text{ a.e. } \textit{for every } n\geqq n_0\ . \tag{53}$$

Then we have that

$$h(x)=0 \quad \text{a.e.} \quad x\in B\,. \tag{54}$$

PROOF. Let $B_p=\{x\in B;\, h(x)>0\}$ and let $B_p^*=\bigcup_{n=n_0}^{\infty} f_{a,b}^n B_p$. Then it is easy to show that

$$B_p^*\cap B_p=\varnothing \quad \text{a.e. and} \quad f_{a,b}^{-n_0}B_p^*\supset B_p^*\cup B_p\,. \tag{55}$$

Using (55) and the assumption of lemma, we obtain that

$$\begin{aligned}\int_{B_p^*} h(x)dx &= \int_{f_{a,b}^{-n_0}B_p^*} h(x)dx \\ &\geqq \int_{B_p^*} h(x)dx+\int_{B_p} h(x)dx\,,\end{aligned} \tag{56}$$

which implies $\int_{B_p} h(x)dx=0$, and so we obtain $B_P=\varnothing$ a.e. We can show that $B_n=\{x\in B;\, h(x)<0\}=\varnothing$ a.e. in the same manner.

LEMMA 3.5. *Let* $(a, b)\in D_1\cup D^*\cup(\bigcup_{k=2}^{\infty} D_k^*)$. *For every interval* $I\subset[0, 1]$ *with positive length, there exists an* n *which satisfies*

$$f_{a,b}^n I=[0, 1]\,. \tag{57}$$

PROOF. It is sufficient to prove that $f_{a,b}^m I\ni b/(b+1)$ for some m, since it is easy to see that $f_{a,b}^n I=[0, 1]$ for some $n\geqq m$ in this case. We can easily show that if, for some interval J, $1-1/b\in J$ then

$$|f_{a,b}J|\geqq\frac{ab}{a+b}|J|\,, \tag{58}$$

where $|\ |$ denote the length of interval.

In the case D_1, we have that

$$|f_{a,b}^2 I|\geqq\min\left\{\frac{ab^2}{a+b},\, ab,\, b^2\right\}|I|=\frac{ab^2}{a+b}|I| \tag{59}$$

except in the case when

$$I\cap f_{a,b}I\ni 1-\frac{1}{b} \quad \text{or} \quad f_{a,b}I\cap f_{a,b}^2 I\ni 1-\frac{1}{b} \tag{60}$$

is satisfied. Using (59) repeatedly we get the desired conclusion if we notice that $ab^2/(a+b)>1$. (Note that $b/(b+1)>(a+b-ab)/b$.) In the case of (60), it is easy to see that $f_{a,b}^2 I\ni b/(b+1)$.

In the case D_k^*, we have that

$$(61)\qquad |f_{a,b}^{k+1}I| \geqq \min\left\{\frac{a^k b^2}{a+b}, a^k b, a^{k-1}b^2, \cdots, ab^k, b^{k+1}\right\}|I| = \frac{a^k b^2}{a+b}|I| ,$$

if at most one interval among $I, f_{a,b}I, \cdots, f_{a,b}^k I$ contains $1-1/b$. If $f_{a,b}^m I$ and $f_{a,b}^{m+i} I$ contain $1-1/b$ for some $0 \leqq m < m+i \leqq k$, then we can show that $f_{a,b}^{m+i+1} I \ni b/(b+1)$. Using (61) repeatedly we get the desired conclusion if we notice that $a^k b^2/(a+b) > 1$ in the case D_k^*.

In the case D^*, we can prove the lemma in the same manner.

THEOREM 3.1. *Let* $(a, b) \in D_1 \cup (\bigcup_{k=2}^{\infty} D_k^*) \cup D^*$. *Then* $h_{a,b}$ *is the density function of an invariant measure for* $f_{a,b}$ *and* $h_{a,b}(x) > 0$ a.e. $x \in [0, 1]$.

PROOF. From Lemmas 3.2 and 3.5, it is sufficient to prove that $h_{a,b}(x) > 0$ on $[0, \varepsilon]$ for some $\varepsilon > 0$. By the definition of $h_{a,b}$, we have $h_{a,b}(0) > 0$. In the case when 0 is periodic for $f_{a,b}$, we have $h_{a,b}(x) = h_{a,b}(0)$ on $[0, \varepsilon]$ for sufficiently small ε. Otherwise, let $h_{a,b}(0) = s$. By Lemma 3.1, we have that, for some n_0,

$$(62)\qquad \sum_{n=n_0}^{\infty} \left(\frac{1}{a}\right)^{N_0(0,n)} \left(\frac{1}{b}\right)^{N_1(0,n)} < \frac{s}{2} .$$

So if we pick a positive ε satisfying $\varepsilon < f_{a,b}^n(0)$ for $1 \leqq n < n_0$, we can show $h_{a,b}(x) > s/2$ on $[0, \varepsilon]$.

THEOREM 3.2. *Let* $(a, b) \in D_1 \cup (\bigcup_{k=2}^{\infty} D_k^*) \cup D^*$. *Then the dynamical system* $(f_{a,b}, h_{a,b}(x)dx)$ *is weak Bernoulli.*

PROOF. Using Lemma 3.5, it is easy to see that $f_{a,b}^2$(resp. $f_{a,b}^k, f_{a,b}$) satisfies the condition of Bowen [1] in the case of D_1(resp. D_k^*, D^*). So we can apply the result of Bowen to get the desired conclusion.

Now let us investigate the support of $h_{a,b}$ in the case D_0. Let $(a, b) \in D_0^{(m)}$ for some $m \geqq 1$ and denote by A_i for $0 \leqq i \leqq 2^m - 1$ the intervals defined by

$$(63)\qquad A_i = \begin{cases} [f_{a,b}^{2^m+i}(1), f_{a,b}^i(1)] & \text{if } N_1(1, i) \text{ is even} \\ [f_{a,b}^i(1), f_{a,b}^{2^m+i}(1)] & \text{if } N_1(1, i) \text{ is odd} . \end{cases}$$

As in part I, we can show that A_i's are disjoint and that

$$(64)\qquad f_{a,b}A_i = A_{i+1} \text{ for } 0 \leqq i \leqq 2^m - 2 , \quad f_{a,b}A_{2^m-1} = A_0 .$$

COROLLARY 3.1. *Let* $(a, b) \in D_0^{(m)}$ *for some* $m \geqq 1$. *Then*

(i) $h_{a,b}$ *is the density function of an invariant measure for* $f_{a,b}$ *and the support of* $h_{a,b}$ *is equal to* $\bigcup_{i=0}^{2m-1} A_i$.

(ii) *The dynamical system* $(f_{a,b}, h_{a,b}(x)dx)$ *is ergodic but not weakly mixing.*

PROOF. This corollary follows from Theorems 1.2, 3.1, 3.2, Lemmas 3.2, 3.3, 3.4 and (64).

COROLLARY 3.2. *Let* $(a, b) \in D_k^{(2)}$ *for some* $k \geqq 2$. *Then* $h_{a,b}$ *is the density function of an invariant measure for* $f_{a,b}$ *and*

(i) *if* $a+b<a^{2k}b^3$, *then the support of* $h_{a,b}$ *is equal to* $\bigcup_{i=0}^{k} J_i$, *where* J_i *is defined in Theorem* 2.3.

(ii) *If* $a+b \geqq a^{2k}b^3$, *then the support of* $h_{a,b}$ *is equal to* $\bigcup_{i=0}^{k} (J_{i,1} \cup J_{i,2})$ *for some sub-intervals* $J_{i,1}$ *and* $J_{i,2}$ *of* $J_i (0 \leqq i \leqq k)$ *which satisfy*

$$f_{a,b}^{k+1} J_{i,1} = J_{i,2} \quad \text{and} \quad f_{a,b}^{k+1} J_{i,2} = J_{i,1} . \tag{65}$$

And the dynamical system $(f_{a,b}, h_{a,b}(x)dx)$ *is ergodic but not weakly mixing.*

PROOF. This corollary follows from Theorems 2.3, 3.1, 3.2, Lemmas 3.2, 3.3 and 3.4.

References

[1] R. BOWEN, Bernoulli maps of the intervals, Israel J. Math., **28** (1977), 161-168.

[2] L. JONKER and D. A. RAND, A Lower Bound for the Entropy of Certain Maps of the Unit Interval, preprint.

[3] A. LASOTA and J. A. YORKE, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc., **186** (1973), 481-488.

[4] T. Y. LI and J. A. YORKE, Period three implies chaos, Amer. Math. Monthly, **82** (1975), 895-922.

[5] T. Y. LI and J. A. YORKE, Ergodic transformations from an interval into itself, Trans. Amer. Math. Soc., **235** (1978), 183-192.

[6] R. M. MAY, Simple mathematical models with very complicated dynamics, Nature, **261** (1976), 459-567.

[7] R. M. MAY and G. F. OSTER, Bifurcations and dynamic complexity in simple ecological models, Amer. Naturalist, **110** (1976), 573-599.

[8] P. STEFAN, A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the line, Comm. Math. Phys. **54** (1977), 237-248.

[9] G. WAGNER, The ergodic behavior of piecewise monotonic transformations, Z. Wahrscheinlichkeitstheorie und Verw Gebiete, **46** (1979), 317-324.

Present Address:
DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
TSUDA-MACHI, KODAIRA

TOKYO 187
and
DEPARTMENT OF MATHEMATICS
KEIO UNIVERSITY
HIYOSHI-CHO, KOHOKU-KU,
YOKOHAMA 223