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On a Family of Continued-Fraction Transformations and Their Ergodic Properties

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Introduction

The simple continued-fraction expansion of real numbers is an important concept in the theory of numbers. And the continued-fraction expansion defined by Hurwitz is also important because it is the expansion by the nearest integers. These two continued-fraction expansions give rise to many interesting problems not only in the theory of numbers but also in ergodic theory. More precisely, many people, starting with Gauss and Lévy, treated endomorphisms from an interval into itself induced from these continued-fraction expansions, and obtain many interesting results; see, for example, Lévy [1], Kuzmin [2], Hurwitz [3], Shiokawa [5] and Nakada-Ito-Tanaka [6].

In this paper we treat a one-parameter family of continued-fraction expansions, which we shall call α-continued-fraction expansions. We note that these α-continued-fraction expansions reduce to the simple continued-fraction expansions in case $\alpha=1$ and to the continued-fraction expansions of Hurwitz in case $\alpha=1/2$. We treat the following three problems:

(1) To investigate the rate of approximation by the nth approximants.

(2) To determine the form of the density function of the invariant measure for the endomorphism induced from an α-continued-fraction expansion, which we shall call an α-continued-fraction transformation.

(3) To investigate how the ergodic properties of α-continued-fraction transformations change when the parameter α changes.

As for (1), we give in § 2 the following result:

(i) In case $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$, we have

$$\left|x-\frac{p_n(x,\alpha)}{q_n(x,\alpha)}\right|\leqq\frac{2}{\sqrt{5}}|q_n(x,\alpha)|^{-2}, \qquad |q_n(x,\alpha)|\geqq\left(\frac{\sqrt{5}-1}{2}\right)^{-n}.$$

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(ii) In case $(\sqrt{5}-1)/2<\alpha<1$, we have

$$\left|x-\frac{p_n(x,\alpha)}{q_n(x,\alpha)}\right|\leqq|q_n(x,\alpha)|^{-2}, \qquad |q_n(x,\alpha)|\geqq\left(\frac{\sqrt{5}-1}{2}\right)^{-n}.$$

Here $p_n(x,\alpha)/q_n(x,\alpha)$ is the nth approximant of the α-continued-fraction expansion. In this section we also derive some relation between an α-continued-fraction expansion and the simple continued-fraction expansion.

Concerning (2), we give the explicit form of the density function of the invariant measure for α-continued-fraction transformation in § 3 (See Theorems 3.1 and 3.2.) in case $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$. The essential idea used in this derivation is the same as that of [6]. We have not succeeded in deriving the form of density function in the case of $(\sqrt{5}-1)/2<\alpha<1$.

In § 4, we investigate ergodic properties of α-continued-fraction transformations. It is almost clear that these transformations are ergodic and also exact (see [7]). In this paper we only give the result stating that in the case of $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$ the values of the entropy of these transformations are independent of α and are given by $\pi^2/(6\log((\sqrt{5}-1)/2))$.

Recently, H. Nakada obtained some interesting results concerning a one-parameter family of transformations given by

$$T_\alpha x=\left|\frac{1}{x}\right|-\left[\left|\frac{1}{x}\right|\right]_\alpha$$

for $1/2\leqq\alpha\leqq1$. These transformations are closely related to, but slightly different from, the transformations we consider in this note.

In concluding these introductory remarks, we would like to thank Professors Yuji Ito and Hitoshi Nakada for their interest in the problem and for valuable advice.

§ 1. Definition and fundamental properties of α-continued-fraction transformations.

Let X_α be the interval $[\alpha-1,\alpha)$ where α is a fixed real number between 1/2 and 1 and let $[x]_\alpha=[x-(\alpha-1)]$ for any real number x. We then define a transformation S_α on X_α, which we call an α-continued-fraction transformation, by

$$(1)\qquad S_\alpha x=\begin{cases}\dfrac{1}{x}-\left[\dfrac{1}{x}\right]_\alpha & \text{for } x\neq0,\\ 0 & \text{for } x=0.\end{cases}$$

If we write, for any real number x,

$$a_0(x, \alpha)=[x]_\alpha \tag{2}$$

and

$$a(x, \alpha)=\begin{cases}\left[\dfrac{1}{x}\right]_\alpha & \text{for } x\neq 0\\ \infty & \text{for } x=0\,,\end{cases} \tag{3}$$

$$a_n(x, \alpha)=a(S_\alpha^{n-1}x', \alpha) \quad (n\geqq 1)\,, \tag{4}$$

where $x'=x-[x]_\alpha$, then we obtain the continued-fraction expansion of the following form:

$$\begin{aligned} x&=a_0(x, \alpha)+\frac{1\ |}{|\,a_1(x, \alpha)}+\frac{1\ |}{|\,a_2(x, \alpha)}+\cdots+\frac{1\ |}{|\,a_n(x, \alpha)+S_\alpha^n x'}\\ &=a_0(x, \alpha)+\frac{1\ |}{|\,a_1(x, \alpha)}+\frac{1\ |}{|\,a_2(x, \alpha)}+\cdots+\frac{1\ |}{|\,a_{n-1}(x, \alpha)}+\frac{1\ |}{|\,\omega_n(x, \alpha)}\end{aligned} \tag{5}$$

where $\omega_n(x, \alpha)=a_n(x, \alpha)+S_\alpha^n x'=1/S_\alpha^{n-1}x'$. As usual, we put

$$\frac{p_n(x, \alpha)}{q_n(x, \alpha)}=a_0(x, \alpha)+\frac{1\ |}{|\,a_1(x, \alpha)}+\cdots+\frac{1\ |}{|\,a_{n-1}(x, \alpha)} \tag{6}$$

and obtain the following formulae for all $n\geqq 1$:

$$\begin{cases} p_n(x, \alpha)=a_{n-1}(x, \alpha)p_{n-1}(x, \alpha)+p_{n-2}(x, \alpha)\,,\\ q_n(x, \alpha)=a_{n-1}(x, \alpha)q_{n-1}(x, \alpha)+q_{n-2}(x, \alpha)\,,\\ p_n(x, \alpha)q_{n-1}(x, \alpha)-p_{n-1}(x, \alpha)q_n(x, \alpha)=(-1)^n\,,\end{cases} \tag{7}$$

$$x=\frac{p_n(x, \alpha)+S_\alpha^{n-1}x' p_{n-1}(x, \alpha)}{q_n(x, \alpha)+S_\alpha^{n-1}x' q_{n-1}(x, \alpha)}\,, \tag{8}$$

where we set $p_{-1}(x, \alpha)=0$, $p_0(x, \alpha)=1$, $q_{-1}(x, \alpha)=1$ and $q_0(x, \alpha)=0$.

Let $Z(\alpha)$ be the set of all integers which appear in the α-continued-fraction expansion, that is,

$$Z(\alpha)=\{i\in \boldsymbol{Z}\cup\{\infty\};\ i=a_k(x, \alpha) \text{ for some } x\in X_\alpha \text{ and } k\in \boldsymbol{N}\} \tag{9}$$

and let us define the mapping ψ_α from X_α to $\prod_1^\infty Z(\alpha)$ by

$$\psi_\alpha(x)=(a_1(x, \alpha), a_2(x, \alpha), \cdots, a_n(x, \alpha), \cdots)\,. \tag{10}$$

We call elements of $\psi_\alpha(X_\alpha)$ the α-admissible sequences. Let σ be the shift operator on $\prod_1^\infty Z(\alpha)$ and let ρ be the formal mapping on $\prod_1^\infty Z(\alpha)$

defined by

$$\rho(\omega_1, \omega_2, \cdots, \omega_n, \cdots) = \frac{1|}{|\omega_1} + \frac{1|}{|\omega_2} + \cdots + \frac{1|}{|\omega_n} + \cdots . \tag{11}$$

Then we obtain the following:

PROPERTY 1.1. (i) $\rho \circ \psi_\alpha =$ identity map on X_α.

(ii) $\sigma \circ \psi_\alpha = \psi_\alpha \circ S_\alpha$ on X_α.

(iii) $\rho \circ \sigma = S_\alpha \circ \rho$ on $\psi_\alpha(X_\alpha)$.

We will give the proof of (i) in Remark 2.2 of §2. If we assume that (i) is valid, then (ii) and (iii) are almost clear. By (ii) and (iii), we obtain the equivalence between (X_α, S_α) and $(\psi_\alpha(X_\alpha), \sigma)$, the symbolical dynamical system associated with the α-continued-fraction transformation.

For any integer a and any element ω of $\prod_1^\infty Z(\alpha)$, let $a \cdot \omega$ be the element of $\prod_1^\infty Z(\alpha)$ defined by

$$(a \cdot \omega)(n) = \begin{cases} a & n = 1 , \\ \omega(n-1) & n \geqq 2 . \end{cases} \tag{12}$$

Let ω_α and $\omega_{\alpha-1}$ be the elements of $\prod_1^\infty Z(\alpha)$ defined by

$$\begin{cases} \omega_\alpha = \left[\frac{1}{\alpha}\right]_\alpha \cdot \psi_\alpha\left(\frac{1}{\alpha} - \left[\frac{1}{\alpha}\right]_\alpha\right) , \\ \omega_{\alpha-1} = \psi_\alpha(\alpha - 1) . \end{cases} \tag{13}$$

Then we obtain the following:

PROPERTY 1.2. $\rho(\omega_\alpha) = \alpha$.

PROOF. It is clear that $1/\alpha - [1/\alpha]_\alpha \in X_\alpha$. So, by (i) of Property 1.1, we get $\rho(\psi_\alpha(1/\alpha - [1/\alpha]_\alpha) = 1/\alpha - [1/\alpha]_\alpha$, and we have

$$\rho(\omega_\alpha) = \frac{1}{\left[\frac{1}{\alpha}\right]_\alpha + \rho\left(\psi_\alpha\left(\frac{1}{\alpha} - \left[\frac{1}{\alpha}\right]_\alpha\right)\right)} = \alpha .$$

It is convenient to define the order $\prec$ in the set $Z(\alpha)$ as follows:

$$\omega_{\alpha-1}(1) \prec \omega_{\alpha-1}(1) - 1 \prec \cdots \prec -k \prec -k-1 \prec \cdots \prec \infty \prec \cdots$$
$$\prec l \prec l-1 \prec \cdots \prec \omega_\alpha(1) + 1 \prec \omega_\alpha(1) .$$

And let us define the order in the set $\prod_1^\infty Z(\alpha)$ in the following manner. Let ω and ω' be two elements of $\prod_1^\infty Z(\alpha)$, then $\omega < \omega'$ will mean that there exists a natural number n such that $\omega(i) = \omega'(i)$ for $i = 1, \cdots, n-1$

and that $\omega(n) \succ \omega'(n)$ (resp. $\omega(n) \prec \omega'(n)$) if n is even (resp. odd). Then we obtain the following two lemmas:

LEMMA 1.1. *An element ω of $\prod_1^\infty Z(\alpha)$ is α-admissible if and only if the values $\rho(\sigma^i\omega)$ are well-defined and lie in X_α for all $i=0, 1, 2, \cdots$.*

PROOF. The "only if" part is clear. Suppose that the latter condition is satisfied, and let $x=\rho(\omega)$. Then $x=1/(a_1(x, \alpha)+S_\alpha x)$, $\rho(\omega)=1/(\omega(1)+\rho(\sigma\omega))$ and $\rho(\sigma\omega) \in X_\alpha$; so we get $a_1(x, \alpha)=\omega(1)$. In the same manner we can show $a_n(x, \alpha)=\omega(n)$ for any $n \geqq 1$, and so we obtain $\psi_\alpha(x)=\omega$, which means that ω is α-admissible.

LEMMA 1.2. *Let ω be an α-admissible element of $\prod_1^\infty Z(\alpha)$ and let a be an integer which satisfies $\omega_{\alpha-1} \leqq a \cdot \omega < \omega_\alpha$. Then $a \cdot \omega$ is also α-admissible.*

PROOF. By Lemma 1.1, it is sufficient to show that $\alpha-1 \leqq \rho(a \cdot \omega) < \alpha$. Let us show that $\rho(a \cdot \omega) < \alpha$. If $a \prec \omega_\alpha(1)$, then $a \prec \omega_\alpha(1)+1$; so

$$\rho(a \cdot \omega)=\frac{1}{a+\rho(\omega)} \leqq \frac{1}{\omega_\alpha(1)+1+\rho(\omega)} ,$$

and from $\rho(\omega) \geqq \alpha-1 > \rho(\sigma\omega_\alpha)-1$ it follows that

$$\rho(a \cdot \omega) < \frac{1}{\omega_\alpha(1)+\rho(\sigma\omega_\alpha)}=\alpha .$$

On the other hand, if $a=\omega_\alpha(1)$, then $\omega > \sigma\omega_\alpha$; so we get $\rho(\omega) > \rho(\sigma\omega_\alpha)$ because ω and $\sigma\omega_\alpha$ are both α-admissible. Therefore, we have

$$\rho(a \cdot \omega)=\frac{1}{a+\rho(\omega)} < \frac{1}{\omega_\alpha(1)+\rho(\sigma\omega_\alpha)}=\alpha .$$

We can show by essentially the same manner that $\alpha-1 \leqq \rho(a \cdot \omega)$.

It is convenient to extend the notion of α-admissibility to the set $\prod_{-\infty}^\infty Z(\alpha)$. We call the element ω of the set $\prod_{-\infty}^\infty Z(\alpha)$ α-admissible if and only if $(\omega(k), \omega(k+1), \cdots)$ is α-admissible as the element of $\prod_1^\infty Z(\alpha)$ for any $k \in \boldsymbol{Z}$. We use this notion in § 3.

§ 2. Approximation theory.

In this section, we discuss the approximation by α-continued-fraction expansions, in analogy with the approximation by the usual continued-fraction expansion (the case of $\alpha=1$). Let x be any fixed real number and let $a_n(\alpha)$, $p_n(\alpha)$ and $q_n(\alpha)$ be the values defined in § 1. (To simplify the matter, we omit x in these notations.) Also let $x_n(\alpha)=S_\alpha^n x'$. Moreover, in the case of $\alpha=1$, we also omit α and write simply a_n, p_n, q_n and x_n. We then consider the following several statements concerning these quantities for any pair of non-negative integers n and m:

$$A_{n,m}^{\pm}: \begin{cases} p_n(\alpha)=\pm p_m , & p_{n+1}(\alpha)=\pm p_{m+1} , \\ q_n(\alpha)=\pm q_m , & q_{n+1}(\alpha)=\pm q_{m+1} , \\ x_n(\alpha)=x_m \in [0, \alpha) , \end{cases}$$

$$B_{n,m}^{\pm}: \begin{cases} p_n(\alpha)=\pm p_m , & p_{n+1}(\alpha)=\pm p_{m+2} , \\ q_n(\alpha)=\pm q_m , & q_{n+1}(\alpha)=\pm q_{m+2} , \\ x_n(\alpha)=x_m-1 \in [\alpha-1, 0) , \end{cases}$$

$$C_{n,m}^{\pm}: \begin{cases} p_n(\alpha)=\pm p_{m+1} , & p_{n+1}(\alpha)=\mp p_{m+2} , \\ q_n(\alpha)=\pm q_{m+1} , & q_{n+1}(\alpha)=\mp q_{m+2} , \\ x_n(\alpha)=-x_{m+1} \in [\alpha-1, 0) , \end{cases}$$

$$D_{n,m}^{\pm}: \begin{cases} p_n(\alpha)=\pm p_{m+1} , & p_{n+1}(\alpha)=\mp (p_{m+2}+p_{m+1}) , \\ q_n(\alpha)=\pm q_{m+1} , & q_{n+1}(\alpha)=\mp (q_{m+2}+q_{m+1}) , \\ x_n(\alpha)=1-x_{m+1} \in (0, \alpha) . \end{cases}$$

Now we have the following

LEMMA 2.1. (i) *$A_{n,m}^{\pm}$ must be followed by either $A_{n+1,m+1}^{\pm}$ or $B_{n+1,m+1}^{\pm}$.*
(ii) *$B_{n,m}^{\pm}$ must be followed by either $C_{n+1,m+1}^{\pm}$ or $D_{n+1,m+1}^{\pm}$.*
(iii) *$C_{n,m}^{\pm}$ must be followed by either $C_{n+1,m+1}^{\mp}$ or $D_{n+1,m+1}^{\mp}$.*
(iv) *$D_{n,m}^{\pm}$ must be followed by one of the following six statements: $A_{n+2,m+2}^{\mp}$, $B_{n+2,m+2}^{\mp}$, $C_{n+2,m+2}^{\mp}$, $D_{n+2,m+2}^{\mp}$, $A_{n+1,m+3}^{\mp}$, $B_{n+1,m+3}^{\mp}$.*

PROOF. We only prove the statements concerning $p_n(\alpha)$ and p_m, because those for $q_n(\alpha)$ and q_m can be proved in the same manner.

(i) If $A_{n,m}^{\pm}$ is satisfied, then we have

$$x_{m+1}=\frac{1}{x_m}-a_{m+1} , \qquad x_{n+1}(\alpha)=\frac{1}{x_m}-a_{n+1}(\alpha) .$$

In case $0\leqq x_{m+1}<\alpha$, we have

$$a_{n+1}(\alpha)=a_{m+1} , \qquad x_{n+1}(\alpha)=x_{m+1} ,$$
$$p_{n+2}(\alpha)=\pm a_{m+1}p_{m+1}\pm p_m=\pm p_{m+2} ,$$

so we get $A_{n+1,m+1}^{\pm}$. In case $\alpha\leqq x_{m+1}<1$, we have

$$a_{n+1}(\alpha)=a_{m+1}+1\ ,\qquad x_{n+1}(\alpha)=x_{m+1}-1\ ,$$
$$a_{m+2}=1\ ,\qquad p_{m+3}=p_{m+2}+p_{m+1}\ ,$$
$$p_{n+2}(\alpha)=\pm(a_{m+1}+1)p_{m+1}\pm p_m=\pm(a_{m+1}p_{m+1}+p_m)\pm p_{m+1}\ ,$$
$$=\pm(p_{m+2}+p_{m+1})=\pm p_{m+3}\ ,$$

so we get $B^{\pm}_{n+1,m+1}$.

We omit the proof of (ii) and (iii), for these cases can be proved in the same way as (i).

(iv) Suppose that $D^{\pm}_{n,m}$ is satisfied. In case $1-\alpha<x_{m+1}<1-1/(\alpha+1)$, we have

$$a_{n+1}(\alpha)=1\ ,\qquad p_{n+2}(\alpha)=\mp(p_{m+2}+p_{m+1})\pm p_{m+1}=\mp p_{m+2}\ ,$$
$$x_{n+1}(\alpha)=\frac{1}{1-x_{m+1}}-1=\frac{x_{m+1}}{1-x_{m+1}}\ ,$$
$$x_{n+2}(\alpha)=\frac{1-x_{m+1}}{x_{m+1}}-a_{n+2}(\alpha)=\frac{1}{x_{m+1}}-1-a_{n+2}(\alpha)\ ,$$
$$x_{m+2}=\frac{1}{x_{m+1}}-a_{m+2}\ .$$

Then, if $0\leqq x_{m+2}<\alpha$, we have

$$a_{n+2}(\alpha)=a_{m+2}-1\ ,\qquad x_{n+2}(\alpha)=x_{m+2}\ ,$$
$$p_{n+3}(\alpha)=\mp(a_{m+2}-1)p_{m+2}\mp(p_{m+2}+p_{m+1})=\mp p_{m+3}\ ,$$

so we get $A^{\mp}_{n+2,m+2}$; and, if $\alpha\leqq x_{m+2}<1$, we have

$$a_{m+3}=1\ ,\qquad p_{m+4}=p_{m+3}+p_{m+2}\ ,$$
$$a_{n+2}(\alpha)=a_{m+2}\ ,\qquad x_{n+2}(\alpha)=x_{m+2}-1\ ,$$
$$p_{n+3}(\alpha)=\mp a_{m+2}p_{m+2}\mp(p_{m+2}+p_{m+1})=\mp p_{m+4}\ ,$$

so we get $B^{\mp}_{n+2,m+2}$. In case $1-1/(\alpha+1)\leqq x_{m+1}\leqq 1/2$ (in this case we have $1/2\leqq x_n(\alpha)\leqq 1/(\alpha+1)$), we have

$$a_{m+2}=2\ ,\qquad a_{n+1}(\alpha)=2\ ,$$
$$x_{m+2}=\frac{1}{x_{m+1}}-2=\frac{1-2x_{m+1}}{x_{m+1}}\ ,\qquad x_{n+1}(\alpha)=\frac{1}{1-x_{m+1}}-2=\frac{2x_{m+1}-1}{1-x_{m+1}}\ ,$$
$$p_{m+3}=2p_{m+2}+p_{m+1}\ ,\qquad p_{n+2}(\alpha)=\mp 2(p_{m+2}+p_{m+1})\pm p_{m+1}=\mp p_{m+3}\ ,$$
$$x_{m+3}=\frac{x_{m+1}}{1-2x_{m+1}}-a_{m+3}\ ,$$
$$x_{n+2}(\alpha)=\frac{1-x_{m+1}}{2x_{m+1}-1}-a_{n+2}(\alpha)=-\frac{x_{m+1}}{1-2x_{m+1}}-1-a_{n+2}(\alpha)\ .$$

Then, if $0 \leqq x_{m+3} \leqq 1-\alpha$, we have

$$a_{n+2}(\alpha) = -(a_{m+3}+1)\,, \qquad x_{n+2}(\alpha) = -x_{m+3}\,,$$
$$p_{n+3}(\alpha) = \pm(a_{m+3}+1)p_{m+3} \mp (p_{m+2}+p_{m+1})$$
$$= \pm(a_{m+3}p_{m+3}+p_{m+2}) = \pm p_{m+4}\,,$$

so we get $C^{\pm}_{n+2,m+2}$; and, if $1-\alpha < x_{m+3} < 1$, we have

$$a_{n+2}(\alpha) = -(a_{m+3}+2)\,, \qquad x_{n+2}(\alpha) = 1-x_{m+3}\,,$$
$$p_{n+3}(\alpha) = \pm(p_{m+4}+p_{m+3})\,,$$

as above, and so we get $D^{\mp}_{n+2,m+2}$. In case $1/2 < x_{m+1} < 1$, we have

$$a_{m+2} = 1\,, \qquad x_{m+2} = \frac{1}{x_{m+1}} - 1\,,$$
$$p_{m+3} = p_{m+2}+p_{m+1} \quad (\text{so, } p_{n+1}(\alpha) = \mp p_{m+3})\,,$$
$$x_{m+3} = \frac{x_{m+1}}{1-x_{m+1}} - a_{m+3}\,,$$
$$x_{n+1}(\alpha) = \frac{1}{1-x_{m+1}} - a_{n+1}(\alpha) = \frac{x_{m+1}}{1-x_{m+1}} + 1 - a_{n+1}(\alpha)\,.$$

Then, if $0 \leqq x_{m+3} < \alpha$, we have

$$a_{n+1}(\alpha) = a_{m+3}+1\,, \qquad x_{n+1}(\alpha) = x_{m+3}\,,$$
$$p_{n+2}(\alpha) = \mp(a_{m+3}+1)p_{m+3} \pm p_{m+1} = \mp p_{m+4}\,,$$

so we get $A^{\mp}_{n+1,m+3}$; and if $\alpha \leqq x_{m+3} < 1$, we have

$$a_{n+1}(\alpha) = a_{m+3}+2\,, \qquad x_{n+1}(\alpha) = x_{m+3}-1\,,$$
$$a_{m+4} = 1\,, \qquad p_{m+5} = p_{m+4}+p_{m+3}\,,$$
$$p_{n+2}(\alpha) = \mp(p_{m+4}+p_{m+3}) = \mp p_{m+5}\,,$$

so we get $B^{\mp}_{n+1,m+3}$.

LEMMA 2.2. *For any non-negative integer n, there exists an integer $m \geqq n$ such that either*

(14) $$p_n(\alpha) = \pm p_m\,, \qquad q_n(\alpha) = \pm q_m\,,$$

or

(15) $$p_n(\alpha) = \pm(p_{m+1}+p_m)\,, \qquad q_n(\alpha) = \pm(q_{m+1}+q_m)\,,$$

is valid.

PROOF. In case $0 \leqq x - a_0 < \alpha$, we have

$$a_0(\alpha) = a_0\,, \qquad x_0(\alpha) = x_0\,, \qquad p_1(\alpha) = p_1\,, \qquad q_1(\alpha) = q_1\,,$$

so we get $A^{+}_{0,0}$. In case $\alpha \leqq x - a_0 < 1$, we have

$$a_1 = 1\,, \qquad p_2 = a_0 + 1\,, \qquad q_2 = 1\,,$$
$$a_0(\alpha) = a_0 + 1\,, \qquad x_0(\alpha) = x_0 - 1\,,$$
$$p_1(\alpha) = a_0 + 1 = p_2\,, \qquad q_1(\alpha) = 1 = q_2\,,$$

so we get $B^{+}_{0,0}$. So, by using Lemma 2.1 inductively, we obtain the conclusion.

REMARK 2.1. (i) In case $\alpha = 1/2$, we can show in Lemma 2.1 that from $D^{\pm}_{n,m}$ only two case $A^{\mp}_{n+1,m+3}$ and $B^{\mp}_{n+1,m+3}$ follow; so we can show in Lemma 2.2 that only (14) is valid in this case.

(ii) In case $1/2 < \alpha < (\sqrt{5} - 1)/2$, we can show in Lemma 2.1 that from $D^{\pm}_{n,m}$ follow only four cases, $C^{\mp}_{n+2,m+2}$, $D^{\mp}_{n+2,m+2}$, $A^{\mp}_{n+1,m+3}$ and $B^{\mp}_{n+1,m+3}$, so we can show, in addition to Lemma 2.2, that $|p_n(\alpha)|$ and $|q_n(\alpha)|$ are increasing in n.

In the usual continued-fraction expansion we have

$$\left| x - \frac{p_n}{q_n} \right| \leqq \frac{2}{\sqrt{5}} q_n^{-2}\,, \qquad q_n > \left(\frac{\sqrt{5} - 1}{2} \right)^{-n} \tag{16}$$

(see [4]). Using these relations, we have the following

PROPOSITION 2.1. (i) *In case* $1/2 \leqq \alpha \leqq (\sqrt{5} - 1)/2$, *we have*

$$\left| x - \frac{p_n(\alpha)}{q_n(\alpha)} \right| \leqq \frac{2}{\sqrt{5}} |q_n(\alpha)|^{-2}\,, \qquad |q_n(\alpha)| > \left(\frac{\sqrt{5} - 1}{2} \right)^{-n}. \tag{17}$$

(ii) *In case* $(\sqrt{5} - 1)/2 < \alpha < 1$, *we have*

$$\left| x - \frac{p_n(\alpha)}{q_n(\alpha)} \right| \leqq |q_n(\alpha)|^{-2}\,, \qquad |q_n(\alpha)| > \left(\frac{\sqrt{5} - 1}{2} \right)^{-n}. \tag{18}$$

PROOF. In the case when (14) is valid, we have

$$\left| x - \frac{p_n(\alpha)}{q_n(\alpha)} \right| = \left| x - \frac{p_m}{q_m} \right| \leqq \frac{2}{\sqrt{5}} q_m^{-2} = \frac{2}{\sqrt{5}} |q_n(\alpha)|^{-2}$$

and

$$|q_n(\alpha)| = q_m > \left(\frac{\sqrt{5} - 1}{2} \right)^{-m} \geqq \left(\frac{\sqrt{5} - 1}{2} \right)^{-n}.$$

In the case when (15) is valid, we have

$$\left|x-\frac{p_n(\alpha)}{q_n(\alpha)}\right|=\left|\frac{p_{m+1}+x_m p_m}{q_{m+1}+x_m q_m}-\frac{p_{m+1}+p_m}{q_{m+1}+q_m}\right|$$
$$=\frac{1-x_m}{(q_{m+1}+x_m q_m)(q_{m+1}+q_m)}\,.$$

But, in this case we have $x_m\in(1-\alpha,1)$, so

$$\frac{1-x_m}{q_{m+1}+x_m q_m}\leqq\frac{\alpha}{q_{m+1}+(1-\alpha)q_m}\leqq\frac{\alpha}{(1-\alpha/2)(q_{m+1}+q_m)}\,.$$

So we get

$$\left|x-\frac{p_n(\alpha)}{q_n(\alpha)}\right|\leqq\frac{\alpha}{1-\alpha/2}|q_n(\alpha)|^{-2}$$

and

$$|q_n(\alpha)|=q_{m+1}+q_m\geqq q_m>\left(\frac{\sqrt{5}-1}{2}\right)^{-m}\geqq\left(\frac{\sqrt{5}-1}{2}\right)^{-n}.$$

It is sufficient to note that from $\alpha\leqq(\sqrt{5}-1)/2$ it follows that $\alpha/(1-\alpha/2)\leqq 2/\sqrt{5}$, and from $\alpha\leqq1$ it follows that $\alpha/(1-\alpha/2)\leqq1$.

REMARK 2.2. Now we can prove (i) of Property 1.1, noting that

$$x=\lim_{n\to\infty}\frac{p_n(\alpha)}{q_n(\alpha)}=\rho(\psi_\alpha(x))\quad\text{for any}\quad x\in X_\alpha\,.$$

§ 3. The density function of the S_α-invariant measure.

For any Borel set A and any integer a, let $1/(a+A)$ be the set of real numbers $1/(a+x)$, $x\in A$. The following sublemma is essential for the calculation of the density function.

SUBLEMMA. *For any Borel set A, an integer a and a real number x, we have*

$$\frac{1}{(a+x)^2}\int_A\frac{dy}{\left(1+\dfrac{1}{a+x}y\right)^2}=\int_{1/(a+A)}\frac{dy}{(1+xy)^2}\,. \tag{19}$$

PROOF. It is sufficient to establish the following identity for any real numbers s and t $(s<t)$, which is proved straight-forwardly:

$$\frac{1}{(a+x)^2}\int_s^t\frac{dy}{\left(1+\dfrac{1}{a+x}y\right)^2}=\int_{1/(a+t)}^{1/(a+s)}\frac{dy}{(1+xy)^2}\,.$$

Let us define the set $R_\alpha(x)$ of real numbers for any $x \in X_\alpha$ as follows:

$$R_\alpha(x)=\left\{\frac{1|}{|a_0}+\frac{1|}{|a_{-1}}+\frac{1|}{|a_{-2}}+\cdots;\ (\cdots a_{-2}a_{-1}a_0\cdot\psi_\alpha(x))\ \text{is } \alpha\text{-admissible}\right\}. \tag{20}$$

Then we have the following

LEMMA 3.1. *If $R_\alpha(x)$ is a Borel set and satisfies*

$$R_\alpha(x)=\sum_{a\cdot\psi_\alpha(x):\alpha\text{-admissible}} \frac{1}{a+R_\alpha\left(\dfrac{1}{a+x}\right)} \quad a.e. \tag{21}$$

for every $x \in X_\alpha$ (here symbol $\sum$ means the disjoint union), then

$$f(x)=\int_{R_\alpha(x)} \frac{dy}{(1+xy)^2}$$

gives the density function of an S_α-invariant measure.

PROOF. It is sufficient to show that

$$f(x)=\sum_{a\cdot\psi_\alpha(x):\alpha\text{-admissible}} \frac{1}{(a+x)^2} f\left(\frac{1}{a+x}\right); \tag{22}$$

but this can easily be proved by using sublemma and the assumption of this lemma.

Now let us calculate the density function of the S_α-invariant probability measure for several α's.

(I) Tne case of $1/2 \leqq \alpha \leqq 2-\sqrt{2}$.

LEMMA 3.2. *If $1/2 \leqq \alpha \leqq 2-\sqrt{2}$, then*

$$\begin{cases} \omega_\alpha=(2, b_2, b_3, b_4, \cdots), \\ \omega_{\alpha-1}=(-2, b_2+1, b_3, b_4, \cdots), \end{cases} \tag{23}$$

where $-3 < b_2 < \infty$ ($b_2=\infty$ in the particular case $\alpha=1/2$; $b_2=-3$ in case $\alpha=2-\sqrt{2}$).

PROOF. It is clear from

$$\alpha-1 \leqq \frac{1}{\alpha}-2 < \alpha \Longleftrightarrow \sqrt{2}-1 < \alpha \leqq \frac{\sqrt{5}-1}{2}, \tag{24}$$

$$\alpha-1\leqq\frac{1}{\alpha-1}+2<\alpha \Longleftrightarrow \frac{\sqrt{5}-3}{2}<\alpha\leqq 2-\sqrt{2}\ , \tag{25}$$

that $\omega_\alpha(1)=2$ and $\omega_{\alpha-1}(1)=-2$ if and only if $1/2\leqq\alpha\leqq 2-\sqrt{2}$. Now let $\omega_\alpha=(2, b_2, b_3, b_4, \cdots)$, then we have $\rho(\omega_\alpha)=\alpha$, and from the relation

$$a+\frac{1|}{|2}+\frac{1|}{|b+x}=(a+1)+\frac{1|}{|-2}+\frac{1|}{|(b+1)+x}\ , \tag{26}$$

(which can easily be shown), we get $1+\rho(-2, b_2+1, b_3, \cdots)=\alpha$, that is, $\rho(-2, b_2+1, b_3, \cdots)=\alpha-1$. But, from this, we obtain $\rho(b_2+1, b_3, \cdots)=1/(\alpha-1)+2$, so, by using Lemma 1.2, we see that the sequence $(-2, b_2+1, b_3, \cdots)$ is α-admissible. So we have $\omega_{\alpha-1}=(-2, b_2+1, b_3, \cdots)$. The inequality $-3<b_2<\infty$ can be shown easily.

From here to Theorem 3.1, we assume that $1/2\leqq\alpha\leqq 2-\sqrt{2}$. Let us define two functions $m_1(x)$ and $m_2(x)$ as follows:

$$m_1(x)=\begin{cases}1-\beta & \text{if } \psi_\alpha(x)\leqq\sigma\omega_\alpha \\ \beta & \text{if } \psi_\alpha(x)>\sigma\omega_\alpha\ ,\end{cases} \tag{27}$$

$$m_2(x)=\begin{cases}-\beta & \text{if } \psi_\alpha(x)\leqq\sigma\omega_{\alpha-1} \\ \beta-1 & \text{if } \psi_\alpha(x)>\sigma\omega_{\alpha-1}\ ,\end{cases} \tag{28}$$

where $\beta=(\sqrt{5}-1)/2$. Then we obtain the following Lemmas 3.3–3.6.

LEMMA 3.3. *Let $a\cdot\psi_\alpha(x)$ and $(a+1)\cdot\psi_\alpha(x)$ be both α-admissible. Then we have*

$$\frac{1}{a+m_1\left(\frac{1}{a+x}\right)}=\frac{1}{(a+1)+m_2\left(\frac{1}{(a+1)+x}\right)}\ . \tag{29}$$

(*Here we note that $\psi_\alpha(1/(a+x))=a\cdot\psi_\alpha(x)$.*)

RROOF. It is sufficient to show that

$$m_1\left(\frac{1}{a+x}\right)=1+m_2\left(\frac{1}{(a+1)+x}\right)\ .$$

If $a\succ b_2$ or if $a=b_2$ and $\psi_\alpha(x)<\sigma^2\omega_\alpha$ $(=\sigma^2\omega_{\alpha-1})$, then we have $a\cdot\psi_\alpha(x)>\sigma\omega_\alpha$ and $(a+1)\cdot\psi_\alpha(x)>\sigma\omega_{\alpha-1}$; so we get $m_1(1/(a+x))=\beta$ and $m_2(1/((a+1)+x))=\beta-1$. On the other hand, if $a=b_2$ and $\psi_\alpha(x)\geqq\sigma^2\omega_\alpha$ $(=\sigma^2\omega_{\alpha-1})$ or if $a\prec b_2$, then we have $a\cdot\psi_\alpha(x)\leqq\sigma\omega_\alpha$ and $(a+1)\cdot\psi_\alpha(x)\leqq\sigma\omega_{\alpha-1}$; so we get $m_1(1/(a+x))=1-\beta$ and $m_2(1/((a+1)+x))=-\beta$.

Lemma 3.4. *If $a_n \cdots a_2 a_1 \cdot \psi_\alpha(x)$ is α-admissible, then we have*

$$m_2(x) \leqq \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots + \frac{1|}{|a_n} \leqq m_1(x) \ . \tag{30$_n$}$$

Proof. Let us prove this lemma by induction on n. Let $a_1 \cdot \psi_\alpha(x)$ be α-admissible. Then $\psi_\alpha(x) \leqq \sigma\omega_\alpha$ (or $\psi_\alpha(x) > \sigma\omega_\alpha$) yields $a_1 < 3$ (or $a_1 < 2$, respectively) and $\psi_\alpha(x) \leqq \sigma\omega_{\alpha-1}$ (or $\psi_\alpha(x) > \sigma\omega_{\alpha-1}$) yields $a_1 > -2$ (or $a_1 > -3$, respectively), so we get $(30)_1$, that is,

$$m_2(x) \leqq \frac{1}{a_1} \leqq m_1(x) \ .$$

Now let us assume that $(30)_n$ is valid for some $n \geqq 1$. Let $a_{n+1} a_n \cdots a_2 a_1 \cdot \psi_\alpha(x)$ be α-admissible, then by the induction hypothesis we have

$$m_2\left(\frac{1}{a_1+x}\right) \leqq \frac{1|}{|a_2} + \cdots + \frac{1\ |}{|a_{n+1}} \leqq m_1\left(\frac{1}{a_1+x}\right) ,$$

from which it follows that

$$\frac{1}{a_1 + m_1\left(\dfrac{1}{a_1+x}\right)} \leqq \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots + \frac{1\ |}{|a_{n+1}} \leqq \frac{1}{a_1 + m_2\left(\dfrac{1}{a_1+x}\right)} \ . \tag{31}$$

If $\psi_\alpha(x) \leqq \sigma\omega_{\alpha-1}$, then, by Lemma 3.3 we have

$$\frac{1}{a_1 + m_1\left(\dfrac{1}{a_1+x}\right)} \geqq \frac{1}{-2 + m_1\left(\dfrac{1}{-2+x}\right)} ,$$

but, since $(-2) \cdot \psi_\alpha(x) < \sigma\omega_\alpha$, we have

$$\frac{1}{-2 + m_1\left(\dfrac{1}{-2+x}\right)} = \frac{1}{-2+1-\beta} = -\beta = m_2(x) \ .$$

On the other hand if $\psi_\alpha(x) > \sigma\omega_{\alpha-1}$, then we have

$$\frac{1}{a_1 + m_1\left(\dfrac{1}{a_1+x}\right)} \geqq \frac{1}{-3 + m_1\left(\dfrac{1}{-3+x}\right)} \ .$$

In this case we can show $(-3) \cdot \psi_\alpha(x) \leqq \sigma\omega_\alpha$ in the following manner: If $1/2 \leqq \alpha < 2 - \sqrt{2}$, then $b_2 > -3$, so we have $(-3) \cdot \psi_\alpha(x) \leqq \sigma\omega_\alpha$. If $\alpha = 2 - \sqrt{2}$, then $b_2 = -3$, but this time we have $\sigma\omega_{\alpha-1} = \sigma^2\omega_\alpha$, so $\psi_\alpha(x) > \sigma\omega_{\alpha-1} = \sigma^2\omega_\alpha$

implies $(-3)\cdot\psi_\alpha(x)<\sigma\omega_\alpha$. So, we have

$$\frac{1}{-3+m_1\left(\dfrac{1}{-3+x}\right)}=\frac{1}{-3+1-\beta}=\beta-1=m_2(x)\ .$$

In the same manner we can show that

$$\frac{1}{a_1+m_2\left(\dfrac{1}{a_1+x}\right)}\leqq m_1(x)\ ,$$

so from (31) we get $(30)_{n+1}$, which completes the proof.

LEMMA 3.5. *If* $\cdots a_n\cdots a_2\,a_1\cdot\psi_\alpha(x)$ *is* α*-admissible, then the function* $\rho(a_1, a_2, \cdots, a_n, \cdots)$ *is well-defined, that is, there exists a limit of the sequence*

$$\left\{\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_n};\ n=1, 2, \cdots\right\}\ . \tag{32}$$

PROOF. As in the proof of Lemma 3.4, we can show that, for any natural number n and m,

$$\frac{1|}{|a_1}+\cdots+\frac{1|}{|a_n+x'}\leqq\frac{1|}{|a_1}+\cdots+\frac{1|}{|a_{n+m}}\leqq\frac{1|}{|a_1}+\cdots+\frac{1|}{|a_n+x''}\ , \tag{33}$$

where

$$x'=m_1\left(\frac{1|}{|a_n}+\cdots+\frac{1|}{|a_1+x}\right)\ ,\qquad x''=m_2\left(\frac{1|}{|a_n}+\cdots+\frac{1|}{|a_1+x}\right)$$

or

$$x'=m_2\left(\frac{1|}{|a_n}+\cdots+\frac{1|}{|a_1+x}\right)\ ,\qquad x''=m_1\left(\frac{1|}{|a_n}+\cdots+\frac{1|}{|a_1+x}\right)\ .$$

But the distance between the upper bound and lower bound of (33) is bounded above by $|x'-x''|/|q_n|^2$, and this value converges to 0 as $n\to\infty$. So the sequence (32) converges to some real number.

LEMMA 3.6. (i) *For any* $x\in X_\alpha$, *the set* $R_\alpha(x)$ *coincides with the interval* $[m_2(x), m_1(x)]$ *modulo rational numbers.*

(ii) *For any* $x\in X_\alpha$, *we have* (21).

PROOF. First of all, we show that $R_\alpha(x)\ni m_1(x), m_2(x)$. If $\psi_\alpha(x)\leqq\sigma\omega_\alpha$, then we have

$$m_1(x)=1-\beta=\frac{1|}{|3}+\frac{1\ |}{|-3}+\frac{1|}{|3}+\frac{1\ |}{|-3}+\cdots,$$
$$\cdots(-3)3\,(-3)3\cdot\psi_\alpha(x)\quad\text{is } \alpha\text{-admissible},$$

from which it follows that $m_1(x)\in R_\alpha(x)$. On the other hand if $\psi_\alpha(x)>\sigma\omega_\alpha$, then we have

$$m_1(x)=\beta=\frac{1|}{|2}+\frac{1\ |}{|-3}+\frac{1|}{|3}+\frac{1\ |}{|-3}+\cdots,$$
$$\cdots(-3)3\,(-3)2\cdot\psi_\alpha(x)\quad\text{is } \alpha\text{-admissible},$$

whence follows that $m_1(x)\in R_\alpha(x)$. In the same manner, we can show that $m_2(x)\in R_\alpha(x)$. Now it is easy to show that, if $a_n\cdots a_1\cdot\psi_\alpha(x)$ is α-admissible, then values

$$\frac{1|}{|a_1}+\cdots+\frac{1\ \ |}{|a_n+x'}\quad\text{and}\quad\frac{1|}{|a_1}+\cdots+\frac{1\ \ |}{|a_n+x''}$$

are contained in $R_\alpha(x)$, where x' and x'' are as in the proof of Lemma 3.5. And as in the proof of Lemma 3.5, the distance between these two values converges to 0 as $n\to\infty$, so we get (i) by using the results of Lemma 3.3. By using (i) and Lemma 3.3, we can easily show (ii).

Now let us give the form of the density function $f_\alpha(x)$ of the S_α-invariant probability measure.

THEOREM 3.1. *Let* $1/2\leqq\alpha\leqq2-\sqrt{2}$. *Then we have*

$$f_\alpha(x)=\begin{cases}\dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+2}-\dfrac{1}{x-\beta-1}\right) & \text{if}\quad \alpha-1\leqq x\leqq\dfrac{1}{\alpha-1}+2\\[2ex] \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+2}-\dfrac{1}{x-\beta-2}\right) & \text{if}\quad \dfrac{1}{\alpha-1}+2<x\leqq\dfrac{1}{\alpha}-2\\[2ex] \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+1}-\dfrac{1}{x-\beta-2}\right) & \text{if}\quad \dfrac{1}{\alpha}-2<x<\alpha\,.\end{cases}\tag{34}$$

PROOF. By Lemmas 3.1 and 3.6, it is clear that the function

$$f(x)=\int_{m_2(x)}^{m_1(x)}\frac{dy}{(1+xy)^2}=\frac{1}{x+\dfrac{1}{m_1(x)}}-\frac{1}{x+\dfrac{1}{m_2(x)}}\tag{35}$$

is the density function of the S_α-invariant measure. So it is sufficient to calculate the normalizing constant C. But we get

$$(36)\quad C^{-1}=\int_{\alpha-1}^{\alpha}\left(\frac{1}{x+\dfrac{1}{m_1(x)}}-\frac{1}{x+\dfrac{1}{m_2(x)}}\right)dx$$

$$=\int_{\alpha-1}^{1/\alpha-2}\frac{dx}{x+\beta+2}+\int_{1/\alpha-2}^{\alpha}\frac{dx}{x+\beta+1}-\int_{\alpha-1}^{1/(\alpha-1)+2}\frac{dx}{x-\beta-1}-\int_{1/(\alpha-1)+2}^{\alpha}\frac{dx}{x-\beta-2}$$

$$=\log\left|\frac{\left(\dfrac{1}{\alpha}+\beta\right)(\alpha+\beta+1)(\alpha-\beta-2)\left(\dfrac{1}{\alpha-1}-\beta\right)}{(\alpha+\beta+1)\left(\dfrac{1}{\alpha}+\beta-1\right)\left(\dfrac{1}{\alpha-1}-\beta+1\right)(\alpha-\beta-2)}\right|$$

$$=\log(\beta+2)\ .$$

(II) The case of $2-\sqrt{2}<\alpha\leqq(\sqrt{5}-1)/2$.

LEMMA 3.7. *If* $2-\sqrt{2}<\alpha<(\sqrt{5}-1)/2$, *then*

$$(37)\qquad \begin{cases}\omega_\alpha=(2,\ -3,\ b_3,\ b_4,\ \cdots)\\ \omega_{\alpha-1}=(-3,\ 2,\ b_3-1,\ b_4,\ \cdots)\end{cases}$$

where $-3<b_3<3$. *If* $\alpha=(\sqrt{5}-1)/2$, *then*

$$(38)\qquad \begin{cases}\omega_\alpha=(2,\ -3,\ 3,\ -3,\ \cdots)\\ \omega_{\alpha-1}=(-3,\ 3,\ -3,\ 3,\ \cdots)\ .\end{cases}$$

PROOF. It is clear from (24) and

$$(39)\qquad \alpha-1\leqq\frac{1}{\alpha-1}+3<\alpha\iff 2-\sqrt{2}<\alpha\leqq\frac{5-\sqrt{13}}{2}\ ,$$

that $\omega_\alpha(1)=2$ and $\omega_{\alpha-1}(1)=-3$ if and only if $2-\sqrt{2}<\alpha\leqq(\sqrt{5}-1)/2$. And also we have

$$(40)\qquad \alpha-1\leqq\frac{1}{\dfrac{1}{\alpha}-2}+3<\alpha\iff 2-\sqrt{2}\leqq\alpha<\frac{3-\sqrt{3}}{2}\ ,$$

$$(41)\qquad \alpha-1\leqq\frac{1}{\dfrac{1}{\alpha-1}+3}-2<\alpha\iff\frac{\sqrt{3}}{3}\leqq\alpha<\frac{\sqrt{5}-1}{2}\ ,$$

so, except for the case $\alpha=(\sqrt{5}-1)/2$, we have $\omega_\alpha(2)=-3$ and $\omega_{\alpha-1}(2)=2$. Now let $\omega_\alpha=(2,\ -3,\ b_3,\ b_4,\ \cdots)$, then by using the relation (26) twice, we obtain

$$\begin{aligned}\alpha&=\rho(2,\ -3,\ b_3,\ b_4,\ \cdots)\\ &=1+\rho(-2,\ -2,\ b_3,\ b_4,\ \cdots)\\ &=1+\rho(-3,\ 2,\ b_3-1,\ b_4,\ \cdots)\ ,\end{aligned}$$

that is

$$\rho(-3, 2, b_3-1, b_4, \cdots)=\alpha-1 .$$

But by using Lemma 1.2, we can show that the sequence $(-3, 2, b_3+1, b_4, \cdots)$ is α-admissible, so we have (37). And $-3<b_3<3$ is almost clear. If $\alpha=(\sqrt{5}-1)/2$, then $1/\alpha-2=\alpha-1$ and $1/(\alpha-1)+3=-(\alpha-1)$; so (38) is clear.

From here to Theorem 3.2, we assume that $2-\sqrt{2}<\alpha\leqq(\sqrt{5}-1)/2$. Let us now define $m_1(x)$ and $m_2(x)$ as follows

$$m_1(x)=\begin{cases}1-\beta & \text{if} \quad \psi_\alpha(x)\leqq\sigma\omega_\alpha \\ 1-\gamma & \text{if} \quad \sigma\omega_\alpha<\psi_\alpha(x)<\sigma^2\omega_{\alpha-1} \\ \beta & \text{if} \quad \sigma^2\omega_{\alpha-1}\leqq\psi_\alpha(x) ,\end{cases} \tag{42}$$

$$m_2(x)=\begin{cases}-\gamma & \text{if} \quad \psi_\alpha(x)<\sigma^2\omega_\alpha \\ \beta-1 & \text{if} \quad \sigma^2\omega_\alpha\leqq\psi_\alpha(x)\leqq\sigma\omega_{\alpha-1} \\ \dfrac{\gamma-1}{2} & \text{if} \quad \sigma\omega_{\alpha-1}<\psi_\alpha(x) ,\end{cases} \tag{43}$$

where $\gamma=\sqrt{2}-1$. Then we obtain the following Lemmas 3.8 and 3.9.

LEMMA 3.8. *Let $a\cdot\psi_\alpha(x)$ and $(a+1)\cdot\psi_\alpha(x)$ be both α-admissible. Then we have*

$$\frac{1}{a+m_1\left(\frac{1}{a+x}\right)}=\frac{1}{(a+1)+m_2\left(\frac{1}{(a+1)+x}\right)} . \tag{44}$$

PROOF. As before, we prove that

$$m_1\left(\frac{1}{a+x}\right)=1+m_2\left(\frac{1}{(a+1)+x}\right) .$$

If $a>b_3-1$ or if $a=b_3-1$ and $\psi_\alpha(x)\leqq\sigma^3\omega_\alpha$ $(=\sigma^3\omega_{\alpha-1})$, then we have $a\cdot\psi_\alpha(x)\geqq\sigma^2\omega_{\alpha-1}$ and $\sigma\omega_{\alpha-1}>(a+1)\cdot\psi_\alpha(x)\geqq\sigma^2\omega_\alpha$, so we get $m_1(1/(a+x))=\beta$ and $m_2(1/((a+1)+x))=\beta-1$. On the other hand, if $a=b_3-1$ and $\psi_\alpha(x)>\sigma^3\omega_\alpha$ $(=\sigma^3\omega_{\alpha-1})$ or if $a<b_3-1$, then we have $\sigma\omega_\alpha<a\cdot\psi_\alpha(x)<\sigma^2\omega_{\alpha-1}$ and $(a+1)\cdot\psi_\alpha(x)<\sigma^2\omega_\alpha$, so we have $m_1(1/(a+x))=1-\gamma$ and $m_2(1/((a+1)+x))=-\gamma$.

LEMMA 3.9. *If $a_n \cdots a_2 a_1\cdot\psi_\alpha(x)$ is α-admissible, then we have*

$$m_2(x)\leqq\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_n}\leqq m_1(x) . \tag*{(45)$_n$}$$

PROOF. We can show this lemma in essentially the same manner as in Lemma 3.5. First we prove that

$$\frac{1}{a+m_2\left(\frac{1}{a+x}\right)}\leqq m_1(x)\ , \qquad \frac{1}{a+m_1\left(\frac{1}{a+x}\right)}\geqq m_2(x)\ ,$$

when $a\cdot\psi_\alpha(x)$ is α-admissible. We only show the former inequality. If $\psi_\alpha(x)\leqq\sigma\omega_\alpha$, then we have $a<3$, so by Lemma 3.8, we have

$$\frac{1}{a+m_2\left(\frac{1}{a+x}\right)}\leqq\frac{1}{3+m_2\left(\frac{1}{3+x}\right)}\ .$$

In this case we can show $\sigma^2\omega_\alpha\leqq 3\cdot\psi_\alpha(x)\leqq\sigma\omega_{\alpha-1}$ in the following manner. $3\cdot\psi_\alpha(x)\leqq\sigma\omega_{\alpha-1}$ is clear because $\sigma\omega_{\alpha-1}=(2, b_3-1, b_4, \cdots)$. And we can show that $\sigma^2\omega_\alpha\leqq 3\cdot\sigma\omega_\alpha$ if $\alpha\leqq(\sqrt{5}-1)/2$, so we have $\sigma^2\omega_\alpha\leqq 3\cdot\sigma\omega_\alpha\leqq 3\cdot\psi_\alpha(x)$. So we have

$$\frac{1}{3+m_2\left(\frac{1}{3+x}\right)}=\frac{1}{3+\beta-1}=1-\beta=m_1(x)\ .$$

If $\sigma\omega_\alpha<\psi_\alpha(x)<\sigma^2\omega_{\alpha-1}$, we have $a<2$ and $\sigma\omega_{\alpha-1}<2\cdot\psi_\alpha(x)$, so we have

$$\frac{1}{a+m_2\left(\frac{1}{a+x}\right)}\leqq\frac{1}{2+m_2\left(\frac{1}{2+x}\right)}=\frac{1}{2+\frac{\gamma-1}{2}}=1-\gamma=m_1(x)\ .$$

If $\sigma^2\omega_{\alpha-1}\leqq\psi_\alpha(x)$, we have $a<2$ and $\sigma^2\omega_\alpha\leqq 2\cdot\psi_\alpha(x)\leqq\sigma\omega_{\alpha-1}$, so we have

$$\frac{1}{a+m_2\left(\frac{1}{a+x}\right)}\leqq\frac{1}{2+m_2\left(\frac{1}{2+x}\right)}=\frac{1}{2+\beta-1}=\beta=m_1(x)\ .$$

Now we can prove Lemmas 3.5 and 3.6 also in the case of $2-\sqrt{2}<\alpha\leqq(\sqrt{5}-1)/2$ by using Lemmas 3.7 and 3.8. And then we obtain the following

THEOREM 3.2. *Let* $2-\sqrt{2}<\alpha\leqq(\sqrt{5}-1)/2$.

(i) *If* $2-\sqrt{2}<\alpha<(7+\sqrt{13})/18$, *then we get*

$$\omega_{\alpha-1}<\sigma^2\omega_\alpha<\sigma\omega_\alpha<\sigma^2\omega_{\alpha-1}<\sigma\omega_{\alpha-1}<\omega_\alpha\ , \tag{46}$$

so we have

$$(47)\quad f_\alpha(x)=\begin{cases}\dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+2}-\dfrac{1}{x-\gamma-2}\right) & \textit{if}\quad x\in[\alpha-1,\ \rho(\sigma^2\omega_\alpha))\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+2}-\dfrac{1}{x-\beta-2}\right) & \textit{if}\quad x\in[\rho(\sigma^2\omega_\alpha),\ \rho(\sigma\omega_\alpha)]\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\dfrac{\gamma+3}{2}}-\dfrac{1}{x-\beta-2}\right) & \textit{if}\quad x\in(\rho(\sigma\omega_\alpha),\ \rho(\sigma^2\omega_{\alpha-1}))\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+1}-\dfrac{1}{x-\beta-2}\right) & \textit{if}\quad x\in[\rho(\sigma^2\omega_{\alpha-1}),\ \rho(\sigma\omega_{\alpha-1})]\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+1}-\dfrac{1}{x-\gamma-3}\right) & \textit{if}\quad x\in(\rho(\sigma\omega_{\alpha-1}),\ \alpha)\,.\end{cases}$$

(ii) *If* $(7+\sqrt{13})/18<\alpha<(10-\sqrt{2})/14$, *then we get*

$$(48)\qquad \omega_{\alpha-1}<\sigma\omega_\alpha<\sigma^2\omega_\alpha<\sigma^2\omega_{\alpha-1}<\sigma\omega_{\alpha-1}<\omega_\alpha\,,$$

so we have

$$(49)\quad f_\alpha(x)=\begin{cases}\dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+2}-\dfrac{1}{x-\gamma-2}\right) & \textit{if}\quad x\in[\alpha-1,\ \rho(\sigma\omega_\alpha)]\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\dfrac{\gamma+3}{2}}-\dfrac{1}{x-\gamma-2}\right) & \textit{if}\quad x\in(\rho(\sigma\omega_\alpha),\ \rho(\sigma^2\omega_\alpha))\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\dfrac{\gamma+3}{2}}-\dfrac{1}{x-\beta-2}\right) & \textit{if}\quad x\in[\rho(\sigma^2\omega_\alpha),\ \rho(\sigma^2\omega_{\alpha-1}))\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+1}-\dfrac{1}{x-\beta-2}\right) & \textit{if}\quad x\in[\rho(\sigma^2\omega_{\alpha-1}),\ \rho(\sigma\omega_{\alpha-1})]\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+1}-\dfrac{1}{x-\gamma-3}\right) & \textit{if}\quad x\in(\rho(\sigma\omega_{\alpha-1}),\ \alpha)\,.\end{cases}$$

(iii) *If* $(10-\sqrt{2})/14<\alpha<(\sqrt{5}-1)/2$, *then we get*

$$(50)\qquad \omega_{\alpha-1}<\sigma\omega_\alpha<\sigma^2\omega_\alpha<\sigma\omega_{\alpha-1}<\sigma^2\omega_{\alpha-1}<\omega_\alpha\,,$$

so we have

$$\begin{cases}\dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+2}-\dfrac{1}{x-\gamma-2}\right) & \textit{if}\quad x\in[\alpha-1,\ \rho(\sigma\omega_\alpha)]\\ \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\dfrac{\gamma+3}{2}}-\dfrac{1}{x-\gamma-2}\right) & \textit{if}\quad x\in(\rho(\sigma\omega_\alpha),\ \rho(\sigma^2\omega_\alpha))\end{cases}$$

$$
(51)\quad f_\alpha(x)=\begin{cases}\dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\dfrac{\gamma+3}{2}}-\dfrac{1}{x-\beta-2}\right) & \text{if } x\in[\rho(\sigma^2\omega_\alpha),\ \rho(\sigma\omega_{\alpha-1})]\\[2ex] \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\dfrac{\gamma+3}{2}}-\dfrac{1}{x-\gamma-3}\right) & \text{if } x\in(\rho(\sigma\omega_{\alpha-1}),\ \rho(\sigma^2\omega_{\alpha-1}))\\[2ex] \dfrac{1}{\log(\beta+2)}\left(\dfrac{1}{x+\beta+1}-\dfrac{1}{x-\gamma-3}\right) & \text{if } x\in[\rho(\sigma^2\omega_{\alpha-1}),\ \alpha)\ .\end{cases}
$$

(iv) *The case* $\alpha=(7+\sqrt{13})/18$ (*resp.* $(10-\sqrt{2})/14$, $(\sqrt{5}-1)/2$) *can be considered as the degenerate case in* (i) *and* (ii) (*resp.* (ii) *and* (iii), (iii)).

PROOF. It can easily be shown that the function (35) is also the density function of S_α-invariant measure in this case. And we get the same value $1/(\log(\beta+2))$ as the normarizing constant. If we consider the proof case by case according to the order of ω_α, $\sigma\omega_\alpha$, $\sigma^2\omega_\alpha$, $\omega_{\alpha-1}$, $\sigma\omega_{\alpha-1}$ and $\sigma^2\omega_{\alpha-1}$, then we obtain (i), (ii) and (iii). (iv) is almost clear.

§ 4. The entropy of S_α.

It is well-known that the entropy $h(S_1)$ of S_1, the usual continued-fraction transformation, is given by

$$
(52)\qquad h(S_1)=2\int_0^1 \log x\,\frac{1}{\log 2}\,\frac{1}{1+x}dx=\frac{\pi^2}{6\log 2}
$$

(see [8]). In the same manner we can show that the entropy $h(S_\alpha)$ is given by

$$
(53)\qquad h(S_\alpha)=2\int_{\alpha-1}^{\alpha}\log|x|f_\alpha(x)dx\ .
$$

As for the value of this entropy, we obtain the following

THEOREM 4.1. *For all* $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$, $h(S_\alpha)$ *takes the same value* $\pi^2/(6\log(\beta+1))$, *independent of* α.

PROOF. In the first place, we show that $h(S_\alpha)$ is independent of α. Let $1/2\leqq\alpha\leqq 2-\sqrt{2}$, then we have

$$
\begin{aligned}(54)\qquad Ch(S_\alpha)=&\int_{\alpha-1}^{1/\alpha-2}\frac{1}{x+\beta+2}\log|x|dx+\int_{1/\alpha-2}^{\alpha}\frac{1}{x+\beta+1}\log|x|dx\\ &-\int_{\alpha-1}^{1/(\alpha-1)+2}\frac{1}{x-\beta-1}\log|x|dx-\int_{1/(\alpha-1)+2}^{\alpha}\frac{1}{x-\beta-2}\log|x|dx\end{aligned}
$$

where $C=\log(\beta+2)$. So we get

$$
\begin{aligned}
(55)\quad C\frac{d}{d\alpha}h(S_\alpha) &= -\frac{1}{\alpha^2}\frac{\log\left|\frac{1-2\alpha}{\alpha}\right|}{\frac{1}{\alpha}+\beta}-\frac{\log|\alpha-1|}{\alpha+\beta+1}+\frac{\log\alpha}{\alpha+\beta+1} \\
&\quad +\frac{1}{\alpha^2}\frac{\log\left|\frac{1-2\alpha}{\alpha}\right|}{\frac{1}{\alpha}+\beta-1}+\frac{1}{(\alpha-1)^2}\frac{\log\left|\frac{2\alpha-1}{\alpha-1}\right|}{\frac{1}{\alpha-1}-\beta+1}+\frac{\log|\alpha-1|}{\alpha-\beta-2} \\
&\quad -\frac{\log\alpha}{\alpha-\beta-2}-\frac{1}{(\alpha-1)^2}\frac{\log\left|\frac{2\alpha-1}{\alpha-1}\right|}{\frac{1}{\alpha-1}-\beta} \\
&=\log\alpha\left\{\frac{1}{\alpha^2}\frac{1}{\frac{1}{\alpha}+\beta}+\frac{1}{\alpha+\beta+1}-\frac{1}{\alpha^2}\frac{1}{\frac{1}{\alpha}+\beta-1}-\frac{1}{\alpha-\beta-2}\right\} \\
&\quad +\log|\alpha-1|\left\{-\frac{1}{\alpha+\beta+1}-\frac{1}{(\alpha-1)^2}\frac{1}{\frac{1}{\alpha-1}-\beta+1}\right. \\
&\quad \left.+\frac{1}{\alpha-\beta-2}+\frac{1}{(\alpha-1)^2}\frac{1}{\frac{1}{\alpha-1}-\beta}\right\} \\
&\quad +\log|2\alpha-1|\left\{-\frac{1}{\alpha^2}\frac{1}{\frac{1}{\alpha}+\beta}+\frac{1}{\alpha^2}\frac{1}{\frac{1}{\alpha}+\beta-1}\right. \\
&\quad \left.+\frac{1}{(\alpha-1)^2}\frac{1}{\frac{1}{\alpha-1}-\beta+1}-\frac{1}{(\alpha-1)^2}\frac{1}{\frac{1}{\alpha-1}-\beta}\right\} \\
&=0\,.
\end{aligned}
$$

In the same manner we can show $(d/d\alpha)h(S_\alpha)=0$ in case $2-\sqrt{2}\leqq\alpha\leqq(\sqrt{5}-1)/2$. So, the value of $h(S_\alpha)$ is independent of α if $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$.

Now let us calculate the value $h(S_{1/2})$. It is well known that $h(S_1)$ is given by

$$
(56)\qquad h(S_1)=2\lim_{n\to\infty}\frac{1}{n}\log q_n(x,1)\quad \text{a.e.}\quad x\in X_1
$$

(see [8]). In the same manner we can show that

$$h(S_{1/2})=2\lim_{n\to\infty}\frac{1}{n}\left|\log q_n\left(x,\frac{1}{2}\right)\right| \quad \text{a.e.} \quad x\in X_{1/2}\,. \tag{57}$$

Let $x\in[0,1/2)$ and let

$$\begin{cases}\psi_1(x)=(a_1,a_2,\cdots,a_n,\cdots)\\ \psi_{1/2}(x)=(b_1,b_2,\cdots,b_n,\cdots)\,,\end{cases} \tag{58}$$

then we obtain the following

LEMMA 4.1. *Let us define* $\{\tau_k, k\geqq 0\}$ *by*

$$\begin{cases}\tau_0=-1\\ \tau_k=\min\{n\geqq\tau_{k-1}+2;\ a_n=1\}\quad(k\geqq 1)\end{cases} \tag{59}$$

and for each natural number n *which does not coincide with any* τ_k $(k\geqq 0)$, *let us define* $N(n)$ *by*

$$N(n)=n-k \quad \textit{if} \quad \tau_k<n<\tau_{k+1}\,. \tag{60}$$

Then we have, for each n *as above, and a.e.* x,

$$b_{N(n)}=\begin{cases}(-1)^k a_n & \textit{if}\quad \tau_k+1<n<\tau_{k+1}-1\\ (-1)^k(a_n+1) & \textit{if}\quad \tau_k+1=n<\tau_{k+1}-1\quad \textit{or}\\ & \textit{if}\quad \tau_k+1<n=\tau_{k+1}-1\\ (-1)^k(a_n+2) & \textit{if}\quad \tau_k+1=n=\tau_{k+1}-1\,,\end{cases} \tag{61}$$

and

$$q_{N(n)}\left(x,\frac{1}{2}\right)=q_n(x,1) \quad \textit{or} \quad -q_n(x,1)\,. \tag{62}$$

PROOF. From the definition of $S_{1/2}$ and from the relations

$$\frac{1|}{|a}+\frac{1|}{|1}+\frac{1\ \ |}{|b+t}=\frac{1\ \ |}{|a+1}+\frac{1\qquad |}{|-(b+1)-t}\,, \tag{63}$$

$$\frac{1\ \ |}{|-a}+\frac{1\ \ |}{|-1}+\frac{1\qquad |}{|-b-t}=\frac{1\qquad |}{|-(a+1)}+\frac{1\qquad |}{|(b+1)+t}\,, \tag{64}$$

we obtain (61). Using Lemma 2.1 and Remark 2.1, we obtain (62).

By the definition of $N(n)$, we can easily show that

$$N(n)=n+\sum_{j=1}^{\infty}(-1)^j N_j(n) \tag{65}$$

for any $n\notin\{\tau_k;\ k\geqq 0\}$, where $N_j(n)$ is given by

$$N_j(n)=\#\{1\leqq i\leqq n-j+1;\ a_i=a_{i+1}=\cdots=a_{i+j-1}=1\}\ . \tag{66}$$

By the ergodicity of S_1, we obtain that

$$\lim_{n\to\infty}\frac{N_j(n)}{n}=\begin{cases}\displaystyle\int_{\xi_{j+1}}^{\xi_j}\frac{1}{\log 2}\frac{1}{1+x}dx & \text{if } j \text{ is odd}\\ \displaystyle\int_{\xi_j}^{\xi_{j+1}}\frac{1}{\log 2}\frac{1}{1+x}dx & \text{if } j \text{ is even}\ ,\end{cases} \tag{67}$$

where $\xi_j=\rho(1,1,\cdots,1,\infty,\infty,\cdots)$ with $(\underbrace{1,1,\cdots,1}_{j},\infty,\infty,\cdots)$. If we notice that $\xi_2<\xi_4<\cdots<\xi_3<\xi_1$ and that $\lim_{j\to\infty}\xi_j=\rho(1,1,1,\cdots)=\beta$, we obtain

$$\lim_{n\to\infty}\frac{N(n)}{n}=\int_0^{\beta}\frac{1}{\log 2}\frac{1}{1+x}dx=\frac{\log(\beta+1)}{\log 2}\ . \tag{68}$$

From (56), (57), (62) and (68), we obtain that

$$h(S_{1/2})=\frac{\pi^2}{6\log 2}\frac{\log 2}{\log(\beta+1)}=\frac{\pi^2}{6\log(\beta+1)}\ . \tag{69}$$

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