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On Normal Projective Surfaces with Trivial Dualizing Sheaf

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Let X be a normal projective surface defined over an algebraically closed field k of characteristic $\neq 2, 3$, whose dualizing sheaf ω_X is isomorphic to the structure sheaf $\mathcal{O}_X$. In this paper we shall investigate such a surface, especially when it has at least one singular point with positive geometric genus (see Definition 1).

Normal surface singularities with small geometric genera have been studied by many authors (cf. Artin [2], Laufer [7] and Yau [11]). Their results show that such singularities have rather simple properties, and it seems that the geometric genus is an important invariant for studies of singularities.

We will determine the geometric genus of singular points on X by means of the irregularity q of a non-singular model of X (Theorem 1), and estimate q in terms of the dimension of the projective space in which X is embedded (Theorem 3). On the other hand, as a corollary to Theorem 1, we have $H^1(X, \mathcal{O}_X)=0$ if X is not an abelian surface, and hence such an X has properties similar to a $K3$ surface; a characterization of X (when X is not an abelian surface) is given in Theorem 2.

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§ 1. Preliminaries.

In this section we summarize some local properties of singularities on a normal surface. Let X be a normal surface with a singular point x. Let $\pi: \tilde{X} \to X$ be the minimal resolution of x and let $A=\bigcup_{i=1}^{n} A_i$ denote the exceptional set $\pi^{-1}(x)$, where A_i's are the irreducible components of A.

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PROPOSITION 1 (Mumford [8]). *The intersection matrix* $((A_i, A_j))$ *is negative definite.*

DEFINITION 1 (Wagreich [12]). The geometric genus of x is defined to be $\dim_k (R^1\pi_*\mathcal{O}_{\tilde{X}})_x$ and denoted by $p_g(x)$.

PROPOSITION 2 (Artin [2]). *There exists a unique divisor* Z *on* X *satisfying the following properties*:

(i) $Z\geqq 0$, $Z\neq 0$,

(ii) $(Z, A_i)\leqq 0$ *for every* A_i,

(iii) *if another divisor* Z' *satisfies the conditions above, then* $Z\leqq Z'$.

Further, we have $\operatorname{supp}(Z)=A$.

DEFINITION 2 (Artin [2]). We call Z, determined as in Proposition 2, the fundamental divisor of A.

In what follows Z denotes the fundamental divisor of A.

DEFINITION 3 (Artin [2], Laufer [7] and Saito [10]). The point x is said to be rational if $p_g(x)=0$.

The singular point x is minimally elliptic if $p_a(Z)=1$ and any one-dimensional connected proper subvariety of A is the exceptional set for a rational singular point, where $p_a(Z)$ denotes the arithmetic genus of Z.

The singular point x is simple elliptic if A consists of a non-singular elliptic curve.

The following three propositions are well known and not difficult to prove.

PROPOSITION 3. *We have*

$$p_a(A_i)\leqq p_a(Z) \quad \textit{for every} \quad A_i\ ,$$

and

$$p_a(Z)\leqq \sup_{\substack{D>0 \\ \operatorname{supp}(D)\subseteq A}} p_a(D)\leqq p_g(x)\ .$$

PROPOSITION 4. *The following conditions are equivalent*:

(i) *The local ring* $\mathcal{O}_{x,X}$ *is a Gorenstein ring.*

(ii) *The dualizing sheaf* ω_X *is invertible at* x.

(iii) *There exists an open neighborhood* U *of* x *such that* $\omega_{X|U-\{x\}}\cong \mathcal{O}_{U-\{x\}}$.

PROPOSITION 5. *Assume that* $\mathcal{O}_{x,X}$ *is Gorestein. Then we have* $\omega_{\tilde{X}}\cong \pi^*\omega_X\otimes\mathcal{O}_{\tilde{X}}(-Z')$ *where* Z' *is an effective divisor supported on* A. *Moreover we have*

(i) $Z'=0$ *if and only if* x *is a rational double point.*

(ii) *If* x *is not a rational double point, then* $\operatorname{supp}(Z')=A$.

PROPOSITION 6 (Hidaka and Watanabe [6]). *Assume that* $\mathcal{O}_{x,X}$ *is Gorenstein. Then, if* $p_a(Z)\geqq 2$, *it follows that* $p_g(x)\geqq p_a(Z)+1$.

PROPOSITION 7 (Artin [1], [2] and Laufer [7]). *The singular point* x *is a rational double point if and only if* x *is a rational singularity and* $\mathcal{O}_{x,X}$ *is Gorenstein.*

The point x *is a minimally elliptic singular point if and only if* $p_g(x)=1$ *and* $\mathcal{O}_{x,X}$ *is Gorenstein.*

§ 2. Singularities on a surface X with $\omega_X\cong\mathcal{O}_X$.

Throughout this section, X denotes a normal projective surface with $\omega_X\cong\mathcal{O}_X$ and $\pi\colon\tilde{X}\to X$ denotes the minimal resolution of X. We note that every local ring $\mathcal{O}_{x,X}$ $(x\in X)$ is a Gorenstein ring. Let us denonte by S the set of singular points on X which are not rational double points. Put $q=\dim_k H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$. For any non-singular surface Y, K_Y denotes the canonical divisor class of Y.

PROPOSITION 8. (i) *If* X *is non-singular, then* X *is either a* $K3$ *surface or an abelian surface.*

(ii) *If* X *has singular points and* $S=\emptyset$, *then* $\tilde{X}$ *is a* $K3$ *surface.*

(iii) *If* $S\neq\emptyset$, *then* $\tilde{X}$ *is birationally equivalent to a ruled surface.* (*By a ruled surface, we mean the projectivization of a vector bundle of rank* 2 *over a non-singular complete curve.*)

PROOF. By Proposition 5 and the assumption of $\omega_X\cong\mathcal{O}_X$, we have that $|-K_{\tilde{X}}|\neq\emptyset$ and that $-K_{\tilde{X}}=0$ if and only if $S=\emptyset$. The classification theory of surfaces (cf. Bombieri and Husemoller [4]) shows that $\tilde{X}$ is a $K3$ surface or an abelian surface if $-K_{\tilde{X}}=0$ and is birationally equivalent to a ruled surface if $|-K_{\tilde{X}}|\neq\emptyset$ and $-K_{\tilde{X}}\neq 0$. Assume that $\tilde{X}$ is an abelian surface. Then $\tilde{X}$ has no curve of negative self-intersection number, and hence, by Proposition 1, we have that $\tilde{X}=X$. Q.E.D.

In what follows we are mainly concerned with the case of $S\neq\emptyset$.

THEOREM 1. *Assume that* $S\neq\emptyset$. *Then we have*

(i) *if* $q\neq 1$, *then* S *consists of one point with* $p_g=q+1$,

(ii) *if* $q=1$, *then* S *consists of either one point with* $p_g=2$ *or two points with* $p_g=1$.

Moreover, in the second case of (ii), *both of the two points are simple* *elliptic.*

Before proving the theorem, we prepare some lemmas.

LEMMA 1. *Let* $\tilde{X}=X_0\xrightarrow{\mu_1}X_1\xrightarrow{\mu_2}\cdots\xrightarrow{\mu_n}X_n=\bar{X}$ *be a sequence of blow-downs obtaining a relatively minimal model* $\bar{X}$ *of* $\tilde{X}$. *Then there exists* $D_i\in|-K_{X_i}|$ $(0\leqq i\leqq n)$ *such that*

(i) $\mathrm{supp}(D_0)$ *is the union of the exceptional sets of* π *which correspond to the singular points in* S,

(ii) μ_i *is the blow-up with center at a point on* $\mathrm{supp}(D_i)$ *for* $1\leqq i\leqq n$,

(iii) $\mu_i(D_{i-1})=D_i$ *for* $1\leqq i\leqq n$.

PROOF. The statement (i) is a consequence of the assumption $\omega_X\cong\mathcal{O}_X$ and Proposition 5. Assume we have $D_0,\cdots,D_{i-1}$ $(0<i\leqq n)$ as in the statement of the lemma. Then $-D_{i-1}$ is linearly equivalent to $\mu_i^*K_{X_i}+E$ where E is the exceptional curve of the blow-up μ_i. Since μ_i is a birational morphism, we can find a divisor D_i linearly equivalent to $-K_{X_i}$ such that $D_{i-1}=\mu_i^*D_i-E$. Since D_{i-1} is effective by our assumption, we conclude tnat D_i is also effective and the center of the blow-up μ_i must lie on $\mathrm{supp}(D_i)$. Therefore, by induction on i, we have proved Lemma 1.

Let $\bar{X}$ be a ruled surface over a non-singular curve C of genus q and let $\varpi:\bar{X}\to C$ denote the canonical surjection. Then we can find a locally free sheaf $\mathcal{E}$ of rank 2 over C such that $\bar{X}\cong\boldsymbol{P}(\mathcal{E})$ over C and that $\mathcal{E}$ is normalized (i.e., $H^0(\mathcal{E})\neq 0$ and $H^0(\mathcal{E}\otimes\mathcal{L})=0$ for any invertible sheaf $\mathcal{L}$ on C such that $\deg\mathcal{L}<0$). Then there exists a section $\sigma: C\to\bar{X}$ with image C_0 such that $\mathcal{O}_{\bar{X}}(C_0)\cong\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)$. We fix such a C_0. Put $e=-(C_0^2)$. Then we have $-K_{\bar{X}}\equiv 2C_0+(e-2q+2)f$ where $\equiv$ means numerical equivalence and f denotes a fibre of ϖ.

LEMMA 2. *Assume* $q\geqq 1$. *Then* $|-K_{\bar{X}}|$ *contains no irreducible curve.*

PROOF. If $q\geqq 2$, the lemma is trivial because $p_a(-K_{\bar{X}})=1$.

We now assume that C is an elliptic curve. Suppose that there is an irreducible element C' in $|-K_{\bar{X}}|$. Since $p_a(C')=1$, C' is a non-singular elliptic curve and $p=\varpi\circ i: C'\to C$ is an étale morphism of degree 2, where $i: C'\to\bar{X}$ is the inclusion. Performing elementary transformations at points of $C'\cap C_0$, if necessary, we may assume that $C'\cap C_0=\varnothing$. Set $\bar{X}'=\boldsymbol{P}(p^*\mathcal{E})$ and let $\varpi':\bar{X}'\to C'$, $\tilde{p}:\bar{X}'\to\bar{X}$ be the induced morphisms. Then it follows easily that $\tilde{p}^*(C'+C_0)$ is the sum of three disjoint elliptic curves on $\bar{X}'$, each of which is isomorphic to C' via ϖ'. Consequently we obtain that $p^*\mathcal{E}\cong\mathcal{O}_{C'}\oplus\mathcal{O}_{C'}$.

Now there is an exact sequence of $\mathcal{O}_C$-modules:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

with an invertible sheaf $\mathcal{L}$ on C because $\mathcal{E}$ is normalized. Hence on C',

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{O}_{C'} \oplus \mathcal{O}_{C'} \longrightarrow p^*\mathcal{L} \longrightarrow 0$$

is exact, and so $p^*\mathcal{L} \cong \mathcal{O}_{C'}$. This implies that $\mathcal{L}^2 \cong \mathcal{O}_C$.

If $\mathcal{L} \cong \mathcal{O}_C$, then we infer that the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

is non-trivial by the existence of C' on $\bar{X}$. From the result of M. F. Atiyah [3], we get

$$\begin{aligned}\dim|-K_{\bar{X}}| &= \dim H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-K_{\bar{X}})) - 1 \\ &= \dim H^0(C, S^2\mathcal{E}) - 1 = 0 .\end{aligned}$$

But $|-K_{\bar{X}}|$ contains at least two elements, namely C' and $2C_0$, a contradiction.

If $\mathcal{L} \not\cong \mathcal{O}_C$ and $\mathcal{L}^2 \cong \mathcal{O}_C$, then $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$. But this is impossible, too. Indeed,

$$\begin{aligned}\dim|-K_{\bar{X}}| &= \dim H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-K_{\bar{X}})) - 1 \\ &= \dim H^0(C, S^2\mathcal{E} \otimes \mathcal{L}) - 1 = 0 ,\end{aligned}$$

and $|-K_{\bar{X}}| \ni C', C_0 + C_1$, where C_1 is the image of the section $\sigma_1 \colon C \to \bar{X}$ corresponding to the surjection $\mathcal{E} \to \mathcal{O}_C \to 0$. Q.E.D.

LEMMA 3. *Assume $q \geqq 1$. Then, if D is an effective divisor in $|-K_{\bar{X}}|$, we have either*

(i) $D = 2C_0 + (e - 2q + 2)$*-fibres*

or

(ii) $q = 1$ *and D is the sum of two disjoint sections of* ϖ.

PROOF. If $D \geqq 2C_0$, then D is clearly of type (i) of the Lemma. Therefore we may assume that $D \not\geqq 2C_0$.

Assume that D contains a section C_1 of ϖ as a component. Then we have

$$D = C_1 + C_2 + \sum_{i=1}^{r} f_i ,$$

where C_2 is another section and f_i's are fibres. Then we get

$$1 = p_a(D) = p_a(C_1) + p_a(C_2) + (C_1, C_2) + r - 1 ,$$

and so

$$2q=p_a(C_1)+p_a(C_2)=2-r-(C_1, C_2) .$$

Since $C_1\neq C_0$ or $C_2\neq C_0$, we have $(C_1, C_2)\geqq 0$. Therefore we conclude

$$q=1 , \quad r=0 \quad \text{and} \quad (C_1, C_2)=0 .$$

Assume that D contains no section as a component. Then the irreducible decomposition of D is as follows:

$$D=D_1+\sum_{i=1}^{r} f_i ,$$

where f_i's are fibres. Then we have

$$1=p_a(D)=p_a(D_1)+r .$$

From Lemma 2, we see $r\geqq 1$, and hence $p_a(D_1)=0$. This implies that D_1 is contained in a fibre, which is absurd. Q.E.D.

PROOF OF THEOREM 1. From the Leray spectral sequence of the map $\pi: \tilde{X}\to X$, we have the exact sequence:

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^0(R^1\pi_*\mathcal{O}_{\tilde{X}}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) .$$

Since

$$H^2(X, \mathcal{O}_X)\cong H^0(X, \omega_X)\cong H^0(X, \mathcal{O}_X)\cong k$$

and

$$H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})\cong H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}))=H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-D_0))=0 ,$$

we have

$$(*) \qquad \sum_{x\in X} p_g(x)=q+1-\dim H^1(X, \mathcal{O}_X) .$$

If $q=0$, then we get $\sum_{x\in X} p_g(x)\leqq 1$. Therefore our theorem is clear in this case by Proposition 7. In what follows we assume that $q\geqq 1$ and use the notations in Lemma 1. Then, by Proposition 8, $\bar{X}$ is a ruled surface over some non-singular curve C of genus q.

In order to determine the geometric genus of a singular point on X, we look at the exceptional set of it. But by virture of Lemma 1, it suffices to consider the configuration of supp(D_n) and the multiplicities of the components of D_n. In particular, we note that the number of connected components of D_n is equal to that of D_0, the number of singular points in S.

Assume that $q \geqq 2$. Then by Lemma 3, $\mathrm{supp}(D_n)$ is connected and hence S consists of one point x. Moreover, again from Lemma 3, D_n contains a curve of genus q as a component, and hence so does D_0. Therefore, from Propositions 3 and 6, we get $p_g(x) \geqq q+1$, and so $p_g(x) = q+1$ by (*).

Finally, we consider the case of $q=1$. If D_n is the sum of two disjoint elliptic curves, then so is D_0, and hence S consists of two simple elliptic singular points. Otherwise, we have $D_n = 2C_0 + e$-fibres (Lemma 3) and S consists of one point x. We shall prove $p_g(x)=2$. By (*), we have only to show $p_g(x) \geqq 2$, and from Proposition 7, it suffices to prove that x is not a minimally elliptic singularity. If it were, then either every component of D_0 is a non-singular rational curve or D_0 is some multiple of a non-singular elliptic curve. In our situation only the latter case could occur, so we have $D_n = 2C_0$. But since $(D_n^2)=0$, D_n is not contractable by Proposition 1. From the diagram of Lemma 1, we conclude that D_0 contains a non-singular rational curve, and we get a contradiction. This completes the proof of Theorem 1.

COROLLARY 1. *If X is not an abelian surface, then we have*

$$H^1(X, \mathcal{O}_X) = 0 .$$

PROOF. This follows immediately from Theorem 1 and (*).

COROLLARY 2. *If $q \geqq 1$, then any rational double point on X is of type A_n* (cf. Saito [10]).

PROOF. Assume x is a rational double point on X. Then $\pi^{-1}(x)$ consists of non-singular rational curves (Proposition 3) which do not meet D_0 (notations are as in Lemma 1). By $q \geqq 1$, $\bar{X}$ is a ruled surface of genus q, and so any rational curve on $\tilde{X}$ must be either the proper transformation of a fibre on $\bar{X}$ or that of the exceptional curve of the blow-up μ_i for some i. Therefore each rational curve on $\tilde{X}$, except the components of D_0, meets at most two of other rational curves. Q.E.D.

§ 3. Surface X with $\omega_X \cong \mathcal{O}_X$ in a projective space.

In this section we will prove two theorems concerning with the embeddings of a normal surface X with $\omega_X \cong \mathcal{O}_X$ in projective spaces. We restrict ourselves to the case in which the embeddings are defined by complete linear systems on X, i.e., we treat only very ample invertible sheaves on X.

THEOREM 2. *Let X be a normal projective surface. Then the following conditions are equivalent*:

(i) $\omega_X \cong \mathcal{O}_X$ *and X is not an abelian surface.*

(ii) *For any very ample invertible sheaf $\mathcal{H}$ on X, a general member of $|\mathcal{H}|$ is a canonical curve of genus equal to* $\dim|\mathcal{H}|$.

(iii) *There exists a very ample invertible sheaf $\mathcal{H}$ on X such that a general member of $|\mathcal{H}|$ is a canonical curve of genus equal to* $\dim|\mathcal{H}|$.

PROOF. We first prove the implication (i)⇒(ii). Given any very ample invertible sheaf $\mathcal{H}$ on X, take a general member H of $|\mathcal{H}|$ so that H is non-singular, irreducible and disjoint from the singular points of X. Put $g=\dim|\mathcal{H}|$. From Corollary 1 and the exact cohomology sequence associated to the sheaf exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0\ ,$$

it is deduced that the genus of H is equal to g. Moreover we have

$$\omega_H \cong \omega_X \otimes \mathcal{O}_X(H)_{|H} \cong \mathcal{O}_X(H)_{|H}\ .$$

Hence H is a canonical curve of genus g.

The implication (ii)⇒(iii) is trivial.

To complete the proof, we prove the implication (iii)⇒(i). Let U denote $X-\mathrm{Sing}(X)$. To prove $\omega_X \cong \mathcal{O}_X$, it suffices to show $\omega_{X|U} \cong \mathcal{O}_U$ (Proposition 4). Fix a general member H of $|\mathcal{H}|$ as before and put $g=\dim|\mathcal{H}|$. By our assumption, H is a canonical curve of genus g. For each positive integer n we have an exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{O}_X((n-1)H)) \longrightarrow H^0(X, \mathcal{O}_X(nH)) \xrightarrow{\alpha_n} H^0(H, \mathcal{O}_H(nH)) \\ \longrightarrow H^1(X, \mathcal{O}_X((n-1)H)) \longrightarrow H^1(X, \mathcal{O}_X(nH))\ . \end{aligned}$$

Since $\dim H^0(X, \mathcal{O}_X(H))=g+1$ and $\dim H^0(H, \mathcal{O}_H(H))=g$, α_1 is surjective. A canonical curve is projectively normal (Saint-Donat [9]), thus we infer that α_n is surjective for all n. Hence we get

$$H^1(X, \mathcal{O}_X) \subseteqq H^1(X, \mathcal{O}_X(H)) \subseteqq H^1(X, \mathcal{O}_X(2H)) \subseteqq \cdots\ .$$

Therefore, by Serre's theorem (cf. Grothendieck [5]), we have

$$H^1(X, \mathcal{O}_X(nH))=0 \quad \text{for} \quad n \geqq 0\ .$$

On the other hand, we have $\omega_X \otimes \mathcal{O}_H \cong \mathcal{O}_H$ by the adjunction formula. Therefore we get the exact sequence:

$$0 \longrightarrow \omega_X \otimes \mathcal{O}_X(-H) \longrightarrow \omega_X \longrightarrow \mathcal{O}_H \longrightarrow 0\ .$$

The exact cohomology sequence associated to it, together with $H^1(X, \mathcal{O}_X(H))=0$, shows that $H^0(X, \omega_X)\neq 0$. Hence we obtatin $H^0(U, \omega_X)\neq 0$ since ω_X has no torsion. Now non-zero element φ in $H^0(U, \omega_X)$ defines an effective Weil divisor on X which does not meet H because $\omega_X\otimes\mathcal{O}_H=\mathcal{O}_H$. But H is very ample, which implies that φ nowhere vanishes on U. This proves that $\omega_{X|U}\cong\mathcal{O}_U$.

We have seen $H^1(X, \mathcal{O}_X)=0$, and hence X is not an abelian surface. Q.E.D.

THEOREM 3. *Let X be a normal projective surface with $\omega_X\cong\mathcal{O}_X$, and let $\pi: \tilde{X}\to X$ and q be as in § 2. Assume that there exists on X a very ample invertible sheaf $\mathcal{H}$ of* $\dim|\mathcal{H}|=g$. *Then we have*

$$0\leqq q\leqq g/2 \quad or \quad q=g\ .$$

Moreover, $q=g$ if and only if X is a cone over a canonical curve of genus g.

PROOF. We may assume that there is a singular point on X which is not a rational double point. The notations are the same as in Lemma 1. Take an element H in $|\mathcal{H}|$ sufficiently general so that H is a non-singular irreducible curve of genus g (cf. Theorem 2) not passing any singular point on X. Put $\tilde{H}=H_0=\pi^{-1}(H)$, $H_i=\mu_i\circ\cdots\circ\mu_1(\tilde{H})$ $(i=1, \cdots, n)$ and $\bar{H}=H_n$.

We show first that for all i, all exceptional curves of the first kind on X_i must meet H_i. Indeed, if this were not true for some i, also on $\tilde{X}$ there exists an exceptional curve of the first kind E such that $E\cap\tilde{H}=\varnothing$. Since $\tilde{H}$ meets all curves on $\tilde{X}$ except the components of the exceptional sets of π, E is contained in some exceptional set. But this is impossible because $\tilde{X}$ is the minimal resolution of X.

Now let us prove that $0\leqq q\leqq g/2$ or $q=g$; suppose to the contrary that $(g+1)/2\leqq q\leqq g-1$. Then we see first that $q\geqq 2$, and hence $\tilde{X}$ is a ruled surface over some non-singular curve C of genus q. Let $\mathcal{E}$, C_0 and e be as in § 2. Secondly, we deduce that $2g-2\leqq 2(2q-2)$ and that $\tilde{H}$ is not isomorphic to C. Therefore the induced map $\tilde{H}\to C$ is a surjective morphism of degree 2 between non-singular curves of genus g and q respectively.

We claim that $\bar{H}$ is non-singular. If not, we may assume that μ_n is the blow-up with center at a singular point of $\bar{H}$, say $\bar{x}$. Let $\bar{f}$ denote the fibre through $\bar{x}$ and $\bar{f}'$ the proper transformation of $\bar{f}$ by μ_n. Since $(\bar{f}, \bar{H})=2$, we have $\mathrm{mult}_{\bar{x}}(\bar{H})=2$, and hence $(\bar{f}', H_{n-1})=0$. This does not happen because $\bar{f}'$ is an exceptional curve of the first kind. Thus we conclude that $\bar{H}$ is non-singular and is isomorphic to $\tilde{H}$.

Hence, writing $\bar{H}\equiv 2C_0+bf$ for some integer b, f being a fibre, we obtain

$$g=p_a(\bar{H})=b+2q-e-1 .$$

Then by our hypothesis we have

$$\begin{aligned} b&=g+1-2q+e \\ &=2\left(\frac{g+1}{2}-q\right)+e\leqq e . \end{aligned}$$

Therefore, by $0\leqq(C_0, \bar{H})=-2e+b$, we get $e\leqq 0$. But by Lemma 3, we have that $e\geqq 2q-2\geqq 2$ when $q\geqq 2$, a contradiction. This proves the inequality of the theorem.

Assume that C is a canonical curve. Then a cone over C is a normal surface since C is projectively normal and its minimal resolution is a ruled surface over C. Hence "if" part of the second assertion of our theorem is proved.

Suppose that $q=g$. Then $\bar{X}$ is a ruled surface over a curve C of genus g. Again we use the notations $\mathscr{E}$, C_0 and e as in § 2. We note that $\bar{H}\to C$ is an isomorphism. Suppose that $\bar{X}\neq\tilde{X}$ and let $\bar{x}$ denote the center of μ_n. If $\bar{x}\in\bar{H}$, then the proper transformation of the fibre through $\bar{x}$ by μ_n is an exceptional curve of the first kind on X_{n-1} which does not meet H_{n-1}. If $\bar{x}\notin\bar{H}$, then $\mu_n^{-1}(\bar{x})$ has the same property. Anyway, we get a contradiction, thus we obtain that $\bar{X}=\tilde{X}$. Since $q\geqq 2$, we have $D_0=2C_0+\varpi^*(-\mathfrak{k}-\mathfrak{e})$ by Lemma 3, where $\mathfrak{k}$ is the canonical divisor of C and $\mathfrak{e}$ is a divisor on C such that

$$0\longrightarrow\mathcal{O}_C\longrightarrow\mathscr{E}\longrightarrow\mathcal{O}_C(\mathfrak{e})\longrightarrow 0$$

is exact. Since $\mathrm{supp}(D_0)\cap\mathrm{supp}(\tilde{H})=\emptyset$, we get $\mathfrak{e}=-\mathfrak{k}$ and $\mathscr{E}$ is decomposable, i.e.,

$$\tilde{X}=\boldsymbol{P}(\mathcal{O}_C\oplus\mathcal{O}_C(-\mathfrak{k})) , \qquad D_0=2C_0 .$$

It follows that X is a cone over a canonical curve of genus g. Q.E.D.

§ 4. Examples.

4.1. A cone X over a canonical curve of genus g is an example of a normal surface with $\omega_X\cong\mathcal{O}_X$ for any $g\geqq 3$. The vertex of the cone is the unique singular point and its geometric genus is equal to $g+1$ (cf. Theorems 1 and 3).

4.2. A normal quartic hypersurface X in $\boldsymbol{P}^3$ satisfies the condition of $\omega_X \cong \mathcal{O}_X$.

EXAMPLE 1. $X=\{X_0X_1^3+X_0X_2^3+X_3^4=0\}\subseteqq \boldsymbol{P}^3$ has only one singular point $(1:0:0:0)$ which is minimally elliptic (cf. Laufer [7]).

EXAMPLE 2. $X=\{X_0^2X_1^2+X_2^4+X_3^4=0\}\subseteqq \boldsymbol{P}^3$ has exactly two singular points; $(1:0:0:0)$ and $(0:1:0:0)$. Both of them are simple elliptic (cf. Saito [10]).

Also the normal complete intersection of a cubic and a quadric (resp. three quadrics) in $\boldsymbol{P}^4$ (resp. in $\boldsymbol{P}^5$) has trivial dualizing sheaf.

4.3. Finally, let us construct a normal projective surface X with $\omega_X \cong \mathcal{O}_X$ which is not a cone over a canonical curve and has a singular point with $p_g=q+1$ for each $q\geqq 1$.

Let C be a non-singular curve of genus q and let $\mathfrak{k}$ denote the canonical divisor of C. Take a divisor $\mathfrak{a}$ on C such that $a=\deg \mathfrak{a}\geqq 2q+1$. Set $\mathscr{E}=\mathcal{O}_C\oplus\mathcal{O}_C(-\mathfrak{k}-\mathfrak{a})$ and $\bar{X}=\boldsymbol{P}(\mathscr{E})$. Let $\varpi:\bar{X}\to C$ denote the canonical surjection and let C_0 be the unique element of $|\mathcal{O}_{\boldsymbol{P}(\mathscr{E})}(1)|$. Then we can find $C_1, C_2, C_3\in|C_0+\varpi^*(\mathfrak{k}+\mathfrak{a})|$ and $\mathfrak{a}'\in|\mathfrak{a}|$ such that $\operatorname{supp}(C_i)\cap\operatorname{supp}(C_0)=\varnothing$ $(i=1, 2, 3)$ and that $B=\operatorname{supp}(\varpi^*\mathfrak{a}')\cap\operatorname{supp}(C_1+C_2+C_3)$ consists of distinct $3a$ points. Let $\tilde{X}$ denote the surface obtained by blowing-up all points of B. For any divisor D on $\bar{X}$, we write $\tilde{D}$ the proper transformation of D on $\tilde{X}$. Then we see that $D_0=2\tilde{C}_0+\widetilde{\varpi^*\mathfrak{a}'}$ is the anti-canonical divisor of $\tilde{X}$ and that $\operatorname{supp}(D_0)\cap\operatorname{supp}(\tilde{C}_1+\tilde{C}_2+\tilde{C}_3)=\varnothing$. Moreover one can check that $2(\tilde{C}_1+\tilde{C}_2+\tilde{C}_3)-D_0$ is an ample divisor. Hence, for large N, $N(\tilde{C}_1+\tilde{C}_2+\tilde{C}_3)$ defines a contraction of $\operatorname{supp}(D_0)$ and the image X of the contraction satisfies the desired properties.

ADDED IN PROOF. We thank Professor Kimiko Watanabe for pointing out to us the following fact: There exist normal quartic surfaces with a singular point of geometric genus equal to two; an example of such surfaces is given by the surface defined by $(X_0X_1-X_2^2)^2+X_1^4+X_3^4=0$. This fact and examples in §4 show that all possible singular surfaces described in Theorems 1 and 3 really exist when $g=3$.

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