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On Projective Normality of Space Curves on a Non-Singular Cubic Surface in P^3

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Introduction

The purpose of this paper is to give a necessary and sufficient condition for space curves on a non-singular cubic surface in P^3 to be arithmetically Cohen-Macaulay. It is known that arithmetically Cohen-Macaulay curves form a smooth open subset in the Hilbert scheme $\mathbf{Hilb}_{P^3}^{p(z)}$ parametrizing curves in P^3 ([2], Théorèm 2). Also the dimension of the Hilbert scheme at a point corresponding to such a curve is calculated in [2], using the free resolution of the curve. There are essentially twelve types of arithmetically Cohen-Macaulay curves on a non-singular cubic surface in P^3. We shall prove this in §2 and §3. In §4, we shall determine free resolutions of these curves. We can determine the arithmetic genus and the degree of the curve by the free resolution.

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§1. Statement of the Result.

Let X be a non-singular cubic surface in the projective 3-space P^3 over an algebraically closed field of arbitrary characteristic. Then X is obtained from P^2 by blowing-up six points $P_1, \cdots, P_6$ which are not on a conic and no three of which are collinear. We denote by E_i the exceptional curve corresponding to P_i $(i=1, \cdots, 6)$, and L the total transform of a line in P^2. Then $\operatorname{Pic} X \cong Z^7$, with free basis $[L], [E_1], \cdots, [E_6]$ where $[L]$, $[E_i]$ are the linear equivalence class of L, E_i respectively, with intersection numbers

$$\begin{cases} E_i \cdot E_j = -\delta_{ij} & 1 \leqq i, j \leqq 6 \\ L^2 = 1 \\ L \cdot E_i = 0 & 1 \leqq i \leqq 6 . \end{cases}$$

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DEFINITION 1.1. Let D be a divisor on X, and $\mathscr{L}=\mathscr{O}_X(D)$ be the corresponding invertible sheaf. We say $\mathscr{L}$ (or D) is of type $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ if $D\sim aL-\sum_{i=1}^{6} b_iE_i$.

DEFINITION 1.2. A curve C in $\boldsymbol{P}^3$ is called arithmetically Cohen-Macaulay when its affine cone is Cohen-Macaulay. It is equivalent to that $H^1(\boldsymbol{P}^3, \mathscr{I}_C(n))$ vanishes for every $n\in\boldsymbol{Z}$.

REMARK 1.3. X is embedded in $\boldsymbol{P}^3$ by the divisor H of type (3, 1, 1, 1, 1, 1, 1), hence $\mathscr{L}(l)$ is of type $(a+3, b_1+l, \cdots, b_6+l)$ if $\mathscr{L}$ is of type $(a, b_1, \cdots, b_6)$.

Then our theorem is

THEOREM 1.4. *Let $\mathscr{L}$ be an invertible sheaf on X of type $(a, b_1, \cdots, b_6)$ with $b_1\geqq b_2\geqq\cdots\geqq b_6$. Then $H^1(X, \mathscr{L}^{-1}(l))=0$ for every $l\in\boldsymbol{Z}$ if and only if $(a, b_1, \cdots, b_6)$ is one of the followings;*

$A_1)$ $(3n, n, n, n, n, n, n)$ $n\in\boldsymbol{Z}$
$A_2)$ $(3n, n+1, n, n, n, n, n)$ $n\in\boldsymbol{Z}$
$A_3)$ $(3n, n, n, n, n, n, n-1)$ $n\in\boldsymbol{Z}$
$A_4)$ $(3n, n+1, n, n, n, n, n-1)$ $n\in\boldsymbol{Z}$
$B_1)$ $(3n+1, n, n, n, n, n, n)$ $n\in\boldsymbol{Z}$
$B_2)$ $(3n+1, n+1, n, n, n, n, n)$ $n\in\boldsymbol{Z}$
$B_3)$ $(3n+1, n+1, n+1, n, n, n, n)$ $n\in\boldsymbol{Z}$
$B_4)$ $(3n+1, n+1, n+1, n+1, n, n, n)$ $n\in\boldsymbol{Z}$
$C_1)$ $(3n+2, n+1, n+1, n+1, n+1, n+1, n+1)$ $n\in\boldsymbol{Z}$
$C_2)$ $(3n+2, n+1, n+1, n+1, n+1, n+1, n)$ $n\in\boldsymbol{Z}$
$C_3)$ $(3n+2, n+1, n+1, n+1, n+1, n, n)$ $n\in\boldsymbol{Z}$
$C_4)$ $(3n+2, n+1, n+1, n+1, n, n, n)$ $n\in\boldsymbol{Z}$.

COROLLARY 1.5. *Let C be a curve on X. Then C is arithmetically Cohen-Macaulay in $\boldsymbol{P}^3$ if and only if C is of type $A_1)$-$C_4)$ in Theorem 1.4.*

Furthermore, in each type $A_1)$-$C_4)$, there exists a non-singular curve if and only if

$$n\geqq\begin{cases}0 & (A_3), B_1), B_2), B_3), C_2), C_3), C_4))\\ 1 & (A_1), A_2), A_4), B_4), C_1))\ .\end{cases}$$

PROOF OF COROLLARY 1.5. Let $\mathscr{I}_C$ be the ideal sheaf of C in $\boldsymbol{P}^3$. Then, we have the following exact commutative diagram.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{O}_X(-C)\otimes\mathcal{O}_X(l) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{P^3}(-X)\otimes\mathcal{O}_{P^3}(l) & \longrightarrow & \mathcal{O}_{P^3}(l) & \longrightarrow & \mathcal{O}_X(l) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_l} & & \| & & \downarrow & & \\
C & \longrightarrow & \mathcal{J}_C(l) & \longrightarrow & \mathcal{O}_{P^3}(l) & \longrightarrow & \mathcal{O}_C(l) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{Coker}\,(\alpha_l) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

By the snake lemma, $\mathcal{O}_X(-C)\otimes\mathcal{O}_X(l)=\mathrm{Coker}\,(\alpha_l)$. Since $H^i(\boldsymbol{P}^3, \mathcal{O}_{P^3}(-X)\otimes\mathcal{O}_{P^3}(l))=0$ for every $l\in\boldsymbol{Z}$ and $i=1, 2$, $\cdot C$ is arithmetically Cohen-Macaulay if and only if $H^1(\boldsymbol{P}^3, \mathcal{J}_C(l))=0$ for every $l\in\boldsymbol{Z}$, and this is equivalent to saying that $H^1(\mathcal{O}_X(-C)\otimes\mathcal{O}_X(l))=0$ for every $l\in\boldsymbol{Z}$.

The last part of the corollary follows from the next lemma.

LEMMA 1.6 ([1], p. 407). *Let D be a divisor on X. Then the followings are equivalent.*

(i) *$|D|$ contains an irreducible curve,*

(ii) *$|D|$ contains a non-singular irreducible curve,*

(iii) *D is one of the followings,*

(a) *D is one of the 27 lines on X,*

(b) *$D\sim L-E_i$ for some i $(1\leqq i\leqq 6)$,*

(c) *$D\cdot L\geqq 0$ for every line L, and $D^2>0$.*

REMARK 1.7. If C is a non-singular curve, then C is arithmetically Cohen-Macaulay if and only if C is projectively normal.

§ 2. Proof of the "only if" part of the Theorem 1.4.

Let $\pi\colon X\to\boldsymbol{P}^2$ be the projection, and $\mathcal{L}$ be an invertible sheaf on X. From the spectral sequece

$$E_2^{pq}=H^p(\boldsymbol{P}^2, R^q\pi_*\mathcal{L}) \Longrightarrow H^{p+q}(X, \mathcal{L}),$$

we have the exact sequence

$$(2.1) \qquad 0 \longrightarrow H^1(P^2, \pi_*\mathscr{L}) \longrightarrow H^1(X, \mathscr{L}) \longrightarrow H^0(P^2, R^1\pi_*\mathscr{L}) \longrightarrow H^2(P^2, \pi_*\mathscr{L}) .$$

Lemma 2.2. *Let* $\mathscr{L}$ *be of type* $(a, b_1, \cdots, b_6)$. *Then for each* i, $(R^1\pi_*\mathscr{L})_{P_i}=0$ *if and only if* $b_i \geqq -1$.

Proof. By the formal function theorem, we have

$$(R^1\pi_*\mathscr{L}))\hat{}_{P_i} = \varprojlim_n H^1(X_n, \mathscr{L}_n)$$

where $X_n = X \times_{P^2} \operatorname{Spec}(\mathscr{O}_{P_i}/\mathscr{M}_{P_i}^n)$, and $\mathscr{L}_n$ is the restsriction of $\mathscr{L}$ to X_n. There is a natural exact sequence

$$0 \longrightarrow \mathscr{O}_X(-E_i)^n/\mathscr{O}_X(-E_i)^{n+1} \longrightarrow \mathscr{O}_{X_{n+1}} \longrightarrow \mathscr{O}_{X_n} \longrightarrow 0 .$$

Furthermore, $\mathscr{O}_X(-E_i)^n/\mathscr{O}_X(-E_i)^{n+1} = \mathscr{O}_{E_i}(n)$. Tensoring with $\mathscr{L}$ we have the exact sequence

$$0 \longrightarrow \mathscr{O}_{E_i}(n+b_i) \longrightarrow \mathscr{L}_{n+1} \longrightarrow \mathscr{L}_n \longrightarrow 0 .$$

From this follows the exact sequence

$$H^1(E_i, \mathscr{O}_{E_i}(n+b_i)) \longrightarrow H^1(X_{n+1}, \mathscr{L}_{n+1}) \longrightarrow H^1(X_n, \mathscr{L}_n) \longrightarrow 0 .$$

If $b_i \geqq -1$, we have $H^1(E_i, \mathscr{O}_{E_i}(n+b_i))=0$ since $E_i \cong P^1$. It follows that $H^1(X_{n+1}, \mathscr{L}_{n+1}) \cong (H^1(X_n, \mathscr{L}_n) \cong \cdots \cong H^1(X_1, \mathscr{L}_1) \cong H^1(E_i, \mathscr{O}_{E_i}(b_i))=0$, and that $(R^1\pi_*\mathscr{L})_{P_i} = (R^1\pi_*\mathscr{L})\hat{}_{P_i}=0$.

Conversely, if $(R^1\pi_*\mathscr{L})_{P_i}=0$, then $H^1(X_n, \mathscr{L}_n)=0$ for every n, since $\{H^1(X_n, \mathscr{L}_n)\}_n$ is a surjective system. In particular, we have $H^1(E_i, \mathscr{O}_{E_i}(b_i))=0$ and hence $b_i \geqq -1$.

Corollary 2.3. *The followings are equivalent.*

(i) $b_i \geqq -1$ *for every* i,

(ii) $R^1\pi_*\mathscr{L}=0$,

(iii) $H^0(P^2, R^1\pi_*\mathscr{L})=0$.

Lemma 2.4. *There is an exact sequence*

$$0 \longrightarrow \pi_*\mathscr{L} \longrightarrow \mathscr{O}_{P^2}(a) \longrightarrow \bigoplus_{i=1}^{6} \mathscr{O}_{b_iP_i} \longrightarrow 0 ,$$

where we put $\mathscr{O}_{b_iP_i} = \mathscr{O}_{P_i}/\mathscr{M}_{P_i}^{b_i}$.

Proof. Since the assertion is local on P^2 and π is an isomorphism outside $\{P_1, \cdots, P_6\}$, we have only to show that

$$0 \longrightarrow \pi_* \mathcal{O}_X(aL - b_i E_i) \longrightarrow \mathcal{O}_{P^2}(a) \longrightarrow \mathcal{O}_{b_i P_i} \longrightarrow 0$$

is exact for each i. When $b_i \geqq 0$, $\pi_* \mathcal{O}_X(aL - b_i E_i) = \mathcal{M}_{P_i}^{b_i} \cdot \mathcal{O}_{P^2}(a)$ and the assertion is obvious. If $b_i < 0$, $\mathcal{O}_{b_i P_i} = 0$ and we have only to show that $\pi_* \mathcal{O}_X(aL - b_i E_i) \cong \mathcal{O}_{P^2}(a)$.

There is an exact sequence

$$0 \longrightarrow \pi_* \mathcal{O}_X(aL - (b_i + 1)E_i) \longrightarrow \pi_* \mathcal{O}_X(aL - b_i E_i) \longrightarrow \pi_* \mathcal{O}_{E_i}(b_i) \ .$$

$\pi_* \mathcal{O}_{E_i}(b_i)$ is only supported at P_i and $(\pi_* \mathcal{O}_{E_i}(b_i))_{P_i} \cong H^0(P_i, \pi^* \mathcal{O}_{E_i}(b_i)) \cong H^0(E_i, \mathcal{O}_{E_i}(b_i)) \cong 0$ if $b_i < 0$. Hence we have $\pi_* \mathcal{O}_X(aL - b_i E_i) \cong \pi_* \mathcal{O}_X(aL - (b_i + 1)) \cong \cdots \cong \pi_* \mathcal{O}_X(aL) \cong \mathcal{O}_{P^2}(a)$.

COROLLARY 2.5. *If $b_i = -1$ or 0 for every i, then*

$$H^1(X, \mathcal{L}) = 0 \ .$$

REMARK 2.6. Let $\mathcal{L}$ be of type A_2) and $\mathcal{G}$ be of type A_3). By the Serre duality, $H^1(X, \mathcal{L}^{-1}(l))$ vanishes for every $l \in \boldsymbol{Z}$ if and only if $H^1(X, \mathcal{G}^{-1}(l))$ vanishes for every $l \in \boldsymbol{Z}$. Similarly $H^1(X, \mathcal{L}^{-1}(l))$ vanishes for every $l \in \boldsymbol{Z}$ if and only if $H^1(X, \mathcal{G}^{-1}(l))$ vanishes for every $l \in \boldsymbol{Z}$, when $\mathcal{L}$ is of type B_i) and $\mathcal{G}$ is of type C_i) $(1 \leqq i \leqq 4)$.

Now, we shall prove the "only if" part of the Theorem. By moving l appropriately, we may assume that $\mathcal{L}(l) = \mathcal{O}_X(D)$ is of type $(a, b_1, \cdots, b_6)$ with $0 \leqq a \leqq 2$. In this case we have $H^2(\boldsymbol{P}^2, \pi_* \mathcal{O}_X(-D)) = H^2(\boldsymbol{P}^2, \mathcal{O}_{P^2}(-a)) = 0$ by Lemma 2.4. Then by (2.1) we see that the followings are equivalent,

(i) $H^1(X, \mathcal{O}_X(-D)) = 0$,

(ii) $H^1(\boldsymbol{P}^2, \pi_* \mathcal{O}_X(-D)) = 0$ and $H^0(\boldsymbol{P}^2, R^1 \pi_* \mathcal{O}_X(-D)) = 0$,

(iii) $H^0(\boldsymbol{P}^2, \mathcal{O}_{P^2}(-a)) \to \bigoplus_{i=1}^6 \mathcal{O}_{P_i} / \mathcal{M}_{P_i}^{b_i} \to 0$ is exact and $-b_i \geqq -1$ for every i.

(See also Corollary 2.3 and Lemma 2.4.)

If $a = 0$, then we have $\sum \dim \mathcal{O}_{P_i} / \mathcal{M}_{P_i}^{-b_i} \leqq h^0(\boldsymbol{P}^2, \mathcal{O}_{P^2}) = 1$, and hence

$$-1 \leqq -b_1 \leqq \cdots \leqq -b_5 \leqq 0 \ , \qquad -b_6 \leqq -1 \ .$$

The type of $\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-1)$ is $(-3, -b_1 - 1, \cdots, -b_6 - 1)$ and $H^1(X, \mathcal{O}_X(-D) \otimes \mathcal{O}_X(-1)) = 0$. Combining (2.1) and Lemma 2.4, this gives

$$\begin{aligned} h^0(\boldsymbol{P}^2, R^1 \pi_* (\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-1))) &\leqq h^2(\boldsymbol{P}^2, \pi_* (\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-1))) \\ &= h^2(\boldsymbol{P}^2, \mathcal{O}_{P^2}(-3)) = 1 \ . \end{aligned}$$

By Lemma 2.2, we have

$$-b_2 - 1, \cdots, -b_6 - 1 \geqq -1 \ .$$

Hence we have

$$1\geqq b_1\geqq 0\ ,\qquad b_2=b_3=b_4=b_5=0\ ,\qquad 0\geqq b_6\geqq -1\ .$$

If $a=1$, then we have $\sum \dim \mathcal{O}_{P_i}/\mathcal{M}_{P_i}^{-b_i}\leqq h^0(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(-1))=0$, and hence

$$-1\leqq -b_1\leqq \cdots \leqq -b_6\leqq 0\ .$$

The type of $\mathcal{O}_X(-D)\otimes\mathcal{O}_X(-1)$ is $(-4, -b_1-1, \cdots, -b_6-1)$ and $H^1(X, \mathcal{O}_X(-D)\otimes\mathcal{O}_X(-1))=0$. As above

$$h^0(\boldsymbol{P}^2, R^1\pi_*(\mathcal{O}_X(-D)\otimes\mathcal{O}_X(-1)))\leqq 3\ ,$$

and hence we have

$$1\geqq b_1=b_2=b_3\geqq 0\ ,\qquad b_4=b_5=b_6=0\ .$$

The proof of the only if part of the Theorem is complete (see Remark 2.6).

§ 3. Proof of the "if" part of the Theorem 1.4.

By Remark 2.6 we may restrict ourselves to check the cases A_1), A_3), A_4) and B_1)-B_4). Furthermore by Remark 1.3, it is sufficient to show that $H^1(X, \mathcal{L}^{-1})$ vanishes for every $n\in \boldsymbol{Z}$, for the sheaf $\mathcal{L}$ of the above type.

LEMMA 3.1 ([1], V, 4.13). *Let $\mathcal{L}$ be an invertible sheaf of type $(a, b_1, \cdots, b_6)$ on X. Then the followings are equivalent.*

(i) *$\mathcal{L}$ is ample,*

(ii) *$\mathcal{L}$ is very ample,*

(iii) *$b_i>0$ for each i, $a>b_i+b_j$ for each i, j and $2a>\sum_{j\neq i} b_j$ for each i.*

Case A_1. Let H be a hyperplane section in general position. Then, H is arithmetically Cohen-Macaulay and hence $H^1(X, \mathcal{O}_X(-H)\otimes\mathcal{O}_X(n))=0$ for every $n\in \boldsymbol{Z}$, and $\mathcal{O}_X(H)\otimes\mathcal{O}_X(n)$ is of type $(3(n+1), n+1, \cdots, n+1)$.

Case A_3. By the Kodaira Vanishing Theorem and Lemma 3.1 we have only to check the case $\mathcal{L}=\mathcal{O}_X(E_1)$. There is a natural exact sequence

$$\begin{aligned}0\longrightarrow H^0(X, \mathcal{O}_X(-E_1))\longrightarrow H^0(X, \mathcal{O}_X)\longrightarrow H^0(E_1, \mathcal{O}_{E_1})\\ \longrightarrow H^1(X, \mathcal{O}_X(-E_1))\longrightarrow 0\ .\end{aligned}$$

Since $h^0(X, \mathcal{O}_X(-E_1))=0$ and $h^0(X, \mathcal{O}_X)=h^0(E_1, \mathcal{O}_{E_1})=1$, this sequence implies $H^1(X, \mathcal{O}_X(-E_1))=0$.

Case A_4. Let $D \sim 3nL-(n+1)-\sum_{i=2}^{5} nE_i-(n-1)E_6$. There are natural exact sequences

$$\begin{aligned}\cdots &\longrightarrow H^1(X, \mathcal{O}_X(-D-E_1)) \longrightarrow H^1(X, \mathcal{O}_X(-D))\\ &\longrightarrow H^1(E_1, \mathcal{O}_{E_1}(-D)) \longrightarrow \cdots\\ \cdots &\longrightarrow H^0(E_6, \mathcal{O}_{E_6}(-D+E_6)) \longrightarrow H^1(X, \mathcal{O}_X(-D))\\ &\longrightarrow H^1(X, \mathcal{O}_X(-D+E_6)) \longrightarrow \cdots .\end{aligned}$$

By the case A_2) and A_3), $H^1(X, \mathcal{O}_X(-D-E_1))=H^1(X, \mathcal{O}_X(-D+E_6))=0$. Furthermore,

$$\begin{aligned}&\mathcal{O}_{E_1}(-D)=\mathcal{O}_{E_1}(-n-1)\,, \quad \text{hence} \quad H^1(E_1, \mathcal{O}_{E_1}(-D))=0 \quad \text{if} \quad n\leqq 0\,,\\ &\mathcal{O}_{E_6}(-D+E_6))=\mathcal{O}_{E_6}(-n)\,, \quad \text{hence} \quad H^1(E_6, \mathcal{O}_{E_6}(-D+E_6))=0 \quad \text{if} \quad n>0\,.\end{aligned}$$

Hence we have $H^1(X, \mathcal{O}_X(-D))=0$ for every $n \in \boldsymbol{Z}$.

Case B_1. By the Kodaira Vanishing Theorem, Lemma 3.1 and Corollary 2.5, we have only to check the case $\mathcal{L}=\mathcal{O}_X(-H+L)$. There is a natural exact sequence

$$\begin{aligned}0 &\longrightarrow H^0(X, \mathcal{O}_X(H-L)) \longrightarrow H^0(X, \mathcal{O}_X(H)) \longrightarrow H^0(L, \mathcal{O}_L(H))\\ &\longrightarrow H^1(X, \mathcal{O}_X(H-L)) \longrightarrow 0\,.\end{aligned}$$

Clearly $h^0(L, \mathcal{O}_L(H))=h^0(L, \mathcal{O}_L(3))=4$. Furthermore, we have $h^0(X, \mathcal{O}_X(H-L))=0$ since $|H-L|=|2L-\sum_{i=1}^{6} E_i|=|2\tilde{L}-\sum_{i=1}^{6} P_i|=\phi$ by the choice of P_is, (where $\tilde{L}$ is a line in $\boldsymbol{P}^2$), and $h^0(X, \mathcal{O}_X(H))\geqq 4$ since $|H|=|3L-\sum_{i=1}^{6} E_i|=|3\tilde{L}-\sum_{i=1}^{6} P_i|$. It follows $H^1(X, \mathcal{O}_X(H-L))=0$.

Case B_2. By the Kodaira Vanishing Theorem, Lemma 3.1 and Corollary 2.5, we have only to check the case $\mathcal{L}=\mathcal{O}_X(-2L-E_1)$. There is a natural exact sequence

$$\cdots \longrightarrow H^1(X, \mathcal{O}_X(2L)) \longrightarrow H^1(X, \mathcal{L}^{-1}) \longrightarrow H^1(E_1, \mathcal{O}_{E_1}(2L+E_1)) \longrightarrow \cdots .$$

Since $-2L$ is of type B_1), $H^1(X, \mathcal{O}_X(2L))=0$. Clearly $h^1(E_1, \mathcal{O}_{E_1}(2L+E_1))=h^1(E_1, \mathcal{O}_{E_1}(-1))=0$, then we have $H^1(X, \mathcal{L}^{-1})=0$.

Case B_3 and B_4. Follow also from the Kodaira Vanishing Theorem, Lemma 3.1 and Corollary 2.5, except the case $n=1$.

In the case B_3, $\mathcal{L}=\mathcal{O}_X(4L-2E_1-2E_2-\sum_{i=3}^{6} E_i)$ for $n=1$. There is a natural exact sequence

$$\begin{aligned}0 &\longrightarrow H^0(X, \mathcal{O}_X(L-E_1-E_2)) \longrightarrow H^0(X, \mathcal{O}_X(L-E_1)) \longrightarrow H^0(E_2, \mathcal{O}_{E_2}(L-E_1))\\ &\longrightarrow H^1(X, \mathcal{O}_X(L-E_1-E_2)) \longrightarrow H^1(X, \mathcal{O}_X(L-E_1)) \longrightarrow \cdots .\end{aligned}$$

The result in the Case B_2 and Remark 2.6 imply $H^1(X, \mathcal{O}_X(L-E_1))=0$. Also we have $h^0(X, \mathcal{O}_X(L-E_1-E_2))=\dim|L-E_1-E_2|+1=\dim|L-P_1-P_2|+$

$1=1$, $h^0(X, \mathcal{O}_X(L-E_1))=\dim|L-E_1|+1=\dim|\tilde{L}-P_1|+1=2$, and $h^0(E_2, \mathcal{O}_{E_2}(L-E_1))=h^0(E_2, \mathcal{O}_{E_2})=1$. Then it follows $H^1(X, \mathcal{O}_X(L-E_1-E_2))=0$. Hence we have $H^1(X, \mathcal{L}^{-1})=H^1(X, \mathcal{O}_X(L-E_1-E_2))'=0$. (Where $'$ denotes a dual vector apace.)

In the Case B_4, $\mathcal{L}=\mathcal{O}_X(4L-\sum_{i=1}^{3} 2E_i-\sum_{j=4}^{6} E_j)$ for $n=1$. By the same methods as above, we have $H^1(X, \mathcal{L}^{-1})=H^1(X, \mathcal{O}_X(L-E_1-E_2-E_3))'=0$.

The proof of the Theorem is complete.

§4. Application.

THEOREM 4.1 ([2], Theorem 2). *Let* **Hilb**^{p} *be the Hilbert scheme parametrizing closed subschemes of codimension 2 in* P^e. *Then,*

(i) *Arithmetically Cohen-Macaulay closed subschemes form a smooth open subset in* **Hilb**^{p}.

(ii) *Let* $e\geqq 3$, *and* X *be a closed subscheme of codimension 2 in* P^e. *If* $\mathcal{O}_X$ *has a free homogeneous resolution of degree 0 of the form*

$$0 \longrightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_{P^e}(-n_i) \longrightarrow \bigoplus_{j=1}^{m} \mathcal{O}_{P^e}(-d_j) \longrightarrow \mathcal{O}_{P^e} \longrightarrow \mathcal{O}_X \longrightarrow 0\,,$$

then we have

$$\dim_{[X]} \mathbf{Hilb}^p=\sum_{n_i\geqq d_j}\binom{n_i-d_j+e}{e}+\sum_{d_j\geqq n_i}\binom{d_j-n_i+e}{e}-\sum_{n_i\geqq n_j}\binom{n_i-n_j+e}{e}$$
$$-\sum_{d_i\geqq d_j}\binom{d_i-d_j+e}{e}+1\,.$$

Where $[X]$ *denotes a point of* **Hilb**^{p} *corresponding to* X.

Now let C be an arithmetically Cohen-Macaulay curve in P^3. Then, as an $\mathcal{O}_{P^3}$-module, $\mathcal{O}_C$ has a free homogenous resolution of length 2,

$$0 \longrightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_{P^3}(-n_i) \xrightarrow{\varphi} \bigoplus_{j=1}^{m} \mathcal{O}_{P^3}(-d_j) \longrightarrow \mathcal{O}_{P^3} \longrightarrow \mathcal{O}_C \longrightarrow 0\,. \tag{4.2}$$

We say that the resolution (4.2) is minimal if each entry of the matrix φ has a positive degree.

PROPOSITION 4.3. *If* C *is on a non-singular cubic surface* X, *the minimal resolution of* $\mathcal{O}_C$ *is one of the followings.*

I) $0\to\mathcal{O}_{P^3}(-(r+s))\to\mathcal{O}_{P^3}(-r)\oplus\mathcal{O}_{P^3}(-s)\to\mathcal{O}_{P^3}\to\mathcal{O}_C\to 0$, *where* $r=3$, $r=s=2$, $r=s=1$, *or* $r=2$ $s=1$.

II) $0\to\mathcal{O}_{P^3}(-(r+2))^{\oplus 2}\to\mathcal{O}_{P^3}(-3)\oplus\mathcal{O}_{P^3}(-r)\oplus\mathcal{O}_{P^3}(-(r+3))\to\mathcal{O}_{P^3}\to\mathcal{O}_C\to$

0, *where* $r\geqq 2$.

III) $0\to\mathcal{O}_{P^3}(-(r+2))\oplus\mathcal{O}_{P^3}(-(r+3))\to\mathcal{O}_{P^3}(-3)\oplus\mathcal{O}_{P^3}(-(r+1))^{\oplus 2}\to\mathcal{O}_{P^3}\to\mathcal{O}_C\to 0$, *where* $r\geqq 2$.

IV) $0\to\mathcal{O}_{P^3}(-(r+2))^{\oplus 3}\to\mathcal{O}_{P^3}(-3)\oplus\mathcal{O}_{P^3}(-(r+1))^{\oplus 3}\to\mathcal{O}_{P^3}\to\mathcal{O}_C\to 0$, *where* $r\geqq 2$.

V) $0\to\mathcal{O}_{P^3}(-3)^{\oplus 2}\to\mathcal{O}_{P^3}(-2)^{\oplus 3}\to\mathcal{O}_{P^3}\to\mathcal{O}_C\to 0$.

Furthermore, the arithmetic genus and the degree of C is as follows.

	$p_a(C)$	*degree*
I)	$1+\frac{1}{2}rs(r+s-4)$	rs
II)	$\frac{1}{2}(r-1)(3r-2)$	$3r-1$
III)	$\frac{1}{2}r(3r-1)$	$3r+1$
IV)	$\frac{3}{2}r(r-1)$	$3r$
V)	0	3.

PROOF. Assume that the resolution (4.2) is minimal. If we denote by $I(C)$ the ideal of C, $I(C)$ is generated by $(m-1)$-minors of the matrix φ ([3], § 3). Since C is on X, $I(C)$ must contain an irreducible cubic form. Hence the minimality of (4.2) implies that the matrix $(\deg\varphi_{ij})_{1\leqq i\leqq m;1\leqq j\leqq m-1}$ is one of the followings,

I) $\begin{pmatrix} r \\ s \end{pmatrix}$ $r=3$, $r=s=2$, $r=s=1$, or $r=2$, $s=1$,

II) $\begin{pmatrix} r-1 & r-1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ $r\geqq 2$,

III) $\begin{pmatrix} r-1 & r \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$ $r\geqq 2$,

IV) $\begin{pmatrix} r-1 & r-1 & r-1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $r\geqq 2$,

V) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The first part of the proposition follows.

For the second part of the proposition, it is known that $2\deg C=\sum_{i=1}^{m-1} n_i^2-\sum_{j=1}^{m} d_j^2$ and $p_a(C)=1+(1/6)\sum_{i=1}^{m-1} n_i^3-\sum_{j=1}^{m} d_j^3-2\deg C$, if $\mathcal{O}_C$ has a resolution of the form (4.2) ([3], Proposition 3.1).

Let C be on a non-singular cubic surface X in $\boldsymbol{P}^3$, and for example of type A_4) in Theorem 1.4. Since $C\sim 3nL-(n+1)E_1-\sum_{i=2}^{5} E_i-(n-1)E_6$, we have $p_a(C)=(3/2)n(n-1)$ and $\deg C=3n$ ([1], V, 4.8). Comparing this with the results of Proposition 4.3, we see that the minimal resolution of $\mathcal{O}_C$ is the type IV) with $r=n$ for $n\geqq 2$, and the type V) for $n=1$. (If $n\leqq 0$, there are no effective members in this class.)

Furthermore, by Theorem 4.1 we have

$$\dim_{[C]}\mathbf{Hilb}^p_{P^3}=\begin{cases}\dfrac{3}{2}n^2+\dfrac{3}{2}n+18 & n\geqq 3\\ 24 & n=2\\ 12 & n=1\,.\end{cases}$$

By the similar methods, we have the following theorem.

THEOREM 4.4. *For each arithmetically Cohen-Macaulay curve on X, we have*

type	existence	minimal resolution	$\mu(C)$	$\dim_{[C]}\mathbf{Hilb}^p_{P^3}$	
A_1)	$n\geqq 1$	I) *with* $r=3$ $s=n$	2	$\frac{3}{2}n^2+\frac{3}{2}n+19$	$n\geqq 4$
				36	$n=3$
				24	$n=2$
				12	$n=1$
A_2)	$n\geqq 1$	II) *with* $r=n$ *for* $n\geqq 2$	3	$\frac{3}{2}n^2+\frac{1}{2}n+19$	$n\geqq 4$
				32	$n=3$
		I) *with* $r=2$ *for* $n=1$		20	$n=2$
		$s=1$	2	8	$n=1$
A_3)	$n\geqq 0$	III) *with* $r=n$ *for* $n\geqq 3$	3	$\frac{3}{2}n^2+\frac{5}{2}n+19$	$n\geqq 3$
B_3)				28	$n=2$
C_2)		I) *with* $r=s=2$ *for* $n=1$	2	16	$n=1$
		I) *with* $r=s=1$ *for* $n=0$	2	4	$n=0$
A_4)	$n\geqq 1$	IV) *with* $r=n$ *for* $n\geqq 2$	4	$\frac{3}{2}n^2+\frac{3}{2}n+18$	$n\geqq 3$
B_4)				24	$n=2$
C_1)		V) *for* $n=1$	3	12	$n=1$

type	existence	minimal resolution	$\mu(C)$	$\dim_{[C]}\mathbf{Hilb}^{p}_{\mathbf{P}^3}$	
$B_1)$	$n\geqq 0$	IV) *with* $r=n+1$ *for* $n\geqq 1$	4	$\frac{3}{2}n^2+\frac{9}{2}n+21$	$n\geqq 2$
$C_4)$				24	$n=1$
		V) *for* $n=0$	3	12	$n=0$
$B_2)$	$n\geqq 0$	II) *with* $r=n+1$ *for* $n\geqq 1$	3	$\frac{3}{2}n^2+\frac{7}{2}n+20$	$n\geqq 3$
$C_3)$				32	$n=2$
				20	$n=1$
		I) *with* $r=2$ *for* $n=0$ $s=1$	2	8	$n=0$

(*Where* $\mu(C)$ *denotes the number of minimal gінerators of* $I(C)$.)

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