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The Microlocal Structure of Weighted Homogeneous Polynomials Associated with Coxeter Systems I

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Introduction

Let E be an l-dimensional Euclidean space with an orthonormal basis $\{e_i\}$ and E^* its dual with the dual basis $\{\xi_i\}$. Let, further, W be a finite group of $\mathrm{GL}(E)$ generated by reflections. Such a group is completely classified and forms a Coxeter system (W, S) for an appropriate set S of generators [1]. Let R be the subalgebra of the symmetric algebra $S(E^*)$ whose elements are invariant under the action of W. As is known, there exist algebraically independent homogeneous elements $x_1, \cdots, x_l$ of R such that $R=\boldsymbol{R}[x_1, \cdots, x_l]$. Let $D(\xi)$ be the product of linear functions defining the hyperplanes of reflections of W. Then $D(\xi)^2$ is represented as a polynomial of $x_1, \cdots, x_l$. We denote it by $f_W(x)$ and call it the generalized discriminant in this paper.

Let us consider the space $X=(E^*/W)^C$, the complexification of the quotient space of E^* by W, whose coordinate ring is $\boldsymbol{C}\otimes R$. Then

$$m_{ij}(x)=\frac{1}{2}\sum_{k=1}^{l}\frac{\partial x_i}{\partial \xi_k}\frac{\partial x_j}{\partial \xi_k} \qquad (1\leqq i,\ j\leqq l)$$

belong to R and the vector fields

$$X_i=\sum_{j=1}^{l} m_{ij}(x)\frac{\partial}{\partial x_j} \qquad (1\leqq i\leqq l)$$

leave $f(x)=f_W(x)$ invariant. More precisely, we have

$$X_i f(x)=c_i(x)f(x)$$

with certain polynomials $c_i(x)\in R$. Furthermore, $X_1, \cdots, X_l$ form a free basis of the Lie algebra of vector fields leaving the set $\{x; f(x)=0\}$ invariant ([7]).

In this paper, we shall study the microlocal structure of the $\mathscr{D}_X$-Module

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$$\mathscr{L}_\alpha=\mathscr{D}_X\Big/\sum_{i=1}^{l}\mathscr{D}_X(X_i-\alpha c_i(x)) \qquad (\alpha\in \boldsymbol{C}),$$

where $\mathscr{D}_X$ is the sheaf of differential operators of finite order whose coefficients are in $\mathscr{O}_X$. Our main result is stated in Theorem 4.1, which gives enough information concerning the microlocal structure of $\mathscr{L}_\alpha$ in terms of the Coxeter systems.

The main reason why we study $\mathscr{L}_\alpha$ stems from the following conjecture:

$$\mathscr{L}_\alpha=\mathscr{D}_X(f(x))^\alpha$$

for any $\alpha\in \boldsymbol{C}$ satisfying $b_f(\alpha-n)\neq 0$ $(n=0, 1, 2, \cdots)$. Here $b_f(s)$ is the b-function of $f(x)$.

The present paper is organized as follows. We first summarize widely known facts concerning the Coxeter systems in Section 1. In Section 2, we introduce a symmetric matrix $M(W)$ whose entries are contained in $\boldsymbol{R}$. The rank of $M(W)$ is connected with the rank of the map of $(E^*)^c$ to X (cf. Proposition 2.1). We define the $\mathscr{D}_X$-Module $\mathscr{L}_\alpha$ and prove a connection of the set

$$\Lambda=\{(x, \eta)\in T^*X;\ \eta\cdot M(W)(x)=0\}$$

with the conjugate classes of certain class of subgroups of W in Section 3. Section 4 is devoted to the proof of Theorem 4.1 which forms the main assertion. We propose two conjectures in Section 5 and in Section 6 we give one example for the general theory developed in the preceding sections.

A concrete treatment for each irreducible Coxeter system will be given elsewhere.

We are grateful to Professor K. Saito: Inspired by his lecture, one of us (T. Y.) was led to prove the simpleness of the module $\mathscr{L}_\alpha$ for some examples of Coxeter systems. We are also indebted to Professor M. Sato, whose formulation of a general treatment for the Coxeter system of type A_l and whose proof of Theorem 4.1 for this case (unpublished) are very useful for us to find the unified treatment for all Coxeter systems.

§1. Coxeter system.

To define a Coxeter system (W_l, S) we introduce a group W_l with a set of generators S to be defined by the fundamental relations

(1.1) $$\text{“}(s_is_j)^{m_{ij}}=1, \quad m_{ii}=1, \quad m_{ij}\geqq 2 \quad \text{if} \quad i\neq j$$
$$(m_{ij}=\infty \text{ is permitted}).\text{”}$$

For a basis $e_1, \cdots, e_l$ of $\boldsymbol{R}^l$, the mapping σ_i of $\boldsymbol{R}^l$ to $\boldsymbol{R}^l$ is defined by $\sigma_i(e_j)=e_j+2(\cos(\pi/m_{ij}))e_i$. Then the representation $\sigma\colon W_l\to GL(\boldsymbol{R}^l)$ defined by $s_i\mapsto\sigma_i$ is injective, and we can identify W_l with $\sigma(W_l)$.

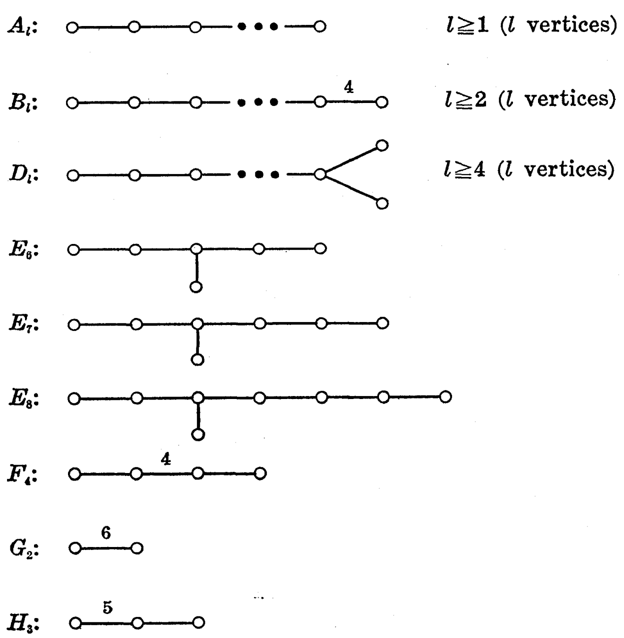
The *graph* of (W_l, S) consists of l vertices $\underset{i}{\circ}$ $(1\leqq i\leqq l)$ and segments with the number m_{ij} $\underset{i}{\circ}\overset{m_{ij}}{\text{———}}\underset{j}{\circ}$ each of which joins vertices $\underset{i}{\circ}$ and $\underset{j}{\circ}$ only when $m_{ij}\geqq 3$, and "m_{ij}" is omitted if $m_{ij}=3$. Remark that $m_{ij}=m_{ji}$.

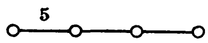
A Coxeter system (W_l, S) is called *irreducible* when its graph is connected. As is known, W_l is a finite group if and only if the matrix $(-\cos(\pi/m_{ij}))_{ij}$ is positive-definite. Hereafter, we consider only finite Coxeter groups. When this is the case, $\sigma(W_l)$ is a subgroup of a real orthogonal group $\boldsymbol{O}(l)$ if we introduce an inner product $\langle,\rangle$ in $\boldsymbol{R}^l$ by $\langle e_i, e_j\rangle=-\cos(\pi/m_{ij})$. We will identify W_l with the corresponding finite subgroup of $\boldsymbol{O}(l)$ and also σ_i with s_i.

Any Coxeter system (W_l, S) is decomposed into irreducible ones, as follows. There are irreducible Coxeter systems (W_{l_i}, S_i) $i=1, \cdots, n$: (W_{l_i}, S_i) consists of a subgroup W_{l_i} of W_l and a subset S_i of S, with $W_l=\prod_{i=1}^n W_{l_i}$ and $S=\bigcup_{i=1}^n S_i$. Furthermore $\boldsymbol{R}^l=\bigoplus_{i=1}^n E_i$, where E_i is the representation space of W_{l_i}.

The following theorems are fundamental (cf. Bourbaki [1]).

THEOREM 1.1. *Any irreducible finite Coxeter system* (W_l, S) *is isomorphic to one of the following Coxeter systems*:



H_4: 

$I_2(m)$: 

($m=5$ or $m\geqq 7$) .

We call $c=s_1\cdots s_l$ a *Coxeter transformation,* and the order of c, which we denote by h, the *Coxeter number.* When $\det(T-c)=\prod_{j=1}^{l}(T-\exp(2\pi\sqrt{-1}m_j/h))$, $0\leqq m_1\leqq\cdots\leqq m_l\leqq h$, we call $(m_1, \cdots, m_l)$ the exponents of W_l. It is known that $\sum_{j=1}^{l}m_j=lh/2$ and that $1=m_1<m_2\leqq\cdots\leqq m_{l-1}<m_l=h-1$.

Let $e_1, \cdots, e_l$ be an orthonormal basis of E. If we define linear forms ξ_i on E by $E\ni\sum_{i=1}^{l}a_ie_i\mapsto a_i\in \boldsymbol{R}$, we can identify the dual E^* of E with $\bigoplus_{i=1}^{l}\boldsymbol{R}\xi_i$. Let $S(E^*)$ denote the symmetric algebra of E^* and $R=S(E^*)^{W_l}$ the subalgebra of $S(E^*)$ whose elements are invariant under the operation of W_l. We may identify $S(E^*)$ with $\boldsymbol{R}[\xi_1, \cdots, \xi_l]$. Furthermore, let $\boldsymbol{H}$ be the set of reflections contained in W_l.

THEOREM 1.2. (1) *$S(E^*)$ is a graded free R-module of* rank $\#W_l$.
(2) *There are homogeneous elements $x_1, \cdots, x_l$ in R such that*

$$R\simeq\boldsymbol{R}[x_1, \cdots, x_l] \quad and \quad k_j=\deg_\xi x_j=m_j+1 .$$

(3) $$\prod_{j=1}^{l}k_j=\#W_l\,, \qquad \#\boldsymbol{H}=\sum_{j=1}^{l}m_j=\frac{1}{2}lh\,.$$

(4) *An element z of $S(E^*)$ is called an anti-invariant if $s(z)=-z$ for any $s\in\boldsymbol{H}$. Put $D(\xi)=\prod_{s\in\boldsymbol{H}}\phi_s(\xi)$, where $\phi_s(\xi)$ denotes a defining function of the hyperplane fixed by s. Then the set of anti-invariants equals $R\cdot D$. (D is called a fundamental anti-invariant.)*

(5) $$\det\left(\frac{\partial x_i}{\partial\xi_j}\right)=\lambda D\,, \qquad \lambda\in\boldsymbol{R}^\times\,.$$

It follows from (4) that $f_{W_l}(x)=D^2$ is invariant by W_l. The statements (2) and (3) show that $f_{W_l}(x)$ is weighted homogeneous of type $(lh; k_1, \cdots, k_l)$. We call f_{W_l} the *generalized discriminant* of W_l in this paper.

Hereafter we put $x_1=\xi_1^2+\cdots+\xi_l^2$ in view of the fact that $k_1=2$, $3\leqq k_2$, and $W_l\subset \boldsymbol{O}(E^*)$.

For later convenience we prepare some notation. Let (W_l, S) be a finite irreducible Coxeter system. We define $\alpha_1, \cdots, \alpha_l$ as follows:

(i) When W_l is a Weyl group of a root system, $\{\alpha_1, \cdots, \alpha_l\}$ is a complete system of positive simple roots corresponding to S.

(ii) When W_l is H_3, H_4 or $I_2(m)$, $\{\alpha_1, \cdots, \alpha_l\}$ is a set of unit vectors such that

$$s_i\alpha_i = -\alpha_i \qquad \text{for} \quad s_i \in S \quad (1 \leqq i \leqq l)\,,$$

and that

$$\langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{m_{ij}}\,.$$

We put

(1.2) $$\alpha(W_l) = \{\alpha_1, \cdots, \alpha_l\}\,.$$

(iii) For a general Coxeter system, we define

(1.3) $$\alpha(W_l) = \bigcup_{i=1}^{m} \alpha(W_i)\,,$$

where $(W_l, S) = \prod_{i=1}^{m} (W_i, S_i)$ is the decomposition into irreducible components.

We define the matrix $P(W_l)$ by

(1.4) $$P(W_l) = (\langle \alpha_i, \alpha_j \rangle)_{ij} \qquad \text{for} \quad \alpha(W_l) = \{\alpha_1, \cdots, \alpha_l\}\,.$$

Then $P(W_l)$ is obviously positive-definite and symmetric, and

(1.5) $$P(W_l) = {}^tQ \cdot Q\,.$$

Here Q is the matrix of the coordinate transformation

(1.6) $$(\alpha_1, \cdots, \alpha_l) = (e_1, \cdots, e_l)Q\,.$$

We denote by H_s the hyperplane fixed by a reflection s, and put

(1.7) $$\mathfrak{H}(W_l) = \{H_s;\ s \text{ is a reflection in } W_l\}\,.$$

We normalize the defining function $\phi_{H_s}(\xi)$ of H_s as follows:

(i) If $\alpha_i = \sum_{j=1}^{l} a_{ij}e_j$, we set

(1.8) $$\phi_i(\xi) = \phi_{H_i}(\xi) = \sum_{j=1}^{l} a_{ij}\xi_j\,.$$

(ii) For a general H_s with $s = \sum b_j\alpha_j$,[1] we set

[1] We identify $\sum b_j\alpha_j$ with the reflection s with respect to the hyperplane orthogonal to $\sum b_j\alpha_j$. Since all coefficients b_j can be taken non-negative (or non-positive) simultaneously, we assume $b_j \geqq 0$ in (1.9).

(1.9) $$\phi_{H_{s_i}}(\xi)=\sum b_j\phi_j(\xi)\ .$$

Here we have written H_i for H_{s_i} $(i=1, \cdots, l)$.

The subset $C=\{\xi\in \boldsymbol{R}^l;\ \phi_{H_i}(\xi)>0,\ 1\leqq i\leqq l\}$ (or $\bar{C}=\{\xi\in \boldsymbol{R}^l;\ \phi_{H_i}(\xi)\geqq 0,\ 1\leqq i\leqq l\}$) is called the open chamber (or the closed chamber) determined by S.

§2. A property of the matrix $M(W_l)$.

In this section, we fix a Coxeter system (W_l, S). Let $x_1, \cdots, x_l$ be the set of fundamental invariants, and let ∇ be the gradient with respect to ξ. The standard inner product $\nabla x_i\cdot\nabla x_j$ is W_l-invariant because of $W_l\subset O(E^*)$. We define the symmetric matrix

(2.1) $$\begin{aligned} M(W_l)&=\left(\frac{1}{2}\nabla x_i\cdot\nabla x_j\right)_{i,j} \\ &=\frac{1}{2}\left[\left(\frac{\partial x_i}{\partial \xi_j}\right)_{i,j}\right]\cdot{}^t\!\left[\left(\frac{\partial x_{i'}}{\partial \xi_{j'}}\right)_{i',j'}\right]. \end{aligned}$$

Then it follows from Theorem 1.2, (5) that

(2.2) $$\det M(W_l)=\frac{\lambda^2}{2^l}f_{W_l}(x)\ .$$

We denote by V and V^* the complexifications of E and E^*, respectively. We also denote by $p(\xi_1, \cdots, \xi_l)=(x_1(\xi), \cdots, x_l(\xi))$ the canonical map of V^* to the quotient space $X=\{(x_1, \cdots, x_l)\in \boldsymbol{C}^l\}$. Outside the set $f_{W_l}^{-1}(0)$, this map is obviously a $\#W_l$-tuple covering.

The purpose of this section is to investigate the rank of $M(W_l)$ on the set $f_{W_l}^{-1}(0)$. In particular, the characterization of it in terms of the Coxeter system is given.

Once for all, let $\mathscr{A}$ be the set of all affine supports of facets (as to the definition of facets, see Bourbaki [1] Chapter V, §1) and let $\mathscr{A}_S$ be the set of all affine supports of facets that belong to the closed chamber $\bar{C}$ defined in §1. An element of $\mathscr{A}_S$ which is given in the form $H_{i_1}\cap\cdots\cap H_{i_k}$ is denoted by $\mathscr{A}(s_{i_1}, \cdots, s_{i_k})$ or $\mathscr{A}(S')$ with $S'=\{s_{i_1}, \cdots, s_{i_k}\}\subset S$. We denote by $\mathring{\mathscr{A}}(S')$ the set of points that belong to the highest dimensional facets of $\mathscr{A}(S')$. According to Bourbaki [1], W_l acts on $\mathscr{A}$ such that $\mathscr{A}=\bigcup_{w\in W_l} w\mathscr{A}_S$. We now define an equivalence relation $\sim$ on $\mathscr{A}_S$ by

(2.3) $$\mathscr{A}(S')\sim\mathscr{A}(S'')\ \text{ if and only if }\ \mathscr{A}(S'')=w\mathscr{A}(S')\quad \text{for some}\ \ w\in W_l\ .$$

Under this equivalence relation we denote by $\overline{\mathscr{A}_S}$ the set of equivalence classes $\mathscr{A}_S/\sim$.

Then we have the following.

PROPOSITION 2.1. *For any subset S' of S and any point x_0 of $p(\mathring{\mathscr{A}}(S'))$, we have*

$$\operatorname{rank}[M(W_l)|_{x=x_0}]=l-(\# S') . \tag{2.4}$$

In order to prove this proposition we prepare two lemmata. We make constant use of the notation (1.8) and (1.9).

LEMMA 2.2. *For any $H\in\mathfrak{H}(W_l)$ and x_i $(1\leqq i\leqq l)$, there is a polynomial $P_{H,i}(\xi)$ such that*

$$\phi_H\Big(\frac{\partial}{\partial\xi}\Big)(x_i(\xi))=\phi_H(\xi)P_{H,i}(\xi) . \tag{2.5}$$

PROOF. Let s be the reflection that fixes H. Put $P(\xi)=\phi_H(\partial/\partial\xi)(x_i(\xi))$. Then we have

$$\begin{aligned} P(s\xi)&=\phi_H\Big(\frac{\partial}{\partial(s\xi)}\Big)(x_i(s\xi))\\ &=-\phi_H\Big(\frac{\partial}{\partial\xi}\Big)(x_i(\xi))\\ &=-P(\xi) . \end{aligned}$$

Therefore, $P(\xi)$ vanishes on H and hence is divided by $\phi_H(\xi)$. Q.E.D.

We now put

$$J(\xi)=\Big(\phi_j\Big(\frac{\partial}{\partial\xi}\Big)(x_i(\xi))\Big)_{ij} . \tag{2.6}$$

Then it follows from (1.6) that

$$J(\xi)=\Big(\frac{\partial x_i}{\partial\xi_j}\Big)\cdot Q . \tag{2.7}$$

LEMMA 2.3. *Let ξ_0 belong to $\mathring{\mathscr{A}}(S')$. Then*

$$\operatorname{rank} J(\xi_0)=l-(\# S') . \tag{2.8}$$

PROOF. We may assume that $S'=\{s_1,\cdots,s_k\}$ $(k=\# S')$ without loss of generality. Let W' denote the subgroup of W_l generated by S'. First, we investigate the left k column vectors

$$\phi_j\left(\frac{\partial}{\partial\xi}\right)(x(\xi))={}^t\left(\phi_j\left(\frac{\partial}{\partial\xi}\right)(x_1(\xi)),\ \cdots,\ \phi_j\left(\frac{\partial}{\partial\xi}\right)(x_l(\xi))\right).$$

We can divide these vectors by $\phi_j(\xi)$ owing to Lemma 2.2. The defining function $\phi_H(\xi)$ corresponding to any H in $\mathfrak{H}(W')$ can be represented by a linear combination of $\phi_1(\xi), \cdots, \phi_k(\xi)$. Therefore $\phi_H(\xi)$ divides a certain non-trivial linear combination of $\phi_1(\partial/\partial\xi)(x(\xi)), \cdots, \phi_k(\partial/\partial\xi)(x(\xi))$. Noting this fact, every $k\times k$ minor of $(\phi_1(\partial/\partial\xi)(x(\xi)), \cdots, \phi_k(\partial/\partial\xi)(x(\xi)))$ vanishes on all H in $\mathfrak{H}(W')$. From Theorem 1.2 (5), (1.6) and (1.8) it follows that

$$\det\left(\phi_i\left(\frac{\partial}{\partial\xi}\right)(x_j(\xi))\right)=\lambda\cdot(\det Q)\prod_{H\in\mathfrak{H}(W_l)}\phi_H(\xi).$$

Therefore, by using Laplace' expansion, some linear combination of $(l-k)\times(l-k)$-minors of $(\phi_{k+1}(\partial/\partial\xi)(x(\xi)), \cdots, \phi_l(\partial/\partial\xi)(x(\xi)))$ becomes $\prod_{H\notin\mathfrak{H}(W')}\phi_H(\xi)$. On the other hand, if $\xi\in\mathring{\mathscr{A}}(S')$, this function does not vanish and (2.8) follows. Q.E.D.

PROOF OF PROPOSITION 2.1. Due to Lemma 2.3, we have only to prove that

$$\operatorname{rank} M(W_l)(\xi_0)=\operatorname{rank} J(\xi_0). \tag{2.9}$$

We also assume $S'=\{s_1, \cdots, s_k\}$ as in the proof of Lemma 2.3. Then the proof of Lemma 2.3 shows that there is an invertible matrix B such that

$$J(\xi_0)=B\begin{pmatrix}0 & 0\\ 0 & I_{l-k}\end{pmatrix}. \tag{2.10}$$

On the other hand, from the definition of $M(W_l)$ and $J(\xi)$, we have

$$M(W_l)(\xi)=\frac{1}{2}J(\xi)\cdot P(W_l)^{-1}\cdot J(\xi). \tag{2.11}$$

We define matrices B', B'' and P' by the formulae

$$\begin{pmatrix}B'\\ B''\end{pmatrix}=B\begin{pmatrix}0\\ I_{l-k}\end{pmatrix}$$

$$P'=(0,\ I_{l-k})\cdot P^{-1}\cdot\begin{pmatrix}0\\ I_{l-k}\end{pmatrix}.$$

Then from (2.10) and (2.11), we have

$$\begin{aligned}2M(W_l)(\xi_0)&=\begin{pmatrix}0 & B'P'\\ 0 & B''P'\end{pmatrix}{}^tB\\ &=\begin{pmatrix}0 & B'\\ 0 & B''\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & P'\end{pmatrix}{}^tB\\ &=J(\xi_0)\begin{pmatrix}1 & 0\\ 0 & P'\end{pmatrix}{}^tB\,.\end{aligned}$$

Since $P(W_l)$ is positive-definite, so is $P(W)^{-1}$ and hence so is P'. Then the assertion (2.9) follows from the invertibility of the matrix $\begin{pmatrix}1 & 0\\ 0 & P'\end{pmatrix}{}^tB$. Q.E.D.

REMARK. The above proof also implies that for $\xi_0 \in \mathring{\mathscr{A}}(S')$

$$\text{rank}\left(\frac{\partial x_i}{\partial \xi_j}\right)\Big|_{\xi=\xi_0}=l-(\# S')\,. \tag{2.12}$$

§3. The singular support of $\mathscr{N}'_\alpha$.

We shall interpret Proposition 2.1 from the analytic viewpoint. In the first place, we review some standard notation and somewhat well-known facts concerning the theory of differential equations, for details see [9], [3].

Let X be a complex manifold of dimension n and let T^*X be the cotangent bundle over X. We denote by π the natural projection of T^*X to X. Let $\mathscr{O}_X$ (resp. $\mathscr{D}_X$) be the sheaf of germs of holomorphic functions on X (resp. the sheaf of differential operators of finite order with coefficients in $\mathscr{O}_X$). We interpret the system of differential equations as a coherent $\mathscr{D}_X$-Module. For a coherent ideal $\mathscr{J}$ of $\mathscr{D}_X$, the singular support of the system $\mathscr{L}=\mathscr{D}_X/\mathscr{J}$ is, by definition, the analytic set $\{(x, \eta)\in T^*X;\ \sigma(P(x, D))(x, \eta)=0$ for all $P(x, D)\in \mathscr{J}\}$, and it is usually denoted by $\check{S}S(\mathscr{L})$. $\check{S}S(\mathscr{L})$ is known to be involutory and $\text{codim}_{T^*X}\check{S}S(\mathscr{L})\leqq n$ for $\mathscr{L}\neq 0$. A system $\mathscr{L}$ is called *holonomic* (or *sub-holonomic*) when $\text{codim}\,\check{S}S(\mathscr{L})\geqq n$ (or $\text{codim}\,\check{S}S(\mathscr{L})\geqq n-1$). An involutory analytic subset of T^*X is a holonomic set if each irreducible component has dimension n. From the definition, $\check{S}S(\mathscr{L})$ is a holonomic set for a holonomic system $\mathscr{L}\neq 0$. For each irreducible component Λ of $\check{S}S(\mathscr{L})$, the *multiplicity* of $\mathscr{L}$ along Λ is denoted by $m_\Lambda(\mathscr{L})$. When $m_\Lambda(\mathscr{L})=1$, we call Λ a *simple holonomic set*. Suppose $\mathscr{L}=\mathscr{D}_X u$ with an unknown function u and Λ is a simple holonomic set of $\check{S}S(\mathscr{L})$. Then the *principal symbol* and the *order* of u on Λ are denoted by $\sigma_\Lambda(u)$ and $\text{ord}_\Lambda(u)$, respectively.

We shall restrict our attention to the study of differential equations governing a complex power of a polynomial or a holomorphic function. For details, see [4], [10], [11]. First we put $\mathscr{D}_X[s]=\mathscr{D}_X\otimes\boldsymbol{C}[s]$ for an indeterminate s that commutes with $\mathscr{D}_X$. We define an ideal

$$\mathscr{J}_f(s)=\{P(s, x, D_x)\in\mathscr{D}_X[s];\ P(s, x, D_X)(f(x))^s=0\},$$

for a holomorphic function $f(x)$ on X. We also define the following $\mathscr{D}_X$ (or $\mathscr{D}_X[s]$)-Modules:

$$\begin{aligned}&\mathscr{N}=\mathscr{D}_X[s](f(x))^s\simeq\mathscr{D}_X[s]/\mathscr{J}_f(s)\,,\\ &\mathscr{M}=\mathscr{D}_X[s](f(x))^s/\mathscr{D}_X[s](f(x))^{s+1}\\ &\qquad\simeq\mathscr{D}_X[s]/(\mathscr{J}_f(s)+\mathscr{D}_X[s]f(x))\,,\\ &\mathscr{N}_\alpha=\mathscr{D}_X/\mathscr{J}_f(\alpha)\qquad(\alpha\in\boldsymbol{C})\,.\end{aligned}\tag{3.1}$$

Here $\mathscr{J}_f(\alpha)=\{P(\alpha, x, D_X)\in\mathscr{D}_x;\ P(s, x, D_x)\in\mathscr{J}_f(s)\}$. Then $\mathscr{N}$, $\mathscr{M}$ and $\mathscr{N}_\alpha$ are coherent $\mathscr{D}_X$-Modules. It is provable that $\mathscr{N}_\alpha\simeq\mathscr{D}_X(f(x))^\alpha$ if and only if $\alpha\in\boldsymbol{C}$ satisfies $b_f(\alpha-n)\neq 0$ for all $n\in\boldsymbol{N}$. Here $b_f(s)$ is the b-function of $f(x)$.

Let $\mathscr{G}_f$ be the Lie algebra

$$\mathscr{G}_f=\{Y:\ Y \text{ is a vector field satisfying } Yf\in\mathscr{O}_Xf\}\,.\tag{3.2}$$

Let $\mathscr{G}_f(s)$ denote the ideal of $\mathscr{D}_X[s]$ generated by $\{Y-s\cdot c(Y);\ Yf=c(Y)f\}$. Let us define the modules

$$\begin{aligned}&\mathscr{N}'=\mathscr{D}_X[s]/\mathscr{G}_f(s)\,,\\ &\mathscr{N}'_\alpha=\mathscr{D}_X/\mathscr{G}_f(\alpha)\qquad(\alpha\in\boldsymbol{C})\,,\end{aligned}\tag{3.3}$$

where $\mathscr{G}_f(\alpha)=\{P(\alpha, x, D_X);\ P(s, x, D_x)\in\mathscr{G}_f(s)\}$. From the definitions (3.1) and (3.3), there follow surjective morphisms of $\mathscr{D}_X[s]$ (or $\mathscr{D}_X$)-Modules:

$$\begin{aligned}&\mathscr{N}'\longrightarrow\mathscr{N}\longrightarrow 0\\ &\mathscr{N}'_\alpha\longrightarrow\mathscr{N}_\alpha\longrightarrow 0\,.\end{aligned}\tag{3.4}$$

It is known that $\mathscr{M}$ and $\mathscr{N}_\alpha$ (or $\mathscr{N}$) are holonomic (or sub-holonomic). More precisely, define

$$\begin{aligned}&W=\text{the closure of } \{(x, s\nabla_x\log f(x));\ x\in X,\ f(x)\neq 0,\ s\in\boldsymbol{C}\}\\ &W_0=(W\cap\{(x,\eta)\in T^*X;\ f(x)=0\})\cup T^*_XX\,.\end{aligned}\tag{3.5}$$

Here ∇_x denotes the gradient with respect to x. Then we have that

$$\check{S}S(\mathcal{N}) \simeq W\,,$$
(3.6)
$$\check{S}S(\mathcal{M}) \subseteq W \cap f^{-1}(0)$$
$$\check{S}S(\mathcal{N}_\alpha) \subset W_0\,.$$

We mention a fundamental result concerning $\mathcal{G}_f$.

THEOREM 3.1 (*K. Saito*). *The following statements are equivalent.*

(1) *$\mathcal{G}_f$ is a locally free $\mathcal{O}_X$-Module.*

(2) *There are n vector fields $X_i = \sum_{j=1}^n m_{ij}(x)\partial/\partial x_j$ $(1 \leqq i \leqq n)$ such that*

$$\det(m_{ij}(x)) \in \mathcal{O}_X^* f(x)\,.^{2)}$$

For a proof, see K. Saito [7].

We shall study the structure of these modules in the case where $f(x) = f_{W_l}(x)$, the generalized discriminant of a Coxeter group W_l. Define the l-tuple of vector fields $X_1, \cdots, X_l$ by

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_l \end{pmatrix} = M(W_l) \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_l \end{pmatrix} = \frac{1}{2}\left(\frac{\partial x_\mu}{\partial \xi_\nu}\right) \begin{pmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \\ \vdots \\ \partial/\partial \xi_l \end{pmatrix}. \tag{3.7}$$

It is easy to see that $X_i D = (1/2)\sum_{k=1}^l (\partial x_i/\partial \xi_k)(\partial D/\partial \xi_k)$ is anti-invariant by W_l, hence $X_i D \in \boldsymbol{R}[x]D$ by Theorem 1.2 (4). Therefore $X_i f_{W_l} = 2(X_i D)D \in \boldsymbol{R}[x] f_{W_l}$. Recalling that $x_1 = \xi_1^2 + \cdots + \xi_l^2$, we have

$$X_1 = \sum_{k=1}^l \xi_k \frac{\partial}{\partial \xi_k} = \sum_{i=1}^l k_i x_i \frac{\partial}{\partial x_i}$$

and hence

$$X_1 f_{W_l} = lh f_{W_l}\,.$$

Theorem 3.1 combined with Theorem 1.2 (5) implies the following.

PROPOSITION 3.2. $\mathcal{G}_{f_{W_l}} = \sum_{i=1}^l \mathcal{O}_X \cdot X_i$.

We shall study, hereafter, the decomposition of $\check{S}S(\mathcal{N}'_\alpha)$ into irreducible components. One of our original concerns is to study the decomposition of $\check{S}S(\mathcal{N}_\alpha)$. Since $\mathcal{N}_\alpha$ is a quotient of $\mathcal{N}'_\alpha$ and as we shall show in the next section that all irreducible components of $\check{S}S(\mathcal{N}'_\alpha)$ are simple, the latter gives us enough information to the former. Moreover it is conjectured that $\mathcal{N}'_\alpha \xrightarrow{\sim} \mathcal{N}_\alpha$ in our case (cf. §5). This isomorphism

[2] $\mathcal{O}_X^*$ denotes the sheaf of invertible elements of $\mathcal{O}_X$.

actually holds for $I_2(m)$. We remark that $\mathcal{N}'_\alpha$ is not isomorphic to $\mathcal{N}_\alpha$ in general (for example, take $f(x)=x_1(x_1-x_2^2x_3)(x_1-x_2^2x_4)$.)

We fix a Coxeter system (W_l, S). When the subgroup of W_l is generated by a subset S' of S, we call it an *S-subgroup* of W_l and denote it by $W_{S'}$. We define an equivalence relation $\sim$ on the set of all S-subgroups such that $W'\sim W''$ if and only if W' is conjugate to W''. Let $\mathscr{C}$ be the set of conjugate classes of S-subgroups. Then

PROPOSITION 3.3. $\bar{\mathscr{A}_S}\simeq\mathscr{C}$.

PROOF. Define the map from $\mathscr{A}_S$ to $\mathscr{C}$ by $\mathscr{A}(S')\to W_{S'}$. Then this map induces the bijection indicated in this proposition because $w\mathscr{A}(S')=\mathscr{A}(S'')$ if and only if $W_{S''}=wW_{S'}w^{-1}$. Q.E.D.

We next set $B(S')=p(\mathscr{A}(S'))$. We remark that $B(S')=B(S'')$ if and only if $W_{S'}$ and $W_{S''}$ are conjugate. Therefore we can restrict our consideration to the conjugate classes of S-subgroups of W_l as far as the set

$$\Lambda=\{(x, \eta)\in T^*X;\ \eta\cdot M(W_l)(x)=0\}$$

is concerned. Equation (2.12) shows that the mapping p is everywhere of maximal rank on $\mathring{\mathscr{A}}(S')$ for each $S'\subset S$, that is,

$$\operatorname{codim}_X B(S')=\# S' . \tag{3.8}$$

We put $\Lambda(S')=T^*_{B(S')}X$. Then $\Lambda(S')$ is a holonomic set and Equation (3.8) is rewritten in the form

$$\operatorname{codim}_X \pi(\Lambda(S'))=\# S' . \tag{3.9}$$

We are then to prove the relation between the singular support of $\mathcal{N}'_\alpha$ and the Coxeter subsystems of W_l.

LEMMA 3.4. *W_0 is contained in Λ.*

PROOF. From the definition of $M(W_l)$, it is easy to see that

$$(\nabla_X \log(f_{W_l}(x)))\cdot M(W_l)=(X_1\log(f_{W_l}(x)),\ \cdots,\ X_l\log(f_{W_l}(x))) .$$

For any element (x_0, η_0) of W_0, there exist an analytic path $t\to x(t)$ on X and a real analytic function $s(t)$ such that

$$x_0=\lim_{t\to 0} x(t)$$
$$\eta_0=\lim_{t\to 0} s(t)\cdot(\nabla_x \log(f_{W_l}(x(t))))$$

with

$$\lim_{t\to 0} s(t)=0 .$$

Noting that

$$X_i f_{W_l}(x)=c_i(x)f_{W_l}(x) \tag{3.10}$$

with $c_i(x)=c(X_i)$, we have

$$\begin{aligned}\eta_0\cdot M(W_l)|_{x=x_0} &= \lim_{t\to 0} s(t)\cdot(\nabla_X \log(f_{W_l}(x(t))))M(W_l)|_{x=x(t)} \\ &= \lim_{t\to 0} s(t)(c_1(x(t)), \cdots, c_l(x(t))) \\ &= 0 .\end{aligned}$$

Q.E.D.

By this lemma and (3.6), we conclude that $\check{S}S(\mathcal{N}_\alpha)$ is contained in Λ. Furthermore

LEMMA 3.5. $\check{S}S(\mathcal{N}'_\alpha)\subseteqq\Lambda$.

PROOF. By an elementary calculation, we have

$$\begin{aligned}\sigma(X_i-\alpha c(X_i)) &= \sigma(X_i) \\ &= \frac{1}{2}\sum_{j=1}^{l}\nabla x_i\cdot\nabla x_j\sigma\left(\frac{\partial}{\partial x_j}\right) .\end{aligned}$$

Therefore, $(x, \eta)\in\check{S}S(\mathcal{N}'_\alpha)$ must satisfy $\eta\cdot M(W_l)(x)=0$. Q.E.D.

These results are summarized in Figure 1.

$$\begin{array}{ccccc} & & \check{S}S(\mathcal{N}'_\alpha) & \subset & \Lambda \\ & & \cup & & \cup \\ \check{S}S(\mathcal{D}_X(f_{W_l}(x))^\alpha) & \subset & \check{S}S(\mathcal{N}_\alpha) & \subset & W_0 \end{array}$$

FIGURE 1

Finally we decompose Λ into irreducible components.

PROPOSITION 3.6. $\Lambda=\bigcup_{S'\subset S}\Lambda(S')$.

PROOF. Lemma 2.2 and the proof of Proposition 2.1 show that the condition $\eta\cdot M(W_l)(x)=0$ is equivalent to the condition $\phi_j(\xi)(\eta\cdot P_j(\xi))=0$ $(1\leqq j\leqq l)$. Here we put

$$P_j(\xi)={}^t(P_{H_j,1}(\xi), \cdots, P_{H_j,l}(\xi)) .$$

(As to the definition of $P_{H_j,i}(\xi)$, see Lemma 2.2.) Therefore the regular part of each irreducible component of Λ is represented by

$$Q(S')=\{(x,\eta)\in T^*X;\ x=x(\xi),\ \xi\in\mathring{\mathscr{A}}(S') \text{ and } \eta\cdot P_j(\xi)=0 \text{ if } s_j\notin S'\} \tag{3.11}$$

for a subset S' of S. Conversely, $Q(S')\subset\Lambda$ for any subset S'. Noting that $\overline{\pi(Q(S'))}=B(S')$ and $\dim Q(S')=l-(\#S')$, we can easily conclude that $\overline{Q(S')}=T^*_{B(S')}X$. Q.E.D.

§4. The microlocal structure of $\mathscr{N}'_\alpha$.

This section is devoted to the proof of the main theorem.

As we have shown in the last part of §2, $B(S')$ and $\Lambda(S')$ depend only on the conjugate class of the S-subgroup $W_{S'}$. We write $[S']$ for the conjugate class of $W_{S'}$ in $\mathscr{C}$, and $B([S'])$ and $\Lambda([S'])$ for $B(S')$ and $\Lambda(S')$, respectively.

We denote by u the generator of $\mathscr{N}'_\alpha$ such that

$$u=1 \bmod \mathscr{G}_{f_{W_l}}(\alpha)\ .$$

THEOREM 4.1.

(1) $\check{S}S(\mathscr{N}'_\alpha)=\bigcup_{[S']}\Lambda([S'])$, *where* $[S']$ *runs all over the set of conjugate classes of S-subgroups.*

(2) *For any subset* S' *of* S, $\Lambda([S'])$ *is an irreducible simple holonomic set and*

$$\operatorname{codim}_X \pi(\Lambda([S']))=\#S'\ . \tag{4.1}$$

$$\operatorname{ord}_{\Lambda([S'])} u=-\frac{1}{2}\sum_i(\#S'_i)h(S'_i)\left(\alpha+\frac{1}{2}\right)\ . \tag{4.2}$$

Here $\prod_i(W'_i, S'_i)$ *is the irreducible decomposition of* $(W_{S'}, S')$ *and* $h(S'_i)$ *is the Coxeter number of* W'_i.

(3) *For a given* S'_i, *we proceed to delete one of its vertices and segments attached to it, and denote by* S'_{ij} $(1\leq j\leq\#S'_i)$ *the resulting graphs. Then* $\Lambda([S'])$ *and each of the holonomic sets* $\Lambda([S'_{ij}])$ *intersect on the common one codimensional analytic subset which we denote by* $I(S', S'_i)$. *Furthermore, for two subsets* S'_i *and* S'_k *of* S', $I(S', S'_i)=I(S', S'_k)$ *if and only if* $S'_i=S'_k$.

REMARK. Let S' and S'' be two subsets of S such that $S'\supset S''$. Assume that $\#S'=\#S''+1$. Then Theorem 4.1 (3) assures that $\Lambda([S'])$ and $\Lambda([S''])$ intersect in an analytic subset of codimension 1. The converse statement at least holds for cases A_l, B_l, G_2, H_3, H_4 and $I_2(m)$. However it does not hold in general.

PROOF. Constant use is made of notation in §1. We introduce linear forms ζ_i $(1\leqq i\leqq l)$ on V defined by

$$\zeta_i\colon \sum_{j=1}^{l} a_i\alpha_i \mapsto a_i\ .$$

Then $\zeta_1, \cdots, \zeta_l$ constitute a basis of V^* and the transformation between ζ_i and ξ_i $(1\leqq i\leqq l)$ is given by

$$Q\begin{pmatrix}\zeta_1\\ \vdots\\ \zeta_l\end{pmatrix}=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_l\end{pmatrix}. \tag{4.3}$$

Hence

$$\left(\frac{\partial}{\partial\zeta_1}, \cdots, \frac{\partial}{\partial\zeta_l}\right)=\left(\frac{\partial}{\partial\xi_1}, \cdots, \frac{\partial}{\partial\xi_l}\right)Q\ . \tag{4.4}$$

(cf. (1.5) and (1.6))

The first step to prove the theorem is to reduce the claims to those for the conormal bundle of the origin for an irreducible Coxeter system. For this purpose, we begin by examining the connection among the generalized discriminants of Coxeter subsystems of a given Coxeter system.

Let (W_l, S) be a Coxeter system, $(W_{S'}, S')$ a Coxeter subsystem of (W_l, S). Then we decompose the fundamental anti-invariant D into two factors, one vanishing on every hyperplane contained in $\mathfrak{H}(W_{S'})$ and the other invertible in a neighborhood of $\mathring{\mathscr{A}}(S')$. That is,

$$\begin{aligned} D(\xi)&=\prod_{H\in\mathfrak{H}(W_l)}\phi_H(\xi)\\ &=\Big(\prod_{H\in\mathfrak{H}(W_l)\setminus\mathfrak{H}(W_{S'})}\phi_H(\xi)\Big)\Big(\prod_{H\in\mathfrak{H}(W_{S'})}\phi_H(\xi)\Big)\ . \end{aligned} \tag{4.5}$$

Choose a point ξ_0 in $\mathring{\mathscr{A}}(S')$. It should be noted that we can restrict our consideration in a neighborhood of ξ_0. We may put $S'=\{s_1, \cdots, s_k\}$ without losing generality. We take a linear local coordinate at ξ_0 by

$$(\xi_0)+\sum_{i=1}^{l-k}\tau_i\varepsilon_i+\sum_{j=1}^{k}\zeta_j\alpha_j\ . \tag{4.6}$$

Here we have identified α_i with its numerical vector with respect to the basis $\{e_1, \cdots, e_l\}$ and have taken $\{\varepsilon_1, \cdots, \varepsilon_{l-k}\}$ as a set of linearly independent vectors which span $\mathscr{A}(S')$. Then the decomposition (4.5) turns out to be

(4.7) $$D(\xi)=\psi(\tau, \zeta)\cdot \prod_{H\in\Phi(W_{S'})} \phi_H\Big(\sum_{j=1}^{k} \zeta_j\alpha_j\Big) .$$

The second factor of the right-hand side of (4.7) is nothing but a fundamental anti-invariant of the Coxeter system $(W_{S'}, S')$. Let $\{z_\mu\}$ be the set of fundamental invariants of $(W_{S'}, S')$. Noting that $p|_{\mathscr{N}(S')}$ has maximal rank at ξ_0 by Lemma 2.3 and that $g=\psi^2$ is invariant by $W_{S'}$, we reach the formula

(4.8) $$f_{W_l}(x)=g(t, z)f_{W_{S'}}(z) ,$$

where t is a local parameter of $B(S')$ at $x_0=p(\xi_0)$ corresponding to τ. Since g is invertible near x_0, we obtain

(4.9) $$P(s, x, D)(gf_{W_{S'}})^s=0 \text{ if and only if } P(s, x, D+s \nabla \log g)(f_{W_{S'}})^s=0 .$$

Set

$$\mathscr{N}''_\alpha=\mathscr{D}_Z/\mathscr{G}_{f_{W_{S'}}}(\alpha) ,$$

$$\mathscr{G}_{f_{W_{S'}}}=\{Y;\ Y \text{ is a vector field on } Z,\ Yf_{W_{S'}}\in\mathscr{O}_Z f_{W_{S'}}\} ,$$

where $Z=C_z^{l-k}$ (cf. (3.2) and (3.3)). Then, (4.8) and (4.9) imply

(4.10) $$\mathscr{G}_{f_{W_l}}(s)=\mathscr{D}_X[s](g\mathscr{G}_{f_{W_{S'}}}(s)g^{-1})+\sum_{i=1}^{l-k}\mathscr{D}_X[s]\Big(g\frac{\partial}{\partial t_i}g^{-1}\Big) .$$

Since

(4.11) $$\mathscr{D}_X\Big/\Big(\mathscr{D}_X\mathscr{G}_{f_{W_{S'}}}(\alpha)+\sum_{i=1}^{l-k}\mathscr{D}_X\Big(\frac{\partial}{\partial t_i}\Big)\Big)\xrightarrow{\sim}\mathscr{O}_U\hat{\otimes}\mathscr{N}''_\alpha$$

for a neighborhood U of the origin in Z, we obtain

(4.12) $$\check{SS}(\mathscr{N}'_\alpha)\simeq T_U^*U\times \check{SS}(\mathscr{N}''_\alpha)$$

in a neighborhood of $\pi^{-1}(x_0)$. If Λ is an irreducible holonomic set of $\check{SS}(\mathscr{N}'_\alpha)$ containing (x_0, η_0), Λ is represented by $T_U^*U\times\Lambda'$ with a certain holonomic set Λ' of $\check{SS}(\mathscr{N}''_\alpha)$. Let $(W_{S'}, S')=\prod_i (W'_i, S'_i)$ be the decomposition into irreducible components. Then $f_{W_{S'}}(z)=\prod_i f_{W'_i}(z^{(i)})$, where $f_{W'_i}(z^{(i)})$ is the generalized discriminant of W'_i. We now put

(4.13) $$\begin{cases}\mathscr{N}''_\alpha=\hat{\otimes}\mathscr{N}'_{\alpha,i}\\ \mathscr{N}'_{\alpha,i}=\mathscr{D}_{Z_i}/\mathscr{G}_{f_{W'_i}}(\alpha) ,\end{cases}$$

where $Z_i=C_{z^{(i)}}^{l_i}$ $(l_i=\#S'_i)$. Then the above argument shows that for each i there is an open neighborhood U_i of the origin in Z_i such that

$$\Lambda \cong T_U^* U \times \prod_i \Lambda_i' \tag{4.14}$$

with $\Lambda_i' = T_{\{0\}}^* U_i$. It follows from the definition of m (cf. p. 201) that

$$m_\Lambda(\mathcal{N}'_\alpha) = \prod_i m_{\Lambda_i'}(\mathcal{N}'_{\alpha,i}) . \tag{4.15}$$

Assuming that (W_l, S) is irreducible, we prove that Λ is simple for $\Lambda = \Lambda(S)$. Since $m_{1j}(x) = m_{j1}(x) = k_j x_j$, $\sum_{j=1}^l \mathcal{O}_{T^*X}\sigma(X_j)$ is a simple ideal on Λ and therefore $X_1 - \alpha c_1(x), \cdots, X_l - \alpha c_l(x)$ form an involutory basis of $\mathcal{G}(\alpha)$ on Λ. (cf. (3.10)) Hence Λ is contained in $\check{S}S(\mathcal{N}'_\alpha)$ and simple. In order to obtain the principal symbol of u (which is the generator of $\mathcal{N}'_\alpha$ such that $u = 1 \bmod \mathcal{G}(\alpha)$) on Λ, we determine $L_{(X_i - \alpha c_i(x))|\Lambda}$ for each i. (As to the definition of L_P, see §2 in [5].) A simple calculation shows that

$$\begin{aligned} L_{(X_1-\alpha c_1(x))|\Lambda} &= -\sum_{j=1}^l k_j \eta_j \frac{\partial}{\partial \eta_j} - lh\left(\alpha + \frac{1}{4}\right) - \frac{l}{2} \\ L_{(X_i-\alpha c_i(x))|\Lambda} &= -k_i \eta_1 \frac{\partial}{\partial \eta_i} \qquad (i = 2, \cdots, l) . \end{aligned} \tag{4.16}$$

Equation (4.16) and the definition of $\sigma_\Lambda(u)$ imply that

$$\sigma_\Lambda(u) = \eta_1^{-(1/2)lh(\alpha+1/2)-l/2} \sqrt{\frac{d\eta_1 \cdots d\eta_l}{dx_1 \cdots dx_l}} . \tag{4.17}$$

(Note that $\sum_{i=1}^l k_i = (1/2)lh + l$.) Since the order of u on Λ is, by definition, the homogeneous degree of $\sigma_\Lambda(u)$ with respect to η, we have

$$\mathrm{ord}_\Lambda u = -\frac{1}{2} lh\left(\alpha + \frac{1}{2}\right) . \tag{4.18}$$

We now proceed to the proof of the theorem. We may assume without loss of generality that (W_l, S) is irreducible. Let $\Lambda(S')$ be an irreducible holonomic set corresponding to a subset S' of S. Then the above argument combined with (4.14) shows that $\Lambda(S')$ is simple and contained in $\check{S}S(\mathcal{N}'_\alpha)$. This and Lemma 3.5 assert (1). Equation (4.1) is nothing other than (3.9). Use the above notation. Let u_i' be the generator of $N'_{\alpha,i}$ for each i. Since $\Lambda(S')$ is simple, we have the following formula by using an elementary property of ord_Λ (see, Proposition 4.2.4 in [9])

$$\mathrm{ord}_\Lambda u = \sum_i \mathrm{ord}_{\Lambda_i'} u_i' . \tag{4.19}$$

Equation (4.2) is, then, an easy consequence of (4.18) and (4.19). Next we prove (3). In view of (4.14), we may assume $\Lambda=\Lambda(S)$. We put $S'=S-\{s_l\}$ for some s_l in S. Then from the definition, for any point $(x_0, \eta_0)\in\Lambda(S)\cap\Lambda(S')$, there exist an analytic path $\xi(t)\in\mathring{\mathscr{A}}(S')$ and a vector $\eta(t)$ such that

$$(4.20)\qquad \begin{aligned} &(x(\xi(t)), \eta(t))\in\Lambda(S') \qquad (\text{if } t\neq 0)\\ &x_0(=0)=\lim_{t\to 0} x(\xi(t))\\ &\eta_0=\lim_{t\to 0}\eta(t)\ . \end{aligned}$$

Then Lemma 2.3 shows that, if $t\neq 0$, $\xi(t)$ and $\eta(t)$ satisfy the equation

$$(4.21)\qquad \eta(t)\cdot P_l(\xi(t))=0$$

(as to the definition of $P_l(\xi)$, see Lemma 2.2 and the proof of Proposition 3.6). The limitation $t\to 0$ implies

$$(4.22)\qquad \eta_0\cdot P_l(0)=0\ .$$

Since we assumed $x_1=\xi_1^2+\cdots+\xi_l^2$, we have $P_{H_l,1}(\xi)=c$ (c is a non-zero constant). On the other hand, for each j $(2\leqq j\leqq l)$, $P_{H_l,j}(\xi)$ is a homogeneous polynomial of ξ with $\deg_\xi P_{H_l,j}(\xi)\geqq 1$. Hence Equation (4.22) means that $\eta_1^0=0$ and $\eta_2^0, \cdots, \eta_l^0$ can take arbitrary values. Here we put $\eta_0=(\eta_1^0, \cdots, \eta_l^0)$. The intersection of $\Lambda(S')$ and $\Lambda(S)$ is, therefore, given by $\{(x, \eta)\in T^*X; x=0, \eta_1=0\}$ and thus we conclude that $\operatorname{codim}\Lambda(S')\cap\Lambda(S)=1$. Hence we have proved (3) except for the last part of it, which is, however, nearly obvious. Q.E.D.

§5. Two conjectures.

We fix a Coxeter system (W_l, S) and put $f(x)=f_{W_l}(x)$. Let $b_f(s)$ be the b-function of $f(x)$, which is, by definition, the monic polynomial of s with the minimal degree such that

$$(5.1)\qquad P(s, x, D_x)(f(x))^{s+1}=b_f(s)(f(x))^s$$

for a differential operator $P(s, x, D_x)$ (cf. [10]). In view of

$$(5.2)\qquad X_1(f(x))^{s+1}=lh(s+1)(f(x))^{s+1}\ ,$$

we can eliminate s from the operator $P(s, x, D_x)$. As is known, we have

$$(5.3)\qquad \mathscr{N}_\alpha=\mathscr{D}_X(f(x))^\alpha$$

for any $\alpha\in\boldsymbol{C}$ if $b_f(\alpha-n)\neq 0$ for any non-negative integer n. As remarked in §3, $\mathcal{N}_\alpha$ is a quotient of $\mathcal{N}'_\alpha$. The microlocal structure of $\mathcal{N}'_\alpha$ has been elaborated in §4. The reason why we have mainly interested in $\mathcal{N}'_\alpha$ instead of $\mathcal{N}_\alpha$ is partly based on the conjecture:

CONJECTURE I. $\mathcal{N}_\alpha=\mathcal{N}'_\alpha$ for any $\alpha\in \boldsymbol{C}$.

Once we assume that

$$S\check{S}(\mathcal{N}_\alpha)=W_0 \tag{5.4}$$[3]

(cf. (3.6)), Theorem 4.1 (1) and Lemma 3.4 easily reduce Conjecture I to

CONJECTURE I′. $\Lambda=W_0$.

We state another conjecture concerning the b-function of $f(x)$:

CONJECTURE II. $b_f(s)=\prod_{i=1}^{l}\prod_{k=1}^{k_i-1}(s+1/2+k/k_i)$.

On assuming Conjecture II, we readily have

$$\deg b_f(s)=\frac{1}{2}lh\ . \tag{5.5}$$

Equation (5.5) is closely connected with the explicit form of the principal symbol (cf. (4.17)).

We remark that these conjectures are true at least for the Coxeter system of type $I_2(m)$ (as proven elsewhere) as well as type A_3 (cf. §6). We also defer the determination of the fundamental invariants and $M(W_l)$ for all irreducible Coxeter systems (except for E_7 and E_8) until a subsequent paper. The succeeding section provides an easy example which illustrates our general formulation and supports Conjectures I and II.

§6. An example.

Let (W, S) be the Coxeter system of type D_3 (which is equal to A_3). We try here to determine the fundamental invariants and the generalized discriminant $f(x)$ and prove that

$$\Lambda=W_0 \tag{6.1}$$

$$b_f(s)=(s+1)^2\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right)\ . \tag{6.2}$$

[3] Professor M. Kashiwara announced (private communication) that he has proved (5.4) though his proof is not yet published. Hence we assume (5.4) in the example discussed in the next section.

Let $E=\boldsymbol{R}e_1+\boldsymbol{R}e_2+\boldsymbol{R}e_3$ be a vector space with an orthonormal basis $\{e_1, e_2, e_3\}$. Then the Coxeter group $W=W(D_3)$ is generated by the reflections s_1, s_2, s_3:

$$s_1: (e_1, e_2, e_3)\mapsto(e_2, e_1, e_3)\ ,$$
$$s_2: (e_1, e_2, e_3)\mapsto(e_1, e_3, e_2)\ ,$$
$$s_3: (e_1, e_2, e_3)\mapsto(e_1, -e_3, -e_2)\ .$$

As is known, W is a semi-direct product of $\mathfrak{S}_3$ by $(\boldsymbol{Z}/2\boldsymbol{Z})^2$, which is isomorphic to $\mathfrak{S}_4$. The Coxeter diagram is of the form:

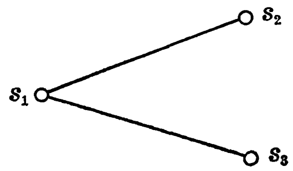


FIGURE 2

$S=\{s_1, s_2, s_3\}$ is a set of generators. Let ξ_1, ξ_2 and ξ_3 be linear forms on E defined by

$$\xi_i(e_j)=\delta_{ij} \qquad (i, j=1, 2, 3)$$

and put

$$E^*=\boldsymbol{R}\xi_1+\boldsymbol{R}\xi_2+\boldsymbol{R}\xi_3\ .$$

Let us identify the symmetric algebra $S(E^*)$ with $\boldsymbol{R}[\xi_1, \xi_2, \xi_3]$. Then $R=S(E^*)^W$ is generated by x_2, x_3, x_4:

$$x_2=\xi_1^2+\xi_2^2+\xi_3^2$$
$$x_3=\xi_1\xi_2\xi_3$$
$$x_4=\xi_2^2\xi_3^2+\xi_3^2\xi_1^2+\xi_1^2\xi_2^2\ .$$

By simple calculation, it follows that

$$M(D_3)=\begin{pmatrix} 2x_2 & 3x_3 & 4x_4 \\ 3x_3 & \frac{1}{2}x_4 & 2x_2x_3 \\ 4x_4 & 2x_2x_3 & 6x_3^2+2x_2x_4 \end{pmatrix},$$

$$f(x)=f_{D_3}(x)=\det M(D_3)$$
$$=-8\Big(x_4-\frac{1}{3}x_2^2\Big)^3-54\Big(x_3^2-\frac{1}{3}x_2x_4+\frac{2}{27}x_2^3\Big)^2\ .$$

In this case, $f(x)$ is the discriminant of the polynomial

$$P(u)=u^3-x_2u^2+x_4u-x_3^2$$
$$=(u-\xi_1^2)(u-\xi_2^2)(u-\xi_3^2)$$

(up to a constant factor). From the above matrix $M(D_3)$, we have

$$X_1=2x_2\frac{\partial}{\partial x_2}+3x_3\frac{\partial}{\partial x_3}+4x_4\frac{\partial}{\partial x_4}\,,$$
$$X_2=3x_3\frac{\partial}{\partial x_2}+\frac{1}{2}x_4\frac{\partial}{\partial x_3}+2x_2x_3\frac{\partial}{\partial x_4}\,,$$
$$X_3=4x_4\frac{\partial}{\partial x_2}+2x_2x_3\frac{\partial}{\partial x_3}+(6x_3^2+2x_2x_4)\frac{\partial}{\partial x_4}\,.$$

Hence

$$[X_1, X_2]=X_2\,,\qquad [X_1, X_3]=2X_3\,,\qquad [X_2, X_3]=x_3X_1$$

and

$$X_1f=12f\,,\qquad X_2f=0\,,\qquad X_3f=4x_2f\,.$$

The conjugate classes of S-subgroups of W are

$$W_{1,2,3}=[S]\,,\qquad W_{1,2}=[\{s_1, s_2\}]$$
$$W_{2,3}=[\{s_2, s_3\}]\,,\qquad W_1=[\{s_1\}]\,,\qquad W_0=[\varnothing]\,.$$

Hence, from Theorem 4.1 (1), it follows that

$$\check{S}S(\mathcal{N}'_\alpha)=\Lambda_{1,2,3}\cup\Lambda_{1,2}\cup\Lambda_{2,3}\cup\Lambda_1\cup\Lambda_0\,.$$

Here we have put $\Lambda_{1,2,3}=\Lambda(W_{1,2,3})$ etc.

PROPOSITION 6.1. $\mathcal{N}_\alpha=\mathcal{N}'_\alpha$.

PROOF. It is sufficient to prove that

$$\Lambda\subset W_0\,,$$

because we have already proved in Theorem 4.1 that $W_0\subset\Lambda$.

We now show that

$$\Lambda_{1,2,3}\subset W_0\,.$$

Put $f_i=\partial f/\partial x_i$ ($i=1, 2, 3$). Then

$$f_2(x)=4(x_2x_4^2+9x_3^2x_4-6x_2^2x_3^2) ,$$
$$f_3(x)=8x_3(9x_2x_4-27x_3^2-2x_2^3) ,$$
$$f_4(x)=4(-6x_4^2+x_2^2x_4+9x_2x_3^2) .$$

We define an analytic path $(x_2(t), x_3(t), x_4(t))$ which coverges to the origin when $t\to 0$, by

$$x_2(t)=3at+bt^2$$
$$x_3(t)=cat^2$$
$$x_4(t)=\frac{3}{2}a^2t^2 ,$$

where a, b, c are arbitrary numbers. Then it is easy to see

$$\lim_{t\to 0}\frac{1}{9a^3+t^5}(f_2(x(t)), f_3(x(t)), f_4(x(t)))$$
$$=(3a^2, -12ac, 4(b+3c^2)) .$$

Since we can regard a, b, c as being arbitrary, this equation and the definition of W_0 imply that

$$\Lambda_{1,2,3}\subset W_0 .$$

We remark in advance that Conjecture I holds for the Coxeter systems of type A_1 or A_2. This follows from a direct calculation. It follows from this remark and (4.14) that $\Lambda_{1,2}$, $\Lambda_{2,3}$, Λ_1 and Λ_0 are contained in W_0. Hence the assertion. Q.E.D.

Next we show

PROPOSITION 6.2. $b_f(s)=(s+1)^2(s+5/6)(s+7/6)(s+3/4)(s+5/4)$.

PROOF. We apply the method expressed in Appendix to the present case and use the notation there. In this case,

$$\mathscr{J}^{(0)}=\mathscr{D}_XX_2+\mathscr{D}_X(3X_3-x_2X_1) ,$$
$$X_0=\frac{1}{12}X_1 .$$

A straightforward calculation leads us to

$$\mathscr{F}=C\Delta_1(x)+C\Delta_2(x) ,$$

where

$$\Delta_1(x)=\delta(x)\ ,$$
$$\Delta_2(x)=\left\{\left(\frac{\partial}{\partial x_2}\right)^3-\frac{3}{4}\left(\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_4}\right)+\frac{1}{16}\left(\frac{\partial}{\partial x_3}\right)^2\right\}\delta(x)\ ,$$

and

$$X_0\Delta_1(x)=-\frac{3}{4}\Delta_1(x)$$
$$X_0\Delta_2(x)=-\frac{5}{4}\Delta_2(x)\ .$$

Thus from Lemma A.2, it follows

$$\tilde{b}_f^3(s)=\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right)\ . \tag{6.3}$$

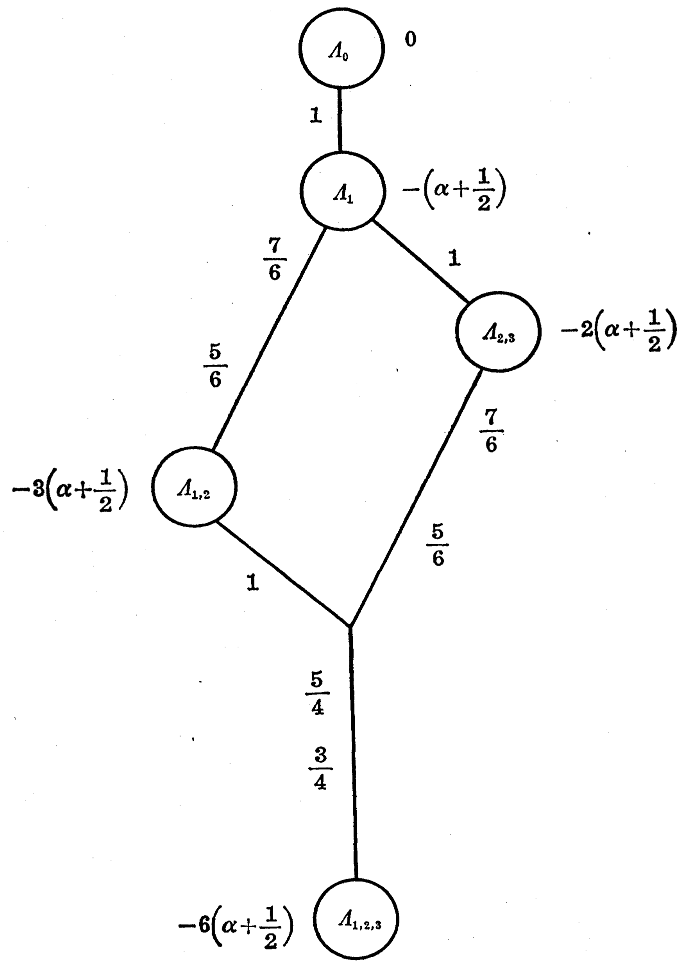


FIGURE 3
THE HOLONOMY DIAGRAM OF $\mathcal{N}_\alpha$

As to $\tilde{b}_f^2(s)$, the corresponding procedure is reduced to the case of the Coxeter systems of type A_2 and $A_1 \times A_1$, which is, however, an easy task (recorded in a subsequent paper [13]). The result is

$$\tilde{b}_f^2(s)=(s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right). \tag{6.4}$$

Hence the proposition follows from (6.3), (6.4) and Lemma A.1.

Q.E.D.

We have thus proved (6.1) and (6.2). Conjectures I and II are, therefore, true in this case. The results mentioned above are summarized in the holonomy diagram of $\mathcal{N}_\alpha$ in the previous page. (As to the holonomy diagram, refer to [3], [8], [11].)

REMARK. One will find an interesting theory written out somewhat a wider scope in K. Saito [6], whose example is in deep connection with ours.

Appendix.

We here record a general method of determining $b_f(s)$ for an analytic function $f(x)$. We keep here the notation (3.1). Put

$$\begin{aligned}\tilde{\mathcal{M}} &:= (s+1)\mathcal{M} \\ &\simeq \mathcal{D}_X[s]/(\mathcal{J}(s)+\mathcal{D}_X[s](\mathfrak{a}+\mathcal{O}_X f)),\end{aligned} \tag{A.1}$$

where

$$\mathfrak{a}=\sum_{i=1}^{n}\mathcal{O}_X\frac{\partial f}{\partial x_i}.$$

Regarding s as an endomorphism of $\tilde{\mathcal{M}}$, we write $\tilde{b}_f(s)$ for the minimal polynomial of s. Then, from the definition, we have

$$b_f(s)=(s+1)\tilde{b}_f(s). \tag{A.2}$$

Assume that there exists a vector field X_0 such that

$$s-X_0\in\mathcal{J}(s). \tag{A.3}$$

Then (A.1) turns out to be

$$\tilde{\mathcal{M}}=\mathcal{D}_X/(\mathcal{J}^{(c)}+\mathcal{D}_X\mathfrak{a}), \tag{A.4}$$

where

$$\mathscr{J}^{(0)} = \mathscr{D}_X \cap \mathscr{J}(s) .$$

We take a regular stratification $X = \bigcup_\alpha X_\alpha$ in the sense of H. Whitney such that

$$\check{S}S(\tilde{\mathscr{M}}) \subset \bigcup_\alpha T^*_{X_\alpha} X . \qquad \text{(A.5)}[4]$$

Then we define $\tilde{b}^k_f(s)$ as the minimal polynomial of the endomorphism s of

$$\bigoplus_{\operatorname{codim} X_\alpha = k} \mathscr{H}om_{\mathscr{D}_X}(\tilde{\mathscr{M}}, \mathscr{B}_{X_\alpha|X})_{x_\alpha} \qquad (x_\alpha \in X_\alpha) ,$$

where $\mathscr{B}_{X_\alpha|X}$ denotes the space of delta functions supported on $X_\alpha \subset X$; $\mathscr{B}_{pt}$ being an abbreviation of $\mathscr{B}_{\{0\}|X}$.

We now recall an interdependence between $\tilde{b}_f$ and $\tilde{b}^k_f$ (cf. Theorem 3.3 in [10]).

LEMMA A.1.[5]

$$\underset{2 \le k \le n}{\mathrm{l.c.m}} (\tilde{b}^k_f) \,|\, \tilde{b}_f \,|\, \prod_{k=2}^{n} \tilde{b}^k_f .$$

In order to determine $\tilde{b}^n_f(s)$, we decompose

$$\mathscr{F} = \mathscr{H}om_{\mathscr{D}_X}(\tilde{\mathscr{M}}, \mathscr{B}_{pt})_0[6]$$

into root subspaces of s. Under the assumption (A.3), a homomorphism $1 \mapsto \Delta(x) \in \mathscr{B}_{pt}$ in $\mathscr{F}$ is an eigenvector of s belonging to an eigenvalue β if and only if the following condition holds.

$$\text{(A.6)} \qquad \begin{aligned} & X_0 \Delta(x) = \beta \Delta(x) , \\ & Q(x, D_x)\Delta(x) = 0 \qquad \text{for all} \quad Q(x, D_x) \in \mathscr{J}^{(0)} , \\ & \frac{\partial f}{\partial x_i} \Delta(x) = 0 \qquad \text{for} \quad i = 1, \cdots, n . \end{aligned}$$

Thus we have

LEMMA A.2. *For a complex number β, $\tilde{b}^n_f(s)$ has the factor $s - \beta$ if and only if there exists a $\Delta(x) \in \mathscr{B}_{pt}$ satisfying* (A.6).

---

4) See, for example, M. Kashiwara, Section 3 in "On the maximally overdetermined system of linear differential equations, I", Publ. of RIMS, Kyoto Univ. 10 (1974/1975), 563-579.

5) l.c.m is an abbreviation of the least common multiple.

6) For the definition of the notation, see [10] and the references there.

We have defined $b_f(s)$ by the existence of $P(s, x, D_x)$ in (5.1). Conversely, if we find out $\tilde{b}_f(s)$ and $\mathscr{J}(s)$, we can construct $P(s, x, D_x)$ in (5.1). We now explain a method of the construction of such an operator under the assumption (A.3) for simplicity. From the definition of $\tilde{b}_f(s)$, it follows

$$\tilde{b}_f(X_0)=Q(x, D_x)+\sum_{i=1}^{n} R_i(x, D_x)\frac{\partial f}{\partial x_i}, \tag{A.7}$$

for some $Q(x, D_x)\in\mathscr{J}^{(0)}$ and $R_i(x, D_x)\in\mathscr{D}_X$ $(i=1, \cdots, n)$. Then the operator in question is given by

$$P(x, D_x)=\sum_{i=1}^{n} R_i(x, D_x)\frac{\partial}{\partial x_i}, \tag{A.8}$$

which works as follows:

$$\begin{aligned} P(x, D_x)f^{s+1} &= (s+1)\sum_{i=1}^{n} R_i(x, D_x)\Big(\frac{\partial f}{\partial x_i}f^s\Big) \\ &= (s+1)\tilde{b}_f(X_0)f^s \\ &= (s+1)\tilde{b}_f(s)f^s \\ &= b_f(s)f^s . \end{aligned}$$

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