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The Structure of Endomorphism Algebras*

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Introduction

Let k be a field and A be an algebra over k with a unity element 1. We denote by $M(A)$ the category of left A-modules. Let Y be an A-module and $E=\mathrm{End}_A(Y)$. We write $M(E)$ for the category of left E-modules and $M'(E)$ for the category of right E-modules.

In this paper we introduce and study an idea of distinguishable modules, which appears quite often in the representation theory of finite groups, by making use of a contravariant representation functor Ψ of $M(A)$ into $M(E)$ (see § 1) and a covariant representation functor Φ of $M(A)$ into $M'(E)$ (see § 3).

DEFINITION (see Definition (2.1)). Assume that an A-module Y is decomposed into a finite number of indecomposable components, say

$$Y=Y_1\oplus Y_2\oplus\cdots\oplus Y_r\ ,$$

and the left A-submodules of soc Y satisfy the D.C.C. Then an indecomposable component Y_ρ, where $1\leqq\rho\leqq r$, is said to be distinguishable (by socle) if soc Y_ρ is multiplicity free and $Y_\rho\cong Y_\sigma$ when soc Y_ρ and soc Y_σ have a same simple submodule up to isomorphism, for any $1\leqq\sigma\leqq r$. When all the indecomposable components $Y_{\rho's}$ are distinguishable, we say that Y has a distinguishable decomposition $Y=Y_1\oplus Y_2\oplus\cdots\oplus Y_r$.

For example when the submodules of Y satisfy the D.C.C. and soc Y is multiplicity free, then Y has a distinguishable decomposition (see [1, Corollary 6.11], [4], [5, Theorem 3.17] and [6, Proposition 2.8 and Corollary 3.5]).

Our main result is as follows

Theorem (see Theorem (2.7)): Let E, Ψ be as above. Assume that

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E is finite dimensional, then Y is decomposed into a finite number of indecomposable components $Y_1, Y_2, \cdots, Y_r$. Assume further that soc Y_1, is also finite dimensional and the left A-submodules of soc Y satisfy the D.C.C. and k is an algebraically closed field. Then

$$hd\Psi(Y_1) \cong \Psi(X)$$

for any simple component X of soc Y_1 if and only if Y_1 is distinguishable.

In § 1 we introduce the functor Ψ and show a necessary and sufficient condition that rad $E=\{f \in E | f(\text{soc } Y)=0\}$* holds (see Theorem 1.5). In § 2 we prove the theorem, then we introduce the other functor Φ in § 3, and show a theorem which is a generalization of [3, Theorem 1] and an example of distinguishable modules in § 4.

The functorial method which appears in this paper has been developed through the research of the modular representations of finite Chevalley groups (see [3] and [6]). One can see further applications of the functor Φ in [7] and [8].

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§ 1. Functor Ψ.

Let k be a field and A be an algebra over k with a unity element 1. We denote by $M(A)$ the category of left A-modules. Let Y be an object in $M(A)$ and we write E for the endomorphism algebra of Y, i.e.,

$$E = \text{End}_A(Y) ,$$

then we denote by $M(E)$ the category of left E-modules.

In this section we study the properties of a contravariant representation functor Ψ of $M(A)$ into $M(E)$ and show a necessary and sufficient condition that the radical of E equals $\{f \in E | f(\text{soc } Y)=0\}$ assuming the left E-submodules of E satisfy the D.C.C. (see Theorem 1.5).

Let $M \in M(A)$ and $\Psi(M)=(M, Y)_A$ (the space of A-homomorphisms from M into Y). Then we can make $\Psi(M)$ into a left E-module by the following operation.

* A formula of this kind had already been studied by [K. Morita, Y. Kawada and H. Tachikawa, Math. Z., **68** (1957), 217-226] in case Y is an injective module.

$$\begin{array}{ccc} E\times\Psi(M) & \longrightarrow & \Psi(M) \\ \cup\!\!\!\!\!\!\;\!\!\in & & \cup\!\!\!\!\!\!\;\!\!\in \\ (\alpha, f) & \longmapsto & \alpha\circ f \end{array}$$

Thus we get a contravariant functor Ψ of $M(A)$ into the category of left E-modules $M(E)$. Notice when $\theta\in(M, M')_A$, then

$$\begin{array}{ccc} \Psi(\theta)\colon \Psi(M') & \longrightarrow & \Psi(M) \\ \cup\!\!\!\!\!\!\;\!\!\in & & \cup\!\!\!\!\!\!\;\!\!\in \\ f & \longmapsto & f\circ\theta \end{array}$$

where $M, M'\in M(A)$.

The following lemma is well-known.

LEMMA 1.1. (i) *The sequence*

$$0\longrightarrow\Psi(M'')\xrightarrow{\Psi(\theta_2)}\Psi(M)\xrightarrow{\Psi(\theta_1)}\Psi(M')$$

is exact for any exact sequence

$$M'\xrightarrow{\theta_1}M\xrightarrow{\theta_2}M''\longrightarrow 0 \text{ in } M(A)\,.$$

(ii) *Assume $M\in M(A)$ be decomposed into a finite number of direct summands in $M(A)$*

$$M=M_1\oplus M_2\oplus\cdots\oplus M_l\,,$$

then

$$\begin{array}{ccc} V\colon \Psi(M) & \cong & \Psi(M_1)\dotplus\cdots\dotplus\Psi(M_l) \\ \cup\!\!\!\!\!\!\;\!\!\in & & \cup\!\!\!\!\!\!\;\!\!\in \\ f & \longmapsto & \sum_{i=1}^{l} f|M_i \end{array}$$

gives rise to an E-isomorphism.

Now let $\Psi(M, M')$ be a map from $(M, M')_A$ to $(\Psi(M'), \Psi(M))_E$ (the space of E-homomorphisms from $\Psi(M')$ into $\Psi(M)$) which takes $\theta\in(M, M')_A$ to $\Psi(\theta)\in(\Psi(M')), \Psi(M))_{E'}$ where M and M' are arbitrary objects in $M(A)$, then $\Psi(M, M')$ is a well-defined k-linear map.

According to the similar arguments of the corresponding items in [3] we can prove the following lemma and proposition.

LEMMA 1.2 (see [3, Lemma (2.1a)]). *If Z is a component of Y as A-module, then the map*

$$\begin{array}{ccc} \Psi(M, Z)\colon (M, Z)_A & \longrightarrow & (\Psi(Z), \Psi(M))_E \\ \cup\!\!\!\!\!\!\;\!\!\in & & \cup\!\!\!\!\!\!\;\!\!\in \\ \theta & \longmapsto & \Psi(\theta) \end{array}$$

is bijective for any $M \in M(A)$.

PROOF. Assume $\Psi(\theta)(f)=0$ for any $f \in \Psi(Z)$. Since Z is a component of Y, $\Psi(\theta)(\iota)=0$ for the embedding

$$\iota \colon Z \hookrightarrow Y .$$

Hence $\iota \circ \theta = \theta = 0$, and we have proved that $\Psi(M, Z)$ is injective.

Next assume $Z=Y$. Let α be an arbitrary element of $(\Psi(Y), \Psi(M))_E$. Since $\Psi(Y)=E$, $\Psi(Y)$ contains a unity element 1_Y of E. Let $f=\alpha(1_Y)$, then

$$f \in \Psi(M)=(M, Y)_A .$$

Therefore

$$\Psi(M, Y)(f)=\Psi(f)=\Psi(\alpha(1_Y)) \in (\Psi(Y), \Psi(M))_E .$$

Since

$$\Psi(\alpha(1_Y))(1_Y)=1_Y \circ \alpha(1_Y)=\alpha(1_Y) ,$$

we have

$$\Psi(\alpha(1_Y))=\alpha .$$

Thus $\Psi(M, Y)$ is bijective.

Now let Z be a component of Y such that $Y=Z \oplus Z'$ for some $Z' \in M(A)$. Let ι be the embedding $\iota \colon Z \hookrightarrow Y$ and π be the projection of Y onto Z. Let t be an element of $(\Psi(Z), \Psi(M))_E$, then $t \circ \Psi(\iota) \in (\Psi(Y), \Psi(M))_E$. Hence there exists $\phi \in (M, Y)_A$ such that $\Psi(\phi)=t \circ \Psi(\iota)$. Finally since $\pi \circ \phi \in (M, Z)_A$ and $\Psi(\pi \circ \phi)=\Psi(\phi) \circ \Psi(\pi)=t \circ \Psi(\iota) \circ \Psi(\pi)=t \circ \Psi(\pi \circ \iota)=t$, thus $\Psi(M, Z)$ is surjective. Q.E.D.

PROPOSITION 1.3 (see [3, Corollary (2.1b)]). *Assume that Y be decomposed into a direct sum of a finite number of indecomposable components $Y_1, Y_2, \cdots, Y_r$. then*

(i) $\Psi(Y) \cong \Psi(Y_1) \dotplus \cdots \dotplus \Psi(Y_r)$ *as left E-modules,*

(ii) $Y_\rho \cong Y_\sigma$ *in* $M(A)$ *if and only if* $\Psi(Y_\rho) \cong \Psi(Y_\sigma)$ *in* $M(E)$, *for all* $1 \leqq \rho, \sigma \leqq r$, *and*

(iii) $\Psi(Y_\rho)$ *is an indecomposable left E-module for all* $1 \leqq \rho \leqq r$.

PROOF. (i) is clear from Lemma 1.1.

(ii) Since Ψ is a functor, $Y_\rho \cong Y_\sigma$ in $M(A)$ implies $\Psi(Y_\rho) \cong \Psi(Y_\sigma)$ in $M(E)$. Conversely if $\Psi(Y_\rho) \cong \Psi(Y_\sigma)$ in $M(E)$, then from Lemma 1.2 there exists $f \in (Y_\rho, Y_\sigma)_A$ such that

$$\Psi(f)\colon \Psi(Y_\sigma)\cong\Psi(Y_\rho)\ .$$

By the same argument there also exists $g\in(Y_\sigma, Y_\rho)_A$ such that $\Psi(g)=\Psi(f)^{-1}$. Therefore

$$\Psi(f)\circ\Psi(g)=\Psi(g\circ f)=1_{\Psi(Y_\rho)}=\Psi(1_{Y_\rho})$$

and

$$\Psi(g)\circ\Psi(f)=\Psi(f\circ g)=1_{\Psi(Y_\sigma)}=\Psi(1_{Y_\sigma})\ .$$

Hence $g\circ f=1_{Y_\rho}$, $f\circ g=1_{Y_\sigma}$ and $Y_\rho\cong Y_\sigma$ in $M(A)$.

(iii) Since

$$\Psi(Y_\rho, Y_\rho)\colon (Y_\rho, Y_\rho)_A \longrightarrow (\Psi(Y_\rho), \Psi(Y_\rho))_E$$

is an anti k-algebra isomorphism, $(\Psi(Y_\rho), \Psi(Y_\rho))_E$ is indecomposable and so is $\Psi(Y_\rho)$ (see, for example [6, Theorem (1.1)]). Q.E.D.

DEFINITION 1.4. Let M be a left A-module. The socle of M, $\mathrm{soc}\, M$, is the sum of all the irreducible submodules of M. Further if the left A-submodules of an algebra A over a field k satisfy the D.C.C., we call $M/(\mathrm{rad}\, A)M$ the head of an A-module M where $\mathrm{rad}\, A$ is the radical of A. We denote by $\mathrm{hd}\, M$ the head of M.

The proof of the following theorem was improved by Professor K. Morita.

THEOREM 1.5. *Assume that the left E-submodules of E satisfy the* D.C.C. *Then Y is decomposed into a finite number of indecomposable components $Y_1, Y_2, \cdots, Y_r$; and we have*

(i) $\mathrm{hd}\,\Psi(Y_\rho)\hookrightarrow\Psi(\mathrm{soc}\, Y_\rho)$ *if $\Psi(\mathrm{soc}\, Y_\rho)$ is semisimple* (i.e., *completely reducible*), *for any $1\leqq\rho\leqq r$, and*

(ii) $\mathrm{rad}\, E=\{f\in E \mid f(\mathrm{soc}\, Y)=0\}$ *if and only if $\Psi(\mathrm{soc}\, Y_\rho)$ is semisimple for all $1\leqq\rho\leqq r$.*

PROOF. It is clear that E is decomposed into a finite number of indecomposable modules (see [2, Theorem (14.2)]). Let ${}_EE=E\pi_1\oplus E\pi_2\oplus\cdots\oplus E\pi_r$ be a decomposition of E into non-zero indecomposable submodules $\{E\pi_\rho\}$ where $\{\pi_\rho\}$ are orthogonal idempotents in E such that $1=\pi_1+\pi_2+\cdots+\pi_r$. Then we have $Y=\pi_1(Y)\oplus\pi_2(Y)\oplus\cdots\oplus\pi_r(Y)$. Notice that $\pi_i|\pi_i(Y)=1_{\pi_i(Y)}$ and $\pi_i|\pi_j(Y)=0$ for $j\neq i$. Since the $E\pi_{\rho'}s$ are indecomposable, the $\pi_\rho(Y)$'s are also indecomposable from a theorem of Fitting (see [6, Theorem (1.1))]).

(i) Assume $\Psi(\mathrm{soc}\, Y_\rho)$ be semisimple, where $1\leqq\rho\leqq r$. Since $\mathrm{soc}\, Y_\rho \overset{\iota}{\hookrightarrow} Y_\rho \overset{\tau}{\rightarrow} Y_\rho/\mathrm{soc}\, Y_\rho\rightarrow 0$ is exact in $M(A)$, the sequence $0\rightarrow\Psi(Y_\rho/\mathrm{soc}\, Y_\rho)\overset{\Psi(\tau)}{\rightarrow}$

$\Psi(Y_\rho)\overset{\Psi(\iota)}{\longrightarrow}\Psi(\text{soc } Y_\rho)$ is also exact in $M(E)$ from Lemma 1.1. Thus we have

$$\Psi(Y_\rho)/\text{Im}\,\Psi(\tau) \hookrightarrow \Psi(\text{soc } Y_\rho)\ .$$

Since ι: soc $Y_\rho \hookrightarrow Y_\rho$ is non trivial, $\Psi(\iota)$ is also a non trivial E-homomorphism from Lemma 1.2. Hence $\Psi(Y_\rho)/\text{Im}\,\Psi(\tau)\cong\text{Im}\,\Psi(\iota)$ is a non zero semisimple E-module. Since $\Psi(Y_\rho)$ is a principal indecomposable module of E (see Proposition (1.3)), we have

$$\text{hd}\,\Psi(Y_\rho)=\Psi(Y_\rho)/\text{Im}\,\Psi(\tau) \hookrightarrow \Psi(\text{soc } Y_\rho)\ .$$

(ii) First assume $\text{rad}\,E = \{f \in E \mid f(\text{soc } Y) = 0\}$. Then since $(\text{rad}\,E)\Psi(\text{soc } Y)=0$, $\Psi(\text{soc } Y_\rho)$ is semisimple for any $1\leqq\rho\leqq r$ (see [2, Exercise 25.4]).

Next assume $\Psi(\text{soc } Y_\rho)$ is semisimple for all $1\leqq\rho\leqq r$. Let f be an element of E such that $f(\text{soc } Y)=0$, then $f\Psi(\text{soc } Y_\rho)=0$ for any $1\leqq\rho\leqq r$. Since $\text{hd}\,\Psi(Y_\rho)\hookrightarrow\Psi(\text{soc } Y_\rho)$ from (i), we have $f\in\text{rad}\,E$ (see [2, Exercise 25.8]). Now let $\alpha\in\text{rad}\,E$. Then since $\Psi(\text{soc } Y_\rho)$ is semisimple, $\alpha\Psi(\text{soc } Y_\rho)=0$. Hence $\alpha(\text{soc } Y_\rho)=0$ for all $1\leqq\rho\leqq r$. Q.E.D.

§ 2. A correspondence theorem.

We first introduce an idea of distinguishable modules.

DEFINITION 2.1. Let A be an algebra over a field k with a unity element 1. Assume that an A-module Y is decomposed into a finite number of indecomposable components, say $Y=Y_1\oplus Y_2\oplus\cdots\oplus Y_r$, and the left A-submodules of soc Y satisfy the D.C.C. Then an indecomposable component Y_ρ, where $1\leqq\rho\leqq r$, is said to be distinguishable (by socle) if soc Y_ρ is multiplicity free (i.e., soc Y_ρ is a direct sum of non-isomorphic simple modules) and $Y_\rho\cong Y_\sigma$ when soc Y_ρ and soc Y_σ have a same simple submodule up to isomorphism, for any $1\leqq\sigma\leqq r$. When all the indecomposable components Y_ρ's are distinguishable, we say that Y has a distinguishable decomposition

$$Y=Y_1\oplus Y_2\oplus\cdots\oplus Y_r\ .$$

Now let $E=\text{End}_A(Y)$ where Y is a left A-module. Throughout this section we assume that the left E-submodules of E satisfy the D.C.C. Then Y is decomposed into a finite number of indecomposable components $Y_1, Y_2, \cdots, Y_r$ as a straight consequence of a theorem of Fitting (see Theorem (1.5)). Thus we have

$$Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_r$$
$$\operatorname{soc} Y = \operatorname{soc} Y_1 \oplus \operatorname{soc} Y_2 \oplus \cdots \oplus \operatorname{soc} Y_r$$

and

$$\Psi(Y) \cong \Psi(Y_1) + \Psi(Y_2) + \cdots + \Psi(Y_r) .$$

In this section we study a condition under which $\Psi(X) \cong \operatorname{hd} \Psi(Y_\rho)$ holds for a given $1 \leqq \rho \leqq r$, where X is a simple component of $\operatorname{soc} Y_\rho$.

LEMMA 2.2. *Let E, Ψ and Y_ρ etc. be as before. Assume that Y_1 is distinguishable and* soc Y_1 *is of finite dimension. Then for any simple component X of* soc Y_1

(i) $\Psi(X) = \sum_{Y_\rho \cong Y_1} \oplus (X, Y_\rho)_A$ *(as k-modules) and*

(ii) $\dim_k \Psi(X) = |\{Y_\rho | Y_\rho \cong Y_1\}| \dim_k (X, X)_A$.

PROOF. (i) Since $\Psi(X) = (X, Y)_A$ by definition, we have $\Psi(X) = \sum_{\rho=1}^{r} \oplus (X, Y_\rho)_A$ as k-modules. Let $f \in (X, Y_\rho)_A$, then $f \neq 0$ implies $Y_1 \cong Y_\rho$. Therefore $(X, Y_\rho)_A \neq 0$ if and only if $Y_1 \cong Y_\rho$.

(ii) from (i) we have $\dim_k \Psi(X) < \infty$ and

$$\Psi(X) = \sum_{Y_\rho \cong Y_1} \oplus (X, \operatorname{soc} Y_\rho)_A$$

Hence $\dim_k \Psi(X) = |\{Y_\rho | Y_\rho \cong Y_1\}| \dim_k (X, X)_A$, because soc Y_1 is multiplicity free. Q.E.D.

From the Schur's lemma we can prove the following corollary.

COROLLARY 2.3. *Under the same assumption of Lemma 2.2, if k is an algebraically closed field, then*

$$\dim_k \Psi(X) = |\{Y_\rho | Y_\rho \cong Y_1\}|$$

where X is a simple component of soc Y_1.

PROPOSITION 2.4. *Let E, Ψ and Y_ρ etc. be as before. Assume that Y_1 is distinguishable and* soc Y_1 *is of finite dimension. Then there exists an injective E-homomorphism of* hd $\Psi(Y_1)$ *into* $\Psi(X)$. i.e.,

$$\operatorname{hd} \Psi(Y_1) \hookrightarrow \Psi(X) ,$$

for any simple component X of soc Y_1. *Hence in this case* hd $\Psi(Y_1)$ *is finite dimensional.*

PROOF. Since $\dim_k \Psi(X)$ is non-zero and finite, we can choose a minimal non-zero submodule X_0 of $\Psi(X)$. Since $X_0 \cong \operatorname{hd} \Psi(Y_\rho)$ for some $1 \leqq \rho \leqq r$ (see [2, Corollary (54.13)]), $(\Psi(Y_\rho), \Psi(X))_E \neq 0$ and so $(X, Y_\rho)_A \neq 0$

for that ρ (see Lemma 1.2). Hence $Y_1 \cong Y_\rho$ from the assumption. Thus

$$\operatorname{hd} \Psi(Y_1) \cong \operatorname{hd} \Psi(Y_\rho) \cong X_0 \hookrightarrow \Psi(X) . \qquad \text{Q.E.D.}$$

COROLLARY 2.5. *Let E, Ψ and Y_ρ etc. be as before. Assume that the decomposition $Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_r$ is distinguishable and the* soc Y *is of finite dimension. Then E is also finite dimensional over k.*

We can prove the following lemma as an application of the Wedderburn's theorem.

LEMMA 2.6. *Let E, Ψ and Y_ρ etc. be as before. Assume that E is finite dimensional and k is algebraically closed, then we have*

$$\dim_k \operatorname{hd} \Psi(Y_\rho) = |\{Y_\sigma | Y_\sigma \cong Y_\rho\}|$$

for any $1 \leqq \rho \leqq r$.

THEOREM 2.7. *Let E, Ψ and Y_ρ etc. be as before. Assume that E and* soc Y_1 *are finite dimensional and the left A-submodules of* soc Y *satisfy the* D.C.C. *and k is an algebraically closed field. Then* $\operatorname{hd} \Psi(Y_1) \cong \Psi(X)$ *for any simple component X of* soc Y_1 *if and only if Y_1 is distinguishable.*

PROOF. First assume $\operatorname{hd} \Psi(Y_1) \cong \Psi(X)$ for any simple component X of soc Y_1. If soc Y_1 is decomposed into a direct sum of simple components $\{X_1, X_2, \cdots, X_t\}$ and $X_1 \cong X_2$, then since

$$\begin{aligned} \Psi(X_1) = (X_1, Y)_A &= \Big(X_1, \sum_{\rho=1}^{r} \oplus Y_\rho\Big)_A \\ &= \{\sum_{Y_\rho \cong Y_1} \oplus (X_1, Y_\rho)_A\} \oplus \{\sum_{Y_\rho \not\cong Y_1} (X_1, Y_\rho)_A\} , \end{aligned}$$

and $\dim_k (X_1, Y_1)_A \geqq 2$, we have $\dim \Psi(X_1) \geqq \sum_{Y_\rho \cong Y_1} \dim (X_1, Y_\rho)_A > |\{Y_\rho | Y_\rho \cong Y_1\}| = \dim_k \operatorname{hd} \Psi(Y_1)$, a contradiction. Hence soc Y_1 is multiplicity free. Let Y_ρ be an indecomposable module from $Y_1, Y_2, \cdots, Y_r$ such that $(X, \operatorname{soc} Y_\rho)_A \neq 0$ for some simple component X of soc Y_1. Then $(X, Y_\rho)_A \neq 0$ and we have $(\Psi(Y_\rho), \Psi(X))_E \neq 0$ from Lemma 1.2. Since $\Psi(X)$ is simple, $\operatorname{hd} \Psi(Y_\rho) \cong \Psi(X) \cong \operatorname{hd} \Psi(Y_1)$. Hence $\Psi(Y_\rho) \cong \Psi(Y_1)$, i.e., $Y_\rho \cong Y_1$ from Proposition 1.3.

Next assume that Y_1 is distinguishable. Then from Proposition 2.4 we have $\operatorname{hd} \Psi(Y_1) \hookrightarrow \Psi(X)$ for any simple component X of soc Y_1. Since $\dim_k \Psi(X) = |\{Y_\rho | Y_\rho \cong Y_1\}|$ from Corollary 2.3 and $\dim_k \operatorname{hd} \Psi(Y_1) = |\{Y_\rho | Y_\rho \cong Y_1\}|$ from Lemma 2.6, we have $\operatorname{hd} \Psi(Y_1) \cong \Psi(X)$. Q.E.D.

COROLLARY 2.8. *Let E, Ψ and Y_ρ etc. be as before. Assume that the left E-submodules of E satisfy the* D.C.C. *and k is an algebraically closed field, and further assume that* soc Y *is finite dimensional. Then*

(i) *The following two statements are equivalent.*

(a) soc Y_ρ *is simple for any* $1\leqq\rho\leqq r$, *and* soc $Y_\rho\cong$soc Y_σ *if and only if* $Y_\rho\cong Y_\sigma$ *for any* $1\leqq\rho, \sigma\leqq r$.

(b) hd $\Psi(Y_\rho)\cong\Psi(\mathrm{soc}\ Y_\rho)$ *for any* $1\leqq\rho\leqq r$.

(ii) *Assume that Y has a distinguishable decomposition* $Y=Y_1\oplus Y_2\oplus\cdots\oplus Y_r$, *then we have*

$$\mathrm{rad}\, E=\{f\in E\,|\,f(\mathrm{soc}\ Y)=0\}\ .$$

PROOF. (i) (a)$\Rightarrow$(b): From Corollary 2.5 E is finite dimensional. Hence hd $\Psi(Y_\rho)\cong\Psi(\mathrm{soc}\ Y_\rho)$, for any $1\leqq\rho\leqq r$, straightly from the theorem.

(b)$\Rightarrow$(a): Since $\dim_k\Psi(\mathrm{soc}\ Y_\rho)$ is finite for any $1\leqq\rho\leqq r$, E is also finite dimensional. Since soc Y_ρ is simple, for any $1\leqq\rho\leqq r$, from Lemma 1.1, soc $Y_\rho\cong$soc Y_σ if and only if $Y_\rho\cong Y_\sigma$ for any $1\leqq\rho, \sigma\leqq r$, from the theorem.

(ii) From Corollary 2.5 E is finite dimensional. Hence we have hd $\Psi(Y_\rho)\cong\Psi(X)$ for any simple component X of soc Y_ρ where $1\leqq\rho\leqq r$. Thus it is clear from Lemma 1.1 and Theorem 1.5. Q.E.D.

§ 3. Functor Φ.

Let k be a field and A be an algebra over k with a unity element 1. We denote by $M(A)$ the category of left A-modules. Let Y be an object in $M(A)$ and $E=\mathrm{End}_A(Y)$, then we write $M'(E)$ for the category of right E-modules. In this section we just introduce a covariant representation functor Φ of $M(A)$ into $M'(E)$ with respect to Y, and its properties which are necessary for later discussion.

Let $M\in M(A)$ and $\Phi(M)=(Y, M)_A$. Then we can make $\Phi(M)$ into a right E-module by the following operation.

$$\begin{array}{ccc}\Phi(M)\times E & \longrightarrow & \Phi(M)\\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ (f, \alpha) & \longmapsto & f\circ\alpha\end{array}$$

If $M, M'\in M(A)$ and $\theta\in(M, M')_A$, we define $\Phi(\theta)$: $\Phi(M)\to\Phi(M')$ to be the mapping of $\Phi(M)$ into $\Phi(M')$ which takes f to $\theta\circ f$ for all $f\in\Phi(M)$. Thus we get a covariant functor Φ of $M(A)$ into the category of right E-modules $M'(E)$.

The following lemma is well-known.

Lemma 3.1. *Φ is a covariant, k-linear and left exact functor from $M(A)$ into $M'(E)$,* i.e.,

(i) $\Phi(\mathrm{id}_M)=\mathrm{id}_{\Phi(M)}$ *for any* $M\in M(A)$, *and* $\Phi(\theta'\circ\theta)=\Phi(\theta')\circ\Phi(\theta)$ *where* $\theta\in(M, M')_A$ *and* $\theta'\in(M', M'')_A$,

(ii) $\Phi(c\theta)=c\Phi(\theta)$ *and* $\Phi(\theta+\theta')=\Phi(\theta)+\Phi(\theta')$ *where* $\theta, \theta'\in(M, M')_A$ *and* $c\in k$,

(iii)
$$0\longrightarrow\Phi(M')\xrightarrow{\Phi(\theta_1)}\Phi(M)\xrightarrow{\Phi(\theta_2)}\Phi(M'')$$

is exact for any exact sequence

$$0\longrightarrow M'\xrightarrow{\theta_1}M\xrightarrow{\theta_2}M''\ \text{in}\ M(A).$$

Now let $\Phi(M, M')$ be a map from $(M, M')_A$ to $(\Phi(M), \Phi(M'))_E$ which takes $\theta\in(M, M')_A$ to $\Phi(\theta)\in(\Phi(M), \Phi(M'))_E$, where M and M' are arbitrary objects in $M(A)$, then $\Phi(M, M')$ is also a well-defined k-linear map.

One can prove the following lemma and proposition by the similar argument of the corresponding items in section 1.

Lemma 3.2. *If Z is a component of Y as A-module, then the map*

$$\begin{array}{ccc}\Phi(Z, M):\ (Z, M) & \longrightarrow & (\Phi(Z), \Phi(M))_E\\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup\\ \theta & \longmapsto & \Phi(\theta)\end{array}$$

is bijective for any $M\in M(A)$.

Proposition 3.3. *Assume that Y be decomposed into a direct sum of finite indecomposable components $Y_1, Y_2\cdots Y_r$. Then*

(i) $\Phi(Y)=\Phi(Y_1)\oplus\Phi(Y_2)\oplus\cdots\oplus\Phi(Y_r)$,

(ii) $Y_\rho\cong Y_\sigma$ *in* $M(A)$ *if and only if* $\Phi(Y_\rho)\cong\Phi(Y_\sigma)$ *in* $M'(E)$, *for all* $1\leqq\rho, \sigma\leqq r$, *and*

(iii) $\Phi(Y_\rho)$ *is an indecomposable right E-module for all* $1\leqq\rho\leqq r$.

§ 4. Quasi-Frobenius endomorphism algebras.

Let $E=\mathrm{End}_A(Y)$, where A is an algebra over a field k with a unity element 1 and Y is a left A-module, as usual. Throughout this section we assume that the left and right E-submodules of E satisfy the D.C.C. Then Y is decomposed into a finite number of indecomposable components $Y_1, Y_2\cdots Y_r$. Thus we have

$$Y=Y_1\oplus Y_2\oplus\cdots\oplus Y_r$$

(4.1) $$\Phi(Y)=\Phi(Y_1)\oplus\Phi(Y_2)\oplus\cdots\oplus\Phi(Y_r)\ ,$$

where Φ is the functor Φ defined in section 3 with respect to Y.

(4.2) Now assume E_E be an injective right E-module (i.e., E is a quasi-Frobenius algebra), then each indecompopable component $\Phi(Y_\rho)$ in (4.1), $1\leqq\rho\leqq r$, (see Proposition 3.3) has a simple socle (see [2, Theorem (58.12)]) and $\Phi(Y_\rho)\cong\Phi(Y_\sigma)$ if and only if $\mathrm{soc}\,\Phi(Y_\rho)\cong\mathrm{soc}\,\Phi(Y_\sigma)$ where $1\leqq\rho$, $\sigma\leqq r$. Hence every simple module in $M'(E)$ is isomorphic to $\mathrm{soc}\,\Phi(Y_\rho)$ for some ρ.

Next we show a theorem, which is a generalization of [3, Theorem 1] and also an example of Y which has a distinguishable decomposition.

THEOREM 4.3 (see [3, Theorem 1]). *Let E, Y_ρ and Φ etc. be as before. Assume that the left and right E-submodules of E satisfy the* D.C.C. *Suppose E_E is an injective right E-module, and assume further for each simple A-module $M\in M(A)$ if M is a component of* soc Y, *then* $\Phi(M)\neq 0$.

Then we have soc Y_ρ *is simple and* $\Phi(\mathrm{soc}\,Y_\rho)=\mathrm{soc}\,\Phi(Y_\rho)$ *for all* $1\leqq\rho\leqq r$.

PROOF. Assume soc Y_ρ not be simple, then there exist simple submodules M, M' of Y_ρ such that $M\cap M'=\{0\}$. Since M, M' are components of soc Y_ρ, we have $\Phi(M)\neq 0$ and $\Phi(M')\neq 0$ from the assumption. Thus $\Phi(Y_\rho)$ contains a submodule $\Phi(M)\oplus\Phi(M')$ with non-zero right E-modules $\Phi(M)$ and $\Phi(M')$. Hence $\mathrm{soc}\,\Phi(Y_\rho)$ is not simple against (4.2). Thus soc Y_ρ is simple for all $\rho\in\{1, 2, \cdots r\}$.

Write $M=\mathrm{soc}\,Y_\rho$. From the above discussion M is simple and $X=\Phi(M)\neq 0$. Since $X\subseteq\Phi(Y_\rho)$ and $\mathrm{soc}\,\Phi(Y_\rho)$ is simple from (4.2), if X is simple, $X=\mathrm{soc}\,\Phi(Y_\rho)=\Phi(\mathrm{soc}\,Y_\rho)$ and the proof is complete.

So we assume that X is not simple. Let X/K be a simple factor module of X. Remark that $0\subsetneqq K\subsetneqq X$. By (4.2) X/K is isomorphic to some submodule of $\Phi(Y)$, hence there exists an E-homomorphism $\beta: X\to\Phi(Y)$ with $\ker\beta=K$. Since $\Phi(Y)$ is injective from the assumption, β can be extended to an E-homomorphism $\beta_1: \Phi(Y_\rho)\to\Phi(Y)$. But Lemma 3.2 shows that $\beta_1=\Phi(\beta_2)$ for some A-homomorphism $\beta_2: Y_\rho\to Y$. Since $\beta_1(f)=\beta_2\circ f$ for all $f\in\Phi(Y_\rho)$ by definition, $\beta_2\circ f=0$ for any f in K. Let f_0 be a non-zero element in K. Since M is simple, $Af_0(Y)=M$ and $\beta_2(M)=A\beta_2(f_0(Y))=0$. Hence $\ker\beta_2\supseteqq M$. Now we have $\beta(X)=\beta_1(X)=\Phi(\beta_2)(X)$, and for all $f\in X$ $(\Phi(\beta_2)(f))(Y)=(\beta_2\circ f)(Y)\subseteqq\beta_2(M)=0$. Hence $\beta(X)=0$ and $\ker\beta\supseteqq X$, which contradicts to our assumption $0\subsetneqq K\subsetneqq X$. Therefore X is simple. Q.E.D.

COROLLARY 4.4. *Under the same assumption of Theorem* 4.3, *we have* soc Y_ρ *is simple for any* $1\leqq\rho\leqq r$, *and* soc $Y_\rho\cong$soc Y_σ *if and only if* $Y_\rho\cong Y_\sigma$, *for any* $1\leqq\rho, \sigma\leqq r$.

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