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Compact Homomorphisms on Function Algebras

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In § 1 we give some characterizations of compact (resp. weakly compact) homomorphisms on function algebras. We also discuss when weakly compact homomorphisms on function algebras become compact. In § 2 we deal with the compactness of some linear operators, in particular, of composition operators on $H^\infty(D)$ as an application of § 1.

§ 1. Compact homomorphisms on function algebras.

Let E and F be Banach spaces and φ be a linear operator of E to F. φ is called a *compact* (resp. *weakly compact*) operator if, for the unit ball S of E, $\varphi(S)$ is relatively compact (resp. relatively weakly compact) in F. We will consider compact (resp. weakly compact) homomorphisms from A to B, that is, compact (resp. weakly compact) operators which are homomorphisms, when A and B are function algebras. We say A is a *function algebra* on a compact Hausdorff space X if A is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates points of X. The family $\{X_i\}_{i=0}^n$ of subsets of a topological space X is said to be a *partition* of X if $X=\bigcup_{i=0}^n X_i$ and X_i are mutually disjoint, closed and open subsets of X for $i=0, 1, 2, \cdots, n$. By A^* and M_A we denote the dual space and the maximal ideal space of a function algebra A respectively. We put $\hat{f}(m)=m(f)$ for $f\in A$, $m\in M_A$.

We begin with characterizations of compact homomorphisms and weakly compact homomorphisms on function algebras (cf. [10]).

THEOREM 1.1. *Let A be a function algebra and B be a function algebra on a compact Hausdorff space X. Suppose that φ is a linear operator from A to B. Then we have*

(a) *φ is a continuous homomorphism if and only if there is a continuous map τ of X to $M_A\cup\{0\}$ with respect to the topology $\sigma(A^*, A)$ such that*

$$(\varphi f)(x)=\hat{f}(\tau x)\ , \quad f\in A \quad and \quad x\in X\ ,$$

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where we put $\hat{f}(\tau x)=0$ *for* $f\in A$ *when* $\tau x=0$.

(b) φ *is a compact homomorphism if and only if there are a partition* $\{X_i\}_{i=0}^n$ *of* X, *a family* $\{P_i\}_{i=1}^n$ *of Gleason parts of* A *and a continuous map* τ_i *of* X_i *to* P_i *for each* i $(i=1, 2, \cdots, n)$ *with respect to the norm topology of* A^* *such that*

$$(*)\qquad (\varphi f)(x)=\begin{cases}\hat{f}(\tau_i x)\,, & f\in A \text{ and } x\in X_i \quad 1\leqq i\leqq n\\ 0\,, & f\in A \text{ and } x\in X_0\,.\end{cases}$$

(c) *Let* X *be a compact metric space. Then* φ *is a weakly compact homomorphism if and only if there are* $\{X_i\}_{i=0}^n$, $\{P_i\}_{i=1}^n$ *as the above* (b) *and a continuous map* τ_i *of* X_i *to* P_i *with respect to the topology* $\sigma(A^*, A^{**})$ *that satisfies* $(*)$ *for each* i, $i=1, 2, \cdots, n$.

PROOF. (a) If φ is a continuous homomorphism from A to B, then φ may be regarded as a continuous linear operator from A to $C(X)$ and so there is a continuous map τ of X to A^* with respect to the topology $\sigma(A^*, A)$ such that $(\varphi f)(x)=\tau x(f)$ for $f\in A$ and $x\in X$ (cf. [1] [2] [9]). Here $\tau x\in M_A\cup\{0\}$ for any $x\in X$. In fact, we know that τx is a complex homomorphism of A: $\tau x(f_1f_2)=(\varphi f_1f_2)(x)=(\varphi f_1)(x)(\varphi f_2)(x)=\tau x(f_1)\tau x(f_2)$ for any $f_1, f_2\in A$. Conversely, if τ is a continuous map of X to $M_A\cup\{0\}$ such that $(\varphi f)(x)=\hat{f}(\tau x)$ for $f\in A$ and $x\in X$, then it is clear that φ is a continuous homomorphism from A to B.

(b) Let φ be a compact homomorphism from A to B. Then φ becomes a compact operator from A to $C(X)$. So there is a continuous map τ of X to A^* with respect to the norm topology of A^* such that $(\varphi f)(x)=\hat{f}(\tau x)$ for $f\in A$ and $x\in X$ (cf. [1] [2] [9]). In the same way as in (a), we see $\tau(X)\subset M_A\cup\{0\}$. Now let P_1 and P_2 be distinct Gleason parts of A. If m_1 is in P_1 and m_2 is in P_2, then $\|m_1-m_2\|=\sup\{|m_1(f)-m_2(f)|: f\in A, \|f\|<1\}=2$. Since $\tau(X)$ is compact with respect to the norm topology of A^*, there is a finite family $\{P_i\}_{i=1}^n$ of Gleason parts of A such that $\tau(X)\subset P_1\cup P_2\cup\cdots\cup P_n\cup\{0\}$. Here we put $X_i=\tau^{-1}(P_i)$ for $i=1, 2, \cdots, n$ and $X_0=\tau^{-1}(\{0\})$. Then $\{X_i\}_{i=0}^n$ is a partition of X. If we put $\tau_i=\tau|_{X_i}$, the restriction of τ to X_i, for $i=1, 2, \cdots, n$, $\{X_i\}_{i=0}^n$, $\{P_i\}_{i=1}^n$ and $\{\tau_i\}_{i=1}^n$ are what we need. The converse is clear.

(c) Let φ be a weakly compact homomorphism from A to B. Then there is a continuous map τ of X to $M_A\cup\{0\}$ with respect to the topology $\sigma(A^*, A^{**})$ such that $(\varphi f)(x)=\hat{f}(\tau x)$ for $f\in A$ and $x\in X$. Now we have to show that $\tau(X)$ is contained in $P_1\cup P_2\cup\cdots\cup P_{n_0}\cup\{0\}$ for a finite family $\{P_i\}_{i=1}^{n_0}$ of Gleason parts of A. Suppose otherwise. Then there exist distinct Gleason parts P_n with $P_n\cap\tau(X)\neq\varnothing$ $(n=1, 2, \cdots)$. Take

m_n in $P_n \cap \tau(X)$ and choose x_n in $\tau^{-1}(m_n)$, and thus we can obtain a sequence $\{x_n\}$ in X. As X is a compact metric space, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to some point x_0 in X. We can assume without loss of generality that x_n converges to x_0. As τ is continuous, $m_n = \tau x_n$ converges to τx_0. Since M_A is closed, τx_0 is in M_A. So τx_0 is contained in some Gleason part P_0. Now we can assume that $P_n \neq P_0$ for $n=1, 2, \cdots$ and $P_n \neq P_m$ for $n \neq m$. As m_n is in P_n and τx_0 is in P_0, we may choose a family $\{f_n\}$ of functions in A such that

$$\|f_n\| < 1 , \quad \hat{f}_n(m_n) = 0 ,$$

and

$$\hat{f}_n(\tau x_0) = 1 - \varepsilon_n , \quad 0 < \varepsilon_n < \frac{1}{n^2} \quad (n = 1, 2, \cdots) .$$

Put $g_n = f_1 f_2 \cdots f_n$ in A. Then

$$\hat{g}_n(\tau x_0) = (1-\varepsilon_1)(1-\varepsilon_2) \cdots (1-\varepsilon_n)$$
$$\hat{g}_n(m_i) = 0 \quad (i \leqq n)$$

and

$$\|g_n\| < 1 \quad (n = 1, 2, 3, \cdots) .$$

As φ is weakly compact and $\|g_n\| < 1$ for $n = 1, 2, 3, \cdots$, $\{\varphi g_n\}$ is relatively weakly compact. Hence there is a subsequence $\{\varphi g_{n_k}\}$ of $\{\varphi g_n\}$ such that φg_{n_k} converges pointwise to some h in $C(X)$ ([1]). We here have

$$(\varphi g_{n_k})(x_i) = \hat{g}_{n_k}(m_i) = 0 , \quad i \leqq n_k$$
$$(\varphi g_{n_k})(x_0) = \hat{g}_{n_k}(\tau x_0) = (1-\varepsilon_1)(1-\varepsilon_2) \cdots (1-\varepsilon_{n_k}) .$$

Thus $h(x_i) = 0$ for $i = 1, 2, 3, \cdots$ and $h(x_0) = (1-\varepsilon_1)(1-\varepsilon_2) \cdots$, where $0 < \varepsilon_n < 1/n^2$. As $\sum \varepsilon_n < \infty$, $h(x_0) \neq 0$. But $h(x_i)$ converges to $h(x_0)$ and $h(x_i) = 0$ for $i = 1, 2, 3, \cdots$. This is absurd. So there is a family $\{P_i\}_{i=1}^{n_0}$ such that $\tau(X)$ is contained in $P_1 \cup P_2 \cup \cdots \cup P_{n_0} \cup \{0\}$. Next we show that $\tau^{-1}(P_i)$ is closed in X. Put $P = P_i$. Let a sequence $\{x_n\}$ be in $\tau^{-1}(P)$ such that x_n converges to x_0. Then τx_n is in P and τx_n converges to τx_0. As M_A is closed, τx_0 must be in M_A. So there is a Gleason part P_0 such that τx_0 is in P_0. If P is different from P_0, we can construct $\{f_n\}$, $\{g_n\}$ in the same way as above and this induces a contradiction. Consequently, x_0 is in $\tau^{-1}(P)$. When we put $X_i = \tau^{-1}(P_i)$ and $X_0 = \tau^{-1}(\{0\})$, $\{X_i\}_{i=0}^{n_0}$ is a partition of X. So (c) can be proved in the same way as in (b).

REMARK. Compact homomorphisms on disc algebras were discussed in [7].

Next we consider when weakly compact homomorphisms from A to B become compact. If $A=C(Y)$, Y is a compact Hausdorff space, and B is a function algebra on a compact metric space, then P_i consists of a single point in Theorem 1.1.(c). So in this case weakly compact homomorphisms are always compact.

Let now A be a function algebra and P be a non-trivial Gleason part of A. A map ρ of a polydisc D^n (a disc if $n=1$) into P is said to be *analytic* if $f\circ\rho$ is an analytic function on D^n for all $f\in A$. We say that P has the condition (α) if P satisfies the following condition; (compare [6; Chap. 4, Theorem 18])

> (α) *for any x in P, there are some open neighborhood $U(x)$ of x in P and an analytic map ρ which is a homeomorphism from a polydisc D^n ($n\geqq 1$, n depends upon $U(x)$) onto $U(x)$.*

EXAMPLES. (1) Let A be the disc algebra or the polydisc algebra $A(D^2)$. Then any non-trivial Gleason part for A satisfies (α).

(2) Let Γ be the unit circle in C and X be the cartesian product of Γ and $I=[0, 1]$. Let A be the function algebra on X generated by polynomials in t and z, where $t\in[0, 1]$ and $z\in\Gamma$. Then any non-trivial Gleason part for A has the property (α).

THEOREM 1.2. *Suppose A is a function algebra and any non-trivial Gleason part P for A satisfies (α). Let B be a function algebra on a compact metric space X. Then any weakly compact homomorphism from A to B is compact.*

PROOF. Let φ be a weakly compact homomorphism from A to B. Then, by Theorem 1.1.(c), there are a partition $\{X_i\}_{i=0}^n$, a family $\{P_i\}_{i=1}^n$ of Gleason parts of A and a continuous map τ_i of X_i to P_i with respect to the topology $\sigma(A^*, A^{**})$ for each $i\geqq 1$ which satisfies $(*)$ in Theorem 1.1.(b). Now if it would be showed that the identity map ψ of P_i onto itself with respect to the norm topology of A^* is continuous, φ should be compact by Theorem 1.1.(b). Hence we only show the continuity of ψ. Let $P=P_i$ be the non-trivial Gleason part and m_0 be in P. By (α), there are a neighborhood $U(m_0)$ and an analytic map ρ of D^n onto $U(m_0)$ that is homeomorphic. Since $\rho^{-1}(m_0)$ is in D^n, for any $\varepsilon>0$ there is a neighborhood V of $\rho^{-1}(m_0)$ in D^n such that

$$|f(z)-f(\rho^{-1}(m_0))|<\varepsilon$$

for any $z\in V$ and any function f which is analytic on D^n with $\|f\|<1$. Here $\rho(V)$ is a neighborhood of m_0 in P and for any $m=\rho(z)$ in $\rho(V)$

$$\begin{aligned}\|m-m_0\| &= \sup\{|\hat{g}(m)-\hat{g}(m_0)|: g\in A,\ \|g\|<1\}\\ &\leqq \sup\{|f(z)-f(\rho^{-1}(m_0))|: f \text{ is analytic on } D^n,\ \|f\|<1\}\\ &\leqq \varepsilon\ .\end{aligned}$$

Hence ψ is continuous.

Let X be a metric space or a locally compact Hausdorff space. By $C_k(X)$ we denote the topological algebra of continuous functions on X with the topology of uniform convergence on compact subsets in X. Let φ be a linear operator from a normed space E to $C_k(X)$. Then φ is compact if and only if there is a continuous map τ of X to the dual space E^* of E with the norm topology such that $(\varphi u)(x)=\tau x(u)$ for $u\in E$, $x\in X$ ([2], [9: Theorem 1]). We obtain the following in the same way as in the proof of Theorem 1.1.

COROLLARY 1.3. *Let φ be a linear operator from a function algebra A to $C_k(X)$. Then φ is a compact homomorphism if and only if there are a partition $\{X_i\}_{i=0}^n$ of X, a family $\{P_i\}_{i=1}^n$ of Gleason parts for A and a continuous map τ_i of X_i to P_i (with respect to the norm topology in A^*) for any $i\geqq 1$ which satisfies $(*)$ in Theorem* 1.1.(b).

§ 2. Examples of compact homomorphisms on function algebras.

(1) Restrictions to Gleason parts.

Let A be a function algebra and P be a non-trivial Gleason part of A. For any f in A, we define $\varphi f=\hat{f}|_P$. Then the linear operator φ from A to $C_k(P)$ is a continuous homomorphism. We assume that P is metric or locally compact as a subspace of M_A and m in P has a unique representing measure.

THEOREM 2.1. *Suppose P satisfies the assumptions above. Then φ is compact if and only if there is an analytic map of a unit open disc D onto P that is homeomorphic.*

PROOF. If there is an analytic map of D onto P that is homeomorphic, the identity map i of P onto P with the norm topology of A^* is continuous as in the proof of Theorem 1.2. Now $(\varphi f)(x)=\hat{f}(x)=\hat{f}(i(x))$ for $f\in A$ and $x\in P$. By Corollary 1.3, φ is a compact homomorphism from A to $C_k(P)$. Conversely, assume φ is compact. Since m in P has a unique representing measure, there is an analytic map ρ of D onto P ([3: Chap. 6, Theorem 7.2]; [6: Chap. 6, Theorem 24]). So it is sufficient to show that ρ is homeomorphic. For s, $t\in D$, let

$$\|t-s\|=\sup\{|g(t)-g(s)|: g\in A(D),\ \|g\|<1\}$$

where $A(D)$ is the disc algebra. For $m_1, m_2 \in P$, we put

$$\|m_1-m_2\|=\sup\{|\hat{f}(m_1)-\hat{f}(m_2)|: f\in A,\ \|f\|<1\}.$$

Then the following is proved (cf. [5]):

$$\|\rho(t)-\rho(s)\|=\|t-s\| \quad \text{for} \quad t, s\in D.$$

As φ is compact, there is a continuous map τ of P to $M_A\cup\{0\}$ with the norm topology of A^* such that $(\varphi f)(x)=\hat{f}(\tau x)$ for f in A and x in P. On the other hand, $(\varphi f)(x)=\hat{f}(x)$. It implies $\tau x=x$ for x in P and τ is the identity map. So the map τ of P onto P with the norm topology of A^* is continuous. Thus by this and the isometric property of ρ, ρ^{-1} is continuous and ρ is a homeomorphism.

(2) Composition operators on $H^\infty(D)$.

We here consider compact composition operators on $H^\infty(D)$ as an application of §1. Let D be a domain in C and $H^\infty(D)$ be the algebra of bounded analytic functions on D with the supremum norm. We assume that the functions in $H^\infty(D)$ separate points on D. For an analytic function ϕ from D to D the *composition operator* C_ϕ on $H^\infty(D)$ is defined by $C_\phi(f)=f\circ\phi$ for $f\in H^\infty(D)$. A composition operator C_ϕ is a continuous homomorphism on $H^\infty(D)$. Let M be the maximal ideal space of $H^\infty(D)$. Then $H^\infty(D)^\wedge$, the image of $H^\infty(D)$ by the Gelfand transform, can be regarded as a function algebra A on M. So C_ϕ may be considered as a continuous homomorphism from A to A. We deal with the case where C_ϕ is compact. Suppose C_ϕ is compact. It follows from Theorem 1.1(b) that there are a partition $\{X_i\}_{i=0}^n$ of M, a family $\{P_i\}_{i=1}^n$ of Gleason parts for A and a continuous map τ_i of X_i to P_i equipped with the norm topology of A^* such that $C_\phi(f)(x)=f(\tau_i x)$ for $f\in A$, $x\in X_i$ $(i\geqq 1)$ and $C_\phi(f)(x)=0$ for $f\in A$, $x\in X_0$. Since $\{X_i\}_{i=0}^n$ is a partition of M and M is a connected set, $X_i=M$ for some i. It is clear that $i=1$, since $H^\infty(D)$ contains the constant function 1. Put $\tau=\tau_1$ and $P=P_1$. Then $f(\phi(x))=C_\phi(f)(x)=f(\tau x)$ for $x\in D$ and $f\in A$. From this $\phi(x)=\tau x$ for $x\in D$. So we have that $\phi(D)=\tau(D)\subset\tau(M)\subset P$. Hence we obtain the following.

THEOREM 2.2. *C_ϕ is compact if and only if ϕ can be extended to a continuous map τ from M to P with respect to the norm topology of A^*.*

PROOF. The "only if" part of the theorem was already proved.

Conversely, if ϕ can be extended to τ from M to P, we put $T(f)(x)=f(\tau x)$ $(x\in M,\ f\in A)$. Then T is a compact homomorphism from A to $C(M)$. Since $M\supset D$ and $T(g)(x)=g(\phi(x))=C_\phi(g)(x)$ $(x\in D,\ g\in H^\infty(D))$, C_ϕ is a compact homomorphism from $H^\infty(D)$ to $H^\infty(D)$.

Next we take a domain D in the Riemann sphere S^2. Let $H^\infty(D)$ be the algebra of bounded analytic functions on D. We assume that $H^\infty(D)$ contains non-constant functions. Theorem 2.2 remains true in this case. The *fiber* M_λ over $\lambda\in\bar{D}$ consists of all homomorphisms $m\in M$ such that $m(f)=f(\lambda)$ for all $f\in H^\infty(D)$ which extend analytically to a neighborhood of λ. The fiber M_λ is a *peak set* for $H^\infty(D)$ if there is some $f\in H^\infty(D)$ whose Gelfand transform $\hat{f}$ is equal to 1 on M_λ while $|\hat{f}(m)|<1$ for all $m\in M\backslash M_\lambda$. See [4] for details on fibers.

From Theorem 2.2, we have the following (cf. [8]).

COROLLARY 2.3. *Let the fiber M_λ is a peak set for $H^\infty(D)$ for any λ in the boundary ∂D of D. Then C_ϕ is compact on $H^\infty(D)$ if and only if $\phi(D)^-\cap\partial D=\varnothing$, where $\phi(D)^-$ is the closure of $\phi(D)$.*

PROOF. It is evident that $D\subset P$. It is not hard to see that $D=P$ from the assumptions of the corollary. From this $\phi(D)=\tau(D)\subset\tau(M)\subset P=D$. So $\phi(D)^-\subset\tau(M)$ since $\tau(M)$ is compact in D. The converse is clear.

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