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On Regular Fréchet-Lie Groups III

A Second Cohomology Class Related to the Lie Algebra of Pseudo-Differential Operators of Order One

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Introduction

Fourier integral operators have been defined by Hörmander [5], and developed extensively by himself and many other authors as a tool of studying fundamental solutions of Cauchy problems of pseudo-differential equations of hyperbolic type. However, if we deal with a Fourier integral operator F defined on a manifold, we see immediately that the expression of F contains usually a huge ambiguity. Phase functions and amplitude functions do not have invariant meanings under the change of local coordinate systems, and the rule of coordinate transformations is usually a very complicated one. Therefore, there arise several difficulties to define a topology, for instance, on the space $\mathscr{F}^0$ of all Fourier integral operators of order 0.

In [11], we gave a sort of global expression of Fourier integral operators and in [12] we defined a "vicinity" $\mathfrak{N}$ of the identity operator in the space $\mathscr{F}^0$ such that $\mathfrak{N}$ satisfies the properties of a topological local group. Moreover we have shown in [11] that $F\in\mathfrak{N}$ can be expressed in an "almost" unique fashion, if we fix a C^∞ riemannian metric on N.

Let us explain this situation at first. Let $\mathscr{D}_\Omega^{(1)}$ be the group of all symplectic transformations of order one on $T^*N-\{0\}$, where T^*N is the cotangent bundle a closed C^∞ riemannian manifold N. It is known that $\mathscr{D}_\Omega^{(1)}$ is isomorphic to the group $\mathscr{D}_\omega(S^*N)$ of all contact transformations on the unit cosphere bundle S^*N. Since $\mathscr{D}_\omega(S^*N)$ is a topological group under the C^∞ topology, we give the same topology on $\mathscr{D}_\Omega^{(1)}$ through

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the above isomorphism. Let $\mathfrak{U}$ be a neighborhood of the identity in $\mathscr{D}_{\Omega}^{(1)}$.

Let $C^\infty(\bar{D}^*N)$ be the Fréchet space of all C-valued C^∞ functions on the closed unit disk bundle $\bar{D}^*N$ in T^*N. Define a diffeomorphism $\tau: D^*N \to T^*N$ by $\tau(x; \theta) = (x; (\tan(\pi/2)|\theta|)(\theta/|\theta|))$, where $(x; \theta)$ indicates the point in T^*N such that the base point is x and $\theta \in T_x^*$ (the fibre of T^*N at x), and D^*N is the open disk bundle in T^*N. We set $\sum_C^0 = \tau^{-1*}C^\infty(\bar{D}^*N)$. $\sum_C^0$ is a Fréchet space through the identification τ.

Let $C^\infty(N\times N)$ be the Fréchet space of all C-valued C^∞ functions on $N\times N$. For each $K \in C^\infty(N\times N)$, we define usually a smoothing operator $K\circ$ with kernel K. A function $\nu(x, y) \in C^\infty(N\times N)$ will be called a *cut off function of breadth* ε, if

(i) $\nu(x, y) \geqq 0$, $\nu(x, y) = \nu(y, x)$,

(ii) $\nu(x, y) \equiv 0$ if $\rho(x, y) \geqq (2/3)\varepsilon$, where ρ is the distance function.

(iii) $\nu(x, y) \equiv 1$ if $\rho(x, y) \leqq (1/3)\varepsilon$.

Usually, we fix a cut off function ν with sufficiently small breadth ε, say $\varepsilon < r_0/50$ where r_0 is the injectivity radius of N.

Under these notations, one can define precisely a "vicinity" $\mathfrak{N}$. Let $\mathfrak{U}$ be a sufficiently small neighborhoods of the identity in $\mathscr{D}_{\Omega}^{(1)}$ and let U_1, V_0 be sufficiently small neighborhoods of 1 in $\sum_C^0$ and of 0 in $C^\infty(N\times N)$ respectively. A Fourier integral operator F is said to be contained in $\mathfrak{N} = \mathfrak{N}(\mathfrak{U}, U_1, V_0)$ if and only if there are $\varphi \in \mathfrak{U}$, $a \in U_1$ and $K \in V_0$ such that F can be written in the form

$$(1) \qquad (Fu)(x) = \int_{T_x^*} a(x; \xi)\widetilde{\nu u}(\varphi(x; \xi))d\xi + (K\circ u)(x),$$

where $d\xi = (1/\sqrt{2\pi})^n d\xi_1 \wedge \cdots \wedge d\xi_n$ using an orthonormal coordinate system $(\xi_1, \cdots, \xi_n)$ on T_x^* and ν is a cut off function mentioned above. $\widetilde{\nu u}$ is defined as follows (cf. [11]):

$$(2) \qquad \widetilde{\nu u}(y; \eta) = \int_N \nu(y, z) e^{-i\langle \eta | Y\rangle} u(z) dz, \quad \cdot_y Y = z \ (\text{i.e., } \mathrm{Exp}_y Y = z),$$

where $dz = (1/\sqrt{2\pi})^n \times$(volume element on N). $\widetilde{\nu u}$ can be regarded as a sort of Fourier transform of u.

The above expression (1) contains almost no ambiguity, for φ and the asymptotic expansion of a are uniquely determined by F. Thus the ambiguity in the expression is contained only in the term of smoothing operators. If $\mathfrak{U}$, U_1, V_0 are chosen to be sufficiently small, then $\mathfrak{N}$ turns out to be a topological local group (cf. [12]).

Though a smoothness structure on $\mathfrak{N}$ will be defined in a forthcoming paper, we call a one parameter family F_t of operators in $\mathfrak{N}$ a *smooth curve*, if there are $\varphi_t \in \mathfrak{U}$, $a_t \in U_1$, $K_t \in V_0$ which are smooth in the variable t such that F_t can be written in the form

$$(3) \qquad (F_t u)(x) = \int_{T_x^*} a_t(x;\xi) \widetilde{\nu u}(\varphi_t(x;\xi))d\xi + (K_t \circ u)(x) .$$

Let $G\mathscr{F}_0{}^0$ be the group generated by a vicinity $\mathfrak{N}$ given by sufficiently small $\mathfrak{U}$, U_1, V_0. Although a manifold-structure on $G\mathscr{F}_0{}^0$ has not yet been defined, we shall define here the tangent space $T_e G\mathscr{F}_0{}^0$ of $G\mathscr{F}_0{}^0$ at the identity as the space of all initial derivatives of smooth curves in $\mathfrak{N}$ starting at the identity. Namely, $P \in T_e G\mathscr{F}_0{}^0$ if and only if there is a smooth curve F_t in $\mathfrak{N}$ such that $F_0 = I$ and $Pu = (d/dt)|_{t=0} F_t u$ for every $u \in C^\infty(N)$. In this paper, we shall prove at first the following:

Proposition A. $T_e G\mathscr{F}_0{}^0 = \sqrt{-1}\mathscr{P}^1$, where $\mathscr{P}^1$ is the space of all pseudo-differential operators of order one with real principal symbols.

For every $P \in \mathscr{P}^1$, there exists, therefore, a smooth curve F_t in $\mathfrak{N}$ such that $F_0 = I$, $(d/dt)|_{t=0} F_t = \sqrt{-1}P$. For every $G \in G\mathscr{F}_0{}^0$, it is easy to see that $t \mapsto GF_tG^{-1}u$ is a C^∞ mapping of $\boldsymbol{R}$ into $C^\infty(N)$ for every $u \in C^\infty(N)$. (In fact, GF_tG^{-1} is again a smooth curve in $\mathfrak{N}$, but this fact will be shown in a forthcoming paper.) We define $\mathrm{Ad}(G)P$ by

$$(4) \qquad \mathrm{Ad}(G)Pu = \frac{1}{\sqrt{-1}} \frac{d}{dt}\Big|_{t=0} GF_tG^{-1}u = GPG^{-1}u .$$

Suppose G_t is another smooth curve in $\mathfrak{N}$ such that $G_0 = I$ and $(d/dt)|_{t=0} G_t = \sqrt{-1}Q$. Then, we see easily that

$$(5) \qquad [\sqrt{-1}Q, \sqrt{-1}P]u = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(G_t)\sqrt{-1}Pu$$

for every $u \in C^\infty(N)$, where $[A, B] = AB - BA$. Note that it is well-known that $[\sqrt{-1}\mathscr{P}^1, \sqrt{-1}\mathscr{P}^1] \subset \sqrt{-1}\mathscr{P}^1$. Hence, we see that the tangent space $T_e G\mathscr{F}_0{}^0$ has a structure of a Lie algebra, which is closely related to the group operations in $G\mathscr{F}_0{}^0$ through (4) and (5). In this sense, we call $\sqrt{-1}\mathscr{P}^1$ the Lie algebra of $G\mathscr{F}_0{}^0$.

For simplicity, we define a bracket product $[\![\ ,\ ]\!]$ on $\mathscr{P}^1$ as follows:

$$(6) \qquad [\![P, Q]\!] = \frac{1}{\sqrt{-1}}[\sqrt{-1}P, \sqrt{-1}Q] .$$

Let $\mathscr{P}^{-m}$, $m\geqq 0$, be the space of all pseudo-differential operators of order $-m$, and let $\mathscr{P}^{-\infty}=\bigcap\mathscr{P}^{-m}$ be the space of all smoothing operators. $\mathscr{P}^{-m}$, $0\leqq m\leqq\infty$, are Lie ideals of $\mathscr{P}^{1}$, and the factor space $\mathscr{P}^{-l}/\mathscr{P}^{-l-1}(l\geqq 0)$ is naturally isomorphic to the abelian Lie algebra $C^{\infty}(S^{*}N)r^{-l}$ of all C-valued C^{∞} functions on $T^{*}N-\{0\}$ of positively homogeneous of degree $-l$. Moreover, $\mathscr{P}^{1}/\mathscr{P}^{0}$ is naturally isomorphic to the Lie algebra $C^{\infty}_{R}(S^{*}N)r$ of all real valued C^{∞} functions on $T^{*}N-\{0\}$ of positively homogeneous of degree 1, where the Lie algebra structure is given by the Poisson bracket $\{\ ,\ \}$. It is well-known that $C^{\infty}_{R}(S^{*}N)r$ is isomorphic to the Lie algebra of all contact vector fields on the unit cosphere bundle $S^{*}N$. Therefore, $\mathscr{P}^{1}/\mathscr{P}^{-1}$ can be regarded as an extension of $\Gamma_{\omega}(TS^{*}N)$ (the Lie algebra of all contact vector fields on $S^{*}N$) or $\mathscr{P}^{1}/\mathscr{P}^{0}$ by the abelian kernel $C^{\infty}(S^{*}N)$ or $\mathscr{P}^{0}/\mathscr{P}^{-1}$, i.e., we have the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{P}^{0}/\mathscr{P}^{-1} & \longrightarrow & \mathscr{P}^{1}/\mathscr{P}^{-1} & \longrightarrow & \mathscr{P}^{1}/\mathscr{P}^{0} & \longrightarrow & 0 \\ & & \wr\| & & & & \|\wr & & \\ & & C^{\infty}(S^{*}N) & & & & C^{\infty}_{R}(S^{*}N)r\cong\Gamma_{\omega}(TS^{*}N)\,. & & \end{array} \tag{7}$$

Hence, the above exact sequence defines a second cohomology class $h\in H^{2}(\Gamma_{\omega}(TS^{*}N);\ C^{\infty}(S^{*}N))$ (cf. [7]). The sequence (7) splits as Lie algebras if and only if $h=0$. For simplicity we use the notation $\cdot_{x}(X,\xi)$ instead of $(\mathrm{Exp}_{x}X;\ (d\,\mathrm{Exp}_{x}^{-1})^{*}_{X}\xi)$. (X,ξ) is regarded as a normal coordinate system of $T^{*}N$ around x. Using this notation, we shall prove the following:

Theorem A. A representative 2-cocycle of h is given by ω_{0} such that for $f, g\in C^{\infty}_{R}(S^{*}N)r$

$$\begin{aligned} -2\omega_{0}(f,g)(x;\xi) &= \frac{\partial^{2}f}{\partial\xi_{i}\partial\xi_{j}}(x;\xi)\frac{\partial^{2}}{\partial X^{i}\partial X^{j}}\Big|_{X=0} g(\cdot_{x}(X,\xi)) \\ &\quad -\frac{\partial^{2}g}{\partial\xi_{i}\partial\xi_{j}}(x;\xi)\frac{\partial^{2}}{\partial X^{i}\partial X^{j}}\Big|_{X=0} f(\cdot_{x}(X,\xi)) \\ &\quad -\frac{1}{3}\xi_{l}(R^{l}_{ijk}+R^{l}_{jik})(x)\Big(\frac{\partial^{2}f}{\partial\xi_{i}\partial\xi_{j}}\frac{\partial g}{\partial\xi_{k}}-\frac{\partial^{2}g}{\partial\xi_{i}\partial\xi_{j}}\frac{\partial f}{\partial\xi_{k}}\Big)(x;\xi)\,, \end{aligned}$$

where R^{l}_{ijk} is the curvature tensor on N.

Moreover, ω_{0} is a coboundary of β which is given by

$$2\beta(f)(\cdot_{x}(X,\xi))=\frac{\partial^{2}f}{\partial\xi_{i}\partial x^{i}}+\begin{Bmatrix} i \\ ik \end{Bmatrix}\frac{\partial f}{\partial\xi_{k}}+\begin{Bmatrix} i \\ jk \end{Bmatrix}\xi_{i}\frac{\partial^{2}f}{\partial\xi_{j}\partial\xi_{k}}\,.$$

Remark. The formula (11.39) written in [10] was false.

The right hand side of the above defining equality of 2β can be understood as $(\partial/\partial\xi_i)(\nabla f/\partial X^i)$ in the following sense: Let $c(t)$ be a smooth curve such that $(d/dt)|_{t=0}c(t)=\partial/\partial X^i$, and let $\xi(t)$ be the parallel displacement of ξ along $c(t)$. Define $\nabla f/\partial X^i$ by $(d/dt)|_{t=0}f(c(t);\xi(t))$. Then, we have easily

$$\frac{\nabla f}{\partial X^i}=\frac{\partial f}{\partial X^i}+\frac{\partial f}{\partial \xi_j}\begin{Bmatrix} k \\ ji \end{Bmatrix}\xi_k \, . \tag{8}$$

Thus, taking the fiber derivative $\partial/\partial\xi_i$ we get the right hand side. Therefore $2\beta(f)$ is well-defined as a global function on $T^*N-\{0\}$.

By the above result, the exact sequence (7) splits as Lie algebras, and hence there is a subalgebra $\mathfrak{G}$ of $\mathscr{P}^1$ such that $\mathfrak{G}\supset\mathscr{P}^{-1}$ and $\mathfrak{G}/\mathscr{P}^{-1}\cong\mathscr{P}^1/\mathscr{P}^0$. Hence, we have an exact sequence

$$\begin{array}{c} 0\longrightarrow\mathscr{P}^{-1}/\mathscr{P}^{-2}\longrightarrow\mathfrak{G}/\mathscr{P}^{-2}\longrightarrow\mathfrak{G}/\mathscr{P}^{-1}\longrightarrow 0 \\ \wr\| \\ \mathscr{P}^1/\mathscr{P}^0\cong\Gamma_\omega(TS^*N)\, . \end{array} \tag{9}$$

Since $\mathscr{P}^{-1}/\mathscr{P}^{-2}$ is abelian, the above sequence defines also a second cohomology class $\tilde{h}\in H^2(\Gamma_\omega(TS^*N);\, C^\infty(S^*N)r^{-1})$ (cf. Losik [7]). However, we have the following:

Theorem B. The above cohomology class $\tilde{h}$ never vanishes on any manifold N such that $\dim N\geqq 2$.

Now, we denote by P_a the pseudo-differential operator with symbol a, that is,

$$(P_a u)(x)=\int_{T_x^*}a(x;\xi)\widetilde{\nu u}(x;\xi)d\xi \, . \tag{10}$$

Define a bilinear mapping $\nabla: C_R^\infty(S^*N)r\times C^\infty(N)\to C^\infty(N)$ by

$$\nabla_f u=\sqrt{-1}(P_f u-\sqrt{-1}P_{\beta(f)}u)\, . \tag{11}$$

Since $C_R^\infty(S^*N)r$ is isomorphic to the Lie algebra of $\mathscr{D}_\omega(S^*N)$, ∇ may be understood as an *invariant connection* (cf. [9] p. 18 or [10] p. 117) on a vector bundle $\mathscr{E}$ over $\mathscr{D}_\omega(S^*N)$ with the fibre $C^\infty(N)$, though we do not give the precise construction of $\mathscr{E}$. Set

$$\sqrt{-1}\mathscr{R}(f,g)=(\nabla_f\nabla_g-\nabla_g\nabla_f-\nabla_{\{f,g\}})u\, , \tag{12}$$

and call it the *curvature* of ∇. By virtue on Jarcobi's identity on $\sqrt{-1}\mathscr{P}^1$,

$\mathscr{R}$ satisfies the following Bianchi's identity:

$$\mathfrak{S}\{[\nabla_f, \mathscr{R}(g, h)] - \mathscr{R}(\{f, g\}, h)\} = 0 , \tag{13}$$

where $\mathfrak{S}$ means the cyclic summation with respect to f, g, h. Moreover, Theorem A shows that $\mathscr{R}(f, g) \in \mathscr{P}^{-1}$, but Theorem B shows that the curvature can never vanish.

§ 1. Lie algebra of $G\mathscr{F}_0^{\ 0}$.

In this section, we shall prove Proposition A. Let F_t be a smooth curve in $\mathfrak{N}$ written in the form (3) such that $\varphi_0 = \mathrm{id}$, $a_0 = 1$ and $K_0 = 0$. We set at first

$$\frac{d}{dt}\bigg|_{t=0} \varphi_t(x; \xi) = \mathfrak{X}(x; \xi) = \mathfrak{X}^i(x; \xi) \frac{\partial}{\partial X^i}\bigg|_{X=0} + \Xi_j(x; \xi) \frac{\partial}{\partial \xi_j} , \tag{14}$$

where $(X^1, \cdots, X^n)$ is a normal chart around x, and $(\xi_1, \cdots, \xi_n)$ is its dual chart on T_x^*. Since φ_t is assumed to be positively homogeneous of order 1, we see that

$$\begin{cases} \mathfrak{X}^i(x; r\xi) = \mathfrak{X}^i(x; \xi) \\ \Xi_j(x; r\xi) = r\Xi_j(x; \xi) \end{cases}, \quad r > 0 . \tag{15}$$

Remark that

$$\frac{d}{dt}\bigg|_{t=0} \widetilde{\nu u}(\varphi_t(x; \xi)) = \mathfrak{X}^i(x; \xi) \frac{\partial}{\partial X^i}\bigg|_{X=0} \widetilde{\nu u}(\cdot_x(X, \xi)) + \Xi_j(x; \xi) \frac{\partial \widetilde{\nu u}}{\partial \xi_j}(x; \xi) , \tag{16}$$

where $\cdot_x(X, \xi) = (\mathrm{Exp}_x X; (d\,\mathrm{Exp}_x^{-1})_X^* \xi)$.

For more precise computations of the right hand members of (16), we need several notations as follows: We have used a brief notation $\cdot_x X$ instead of $\mathrm{Exp}_x X$. If $Y \in T_{\cdot_x X}$ and $\cdot_{\cdot_x X} Y = \cdot_x Z$, then Y can be written by using X and Z, which we shall denote by

$$Y = S(x; Z, X) \quad (\text{cf. } §1. [11]) . \tag{17}$$

We shall use also the following normal coordinate expressions around x:

$$(\cdot_x X; Y) = \cdot_x(X, \tilde{Y}) , \quad (\cdot_x X; \eta) = \cdot_x(X, \xi) ,$$

where $\cdot_x(X, \tilde{Y})$ means $(\mathrm{Exp}_x X; (d\,\mathrm{Exp}_x)_X \tilde{Y})$, and $\cdot_x(X, \xi) = (\mathrm{Exp}_x X; (d\,\mathrm{Exp}_x^{-1})_X^* \xi)$ as was already mentioned. If Y is given by (17), then the normal coordinate expression of S will be denoted by $\tilde{S}(x; Z, X)$.

Using these notations, we see

$$(18)\qquad \begin{cases} \widetilde{\nu u}(x; \xi)=\int_N \nu(x, z)u(z)e^{-i\langle\xi|Z\rangle}dz,\ z={\cdot}_x Z\ . \\ \widetilde{\nu u}({\cdot}_x(X, \xi))=\int_N \nu({\cdot}_x X, z)u(z)e^{-i\langle\xi|\tilde{S}(x;Z,X)\rangle}dz\ . \end{cases}$$

Therefore, we have

$$(19)\qquad \begin{cases} \Xi_j\dfrac{\partial\widetilde{\nu u}}{\partial\xi_j}(x; \xi)=\Xi_j(x; \xi)\int_N \nu(x, z)u(z)\dfrac{\partial}{\partial\xi_j}e^{-\sqrt{-1}\langle\xi|Z\rangle}dz\ , \\ \mathfrak{X}^i\dfrac{\partial}{\partial X^i}\Big|_{X=0}\widetilde{\nu u}({\cdot}_x(X, \xi))=\int_N \mathfrak{X}^i\dfrac{\partial\nu}{\partial X^i}\Big|_{X=0}u(z)e^{-\sqrt{-1}\langle\xi|Z\rangle}dz \\ \qquad -\sqrt{-1}\int_N \mathfrak{X}^l\Big\langle\xi\Big|\dfrac{\partial\tilde{S}}{\partial X^l}(x; Z, 0)\Big\rangle e^{-\sqrt{-1}\langle\xi|Z\rangle}\nu(x, z)u(z)dz\ . \end{cases}$$

Remark that $\partial\tilde{S}^j/\partial X^l(x; 0, 0)=-\delta^j_l$. So, we set

$$(20)\qquad \frac{\partial\tilde{S}^j}{\partial X^l}(x; Z, 0)+\delta^j_l=T^j_{lk}(x; Z)Z^k\ ,$$

and we obtain the following:

$$(21)\qquad \int_{T^*_x}\frac{d}{dt}\Big|_{t=0}\widetilde{\nu u}(\varphi_t(x; \xi))d\xi$$
$$=\int_{T^*_x}\int_N \mathfrak{X}^i(x; \xi)\frac{\partial\nu}{\partial X^i}\Big|_{X=0}u(z)e^{-\sqrt{-1}\langle\xi|Z\rangle}dz+\sqrt{-1}\int_{T^*_x}\mathfrak{X}^l(x; \xi)\xi_l\widetilde{\nu u}(x; \xi)d\xi$$
$$-\iint\Big[\frac{\partial\mathfrak{X}^l}{\partial\xi_k}\xi_jT^j_{lk}+\mathfrak{X}^lT^k_{lk}\Big]e^{-\sqrt{-1}\langle\xi|Z\rangle}\nu(x, z)u(z)dzd\xi$$
$$-\int_{T^*_x}\frac{\partial\Xi_j}{\partial\xi_j}(x; \xi)\widetilde{\nu u}(x; \xi)d\xi\ .$$

Since $\partial\nu/\partial X^i\equiv 0$ for sufficiently small X, we see that the first term of the right hand side is a smoothing operator. We denote this operator by $K_x\circ u$. Set

$$(22)\qquad b(x; \xi)=-\iint\Big[\frac{\partial\mathfrak{X}^l}{\partial\xi_k}(x; \xi+\eta)(\xi_j+\eta_j)T^j_{lk}(x; Z)$$
$$+\mathfrak{X}^l(x; \xi+\eta)T^k_{lk}(x; Z)\Big]e^{-i\langle\eta|Z\rangle}dZd\eta\ ,$$

and we get

$$\int_{T_x^*}\frac{d}{dt}\Big|_{t=0}\widetilde{\nu u}(\varphi_t(x;\xi))d\xi \tag{23}$$
$$=\sqrt{-1}\int_{T_x^*}\mathfrak{X}^l(x;\xi)\xi_l\widetilde{\nu u}(x;\xi)d\xi+\int_{T_x^*}b(x;\xi)\widetilde{\nu u}(x;\xi)d\xi$$
$$-\int_{T_x^*}\frac{\partial \Xi_j}{\partial\xi_j}(x;\xi)\widetilde{\nu u}(x;\xi)d\xi+(K_{\mathfrak{X}}\circ u)(x)\,,$$

and hence

$$\frac{d}{dt}\Big|_{t=0}(F_t u)(x) \tag{24}$$
$$=\sqrt{-1}\int_{T_x^*}\mathfrak{X}^l(x;\xi)\xi_l\widetilde{\nu u}(x;\xi)d\xi$$
$$+\int_{T_x^*}\Big[\frac{d}{dt}\Big|_{t=0}a_t(x;\xi)+b(x;\xi)-\frac{\partial \Xi_j}{\partial\xi_j}(x;\xi)\Big]\widetilde{\nu u}(x;\xi)d\xi$$
$$+\Big(\Big(\frac{d}{dt}\Big|_{t=0}K_t+K_{\mathfrak{X}}\Big)\circ u\Big)(x)\,.$$

The first term (resp. second term) is a pseudo-differential operator of order 1 (resp. 0), and the last term is a smoothing operator. Hence, we get $(d/dt)|_{t=0}F_t\in\sqrt{-1}\mathscr{P}^1$.

Conversely, let $P\in\sqrt{-1}\mathscr{P}^1$, and let $\sqrt{-1}a_1$ be the principal symbol of P. $a_1(x;\xi)$ is real valued and positively homogeneous of degree 1. By the definition of pseudo-differential operators, there are $\tilde{a}(x;\xi)\in\sum_c^0$ and $K\in C^\infty(N\times N)$ such that

$$(Pu)(x)=\int_{T_x^*}(\sqrt{-1}a_1+\tilde{a})\widetilde{\nu u}(x;\xi)d\xi+(K\circ u)(x)\,.$$

(Warning: $\tilde{a}$ and K are not necessarily unique, but the asymptotic expansion of $\tilde{a}$ is uniquely determined by P.) Now, set

$$\mathfrak{X}(x;\xi)=\frac{\partial a_1}{\partial\xi_j}(x;\xi)\frac{\partial}{\partial X^j}\Big|_{X=0}-\frac{\partial}{\partial X^i}\Big|_{X=0}a_1(\cdot_x(X,\xi))\frac{\partial}{\partial\xi_i}\,.$$

Then, $\mathfrak{X}$ is a Hamiltonian vector field on $T^*N-\{0\}$ satisfying (15). Moreover, $\mathfrak{X}^j\xi_j=(\partial a_1/\partial\xi_j)\xi_j=a_1$. $\mathfrak{X}$ generates a one parameter symplectic transformation group $\varphi_t\in\mathscr{D}_\Omega^{(1)}$. Also using $\mathfrak{X}$, we define $b(x;\xi)$ by (22), and $K_{\mathfrak{X}}$ by the first term of (21). Now, set

$$(F_t u)(x)=\int_{T_x^*}\Big(1+t\tilde{a}-tb+t\frac{\partial \Xi_j}{\partial\xi_j}\Big)\widetilde{\nu u}(\varphi_t(x;\xi))d\xi+((tK-tK_{\mathfrak{X}})u)(x)\,.$$

Then the same computation as in (21)~(24) leads us to the conclusion $(d/dt)|_{t=0}F_t=P$. This completes the proof of Proposition A.

It is well-known that $\sqrt{-1}\mathscr{P}^1$ is a Lie algebra under the usual commutator bracket.

REMARK. According to the statement of Theorem A, the exact sequence (7) splits as Lie algebras. However this splitting does not necessarily imply the existence of a splitting of the following exact sequence:

$$(25) \qquad 1 \longrightarrow G\mathscr{P}^0/G\mathscr{P}^{-1} \longrightarrow G\mathscr{F}_0{}^0/G\mathscr{P}^{-1} \longrightarrow \mathscr{D}_\omega(S^*N) \longrightarrow 1\,,$$

where $G\mathscr{P}^{-l}(l \geqq 0)$ be the group of all invertible operators written in the form $I+P$, $P\in\mathscr{P}^{-l}$. If an infinite dimensional analogue of the results of [4] or [13] would hold in this case, then we should see that (25) splits as groups or that $\mathscr{D}_\omega(S^*N)$ is not simply connected.

§ 2. Local cohomology group of the Lie algebra of contact vector fields.

It is easy to see that the cohomology groups related to the exact sequence (7) or (9) can be defined also locally or formally at an arbitrarily fixed point x in N. So, at first, in this section we shall deal with the cohomology group of the Lie algebra of formal symplectic vector fields.

Let $\Phi=C[\![x^1, \cdots, x^n, \xi_1, \cdots, \xi_n]\!]$ be the ring of all formal power series of complex coefficients with variables $x^1, \cdots, x^n, \xi_1, \cdots, \xi_n$, and let Φ' be the subalgebra of Φ consisting of all polynomials of $x^1, \cdots, x^n, \xi_1, \cdots, \xi_n$. Φ and Φ' are Lie algebras under the Poisson bracket $\{\ ,\ \}$ defined as follows:

$$\{f, g\}=\partial^i f\partial_i g-\partial^i g\partial_i f\,,$$

where $\partial^i=\partial/\partial\xi_i$, $\partial_i=\partial/\partial x^i$. We denote by $C^q(\Phi)$ (resp. $C^q(\Phi')$) the space of all q-linear skew-symmetric mapping of $\Phi\times\cdots\times\Phi$ into Φ (resp. $\Phi'\times\cdots\times\Phi'$ into Φ'). We make also the convention $C^0(\Phi)=\Phi$, $C^0(\Phi')=\Phi'$. An element c of $C^q(\Phi)$ will be called a *q-cochain*. For q-cochain $c\in C^q(\Phi)$ (resp. $C^q(\Phi')$), we define $dc\in C^{q+1}(\Phi)$ (resp. $C^{q+1}(\Phi')$) by

$$(26) \qquad \begin{cases} dc(f_1, \cdots, f_{q+1})=\sum_{i=1}^{q+1}(-1)^{i+1}\{f_i, c(f_1, \cdots, \hat{f}_i, \cdots, f_{q+1})\} \\ \quad +\sum_{i<j}(-1)^{i+j}c(\{f_i, f_j\}, f_1, \cdots, \hat{f}_i, \cdots, \hat{f}_j, \cdots, f_{q+1}),\ q\geqq 1 \\ dc(f)=\{c, f\},\ c\in C^0(\Phi) \quad (\text{resp. } C^0(\Phi'))\,. \end{cases}$$

It is well-known that $d^2=0$, and hence the above system $\{C^*(\Phi), d\}$ defines a cohomology group, which will be denoted by $H^*(\Phi)$. Obviously, $H^0(\Phi)=\boldsymbol{C}$ and it is known in [1], [6] that $H^1(\Phi)=\boldsymbol{C}$. The non-trivial first cocycle is a derivation δ_0 given by

$$\delta_0(f)=\xi_j\partial^j f+x^i\partial_i f-2f\,. \tag{27}$$

We shall prove at first the following:

PROPOSITION 2.1. *The second cohomology group $H^2(\Phi)$ is non-trivial.*

REMARK. The above result has been known in [3], [14] by using deformation theories. However, an explicit expression of non-trivial 2-cocycle is not directly given. Here, we have to know an explicit expression. Therefore what we really want is Lemma 2.4 stated below.

Although the above fact will be proved in several lemmas below, we should remark first of all that $H^2(\Phi)$ corresponds to the isomorphism classes of the extensions of the Lie algebra $(\Phi, \{\ ,\ \})$ with the abelian kernel Φ. Therefore, if we have an extension of $(\Phi, \{\ ,\ \})$, then we can find at least a second cocycle.

Now, for $f\in\Phi'$ we denote by P_f the pseudo-differential operator with the symbol $f(x;\xi)$, i.e.,

$$(P_f u)(x)=\int_{R^n}\int_{R^n} f(x;\xi)e^{-\sqrt{-1}\langle\xi|y-x\rangle}u(y)\,\bar{d}y\,\bar{d}\xi\,. \tag{28}$$

Then, by the well-known product formula (cf. [8]), we see that the commutator $[\sqrt{-1}P_f, \sqrt{-1}P_g]$, $f, g\in\Phi'$ is also a pseudo-differential operator $\sqrt{-1}P_h$, $h\in\Phi'$, and h is given as follows:

$$h=\{f, g\}+\sum_{l\geq 2}\left(\frac{1}{\sqrt{-1}}\right)^{l-1}\frac{1}{l!}c_l(f, g) \tag{29}$$

where

$$c_l(f, g)=\partial^{i_1\cdots i_l}f\partial_{i_1\cdots i_l}g-\partial^{i_1\cdots i_l}g\partial_{i_1\cdots i_l}f\,,$$

and

$$\partial^{i_1\cdots i_l}=\partial^{i_1}\cdots\partial^{i_l},\ \partial_{i_1\cdots i_l}=\partial_{i_1}\cdots\partial_{i_l}\,.$$

Obviously, c_l can be extended to a 2-cochain, i.e., $c_l\in C^2(\Phi)$.

LEMMA 2.2. *c_2 is a 2-cocycle. However c_2 is a coboundary of δ_1 defined by $\delta_1(f)=-\partial^k_k f$ where $\partial^k_k=\partial^k\partial_k$.*

Proof. Since the commutator bracket of pseudo-differential operators satisfies Jacobi's identity, c_2 has to be a 2-cocycle. However, we see $c_2(f, g)=(d\delta_1)(f, g)$ by a direct computation.

Now, we set $\nabla_f=\sqrt{-1}(P_f+P_{\sqrt{-1}\delta_1(f)/2})$ for $f\in\Phi'$, and set

$$\sqrt{-1}\mathscr{R}(f, g)=[\nabla_f, \nabla_g]-\nabla_{\{f,g\}}\ , \quad f, g\in\Phi'\ ,$$

and set $\mathscr{R}(f, g)=P_h$,

$$h=\frac{1}{3!}\Omega_3(f, g)+\frac{1}{4!}\Omega_4(f, g)+\cdots\ ,$$

where Ω_k's are the $2k$-homogeneous terms with respect to the order of differentiations. By a direct computation we have the following:

Lemma 2.3. *$\Omega_3(f, g)$ is given by*

$$\Omega_3(f, g)=-c_3(f, g)+\frac{3}{2}c_2(f, \delta_1(g))+\frac{3}{2}c_2(\delta_1(f), g)-\frac{3}{2}\{\delta_1(f), \delta_1(g)\}\ .$$

Moreover, Ω_3 can be extended to a 2-cocycle in $C^2(\Phi)$.

Proof. We have only to check the second statement. By Jacobi's identity, Ω_3 must be a 2-cocycle in $C^2(\Phi')$. However, this 2-cochain can be obviously extended to a cochain in $C^2(\Phi)$.

Lemma 2.4. *Ω_3 can not be a coboundary.*

Proof. Assume for a while that $\Omega_3(f, g)=d\omega(f, g)$ by some $\omega\in C^1(\Phi)$. Then, the above equality should hold for any homogeneous polynomials of degree 3. Moreover, $\Omega_3(f, g)(0)=d\omega(f, g)(0)$. Note that $d\omega(f, g)(0)=-\omega(\{f, g\})(0)$ for such polynomials.

Now, remark that $\{(\xi_1)^3, (x^1)^3\}=\{x^1(\xi_1)^2, 3(x^1)^2\xi_1\}=9(\xi_1)^2(x^1)^2$. Hence $\Omega_3((\xi_1)^3, (x^1)^3)(0)$ must equal $\Omega_3(x^1(\xi_1)^2, 3(x^1)^2\xi_1)(0)$. However, by direct computation, we see that

$$\Omega_3((\xi_1)^3, (x^1)^3)=-c_3((\xi_1)^3, (x^1)^3)=-36\ ,$$

but

$$\begin{aligned}\Omega_3(x^1(\xi_1)^2, 3(x^1)^2\xi_1)&=\frac{-9}{2}\{\delta_1(x^1(\xi_1)^2), \delta_1((x^1)^2\xi_1)\}\\ &=-18\{\xi_1, x^1\}=-18\ .\end{aligned}$$

This completes the proof of Proposition 2.1 also.

Let Φ_2 be the subring of Φ consisting of all f such that the constant term and the linear term vanish. For C-valued skew-symmetric q-linear form $\tilde{c}$ on $\Phi_2\times\cdots\times\Phi_2$, we define $d\tilde{c}$ as follows:

$$d\tilde{c}(f_1, \cdots, f_{q+1})=\sum_{i<j}(-1)^{i+j}\tilde{c}(\{f_i, f_j\}, f_1, \cdots, \hat{f}_i, \cdots, \hat{f}_j, \cdots, f_{q+1})\ .$$

Since $d^2=0$, the above system defines a cohomology group $H^*(\Phi_2, C)$. It is not hard to see that the linear mapping πc defined by

$$(\pi c)(f_1, \cdots, f_q)=c(f_1, \cdots, f_q)(0),\ c\in C^q(\Phi)$$

induces a homomorphism π^* of $H^*(\Phi)$ into $H^*(\Phi_2, C)$. The argument in the proof of the above lemma shows also the following:

COROLLARY 2.5. *$H^2(\Phi_2, C)\neq\{0\}$ and the mapping $\pi^*: H^2(\Phi)\to H^2(\Phi_2, C)$ is not trivial.*

Now, we shall apply the above results to the local cohomology group of the Lie algebra of contact vector fields. Let U be a neighborhood of the origin 0 in R^n. We fix a linear coordinate system $(x^1, \cdots, x^n)$ on U. Denote $R^n-\{0\}$ by R^n_*. We fix on R^n_* the dual coodinate system $(\xi_1, \cdots, \xi_n)$ of $(x^1, \cdots, x^n)$. Let S^* be the unit ball in R^n_* and set $S^*_U=S^*\times U$. By $C^\infty(S^*_U)r^l$(resp. $C^\infty_R(S^*_U)r^l$) we denote the space of all C (resp. R) valued functions on $U\times R^n_*$, positively homogeneous of degree l. It is well-known that $C^\infty(S^*_U)r$ is a Lie algebra under the Poisson bracket, and $C^\infty_R(S^*_U)r$ is its real subalgebra, which is isomorphic to the Lie algebra of contact vector fields on S^*_U.

For every $f(x;\xi)\in C^\infty(S^*_U)r^l$ such that $\mathrm{supp}\, f\subset U\times R^n_*$, we define a pseudo-differential operator P_f by (28). Remark also that (29) and Lemmas 2.2, 2.3 hold also in our situation. Hence, we have a second cocycle Ω_3: $C^\infty_R)S^*_U)r\times C^\infty_R(S^*_U)r\to C^\infty_R(S^*_U)r^{-1}$, defined by the same equality as in Lemma 2.3.

In the last part of this section, we shall prove the following:

PROPOSITION 2.6. *Ω_3 is not a coboundary, if $n\geqq 2$. Especially, $H^2(C^\infty_R(S^*_U)r, C^\infty_R(S^*_U)r^{-1})\neq\{0\}$.*

PROOF. Suppose for a while that $\Omega_3=d\omega$, i.e.,

$$\Omega_3(f, g)=\{f, \omega(g)\}-\{g, \omega(f)\}-\omega(\{f, g\})\ .$$

Let p be a point in $U\times R^n_*$ such that $x^i(p)=0$, $1\leqq i\leqq n$, and $\xi_j(p)=0$, $1\leqq j\leqq n-1$, $\xi_n(p)=1$. (Recall the assumption $n\geqq 2$.) The above equality should hold at p.

Let $r=\sqrt{\sum_{i=1}^{n}(\xi_i)^2}$. If we set $f=(1/r)x^1(\xi_1)^2$, $g=(x^1)^2\xi_1$, then $\{f, \omega(g)\}(p)=\{g, \omega(f)\}(p)=0$, and

$$\{f, g\}=\frac{3}{r}(x^1)^2(\xi_1)^2-\frac{2}{r^3}(x^1)^2(\xi_1)^4 .$$

On the other hand, set $f'=r^{-2}(\xi_1)^3$, $g'=r(x^1)^3$, and we see that $\{f', \omega(g')\}(p)=\{g', \omega(f')\}(p)=0$, and

$$\{f', g'\}=\frac{9}{r}(x^1)^2(\xi_1)^2-\frac{6}{r^3}(x^1)^2(\xi_1)^4 .$$

Thus, $\{f', g'\}=\{3f, g\}$ and hence $3\Omega_3(f, g)(p)$ must equal $\Omega_3(f', g')(p)$. However by direct computations, we see that

$$3\Omega_3(f, g)(p)=-3\cdot\frac{3}{2}\{\delta_1(f), \delta_1(g)\}(p)=-18 ,$$

$$\Omega_3(f', g')(p)=-c_3(f', g')(p)+\frac{3}{2}c_2(f', \delta_1(g'))(p)=-36 .$$

REMARK. Cochains in this section are not assumed to have locality, continuity or differentiability. So, the cohomology in this section is more general than Losik cohomology group.

§ 3. Changing riemannian metrics.

A pseudo-differential operator $\sqrt{-1}P\in\sqrt{-1}\mathscr{P}^1$ on a riemannian manifold N is written in the form

$$(30)\qquad (\sqrt{-1}Pu)(x)=\sqrt{-1}\int_{T_x^*}a(x;\xi)\widetilde{\nu u}(x;\xi)d\xi+(K\circ u)(x) .$$

The above expression contains less ambiguity than any other expression whenever we fix a riemannian metric on N. The asymptotic expansion $a_1+a_0+a_{-1}+\cdots$ of a is determined uniquely by P. However, if we change the riemannian metric, then the above asymptotic expansion changes very seriously. In this section, we shall compute how it will be changed. Remark first of all that all computations in this section can applied to the operators defined locally on an open subset U in N.

Now, let $\dot{g}$, $\mathring{g}$ be two riemannian metrics on N. The exponential mappings with respect to $\dot{g}$, $\mathring{g}$ will be denoted by ${}_{\cdot x}X$, ${}_{\circ x}Y$, respectively. We define ${}_{\circ x}(Y, \tilde{Z})$, ${}_{\circ x}(Y, \xi)$ etc. by the same manner as ${}_{\cdot x}(X, \tilde{Z})$, ${}_{\cdot x}(X, \xi)$ etc. respectively.

We fix a linear coordinate system on the tangent space T_x and its dual coordinate system on the cotangent space T_x^*. Using these coordinate systems $Y \in T_x$, $\xi \in T_x^*$ are expressed by $Y=(Y^1, \cdots, Y^n)$, $\xi=(\xi_1, \cdots, \xi_n)$ respectively. Through the exponential mapping $\circ_x\colon T_x \to N$, the above linear coordinate system can be regarded as a local coordinate system around x. Thus that a point $z \in N$ has a local coordinate $(z^1, \cdots, z^n)$ implies $z = \circ_x Y$ and $z^i = Y^i$, $1 \leqq i \leqq n$. Riemannian metrics $\dot{g}$, $\mathring{g}$ will be expressed $\dot{g}_{ij}(z)$, $\mathring{g}_{ij}(z)$ by using the above local coordinate system. We denote by $|\dot{g}|(z)$, $|\mathring{g}|(z)$ the determinant of $\dot{g}_{ij}(z)$, $\mathring{g}_{ij}(z)$, respectively.

Since the Fourier transform $\widetilde{\nu u}(y; \eta)$ defined in (2) depends on riemannian metrics $\dot{g}$, $\mathring{g}$, we indicate these by $\widetilde{\nu u}^{\cdot}$, $\widetilde{\nu u}^{\circ}$, respectively. Pseudo-differential operators written in the form (30) depend of course on $\dot{g}$, $\mathring{g}$. So, we denote these by $P_a^{\cdot} u$, $P_a^{\circ} u$, where the suffix a indicates the symbol $a(x; \xi)$. Since we are concerned only with the asymptotic expansion of a, we need not to care about the breadth of the cut off function ν and the smoothing operator $K_\circ$. Thus, we set as follows:

$$
(31) \qquad \begin{cases} (P_a^{\cdot} u)(x) = \displaystyle\int_{T_x^*} a(x; \xi) \widetilde{\nu u}^{\cdot}(x; \xi) đ^{\cdot}\xi \,, \\ (P_a^{\circ} u)(x) = \displaystyle\int_{T_x^*} a(x; \xi) \widetilde{\nu u}^{\circ}(x; \xi) đ^{\circ}\xi \,, \end{cases}
$$

where $đ^{\cdot}\xi$, $đ^{\circ}\xi$ are volume elements on T_x^* given by

$$
(32) \qquad \begin{cases} đ^{\cdot}\xi = \dfrac{1}{\sqrt{(2\pi)^n}} \dfrac{1}{\sqrt{|\dot{g}|(x)}} d\xi_1 \wedge \cdots \wedge d\xi_n \\ đ^{\circ}\xi = \dfrac{1}{\sqrt{(2\pi)^n}} \dfrac{1}{\sqrt{|\mathring{g}|(x)}} d\xi_1 \wedge \cdots \wedge d\xi_n \,. \end{cases}
$$

Now, assume $P_a^{\cdot} \equiv P_{\bar{a}}^{\circ}$ modulos moothing operators. Then, $\bar{a}(x; \xi)$ should be written by using $a(x; \xi)$. In what follows, we shall compute out this relation up to the order 0.

Set

$$
đ^{\cdot}z = \frac{1}{\sqrt{(2\pi)^n}} \sqrt{|\dot{g}|(z)}\, dz^1 \wedge \cdots \wedge dz^n, \quad đ^{\circ}z = \frac{1}{\sqrt{(2\pi)^n}} \sqrt{|\mathring{g}|(z)}\, dz^1 \wedge \cdots \wedge dz^n \,.
$$

Recall that

$$
(33) \qquad \begin{cases} (\widetilde{\nu u}^{\cdot})(x; \xi) = \displaystyle\int_N e^{-i\langle \xi | X \rangle} \nu(x, z) u(z) đ^{\cdot}z, \ \cdot_x X = z \,, \\ (\widetilde{\nu u}^{\circ})(x; \xi) = \displaystyle\int_N e^{-i\langle \xi | Y \rangle} \nu(x, z) u(z) đ^{\circ}z, \ \circ_x Y = z \,. \end{cases}
$$

Since $z = \cdot_x X = \circ_x Y$, X depends smoothly on Y whenever Y is sufficiently close to 0. We denote this function by

(34) $$X=\Phi^{\cdot}_{\circ}(x;Y)\ .$$

As $\Phi^{\cdot}_{\circ}(x;0)=0$, we may set

(35) $$\Phi^{\cdot}_{\circ}(x;Y)=\tilde{\Phi}^{\cdot}_{\circ}(x;Y)Y\ ,$$

where $\Phi^{\cdot}_{\circ}(x;Y)\colon T_x\to T_x$ is a linear mapping, which will be written as

(36) $$X^i=\tilde{\Phi}^i_a(x;\ Y)Y^a$$

by using the above linear coordinate system. Using these notations, we see

$$(\widetilde{\nu u^{\cdot}})(x;\ \xi)=\int_N\sqrt{|\dot{g}|(z)/|\mathring{g}|(z)}\,e^{-i\langle\xi\tilde{\Phi}^{\cdot}_{\circ}(x;Y)|Y\rangle}\nu(x,\ z)u(z)đ^{\circ}z\ ,\quad \cdot_x Y=z\ ,$$

where $|\dot{g}|(z)/|\mathring{g}|(z)$ is understood as a function of $(x;\ Y)$. Set $\eta=\xi\tilde{\Phi}^{\cdot}_{\circ}(x,\ Y)$ and we get

$$(P^{\cdot}_a u)(x)=\int_{T^*_x}\int_N\tilde{a}(x;\ \eta,\ Y)e^{-i\langle\eta|Y\rangle}\nu(x,\ z)u(z)đ^{\circ}zđ^{\circ}\eta\ ,\quad z=\circ_x Y\ ,$$

where

(37) $$\begin{cases}\tilde{a}(x;\ \eta,\ Y)=a(x;\ \eta\tilde{\Phi}^{\cdot}_{\circ}(x;Y)^{-1})\dfrac{1}{\det(\tilde{\Phi}^i_a)}H(x;\ Y)\ ,\\ H(x;\ Y)=\sqrt{|\mathring{g}|(x)|\dot{g}|(z)/|\dot{g}|(x)|\mathring{g}|(z)}\ ,\quad z=\circ_x Y\ .\end{cases}$$

Since

$$\tilde{a}(x;\ \eta,\ Y)=\int_{T^*_x}\int_{T_x}\tilde{a}(x;\ \eta,\ Z)e^{-i\langle\zeta|Z-Y\rangle}đ^{\circ}zđ^{\circ}\zeta\ ,$$

we get easily

$$\begin{aligned}(P^{\cdot}_a u)(x)&=\iiiint\tilde{a}(x;\ \eta,\ Z)e^{-i\langle\zeta|Z-Y\rangle-i\langle\eta|Y\rangle}\nu(x,\ z)u(z)đ^{\circ}Zđ^{\circ}\zeta đ^{\circ}zđ^{\circ}\eta\\ &=\iiiint\tilde{a}(x;\ \eta''+\zeta,\ Z)e^{-i\langle\zeta|Z\rangle}đ^{\circ}Zđ^{\circ}\zeta e^{-i\langle\eta''|Y\rangle}\nu(x,\ z)u(z)đ^{\circ}zđ^{\circ}\eta''\ .\end{aligned}$$

Thus, we obtain

(38) $$\bar{a}(x;\ \eta)=\iint\tilde{a}(x;\ \eta+\zeta,\ Z)e^{-i\langle\zeta|Z\rangle}đ^{\circ}Zđ^{\circ}\zeta$$

modulo rapidly decreasing functions in η. By Taylor's theorem, we see

$$\tilde{a}(x;\, \eta+\zeta,\, Z) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} \partial_\eta^\alpha \tilde{a}(x;\, \eta,\, Z)\zeta^\alpha$$

and hence

$$\bar{a}(x;\, \eta) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} (D_Z^\alpha \partial_\eta^\alpha \tilde{a})(x;\, \eta,\, 0)\,, \quad D_Z^i = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial Z^i}\,. \tag{39}$$

Now, we want to compute the right hand side of (39) more precisely by using differential geometrical method. So, let $\left\{ {i \atop jk} \right\}^{\circ}$, $\left\{ {i \atop jk} \right\}^{\bullet}$ be the Christoffel symbols with respect to $\mathring{g}$, $\dot{g}$ respectively. Since $(Y^1, \cdots, Y^n)$ is a normal coordinate system at $x \in N$ with respect to $\mathring{g}$, we have $\left\{ {i \atop jk} \right\}^{\circ}(x)=0$. Then, using the well-known fact $(\partial/\partial z^l)\log\sqrt{|g|} = \left\{ {i \atop il} \right\}$ (cf. [2] p. 18), we get

Lemma 3.1. *The Taylor expansion of $H(x;\, Y)$ of* (37) *is given by*

$$1+\left\{ {i \atop ij} \right\}^{\bullet}(x)Y^j + \frac{1}{2!}\left(\left\{ {i \atop ik} \right\}^{\bullet}_{,l}(x) - \left\{ {i \atop ik} \right\}^{\circ}_{,l}(x) + \left\{ {i \atop ik} \right\}^{\bullet}(x) \left\{ {j \atop jl} \right\}^{\bullet}(x) \right) Y^k Y^l + \cdots\,.$$

Proof. Take the Taylor series of $\log\sqrt{|\mathring{g}|(x)/|\dot{g}|(x)} + \log\sqrt{|\dot{g}|(\cdot_x Y)} - \log\sqrt{\mathring{g}(\cdot_x Y)}$.

Note that $(tY^1, \cdots, tY^n)$ is a geodesic with respect to $\mathring{g}$. Let $(Y^1(t), \cdots, Y^n(t))$ be a geodesic with respect to $\dot{g}$ with the initial vector $X=(X^1, \cdots, X^n)$. Then

$$\frac{d^2}{dt^2}Y^i(t) + \left\{ {i \atop jk} \right\}^{\bullet} \frac{d}{dt}Y^j(t)\frac{d}{dt}Y^k(t) = 0\,, \quad \frac{d}{dt}\Big|_{t=0} Y^i(t) = X^i\,.$$

Hence

$$Y^i(t) = tX^i - \frac{t^2}{2!}\left\{ {i \atop jk} \right\}^{\bullet}(x)X^jX^k - \frac{t^3}{3!}\dot{\Gamma}^i_{jkl}(x)X^jX^kX^l + \cdots\,,$$

where

$$\dot{\Gamma}^i_{jkl} = \frac{1}{3}\mathfrak{S}_{jkl}\left[\left\{ {i \atop jk} \right\}^{\bullet}_{,l} - 2\left\{ {i \atop mj} \right\}^{\bullet} \left\{ {m \atop kl} \right\}^{\bullet} \right] \quad \text{(cf. [2] p. 52)}\,.$$

Therefore, putting $t=1$, we get the coordinate expression of $\cdot_x: T_x \to N$. Namely, setting $Y^i = (\cdot_x X)^i$ (i-th component), we see

$$Y^i = X^i - \frac{1}{2!}\begin{Bmatrix} i \\ jk \end{Bmatrix}^{\cdot}(x)X^jX^k - \frac{1}{3!}\dot{\Gamma}^i_{jkl}(x)X^jX^kX^l + \cdots . \tag{40}$$

Therefore, computing the inverted Taylor series, we obtain the coordinate expression of (34):

$$X^i = Y^i + \frac{1}{2!}\begin{Bmatrix} i \\ jk \end{Bmatrix}^{\cdot}(x)Y^jY^k + \frac{1}{3!}\frac{1}{3}\mathfrak{S}_{jkl}\left[\begin{Bmatrix} i \\ jk \end{Bmatrix}^{\cdot}_{,l}(x) + \begin{Bmatrix} i \\ jm \end{Bmatrix}^{\cdot}(x)\begin{Bmatrix} m \\ kl \end{Bmatrix}^{\cdot}(x)\right]Y^jY^kY^l + \cdots . \tag{41}$$

Hence, we get the following:

LEMMA 3.2.

(i) $$\tilde{\Phi}^i_a(x;\ Y) = \delta^i_a + \frac{1}{2!}\begin{Bmatrix} i \\ ab \end{Bmatrix}^{\cdot}(x)Y^b + \frac{1}{3!}\frac{1}{3}\mathfrak{S}_{abc}\left[\begin{Bmatrix} i \\ ab \end{Bmatrix}^{\cdot}_{,c} + \begin{Bmatrix} i \\ am \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ bc \end{Bmatrix}^{\cdot}\right](x)Y^bY^c + \cdots .$$

(ii) $$(\tilde{\Phi}(x;\ Y)^{-1})^a_j = \delta^a_j - \frac{1}{2}\begin{Bmatrix} a \\ jk \end{Bmatrix}^{\cdot}(x)Y^k - \frac{1}{3!}\frac{1}{3}\mathfrak{S}_{jkl}\begin{Bmatrix} a \\ jk \end{Bmatrix}^{\cdot}_{,l}(x)Y^kY^l + \left[\frac{5}{72}\begin{Bmatrix} a \\ lm \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ jk \end{Bmatrix}^{\cdot}(x) + \frac{5}{72}\begin{Bmatrix} a \\ km \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ jl \end{Bmatrix}^{\cdot}(x) - \frac{1}{18}\begin{Bmatrix} a \\ jm \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ kl \end{Bmatrix}^{\cdot}(x)\right]Y^kY^l + \cdots .$$

(iii) $$(\det(\tilde{\Phi}^{\cdot}_{\circ}))^{-1} = 1 - \frac{1}{2!}\begin{Bmatrix} i \\ ik \end{Bmatrix}^{\cdot}(x)Y^k - \frac{1}{3!}\frac{1}{3}\left(2\begin{Bmatrix} i \\ ik \end{Bmatrix}^{\cdot}_{,l} + \begin{Bmatrix} i \\ kl \end{Bmatrix}^{\cdot}_{,i}\right)(x)Y^kY^l + \left(\frac{1}{8}\begin{Bmatrix} i \\ ik \end{Bmatrix}^{\cdot}\begin{Bmatrix} j \\ jl \end{Bmatrix}^{\cdot} + \frac{1}{72}\begin{Bmatrix} i \\ jk \end{Bmatrix}^{\cdot}\begin{Bmatrix} j \\ il \end{Bmatrix}^{\cdot} - \frac{1}{18}\begin{Bmatrix} i \\ im \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ kl \end{Bmatrix}^{\cdot}\right)(x)Y^kY^l + \cdots .$$

(iv) $$\partial X^i/\partial Y^a = \delta^i_a + \begin{Bmatrix} i \\ ak \end{Bmatrix}^{\cdot}(x)X^k + \frac{1}{2!}\frac{1}{3}\left[\begin{Bmatrix} i \\ ak \end{Bmatrix}^{\cdot}_{,l} + \begin{Bmatrix} i \\ kl \end{Bmatrix}^{\cdot}_{,a} + \begin{Bmatrix} i \\ la \end{Bmatrix}^{\cdot}_{,k} + \begin{Bmatrix} i \\ mk \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ la \end{Bmatrix}^{\cdot} + \begin{Bmatrix} i \\ ml \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ ak \end{Bmatrix}^{\cdot} - 2\begin{Bmatrix} i \\ am \end{Bmatrix}^{\cdot}\begin{Bmatrix} m \\ kl \end{Bmatrix}^{\cdot}\right](x)X^kX^l + \cdots .$$

Now, recall (37) and (39). Using Lemmas 3.1–2, we have the following:

PROPOSITION 3.3. *Suppose $P^{\cdot}_a \equiv P^{\cdot}_{\bar{a}}$ modulo the operators of order $-\infty$. Then*

$$\bar{a}(x;\ \eta) = a(x;\ \eta) - \frac{1}{\sqrt{-1}}\frac{1}{2}\eta_i\begin{Bmatrix} i \\ jk \end{Bmatrix}^{\cdot}(x)\frac{\partial^2 a}{\partial\eta_j\partial\eta_k}(x;\ \eta) + lower\ order\ terms .$$

§ 4. Second cocycles on a closed riemannian manifold.

Let N be a closed manifold with a riemannian metric $\dot{g}$. At the first step in this section, we shall remark the following:

LEMMA 4.1. *For each point x in a riemannian manifold $(N, \dot{g})$, there is a coordinate neighborhood U with a local coordinate system $(y^1, \cdots, y^n)$ such that* $\left\{ {i \atop il} \right\}^{\cdot} \equiv 0$.

PROOF. Let $\sqrt{|\dot{g}|(z)}dz^1 \wedge \cdots \wedge dz^n$ be the volume element of $(N, \dot{g})$. Since $SL(n)$-structure is always integrable, there exists a local coordinate system $y^1, \cdots, y^n$ such that $\sqrt{|\dot{g}|(y)} \equiv 1$. This implies $\left\{ {i \atop il} \right\}^{\cdot} \equiv 0$ by virtue of $(\partial/\partial y_k) \log \sqrt{|\dot{g}|(y)} = \left\{ {i \atop ik} \right\}^{\cdot}$.

At each point we choose by the above lemma a coordinate neighborhood U with a local coordinate system $(y^1, \cdots, y^n)$ such that $\left\{ {i \atop ik} \right\}^{\cdot} \equiv 0$ on U, and let $(\eta_1, \cdots, \eta_n)$ be its dual coordinate system. Define a riemannian metric $\mathring{g}$ on U by $ds^2 = \sum (dy^i)^2$ so that $(y^1, \cdots, y^n)$ is a normal coordinate system with respect to $\mathring{g}$. Now, note that the computations in the previous section are still valid on a coordinate neighborhood U. Remark also $\cdot_x Y = (x^1 + Y^1, \cdots, x^n + Y^n)$ where $(x^1, \cdots, x^n)$ is the coordinate of x and $\left\{ {i \atop jk} \right\}^{\circ} \equiv 0$.

Let $\tilde{a}(x; \eta)$ be a smooth function on $U \times \boldsymbol{R}^n_*$ such that supp $\tilde{a} \subset U \times \boldsymbol{R}^n_*$ and $\tilde{a}$ is of polynomial growth. Since $\cdot_x Y = x + Y = y$, the pseudo-differential operator $P^{\circ}_{\tilde{a}}$ can be written by

$$(P^{\circ}_{\tilde{a}}u)(x) = \int_{\boldsymbol{R}^n_*} \int_{\boldsymbol{R}^n} \tilde{a}(x; \eta) e^{i-\langle \eta | y-x \rangle} \nu(x, y) u(y) \mathring{d}y \mathring{d}\eta \ .$$

Now, let $P^{\cdot}_a, P^{\cdot}_b$ be pseudo-differential operators of order 1 such that supp a, supp $b \subset T^*U - \{0\}$. In this section, we shall compute first of all the bracket product $[\![P^{\cdot}_a, P^{\cdot}_b]\!]$ up to the order 0 (cf. (6)). By Proposition 3.3, we see that $P^{\cdot}_a = P^{\circ}_{\bar{a}}$ such that

$$\bar{a}(y; \eta) = a(y; \eta) + \frac{\sqrt{-1}}{2!} \lambda_0(a) - \frac{1}{3!} \lambda_{-1}(a) + \cdots , \tag{42}$$

where

$$\lambda_0(a) = \eta_i \left\{ {i \atop jk} \right\}^{\cdot} \frac{\partial^2 a}{\partial \eta_j \partial \eta_k} \ . \tag{43}$$

Thus, we see that $[\![P^{\cdot}_a, P^{\cdot}_b]\!] = P^{\circ}_{\bar{h}}$ such that

$$\bar{h}=\{a,\, b\}+\frac{\sqrt{-1}}{2}\lambda_0(\{a,\, b\}) \tag{44}$$
$$+\frac{\sqrt{-1}}{2}[-c_2(a,\, b)+\{a,\, \lambda_0(b)\}+\{\lambda_0(a),\, b\}-\lambda_0(\{a,\, b\})]$$
$$+\text{lower order terms.}$$

If we set $[\![P_a^{\cdot},\, P_b^{\cdot}]\!]=P_h^{\cdot}$, then

$$h=\{a,\, b\}+\sqrt{-1}\,\omega_0(a,\, b)+\text{lower order terms}\ , \tag{45}$$

where

$$-2\omega_0(a,\, b)=c_2(a,\, b)-d\lambda_0(a,\, b)\ . \tag{46}$$

Remark that $\omega_0(a, b)$ must be the cocycle given by the exact sequence (7) in the introduction. However, $\omega_0(a, b)$ in (46) is written by using the local coordinate system $(y^1, \cdots, y^n, \eta_1, \cdots, \eta_n)$. So, we shall rewrite this by using normal coordinate systems.

LEMMA 4.2.

$$\left.\frac{\partial^2}{\partial X^i\partial X^j}\right|_{X=0} b(\cdot_x(X,\, \eta))$$
$$=\frac{\partial^2 b}{\partial y^i\partial y^j}(x;\, \eta)+\frac{\partial^2 b}{\partial\eta_k\partial y^j}\left\{{l \atop ki}\right\}^{\cdot}\eta_l+\frac{\partial^2 b}{\partial\eta_k\partial y^i}\left\{{l \atop kj}\right\}^{\cdot}\eta_l-\frac{\partial b}{\partial y^k}\left\{{k \atop ij}\right\}^{\cdot}$$
$$+\frac{\partial^2 b}{\partial\eta_k\partial\eta_l}\left\{{\alpha \atop ki}\right\}^{\cdot}\eta_\alpha\left\{{\beta \atop lj}\right\}^{\cdot}\eta_\beta+\frac{1}{3}\frac{\partial b}{\partial\eta_k}\Bigg[\left\{{\beta \atop kj}\right\}^{\cdot}_{,i}+\left\{{\beta \atop ki}\right\}^{\cdot}_{,j}+\left\{{\beta \atop ji}\right\}^{\cdot}_{,k}$$
$$+\left\{{\beta \atop \alpha j}\right\}^{\cdot}\left\{{\alpha \atop ik}\right\}^{\cdot}+\left\{{\beta \atop \alpha i}\right\}^{\cdot}\left\{{\alpha \atop jk}\right\}^{\cdot}-2\left\{{\beta \atop k\alpha}\right\}^{\cdot}\left\{{\alpha \atop ij}\right\}^{\cdot}\Bigg]\eta_\beta\ .$$

Proof is little complicated but a direct computation by using (40) and the relation $y^i=x^i+Y^i$.

Compute $(c_2(a, b)-d\lambda_0(a, b))(x;\, \eta)$. Then, we have the following:

$$\begin{cases} \partial^{ij}a\partial_{ij}b+\partial^{ij}a\left\{{l \atop ij}\right\}^{\cdot}_{,k}\eta_l\partial^k b-\left\{{l \atop ij}\right\}^{\cdot}\partial^{ij}a\partial_l b \\ +\partial^{ij}a\eta_l\left\{{l \atop ik}\right\}^{\cdot}\partial^k_j b+\partial^{ij}a\eta_l\left\{{l \atop jk}\right\}^{\cdot}\partial^k_i b \\ -\partial_{ij}a\partial^{ij}b-\eta_l\left\{{l \atop ij}\right\}^{\cdot}_{,k}\partial^k a\partial^{ij}b+\partial_l a\left\{{l \atop ij}\right\}^{\cdot}\partial^{ij}b \\ -\eta_l\left\{{l \atop ik}\right\}^{\cdot}\partial^k_j a\partial^{ij}b-\eta_l\left\{{l \atop jk}\right\}^{\cdot}\partial^k_i a\partial^{ij}b\ , \end{cases} \tag{47}$$

where $\partial^{ab}=\partial^2/\partial\xi_a\partial\xi_b$, $\partial_i^k=\partial^2/\partial y^i\partial\xi_k$ etc. Therefore, using the above lemma, we have

$$\begin{aligned}&\frac{\partial^2 a}{\partial\eta_i\partial\eta_j}\frac{\partial^2}{\partial X^i\partial X^j}\bigg|_{X=0} b(\cdot_x(X,\eta))-\frac{\partial^2 b}{\partial\eta_i\partial\eta_j}\frac{\partial^2}{\partial X^i\partial X^j}\bigg|_{X=0} a(\cdot_x(X,\eta))\\ &\quad-(c_2(a,b)-d\lambda_0(a,b))(x;\eta)\\ &\quad=\frac{1}{3}(\partial^{ij}a\partial^k b-\partial^{ij}b\partial^k a)\left[\left\{{\beta \atop kj}\right\}^{\cdot}_{,i}+\left\{{\beta \atop ki}\right\}^{\cdot}_{,j}-2\left\{{\beta \atop ij}\right\}^{\cdot}_{,k}+\left\{{\beta \atop \alpha j}\right\}^{\cdot}\left\{{\alpha \atop ik}\right\}^{\cdot}\right.\\ &\quad\quad\left.+\left\{{\beta \atop \alpha i}\right\}^{\cdot}\left\{{\alpha \atop jk}\right\}^{\cdot}-2\left\{{\beta \atop k\alpha}\right\}^{\cdot}\left\{{\alpha \atop ij}\right\}^{\cdot}\right]\eta_\beta\,.\end{aligned}$$

Note that the inside of the bracket [ ] on the right hand side of the above equality is equal to

$$\begin{aligned}&\left\{{\beta \atop jk}\right\}^{\cdot}_{,i}-\left\{{\beta \atop ji}\right\}^{\cdot}_{,k}+\left\{{\beta \atop \alpha j}\right\}^{\cdot}\left\{{\alpha \atop ik}\right\}^{\cdot}-\left\{{\beta \atop k\alpha}\right\}^{\cdot}\left\{{\alpha \atop ij}\right\}^{\cdot}\\ &\quad+\left\{{\beta \atop ik}\right\}^{\cdot}_{,j}-\left\{{\beta \atop ij}\right\}^{\cdot}_{,k}+\left\{{\beta \atop \alpha i}\right\}^{\cdot}\left\{{\alpha \atop jk}\right\}^{\cdot}-\left\{{\beta \atop k\alpha}\right\}^{\cdot}\left\{{\alpha \atop ij}\right\}^{\cdot}\,.\end{aligned}$$

Hence, we have the following:

LEMMA 4.3. *$c_2(a,b)-d\lambda_0(a,b)$ is given by*

$$\begin{aligned}&\frac{\partial^2 a}{\partial\eta_i\partial\eta_j}\frac{\partial^2}{\partial X^i\partial X^j}\bigg|_{X=0} b(\cdot_x(X,\eta))-\frac{\partial^2 b}{\partial\eta_i\partial\eta_j}\frac{\partial^2}{\partial X^i\partial X^j}\bigg|_{X=0} a(\cdot_x(X,\eta))\\ &\quad-\frac{1}{3}\left(\frac{\partial^2 a}{\partial\eta_i\partial\eta_j}\frac{\partial b}{\partial\eta_k}-\frac{\partial^2 b}{\partial\eta_i\partial\eta_j}\frac{\partial a}{\partial\eta_k}\right)[R^\beta_{jik}+R^\beta_{ijk}]\eta_\beta\,.\end{aligned}$$

Remark this proves the first half of Theorem A, for by a suitable partition of unity the equality in the above lemma is still valid for any functions a, b on $T^*N-\{0\}$, whenever they are positively homogeneous of degree one.

Now, recall that $c_2(a,b)=d\delta_1(a,b)$ (cf. Lemma 2.2). Therefore, $c_2(a,b)-d\lambda_0(a,b)=d(\delta_1-\lambda_0)(a,b)$. Note that

$$(\delta_1-\lambda_0)(a)(x;\eta)=-\partial_i^i a(x;\eta)-\eta_i\left\{{i \atop jk}\right\}^{\cdot}\partial^{jk}a(x;\eta)\,. \tag{48}$$

LEMMA 4.4. *$\partial_i^i a+\eta_i\left\{{i \atop jk}\right\}^{\cdot}\partial^{jk}a$ is a globally defined function on $T^*N-\{0\}$.*

PROOF. For $a(x;\eta)$, define $\partial_\eta a$ by

(49) $$(\partial_\eta a)(x;\eta)(\xi)=\lim_{\delta\to 0}\frac{1}{\delta}\{a(x;\eta+\delta\xi)-a(x;\eta)\}\ .$$

$\partial_\eta a$ is a fiber preserving mapping of $T^*N-\{0\}$ into TN such that

(50) $$(\partial_\eta a)(x;\eta)=(\partial^i a)(x;\eta)\partial_i\ .$$

Let $y(t)=(y^1(t),\cdots,y^n(t))$ be a smooth curve in U such that $y(0)=x$, and let $\eta(t)$ be the parallel displacement of η along the curve $y(t)$. Suppose $W=(d/dt)|_{t=0}y(t)$. We define $(\nabla\partial_\eta a)(x;\eta)(W)$ by

(51) $$(\nabla\partial_\eta a)(x;\eta)(W)=\frac{\nabla}{dt}\Big|_{t=0}(\partial_\eta a)(y(t);\eta(t))\quad\text{(covariant derivative)}\ .$$

Then, we see easily that

$$\nabla\partial_\eta a(x;\eta)=\left(\partial^i_j a(x;\eta)+\begin{Bmatrix} i\\ jk\end{Bmatrix}^{\cdot}\partial^k a(x;\eta)+\partial^{il}a\begin{Bmatrix} k\\ lj\end{Bmatrix}^{\cdot}\eta_k\right)W^j\partial_i\ .$$

Thus, $\nabla\partial_\eta a$ is a fiber preserving mapping of $T^*N-\{0\}$ into $T^*N\otimes TN$, hence if one write $\nabla\partial_\eta a(x;\eta)=A^i_j\partial^j\otimes\partial_i$, then A^i_i is a globally defined function on $T^*N-\{0\}$. Therefore, we have that $\partial^i_i a+\begin{Bmatrix} i\\ ij\end{Bmatrix}^{\cdot}\partial^j a+\partial^{ij}a\begin{Bmatrix} k\\ ij\end{Bmatrix}^{\cdot}\eta_k$ is a globally defined function. Recall that $\begin{Bmatrix} i\\ ij\end{Bmatrix}^{\cdot}\equiv 0$ by the assumption of a local coordinate system $(y^1,\cdots,y^n)$. Then, we get the desired result.

Set $2\beta(a)=\partial^i_i a+\begin{Bmatrix} k\\ ij\end{Bmatrix}^{\cdot}\eta_k\partial^{ij}a$. Then, we see $\omega_0(a,b)=d\beta(a,b)$. Hence the cocycle ω_0 is in fact a coboundary of β. Since 2β is given by $\nabla_i\partial^i a$, one can write this by using arbitrary local coordinate system, which is obviously given as follows:

$$\begin{aligned}&2\beta(a)(X^1,\cdots,X^n,\xi_1,\cdots,\xi_n)\\&=\frac{\partial^2 a}{\partial X^i\partial\xi_i}+\begin{Bmatrix} i\\ ik\end{Bmatrix}^{\cdot}\frac{\partial a}{\partial\xi_k}+\begin{Bmatrix} i\\ jk\end{Bmatrix}^{\cdot}\xi_i\frac{\partial^2 a}{\partial\xi_j\partial\xi_k}\ .\end{aligned}$$

This completes the proof of Theorem A.

Now, set

(52) $$\nabla^{\cdot}_a=\sqrt{-1}(P^{\cdot}_a-\sqrt{-1}P^{\cdot}_{\beta(a)})\quad(\in\sqrt{-1}\mathscr{P}^1)\ .$$

Since $[\![P^{\cdot}_a,P^{\cdot}_b]\!]=P^{\cdot}_{\{a,b\}}+\sqrt{-1}P^{\cdot}_{\omega_0(a,b)}+\cdots$, and $\omega_0=d\beta$, we see easily that

$$\sqrt{-1}\mathscr{R}^{\cdot}(a,b)=[\nabla^{\cdot}_a,\nabla^{\cdot}_b]-\nabla^{\cdot}_{\{a,b\}}\in\mathscr{P}^{-1}\ .$$

By Jacobi's identity in $\sqrt{-1}\mathscr{P}^1$, we see that $\mathscr{R}^{\cdot}(a, b)$ satisfies the following Bianchi's identity:

$$\mathfrak{S}([\nabla^{\cdot}_a, \mathscr{R}^{\cdot}(b, c)] - \mathscr{R}^{\cdot}(\{a, b\}, c)) = 0 .$$

Let $\tilde{\Omega}^{\cdot}_3(a, b) + \tilde{\Omega}^{\cdot}_4(a, b) + \cdots$ be the asymptotic expansion of the symbol of $\mathscr{R}^{\cdot}(a, b)$, $\tilde{\Omega}^{\cdot}_k(a, b) \in C^\infty(S^*N)r^{-(k-2)}$. Then, by Bianchi's identity, $\tilde{\Omega}^{\cdot}_3$ must be a 2-cocycle.

LEMMA 4.5. *The cohomology class of $\tilde{\Omega}^{\cdot}_3$ in $H^2(C^\infty_R(S^*N)r, C^\infty(S^*N)r^{-1})$ is independent of the choice of riemannian metric $\dot{g}$, whenever the volume element of $\dot{g}$ remains fixed.*

PROOF. Let $\mathring{g}$ be another riemannian metric such that $|\mathring{g}|(z) \equiv |\dot{g}|(z)$. We use the same notations as in § 3, and set

$$P^{\cdot}_a = P^{\circ}_a + P^{\circ}_{\gamma_0(a)} + P^{\circ}_{\gamma_{-1}(a)} + \cdots ,$$

where

$$\gamma_0(a)(x; \eta) = -\frac{1}{2\sqrt{-1}}\eta_i \begin{Bmatrix} i \\ jk \end{Bmatrix}^{\cdot} \frac{\partial^2 a}{\partial\eta_j \partial\eta_k}(x; \eta) .$$

Since $\beta(a)$ depends on the choice of riemannian metric, we indicate this by $\beta^{\cdot}(a)$ or $\beta^{\circ}(a)$. Recall (52), and hence we have

$$\nabla^{\cdot}_a = \nabla^{\circ}_a + P^{\circ}_{(\sqrt{-1}\gamma_0(a) + \beta^{\cdot}(a) - \beta^{\circ}(a))} + P^{\circ}_{\gamma_{-1}(a)} + \cdots . \tag{53}$$

However, $\sqrt{-1}\gamma_0(a) + \beta^{\cdot}(a) - \beta^{\circ}(a) = 0$ by the assumed property of $\dot{g}$, $\mathring{g}$. It follows easily that $\tilde{\Omega}^{\cdot}_3(a, b)$ is cohomologus to $\tilde{\Omega}^{\circ}_3(a, b)$.

Now, we consider $\tilde{\Omega}^{\cdot}_3(a, b)$ on a coordinate neighborhood U with a local coordinate system $(y^1, \cdots, y^n)$ such that $\begin{Bmatrix} i \\ ik \end{Bmatrix}^{\cdot} \equiv 0$ (cf. Lemma 4.1). Let $\mathring{g} = \sum (dy^i)^2$. Then, obviously $\begin{Bmatrix} i \\ jk \end{Bmatrix}^{\circ} \equiv 0$ on U.

LEMMA 4.6. *If we restrict $\tilde{\Omega}^{\cdot}_3$ on U, then $\tilde{\Omega}^{\cdot}_3|U$ is cohomologus to Ω_3 given in Lemma 2.3.*

PROOF. By the above lemma, we see that $\tilde{\Omega}^{\cdot}_3|U$ is cohomologus to $\tilde{\Omega}^{\circ}_3|U$. However, it is easy to see that $\tilde{\Omega}^{\circ}_3|U \equiv \Omega_3$, because of $\begin{Bmatrix} i \\ jk \end{Bmatrix} \equiv 0$.

PROOF OF THEOREM B. Suppose $\tilde{\Omega}^{\cdot}_3(a, b) = d\omega(a, b)$. Since we are considering Losik cohomology class, we may assume that ω is a linear differential operator. Let U be an open coordinate neighborhood such as in the above lemma. Then, $\tilde{\Omega}^{\cdot}_3(a, b)|U = \tilde{\Omega}^{\cdot}_3(a|U, b|U)$, and $\omega(a)|U =$

$\omega(a|U)$. Therefore $\tilde{\Omega}_3(a|U, b|U)=d\omega(a|U, b|U)$. However, by Lemma 4.6 this contradicts the result of Proposition 2.6.

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