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Topological Powers and Reduced Powers

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Introduction

In this paper, we shall deal with first order structures associated to a fixed first order language L with equality. For the sake of simplicity we assume that L has neither individual constant symbols nor function symbols. Also, we shall use standard model-theoretic notions in Chang-Keisler [1]. Let M be a first order L-structure and I be a topological space. Then the topological power of M by I, denoted by $M^{(I)}$, is the substructure of the direct power M^I, whose universe is the set of all the continuous functions from I to the universe $|M|$ of M, where we assume that $|M|$ is endowed with the discrete topology. Topological powers of M are L-structures of the form $M^{(I)}$ for some topological space I; Boolean topological powers of M are L-structures of the form $M^{(I)}$ for some Boolean space I.

Using these notions our main result can be expressed as follows:

THEOREM. *Every topological power of a first order L-structure M is elementarily equivalent to a reduced power of M, and every reduced power of M is elementarily equivalent to a Boolean topological power of M.*

From this theorem we have the following corollary.

COROLLARY 1. *The following three conditions are equivalent for any sentence φ in L.*

1) *Every reduced power of a model of φ is also a model of φ.*
2) *Every topological power of a model of φ is also a model of φ.*
3) *Every Boolean topological power of a model of φ is also a model of φ.*

On the other hand, any topological power of M can be considered as

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global sections of a constant M-sheaf and vice versa (cf. [2]). A sentence φ in L is said to be a global section sentence (respectively, CS-sentence, BCS-sentence) if φ is preserved under sheaves of L-structures (resp. constant sheaves, constant sheaves over Boolean spaces). See section 3 below for details. Therefore we obtain the following Corollary 1′ from Corollary 1.

COROLLARY 1′. *The following three conditions are equivalent for any sentence φ in L.*

1) *φ is a reduced power sentence.*
2) *φ is a CS-sentence.*
3) *φ is a BCS-sentence.*

By using Corollary 1′ and Proposition 6.2.6 (ii) in Chang-Keisler [1], we have the following result:

COROLLARY 2. *Every global section sentence is equivalent to a Horn sentence.*

Corollary 2 is an affirmative answer to the Mansfield's problem [5]. Noted that this Corollary 2 has been already obtained by Volger [7].

In section 1 of this paper we shall state a Feferman-Vaught type theorem and its immediate consequence due to Volger in [6]. By using this fact, we shall give a proof of our theorem in section 2. In section 3, we shall discuss some relationship which exists between topological powers and global sections of sheaves of L-structures, where Corollary 1′ and Corollary 2 will be proved.

§1. Fefermen-Vaught type theorem.

Throughout section 1 and section 2, every universe of an L-structure is nonempty. In the following a Boolean valued L-structure $\mathfrak{M}$ consists of $\langle |\mathfrak{M}|, \mathfrak{M}_B, [\![\varphi]\!]_{\mathfrak{M}} (\varphi \in \mathrm{Form}(L))\rangle$, where $|\mathfrak{M}|$ is an universe, $\mathfrak{M}_B$ is a Boolean algebra, and for every L-formula $\varphi(x_1, \cdots, x_n)$, $[\![\varphi]\!]_{\mathfrak{M}}$ is a function from $\mathfrak{M}^n$ into $\mathfrak{M}_B$ which satisfies Equality Axioms and satisfies

$$[\![\varphi \vee \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} + [\![\psi]\!]_{\mathfrak{M}}\ ,$$

$$[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cdot [\![\psi]\!]_{\mathfrak{M}}\ ,$$

$$[\![\neg \varphi]\!]_{\mathfrak{M}} = \sim [\![\varphi]\!]_{\mathfrak{M}}\ ,$$

$$[\![\forall x \varphi(x)]\!]_{\mathfrak{M}} = \bigwedge_{f \in |\mathfrak{M}|} [\![\varphi(f)]\!]_{\mathfrak{M}}\ ,$$

$$[\![\exists x\varphi(x)]\!]_{\mathfrak{M}} = \bigvee_{f\in|\mathfrak{M}|} [\![\varphi(f)]\!]_{\mathfrak{M}} \, .$$

Let $\mathfrak{M}$ be a Boolean valued L-structure. The two-valued reduct $\widetilde{\mathfrak{M}}$ of $\mathfrak{M}$ is given by $\langle|\mathfrak{M}|, \{\tilde{R}_\zeta \mid R_\zeta$ is a relation symbol in $L\}\rangle$ where $\widetilde{\mathfrak{M}} \models R_\zeta(f_1, \cdots, f_n)$ if and only if $[\![R_\zeta(f_1, \cdots, f_n)]\!]=1$. Hence $\widetilde{\mathfrak{M}}$ is an L-structure.

A Boolean valued L-structure $\mathfrak{M}$ satisfies the maximum principle if for any L-formula $\varphi(x, x_1, \cdots, x_n)$ and for any $f_1, \cdots, f_n \in |\mathfrak{M}|$, there exists $f \in |\mathfrak{M}|$ such that

$$[\![\exists x\varphi(x, f_1, \cdots, f_n)]\!]_{\mathfrak{M}} = [\![\varphi(f, f_1, \cdots, f_n)]\!]_{\mathfrak{M}} \, .$$

A Boolean valued L-structure $\mathfrak{M}$ satisfies the finite completeness property if for any $b \in \mathfrak{M}_B$ and any $a_1, a_2 \in |\mathfrak{M}|$, there exists $a \in |\mathfrak{M}|$ such that $[\![a=a_1]\!] \geqq b$ and $[\![a=a_2]\!] \geqq \sim b$, and satisfies the almost two-valuedness property if $[\![\varphi]\!]=1$ or $[\![\varphi]\!]=0$ for any L-sentence φ.

Let L_{BA} be the first order language of Boolean algebras.

FEFERMAN-VAUGHT TYPE THEOREM ([3], [5], [6]). *For any L-sentence φ, there exists a sequence $\langle\Phi(\zeta_1, \cdots, \zeta_m); \theta_1, \cdots, \theta_m\rangle$ where $\Phi(\zeta_1, \cdots, \zeta_m)$ is an L_{BA}-formula and $\theta_1, \cdots, \theta_m$ are L-sentences, such that for any Boolean valued L-structure $\mathfrak{M}$ satisfying the maximum principle and the finite completeness property, we have $\widetilde{\mathfrak{M}} \models \varphi$ if and only if $\mathfrak{M}_B \models \Phi([\![\theta_1]\!], \cdots, [\![\theta_m]\!])$.*

The following corollary is an immediate consequence of above Feferman-Vaught type theorem.

COROLLARY. *Let $\mathfrak{M}$, $\mathfrak{M}'$ be any Boolean valued L-structures which satisfy the maximum principle, the finite completeness property and the almost two-valuedness property. Suppose that $\mathfrak{M}_B = \mathfrak{M}'_B$ and suppose that for every L-sentence φ $[\![\varphi]\!]_{\mathfrak{M}}=1$ if and only if $[\![\varphi]\!]_{\mathfrak{M}'}=1$.* Then we have $\widetilde{\mathfrak{M}} \equiv \widetilde{\mathfrak{M}}'$.

§2. A proof of the theorem.

LEMMA 1. *For any L-structure M and any topological space I,*

$$\mathfrak{M}^{(I)} = \langle |M^{(I)}|, C(I), [\![\varphi(f_1, \cdots, f_n)]\!]_{\mathfrak{M}^{(I)}} = \{i \in I \mid M \models \varphi(f_1(i), \cdots, f_n(i))\}\rangle$$

is a $C(I)$-valued L-structure which satisfies the maximum principle, the finite completeness property and the almost two-valuedness property where $C(I)$ is the Boolean algebra of all clopen subsets of I. And two-valued reduct $\widetilde{\mathfrak{M}}^{(I)}$ of $\mathfrak{M}^{(I)}$ *coincides with $M^{(I)}$.*

PROOF. For any $f_1, \cdots, f_n \in |M^{(I)}|$, $U_i = f_1^{-1}(f_1(i)) \cap \cdots \cap f_n^{-1}(f_n(i))$ is a clopen subset of I. Since we have $I_0 = \{i \in I \mid M \models \varphi(f_1(i), \cdots, f_n(i))\} = \bigcup_{i \in I_0} U_i$, and $I_1 = \{i \in I \mid M \models \neg \varphi(f_1(i), \cdots, f_n(i))\} = \bigcup_{i \in I_1} U_i$, $I_0 = [\![\varphi(f_1, \cdots, f_n)]\!]$ is a clopen subset of I. $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cdot [\![\psi]\!]$, and $[\![\neg \varphi]\!] = \sim [\![\varphi]\!]$ are easily shown. And, since $[\![\forall x \varphi(x)]\!]$ and $[\![\exists x \varphi(x)]\!]$ are clopen subsets of I, we have $[\![\forall x \varphi(x)]\!] = \bigwedge_{f \in |M^{(I)}|} [\![\varphi(f)]\!]$, $[\![\exists x \varphi(x)]\!] = \bigvee_{f \in |M^{(I)}|} [\![\varphi(f)]\!]$. Obviously $\mathfrak{M}^{(I)}$ satisfies Equality Axioms. Therefore $\mathfrak{M}^{(I)}$ is a $C(I)$-valued L-structure.

For every $i \in I$, if $M \models \exists x \varphi(x, f_1(i), \cdots, f_n(i))$, then fix a_i such that $M \models \varphi(a_i, f_1(i), \cdots, f_n(i))$, otherwise fix $a_i \in |M|$, arbitrarily. We define the relation $\sim$ on I as follows: $i \sim i'$ $(i, i' \in I)$ if and only if $f_1(i) = f_1(i')$, $\cdots$, $f_n(i) = f_n(i')$. This is an equivalence relation. Let $\{i_\alpha\}_\alpha$ be the representatives of the equivalence classes. We define $h: I \to |M|$ as follows; for any $i \in I$, $h(i) = a_{i_\alpha}$ if and only if $i \sim i_\alpha$. For any $M' \subset |M|$, we have $h^{-1}(M') = \bigcup_{a_{i_\alpha} \in M'} U_{i_\alpha}$. Hence h is continuous. Considering that $M \models \exists x \varphi(x, f_1(i), \cdots, f_n(i))$ if and only if $M \models \varphi(h(i), f_1(i), \cdots, f_n(i))$, we have $[\![\exists x \varphi(x, f_1, \cdots, f_n)]\!] = [\![\varphi(h, f_1, \cdots, f_n)]\!]$. Therefore $\mathfrak{M}^{(I)}$ satisfies the maximum principle. It is easy to show that $\mathfrak{M}^{(I)}$ satisfies the finite completeness property and the almost two-valuedness property. From the definition of two-valued reduct $\widetilde{\mathfrak{M}}^{(I)}$ of $\mathfrak{M}^{(I)}$, $\widetilde{\mathfrak{M}}^{(I)}$ coincides with $M^{(I)}$.

Let M be an L-structure, X be an nonempty set and F be a filter on X. We can regard the reduced power M^X/F as a Boolean valued L-structure $\mathfrak{M}^X/F$ in the usual way; i.e., $|\mathfrak{M}^X/F| = |M^X/F|$, $(\mathfrak{M}^X/F)_B = 2^X/F$ and for any L-formula $\varphi(x_1, \cdots, x_n)$,

$$[\![\varphi([f_1], \cdots, [f_n])]\!]_{\mathfrak{M}^X/F} = [\{x \in X \mid M \models \varphi(f_1(i), \cdots, f_n(i))\}]_F ,$$

where $[X']_F$ is an equivalence class of a characteristic function of $X' \subset X$.

It is easy to show that the following Lemma 2 holds.

LEMMA 2. *For any L-structure M, any nonempty set X and any filter on X, $\mathfrak{M}^X/F$ satisfies the maximum principle, the finite completeness property and the almost two-valuedness property. And two-valued reduct $\widetilde{\mathfrak{M}^X/F}$ of $\mathfrak{M}^X/F$ coincides with M^X/F.*

THEOREM. *Every topological power of an L-structure M is elementarily equivalent to a reduced power of M, and every reduced power of M is elementarily equivalent to a Boolean topological power of M.*

PROOF. For any topological space I, there exists an nonempty set X and a filter F on X such that $C(I) \equiv 2^X/F$. From Lemma 1 and Lemma 2, $\mathfrak{M}^{(I)}$ and $\mathfrak{M}^X/F$ are Boolean valued L-structures which satisfy the maximum principle, the finite completeness property and the almost two-valuedness property. And for any L-sentence φ, the conditions $[\![\varphi]\!]_{\mathfrak{M}^{(I)}}=1$, $M\models\varphi$ and $[\![\varphi]\!]_{\mathfrak{M}^X/F}=1$ are mutually equivalent. Therefore, from Corollary of Feferman-Vaught type theorem, we have $\widetilde{\mathfrak{M}}^{(I)}\equiv\widetilde{\mathfrak{M}}^X/F$. Hence $M^{(I)}\equiv M^X/F$ from Lemma 1 and Lemma 2. For any nonempty set X and any filter F on X, from Stone representation theorem, we have $C((2^X/F)^*)\cong 2^X/F$, where $(2^X/F)^*$ is the dual space of $2^X/F$. From Lemma 1, Lemma 2 and Corollary of Feferman-Vaught type theorem, $\widetilde{\mathfrak{M}}^{((2^X/F)^*)}\equiv\widetilde{\mathfrak{M}}^X/F$. Hence we get $M^{((2^X/F)^*)}\equiv M^X/F$.

§3. **Sheaves of L-structures.**

In this section, we admit the empty L-structure. Most of our notations and definitions are taken from Ellerman [4]. Let I be a topological space. We can regard the set $\mathcal{O}(I)$ of open subsets of I as a category in the usual way, i.e., the objects of $\mathcal{O}(I)$ are the open subsets of I, and $\mathcal{O}(I)$ has one morphism from u to v if and only if $u\subset v$. Let M_L be the category of L-structures and homomorphisms (not necessary onto). A functor $P\colon \mathcal{O}(I)^{\mathrm{op}}\to M_L$ (i.e., a contravariant functor from $\mathcal{O}(I)$ to M_L) is called a presheaf of L-structures and denoted by (I, P). A presheaf of L-structures (I, P) is a sheaf of L-structures if the following conditions are satisfied:

(1) For any open set u, any open covering $\{u_\alpha \mid \alpha\in A\}$ of u and any family $\{a_\alpha \mid \alpha\in A\}$ satisfying the condition $a_\alpha\in|P(u_\alpha)|$ for every $\alpha\in A$, if $P^{u_\alpha}_{u_\alpha\cap u_\beta}(a_\alpha)=P^{u_\beta}_{u_\alpha\cap u_\beta}(a_\beta)$ for any $\alpha,\beta\in A$, then there exists exactly one $a\in|P(u)|$ such that $P^u_{u_\alpha}(a)=a_\alpha$ for every $\alpha\in A$.

(2) For any atomic relation $R(x_1,\cdots,x_n)$, any open set u, any open covering $\{u_\alpha\mid\alpha\in A\}$ of u and any $a_1,\cdots,a_n\in|P(u)|$, if $P(u_\alpha)\models R(P^u_{u_\alpha}(a_1),\cdots,P^u_{u_\alpha}(a_n))$ for every $\alpha\in A$, then $P(u)\models R(a_1,\cdots,a_n)$.

Let (I, P) be a sheaf of L-structures. We define the stalk of P at $i\in I$ to be the direct limit $P_i=\varinjlim_{u\ni i}P(u)$. For any atomic relation $R(x_1,\cdots,x_n)$ and $b_1,\cdots,b_n\in P_i$, we have $P_i\models R(b_1,\cdots,b_n)$ if and only if there exists $u\ni i$ and $a_1,\cdots,a_n\in|P(u)|$ such that $P^u_i(a_k)=b_k$ for $k=1,\cdots,n$ and that $P(u)\models R(a_1,\cdots,a_n)$, where P^u_i is the canonical map from $P(u)$ into P_i. Let (I, P) be a sheaf of L-structures. $P(I)$ is said to be global sections of (I, P).

Let M be an L-structure and I be a topological space. We define a constant M-sheaf (I, M) as follows: The universe $|M(u)|$ of $M(u)$ is the set of all continuous functions from u to the universe $|M|$ of M (where we consider that $|M|$ has the discrete topology), for every $u\in\mathcal{O}(I)$. The restriction maps $M^u_v;|M(u)|\to|M(v)|$ $(v\subset u)$ are defined by $M^u_v(f)=f\restriction v$.

If $R(x_1, \cdots, x_n)$ is an atomic relation and $f_1, \cdots, f_n \in |M(u)|$, then $M(u) \models R(f_1, \cdots, f_n)$ if and only if for all $i \in u$ $M \models R(f_1(i), \cdots, f_n(i))$. We remark that every stalk of a constant M-sheaf is isomorphic to M, and for any nonempty L-structure M and any topological space I, global sections $M(I)$ of a constant M-sheaf (I, M) is nonempty and coincides with topological power $M^{(I)}$. Constant sheaves (over Boolean spaces) are sheaves of L-structures of the form (I, M) for some topological space (Boolean space) I and L-structure M.

Let K be a class of sheaves of L-structures. An L-sentence φ is preserved under K if and only if for any $(I, P) \in K$, the conditions $\{i \in I \mid P_i \models \varphi\} = I$ and $|P(I)| \neq \phi$ imply $P(I) \models \varphi$. Let S (resp. CS, BCS) be the class of all sheaves of L-structures (resp. constant sheaves, constant sheaves over Boolean spaces). An L-sentence φ is said to be a global section sentence (resp. CS-sentence BCS-sentence) if φ is preserved under S (resp. CS, BCS).

COROLLARY 1′. *For any L-sentence φ the following three conditions are equivalent.*

1) *φ is a reduced power sentence.*
2) *φ is a CS-sentence.*
3) *φ is a BSC-sentence.*

PROOF. Let φ be a reduced power sentence. From Corollary 1, for any nonempty L-structure M and any topological space I, if $M \models \varphi$, then $M^{(I)} \models \varphi$. Therefore for any constant sheaf (I, M), if $M \models \varphi$ and $|M| \neq \phi$, then $M(I) \models \varphi$. Hence φ is a CS-sentence. Thus, 1) implies 2). Obviously 2) implies 3). Then, the proof will be complete if we will show that 3) implies 1). Let φ be a BCS-sentence. Let M be a nonempty L-structure, X be a nonempty set, and F be a filter on X. The universe $M(2^X/F)^*)$ of global sections of the constant M-sheaf $((2^X/F)^*, M)$ is nonempty. So the relation $M \models \varphi$ implies $M((2^X/F)^*) \models \varphi$. Hence, from Corollary 1, $M \models \varphi$ implies $M^X/F \models \varphi$. So φ is a reduced power sentence.

COROLLARY 2. *Every global section sentence is equivalent to a Horn sentence.*

PROOF. Since a global section sentence is a CS-sentence and also a finite direct product sentence, a global section sentence is a reduced power sentence and also a finite direct product sentence. If an L-sentence φ is a reduced power sentence and a finite direct product sentence, then φ is a reduced product sentence (see Proposition 6.2.6 (ii) in Chang-Keisler [1]). Hence a global section sentence is a reduced product sentence. Therefore a global section sentence is equivalent to a Horn sentence.

Corollary 2 is an affirmative solution of Mansfield's problem [5]. In [7] H. Volger gave a characterization of the global section sentence.

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