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Mixed Problem for Hyperbolic Equations of Second Order in a Domain with a Corner

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Introduction

We consider the mixed problem for hyperbolic equation of second order in domains $\{(t, x, y) | t>0, x>0, y>0\}$ and $\{(t, x, y, z) | t>0, x>0, y>0, z>0\}$. In [4], Kupka and Osher treated the mixed problem for wave equation with zero initial data in a multi-dimensional corner $\{(t, x_1, \cdots, x_n) | t>0, x_k>0\ (k=1, \cdots, n)\}$. Also, in [1], Kojima and Taniguchi considered the mixed problem for wave equation in a domain $\{(t, x, y) | t>0, x>0, y>0\}$, got the semi-group estimate and proved the existence of the classical solution. At that time, boundary operators had constant coefficients. The purpose of this paper is to generalize the results in [1] and [4].

When we treat the mixed problem for hyperbolic equation in a domain with smooth boundaries, we can prove the existence of the classical solution using the energy inequality and functional analysis. But, for the mixed problem in a domain with edges and corners, it seems that we can not yet show the existence of the classical solution by use of the energy inequality and functional analysis. Improving the method in [1], we get the energy inequality and prove the existence of the classical solution for the mixed problem in domains $\{(t, x, y) | t>0, x>0, y>0\}$ and $\{(t, x, y, z) | t>0, x>0, y>0, z>0\}$. The method used in [1] to obtain the energy inequality was that we transformed the mixed problem for wave equation into the one for symmetric hyperbolic system of first order under the boundary condition which was positive definite on one face of the boundary and non-negative on another one. We treated 2×2 or 3×3 hyperbolic system of first order for wave equation in [1]. To use the above method and consider the mixed problem for wave equation with any lower order term of variable coefficients and further a boundary operator of variable coefficients, we concern with $N\times N (N\geqq 4)$

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symmetric hyperbolic system. And this improvement enables us to get a simple proof of Miyatake's result [6] for the mixed problem for hyperbolic equation of second order with variable coefficients in a domain with smooth boundary. When, $n=2$ or 3, we treat the mixed problem with non-zero initial data, which was considered by Kupka and Osher [4] for wave equation with zero initial data. Then, to obtain the energy inequality, we concern with $N\times N(N\geqq 4)$ symmetric hyperbolic system with non-negative boundary condition.

In §1, we explain the notation. In §2, we state problems and results. In §3, we treat the mixed problem for symmetric hyperbolic system in a domain with a corner. In §4, we obtain the energy inequality. In §5 and §6, we prove the existence of the classical solution.

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§1. Notation.

$\boldsymbol{R}^n(\boldsymbol{C}^n)$: n-dimensional real (complex) Euclidean space.

$\boldsymbol{R}^n_+$: the set $\{(x, y)\mid x>0,\ y\in \boldsymbol{R}^{n-1}\}$.

$((\ ,\ ))$: the inner product in $\boldsymbol{C}^N$.

$$\|u\|^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^T dt\int_0^\infty dx\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta\Big(\frac{\partial}{\partial x}\Big)^\gamma\Big(\frac{\partial}{\partial y}\Big)^\delta u\right|^2$$

or

$$=\sum_{\alpha+\beta+\gamma+\delta+\theta=m}\int_0^T dt\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\left|e^{-\mu t}\mu^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta\Big(\frac{\partial}{\partial x}\Big)^\gamma\Big(\frac{\partial}{\partial y}\Big)^\delta\Big(\frac{\partial}{\partial z}\Big)^\theta u\right|^2.$$

$$\langle u\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma=m}\int_0^T dt\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta\Big(\frac{\partial}{\partial y}\Big)^\gamma u\right|^2.$$

$$\langle\!\langle u\rangle\!\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma=m}\int_0^T dt\int_0^\infty dx\left|e^{-\mu t}\mu^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta\Big(\frac{\partial}{\partial x}\Big)^\gamma u\right|^2.$$

$$|||u(t)|||^2_{m,\mu}=\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^\infty dx\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta\Big(\frac{\partial}{\partial x}\Big)^\gamma\Big(\frac{\partial}{\partial y}\Big)^\delta u\right|^2$$

or

$$=\sum_{\alpha+\beta+\gamma+\delta+\theta=m}\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\left|e^{-\mu t}\mu^\alpha\Big(\frac{\partial}{\partial t}\Big)^\beta\Big(\frac{\partial}{\partial x}\Big)^\gamma\Big(\frac{\partial}{\partial y}\Big)^\delta\Big(\frac{\partial}{\partial z}\Big)^\theta u\right|^2.$$

$$D_x=\frac{\partial}{\partial x}\ \text{etc}.$$

$\|u\|_{m,\mu,+}=\|u\|_{m,\mu,\infty}$ etc.

$\mathscr{H}_{m,\mu}[(\boldsymbol{R}^1_+)^n]$:

the space of functions which are obtained by the completion of $C_0^\infty[(\bar{\boldsymbol{R}}^1_+)^n]$ with the norm $\|u\|_{m,\mu,\infty}$.

$$\Lambda^{\pm\theta}_{x,\mu}=\bar{\mathfrak{F}}_x(\mu^2+\xi^2)^{\pm\theta/2}\mathfrak{F}_x\ .$$
$$\Lambda^{\pm\theta}_{y,\mu}=\bar{\mathfrak{F}}_y(\mu^2+\eta^2)^{\pm\theta/2}\mathfrak{F}_y\ .$$

§2. Statement of problems and results.

We consider mixed problems

$$\text{(I)}\begin{cases} L[u]=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}-p(t,x,y)\dfrac{\partial u}{\partial t}-q(t,x,y)\dfrac{\partial u}{\partial x}-r(t,x,y)\dfrac{\partial u}{\partial y} \\ \qquad -s(t,x,y)u=f(t,x,y) \\ u(0,x,y)=u_0(x,y)\ ,\quad u_t(0,x,y)=u_1(x,y) \\ B_1[u]\Big|_{x=0}=\left(\dfrac{\partial u}{\partial x}+b_1(t,y)\dfrac{\partial u}{\partial y}-c_1(t,y)\dfrac{\partial u}{\partial t}+d_1(t,y)u\right)\Big|_{x=0}=g_1(t,y) \\ B_2[u]\Big|_{y=0}=\left(\dfrac{\partial u}{\partial y}+b_2(t,x)\dfrac{\partial u}{\partial x}-c_2(t,x)\dfrac{\partial u}{\partial t}+d_2(t,x)u\right)\Big|_{y=0}=g_2(t,x) \\ \qquad (t,x,y)\in(\boldsymbol{R}^1_+)^3 \end{cases}$$

$$\text{(II)}\begin{cases} L[v]=h(t,x,y) \\ v(0,x,y)=v_0(x,y)\ ,\quad v_t(0,x,y)=v_1(x,y) \\ B_3[v]\Big|_{x=0}=\left(\dfrac{\partial v}{\partial x}-c_3(t,y)\dfrac{\partial v}{\partial t}+d_3(t,y)v\right)\Big|_{x=0}=0 \\ B_4[v]\Big|_{y=0}=\left(\dfrac{\partial v}{\partial y}+d_4(t,x)v\right)\Big|_{y=0}=0 \\ (t,x,y)\in(\boldsymbol{R}^1_+)^3 \end{cases}$$

and

$$\text{(III)}\begin{cases} M[w]=\dfrac{\partial^2 w}{\partial t^2}-\dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial^2 w}{\partial y^2}-\dfrac{\partial^2 w}{\partial z^2}-p_0(t,x,y,z)\dfrac{\partial w}{\partial t} \\ \qquad -p_1(t,x,y,z)\dfrac{\partial w}{\partial x}-p_2(t,x,y,z)\dfrac{\partial w}{\partial y} \\ \qquad -p_3(t,x,y,z)\dfrac{\partial w}{\partial z}-p_4(t,x,y,z)w=k(t,x,y,z) \\ w(0,x,y,z)=w_0(x,y,z),\ w_t(0,x,y,z)=w_1(x,y,z) \\ B_5[w]\Big|_{x=0}=\left(\dfrac{\partial w}{\partial x}+\alpha(t,y,z)w\right)\Big|_{x=0}=0 \\ B_6[w]\Big|_{y=0}=\left(\dfrac{\partial w}{\partial y}+\beta(t,x,z)w\right)\Big|_{y=0}=0 \\ B_7[w]\Big|_{z=0}=\left(\dfrac{\partial w}{\partial z}+\gamma(t,x,y)w\right)\Big|_{z=0}=0 \\ (t,x,y,z)\in(\boldsymbol{R}^1_+)^4\ . \end{cases}$$

Functions $p, q, r, s, b_1, b_2, c_1, c_2, c_3, d_1, d_2, d_3$ and d_4 are smooth complex valued functions and are constant outside a compact set in $(R^1_+)^3$ or $(\bar{R}^1_+)^2$ except b_1, b_2, c_1 and c_2, and same conditions hold for b_1, b_2, c_1 and c_2 in $\{(t, y)|t\geqq 0, y\geqq -\theta\}$ or $\{(t, x)|t\geqq 0, x\geqq -\theta\}$ where θ is a small positive constant. Also, functions $p_0, p_1, \cdots, p_4, \alpha, \beta$ and γ are smooth complex valued functions and are constant outside a compact set in $(R^1_+)^4$ or $(\bar{R}^1_+)^3$.

We assume following conditions for the problem (I):

(C.1)
$$\begin{cases} b_1(t, 0)=\dfrac{1}{b_2(t, 0)}, \quad c_1(t, 0)=\dfrac{c_2(t, 0)}{b_2(t, 0)} \\ d_1(t, 0)=\dfrac{d_2(t, 0)}{b_2(t, 0)}. \end{cases}$$

Here $d_1(t, 0)\neq 0$ for any $t\in\bar{R}^1_+$ or there is a positive constant δ such that $d_1(t, y)\equiv d_2(t, x)\equiv 0$ for any $x, y\in[0, \delta]$ where δ is a positive constant independent of $t\in\bar{R}^1_+$.

The quadratic equation

(C.2)
$$(c_1(t, y)+1)z^2+2b_1(t, y)z+c_1(t, y)-1=0$$

has roots in the domain $\Omega_1=\{z\in C|\ |z|\leqq 1\}$ if they are different and in $\mathring{\Omega}_1$ if they are equal where $(t, y)\in\{(t, y)|t\geqq 0, y\geqq -\theta\}$.

The quadratic equation

(C.3)
$$(c_2(t, x)+b_2(t, x))z^2+2z+c_2(t, x)-b_2(t, x)=0$$

has roots in the domain $\Omega_2=\{z\in C|\operatorname{Re} z\leqq 0\}$ if they are different and in $\mathring{\Omega}_2$ if they are equal where $(t, x)\in\{(t, x)|t\geqq 0, x\geqq -\theta\}$.

The quadratic equation

(C.4)
$$(c_1(t, 0)+1)z^2+2b_1(t, 0)z+c_1(t, 0)-1=0$$

has no roots $z=\pm i$.

We impose on the problem (II) the following condition:

(C.5)
$$\operatorname{Re} c_3(t, y)\geqq 0 .$$

REMARK 1. See [1] and [5] for conditions (C.2)-(C.5).

DEFINITION 1. Let $p, q, r, s, b_1, b_2, c_1$ and c_2 be complex constants in (I).

(i) We say that $\{f, g_1, u_0, u_1\}$ satisfies the compatibility condition of order k in the region $G_1(G_2)$ if the following (C_k) holds

$$(C_k) \qquad B_1^{(m)}(f, u_0, u_1) \equiv \sum_{j=0}^{m} \{B_{1,j}^{(m)}(D_x, D_y)u_j\}(0, y) = (D_t^{m-1}g_1)(0, y) \quad (m=1, 2, \cdots, k)$$

where

$$\sum_{j=0}^{m} B_{1,j}^{(m)}(D_x, D_y)D_t^j u \equiv D_t^{m-1}\{B_1(D_t, D_x, D_y)u\}$$
$$u_{2+i} \equiv \{(D_t^i f)(0, x, y) - (D_t^i L - D_t^{2+i})u\} \quad (i=0, 1, 2, \cdots)$$

and

$$\begin{cases} G_1 = \{y \mid y \geqq 0\} \\ G_2 = \{y \mid y \in \boldsymbol{R}^1\} . \end{cases}$$

(ii) We say that $\{f, g_2, u_0, u_1\}$ satisfies the compatibility condition of order k in the region $G_3(G_4)$ if the following (C_k') holds,

$$(C_k') \qquad B_2^{(m)}(f, u_0, u_1) \equiv \sum_{j=0}^{m} \{B_{2,j}^{(m)}(D_x, D_y)u_j\}(x, 0) = (D_t^{m-1}g_2)(x, 0) \quad (m=1, 2, \cdots, k)$$

where

$$\sum_{j=0}^{m} B_{2,j}^{(m)}(D_x, D_y)D_t^j u \equiv D_t^{m-1}\{B_2(D_t, D_x, D_y)u\}$$
$$u_{2+i} = \{(D_t^i f)(0, x, y) - (D_t^i L - D_t^{2+i})u\} \quad (i=0, 1, 2, \cdots)$$

and

$$\begin{cases} G_3 = \{x \mid x \geqq 0\} \\ G_4 = \{x \mid x \in \boldsymbol{R}^1\} . \end{cases}$$

(iii) We say that $\{g_1, g_2\}$ satisfies the compatibility condition (D_k) $(k=1, 3, 5)$ if the following condition holds:

$$(D_1) \qquad l_1(t, 0) = b_1 \cdot l_2(t, 0)$$

$$(D_3) \qquad b_1 \cdot l_{2xx}(t, 0) = l_{1tt}(t, 0) - l_{1yy}(t, 0) + a l_1(t, 0) - B_1' f_1(t, 0, 0)$$

and

$$(D_5) \qquad b_1 \cdot l_{2xxxx}(t, 0) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + a\right)^2 l_1(t, 0) - B_1'(f_{1tt} - f_{1yy} + f_{1xx} + a f_1)(t, 0, 0)$$

where

$$f_1=e^{-(1/2)(pt-qx-ry)}f(t, x, y)\ ,\quad l_1=e^{-(1/2)(pt-ry)}g_1(t, y)$$
$$l_2=e^{-(1/2)(pt-qx)}g_2(t, x),\quad a=-\frac{1}{4}p^2+\frac{1}{4}q^2+\frac{1}{4}r^2-s$$

and

$$B_1'=\frac{\partial}{\partial x}+b_1\frac{\partial}{\partial y}-c_1\frac{\partial}{\partial t}+\left(d_1-\frac{1}{2}q-\frac{b_1}{2}r-\frac{c_1}{2}p\right).$$

DEFINITION 2. (i) We say that $\{f, g_1, u_0, u_1\}$ has the property (E_k):

(E_k) $\{f, g_1, u_0, u_1\}$ satisfies the (C_k) in G_1 and as a extension $\{\tilde{f}, \tilde{g}_1, \tilde{u}_0, \tilde{u}_1\}$ which satisfies the (C_k) in G_2 and has the same regularity as $\{f, g_1, u_0, u_1\}$.

(ii) We say that $\{f, g_2, u_0, u_1\}$ has the property (E_k'):

(E_k') $\{f, g_2, u_0, u_1\}$ satisfies the (C_k') in G_3 and has a extension $\{\tilde{f}, \tilde{g}_2, \tilde{u}_0, \tilde{u}_1\}$ which satisfies the (C_k') in G_4 and has the same regularity as $\{f, g_2, u_0, u_1\}$.

We state our results,

THEOREM 1. *Assume conditions* (C.1)–(C.4). *Let u be the solution of the problem* (I) *which belongs to* $\mathscr{H}_{2,\mu}[(R_+^1)^3]$.

Then, there exist positive constants C and μ_0 such that the following inequality holds for any $t\in R_+^1$ and any $\mu\geqq\mu_0$.

$$\text{(2.1)}\qquad |||u(t)|||_{1,\mu}^2+\mu\|u\|_{1,\mu,t}^2$$
$$+\mu\sum_{k=0}^{1}\left\{\left\langle \Lambda_{y,\mu}^{-(1/2)}\left(\frac{\partial}{\partial x}\right)^k u\right\rangle_{1-k,\mu,t}^2+\left\langle\!\left\langle \Lambda_{x,\mu}^{-(1/2)}\left(\frac{\partial}{\partial y}\right)^k u\right\rangle\!\right\rangle_{1-k,\mu,t}^2\right\}$$
$$\leqq C\left\{|||u(0)|||_{1,\mu}^2+\frac{1}{\mu}\|f\|_{0,\mu,t}^2\right.$$
$$\left.+\frac{1}{\mu}\langle \Lambda_{y,\mu}^{1/2}g_1\rangle_{0,\mu,t}^2+\frac{1}{\mu}\langle\!\langle \Lambda_{x,\mu}^{1/2}g_2\rangle\!\rangle_{0,\mu,t}^2\right\}.$$

THEOREM 2. *Let $p, q, r, s, b_1, b_2, c_1, c_2, d_1$ and d_2 be complex constants and assume conditions* (C.1)–(C.4). *Let (f, g_1, g_2, u_0, u_1) belongs to* $C_0^\infty[(\bar{R}_+^1)^3]\times[C_0^\infty[(\bar{R}_+^1)^2]]^4$ *and suppose that the following condition (α) or (β) holds,*

(α) *(f, g_1, u_0, u_1) has the property (E_6), (f, g_2, u_0, u_1) satisfies the (C_6') in $\{x\mid x\geqq 0\}$ and conditions (D_1), (D_3) and (D_5) hold.*

(β) *(f, g_2, u_0, u_1) has the property (E_6'), (f, g_1, u_0, u_1) satisfies the (C_6) in $\{y \mid y \geqq 0\}$ and conditions (D_1), (D_3) and (D_5) hold.*

Then, there is a unique classical solution $u \in \mathscr{H}_{5,\mu}[(\boldsymbol{R}_+^1)^3]$ of the problem (I) *which satisfies* (2.1).

REMARK 2. For the existence of the solution, we treated the case where $p \equiv q \equiv r \equiv s \equiv d_1 \equiv d_2 \equiv 0$, and $b_1 = (1/b_2)$ and $c_1 = c_2/b_2$ are real constants in [1].

THEOREM 3. *Assume the condition* (C.5). *Let v be the solution of the problem* (II) *which belongs to $\mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^3]$.*

Then, there exist positive constants C and μ_0 such that the following inequality holds for any $t \in \boldsymbol{R}_+^1$ and any $\mu \geqq \mu_0$

$$|||v(t)|||_{1,\mu}^2 + \mu \| v \|_{1,\mu,t}^2 \leqq C \left\{ |||v(0)|||_{1,\mu}^2 + \frac{1}{\mu} \| h \|_{0,\mu,t}^2 \right\} . \tag{2.2}$$

THEOREM 4. *Let p, q, r, s, c_3, d_3 and d_4 be complex constants and assume the condition* (C.5). *Let (h, v_0, v_1) belongs to $C_0^\infty[(\boldsymbol{R}_+^1)^3] \times [C_0^\infty[(\boldsymbol{R}_+^1)^2]]^2$.*

Then, there is a unique classical solution $v \in \mathscr{H}_{5,\mu}[(\boldsymbol{R}_+^1)^3]$ of the problem (II) *which satisfies* (2.2).

THEOREM 5. *Let w be the solution of the problem* (III) *which belongs to $\mathscr{H}_{2,\mu}[(\boldsymbol{R}_+^1)^4]$.*

Then, there exist positive constants C and μ_0 such that the following inequality holds for any $t \in \boldsymbol{R}_+^1$ and any $\mu \geqq \mu_0$

$$|||w(t)|||_{1,\mu}^2 + \mu \| w \|_{1,\mu,t}^2 \leqq C \left\{ ||| w(0) |||_{1,\mu}^2 + \frac{1}{\mu} \| k \|_{0,\mu,t}^2 \right\} . \tag{2.3}$$

THEOREM 6. *Let $p_0, p_1, p_2, p_3, p_4, \alpha, \beta$ and γ be complex constants, and (k, w_0, w_1) belongs to $C_0^\infty[(\boldsymbol{R}_+^1)^4] \times [C_0^\infty[(\boldsymbol{R}_+^1)^3]]^2$.*

Then, there is a unique classical solution $w \in \mathscr{H}_{5,\mu}[(\boldsymbol{R}_+^1)^4]$ of the problem (III) *which satisfies* (2.3).

REMARK 3. In [4], Kupka and Osher obtained the energy inequality weaker than (2.2) and (2.3).

§ 3. Mixed problem for symmetric hyperbolic system of first order.

We consider the mixed problem

$$(3.1)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=A\dfrac{\partial U}{\partial x}+B\dfrac{\partial U}{\partial y}+K(t, x, y)U+F(t, x, y) \\ U(0, x, y)=U_0(x, y) \\ PU|_{x=0}=(I_{l_1}, \tilde{P})U|_{x=0}=G_1(t, y) \\ QU|_{y=0}=(I_{l_2}, \tilde{Q})T^*U|_{y=0}=G_2(t, x) \\ (t, x, y)\in(\boldsymbol{R}^1_+)^3 \end{cases}$$

where $U={}^t(U_1, \cdots, U_N)$, A and B are $N\times N$ constant Hermite matrices, $\det(AB)\neq 0$, A has distinct eigenvalues and has the form

$$A=\begin{pmatrix} A_I & 0 \\ 0 & A_{II} \end{pmatrix}$$

$$A_I=\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_{l_1} \end{pmatrix}<0,\ A_{II}=\begin{pmatrix} a_{l_1+1} & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix}>0$$

$K, \tilde{P}$ and $\tilde{Q}$ are respectively $N\times N, l_1\times(N-l_1)$ and $l_2\times(N-l_2)$ smooth complex matrices, and are constant outside a compact set $(\bar{\boldsymbol{R}}^1_+)^3$, $\bar{\boldsymbol{R}}^1_+\times\boldsymbol{R}^1$ and $(\bar{\boldsymbol{R}}^1_+)^2$, and T is a smooth unitary matrix such that $B_1=T^*BT$ is diagonal.

We assume the following condition for the problem (3.1):

(C.6) $((AU, U))\geqq 0$ for all $U\in\operatorname{Ker}P(t, y)$ (all $(t, y)\in\bar{\boldsymbol{R}}^1_+\times\boldsymbol{R}^1$)
$((BU, U))\geqq C((U, U))$ for all $U\in\operatorname{Ker}Q(t, x)$ (all $(t, x)\in(\bar{\boldsymbol{R}}^1_+)^2$)

where C is a positive constant.

We extend K to the region $\{(t, x, y)\mid t\geqq 0, x\geqq 0, y<0\}$ as smooth functions. When we set $U(t, x, y)=0$ $(t<0$ or $y<0)$, by the Laplace-Fourier transform of (3.1) with respect to (t, y), we have

$$(3.2)\qquad \begin{cases} \dfrac{d\hat{U}}{dx}=M(\tau, \eta)\hat{U}-A^{-1}\widehat{KU}-A^{-1}\hat{F} \\ \qquad\qquad +A^{-1}B\hat{\hat{U}}(\tau, x, 0)-A^{-1}\tilde{U}_0(x, \eta)\quad (x>0) \\ \widehat{PU}|_{x=0}=\hat{G}_1 \end{cases}$$

where

$$M(\tau, \eta)=A^{-1}(\tau I-i\eta B),\ \tau=\mu+i\sigma,\ \mu>0,\ \sigma\in\boldsymbol{R}^1,$$

$$\hat{\hat{U}}(\tau, x, 0)=\int_{-\infty}^{\infty}e^{-\tau t}U(t, x, 0)dt$$

and

$$\tilde{U}_0(x, \eta)=\int_{-\infty}^{\infty} e^{-iy\cdot\eta} U_0(x, y)dy \ .$$

From now, we treat the estimate of the solution U on the boundary $\{(t, 0, y)|t\geqq 0, y\geqq 0\}$ in the two regions

$$D_1=\{(\tau, \eta)\,|\,|\eta|\leqq\delta_0|\tau|\} \quad \text{and} \quad D_2=\{(\tau, \eta)\,|\,|\tau|\leqq 2/\delta_0|\eta|\}$$

where τ and η are dual variables of t and y, δ_0 is a sufficiently small positive constant.

Firstly, we consider the mixed problem

$$\text{(3.3)} \qquad \begin{cases} \dfrac{\partial W}{\partial t}=A\dfrac{\partial W}{\partial x}+B\dfrac{\partial W}{\partial y}+K(t, x, y)W \\ W(0, x, y)=U_0(x, y) \\ PW|_{x=0}=0 \\ (t, x, y)\in(\boldsymbol{R}^1_+)^2\times\boldsymbol{R}^1 \ . \end{cases}$$

For $U_0(x, y)\in L^2(\boldsymbol{R}^1_+\times\boldsymbol{R}^1)$, we have the solution W of the problem (3.3) which satisfies the following energy inequality

$$\text{(3.4)} \qquad \begin{aligned} &|||W(t)|||^2_{0,\mu}+\mu\|W\|^2_{0,\mu,t}+\mu\langle\Lambda^{-(1/2)}_{y,\mu}W\rangle^2_{0,\mu,t} \\ &\leqq C|||U_0|||^2_{0,\mu} \end{aligned}$$

for any $t\in\boldsymbol{R}^1_+$ and any $\mu\geqq\mu_0$ where C and μ_0 are positive constants. By the Laplace-Fourier transform of (3.3) with respect to (t, y), we obtain

$$\text{(3.5)} \qquad \begin{cases} \dfrac{d\hat{W}}{dx}=M(\tau, \eta)\hat{W}-A^{-1}\widehat{KW}-A^{-1}\tilde{U}_0(x, \eta) \quad (x>0) \\ \widehat{PW}|_{x=0}=0 \ . \end{cases}$$

We set $V=U-W$.

Secondly, we consider the boundary value problem

$$\text{(3.6)} \qquad \begin{cases} \dfrac{d\hat{V}}{dx}=M(\tau, \eta)\hat{V}-A^{-1}\widehat{KV}-A^{-1}\hat{F}+A^{-1}B\hat{\hat{U}}(\tau, x, 0) \quad (x>0) \\ \widehat{PV}|_{x=0}=\hat{G}_1 \end{cases}$$

in $D_1=\{(\tau, \eta)||\eta|\leqq\delta_0|\tau|\}$. Then, we can diagonalize $M(\tau, \eta)$ in D_1 by conditions for A and D_1, and each eigenvalue $\lambda_i(i=1, \cdots, N)$ of $M(\tau, \eta)$ in D_1 satisfies the inequality

$$C\mu \geqq |\operatorname{Re}\lambda_i| \geqq \frac{\mu}{C} \quad (i=1, \cdots, N) \tag{3.7}$$

where C is a positive constant. By the above facts and the integration of (3.6) from zero to infinity in x for fixed $(\tau, \eta) \in D_1$, we get

$$\frac{|\hat{V}(\tau, 0, \eta)|^2}{\sqrt{\mu^2+\eta^2}} \leqq C\left\{\int_0^\infty \left[|\hat{V}(\tau, x, \eta)|^2 + |\widehat{KV}|^2 + \frac{|\hat{\hat{U}}(\tau, x, 0)|^2}{\mu^2+\eta^2} + \frac{1}{\mu^2}|\hat{F}|^2\right]dx\right\} \tag{3.8}$$

where C is a positive constant.

Thirdly, we consider the problem (3.2) in $D_2 = \{(\tau, \eta) \mid |\tau| \leqq (2/\delta_0)|\eta|\}$. Then, we obtain

$$\begin{cases} \dfrac{|\eta|}{c_1} \leqq |\tau| + |\eta| \leqq C_1|\eta| \\ \dfrac{1}{\mu^2+\eta^2} \leqq \dfrac{C_2}{|\tau|^2} \end{cases} \tag{3.9}$$

where C_1 and C_2 are positive constants. Also, we have

$$\begin{aligned} &-\frac{d}{dx}\left(\frac{\hat{U}}{\sqrt{\mu^2+\eta^2}}\right) + \frac{M(\tau, \eta)}{\sqrt{\mu^2+\eta^2}}\hat{U} \\ &= \frac{A^{-1}\widehat{KU}}{\sqrt{\mu^2+\eta^2}} + \frac{A^{-1}\hat{F}}{\sqrt{\mu^2+\eta^2}} - \frac{A^{-1}B\hat{\hat{U}}}{\sqrt{\mu^2+\eta^2}} + \frac{A^{-1}\tilde{U}_0}{\sqrt{\mu^2+\eta^2}}. \end{aligned} \tag{3.10}$$

By (3.9), (3.10) and integration in x, we get

$$\begin{aligned} \frac{|\hat{U}(\tau, 0, \eta)|^2}{\sqrt{\mu^2+\eta^2}} \leqq C\Bigg\{\int_0^\infty \Bigg[&|\hat{U}|^2 + |\widehat{KU}|^2 \\ &+ \frac{|\hat{\hat{U}}(\tau, x, 0)|^2}{\mu^2+\eta^2} + \frac{|U_0(x, \eta)|^2}{|\tau|^2} + \frac{1}{\mu^2}|\hat{F}|^2\Bigg]dx\Bigg\} \end{aligned} \tag{3.11}$$

for fixed $(\tau, \eta) \in D_2$ where C is a positive constant. Using (3.4), (3.8), (3.11) and $U = V + W$, we have

Lemma 3.1. *Assume the condition* (C.6). *Let U be the solution of the problem* (3.1) *which belongs to* $\mathscr{H}_{1,\mu}[(R^3_+)^8]$.

Then, there exist positive constants C and μ_0 such that

$$\langle \Lambda_{\gamma,\mu}^{-(1/2)} U \rangle^2_{0,\mu,+} \leqq C\left\{\frac{1}{\mu}|||U_0|||^2_{0,\mu} + \frac{1}{\mu}\langle\!\langle U \rangle\!\rangle^2_{0,\mu,+} + \|U\|^2_{0,\mu,+} + \frac{1}{\mu^2}\|F\|^2_{0,\mu,+}\right\} \tag{3.12}$$

$$(\text{any} \quad \mu \geqq \mu_0).$$

THEOREM 3.2. *Assume the condition* (C.6). *Let* U *be the solution of the problem* (3.1) *which belongs to* $\mathscr{H}_{1,\mu}[(R^1_+)^3]$.

Then, there exist positive constants C *and* μ_0 *such that the following inequality holds for any* $t \in R^1_+$ *and any* $\mu \geqq \mu_0$,

$$(3.13)\quad |||U(t)|||^2_{0,\mu} + \mu\|U\|^2_{0,\mu,t} + \mu\langle \Lambda^{-(1/2)}_{y,\mu} U\rangle^2_{0,\mu,t} + \langle\!\langle U\rangle\!\rangle^2_{0,\mu,t} \leqq C\left\{|||U_0|||^2_{0,\mu} + \frac{1}{\mu}\|F\|^2_{0,\mu,t} + \frac{1}{\mu}\langle \Lambda^{1/2}_{y,\mu} G_1\rangle^2_{0,\mu,t} + \langle\!\langle G_2\rangle\!\rangle^2_{0,\mu,t}\right\}.$$

PROOF.

$$(3.14)\quad \frac{d}{dt}(e^{-\mu t}U(t), e^{-\mu t}U(t))$$
$$= -2\mu(e^{-\mu t}U, e^{-\mu t}U) + (e^{-\mu t}(AU_x + BU_y + KU + F), e^{-\mu t}U) + (e^{-\mu t}U, e^{-\mu t}(AU_x + BU_y + KU + F))$$
$$\leqq -C_1\mu(e^{-\mu t}U, e^{-\mu t}U) + \frac{C_2}{\mu}(e^{-\mu t}F, e^{-\mu t}F) - [Ae^{-\mu t}U, e^{-\mu t}U] - [[Be^{-\mu t}U, e^{-\mu t}U]]$$

where C_1 and C_2 are positive constants. By the same method in [7] and the condition (C.6), we have

$$(3.15)\quad [Ae^{-\mu t}U, e^{-\mu t}U] \geqq -\delta\mu[\Lambda^{-1/2}_{y,\mu}e^{-\mu t}U, \Lambda^{-1/2}_{y,\mu}e^{-\mu t}U], - \frac{C_1}{\mu}[\Lambda^{1/2}_{y,\mu}e^{-\mu t}G_1, \Lambda^{-1/2}_{y,\mu}e^{-\mu t}G_1],$$

and

$$(3.16)\quad [\![Be^{-\mu t}U, e^{-\mu t}U]\!] \geqq C_2[\![e^{-\mu t}U, e^{-\mu t}U]\!] - C_3[\![e^{-\mu t}G_2, e^{-\mu t}G_2]\!]$$

where δ is a sufficiently small positive constant. By (3.12), (3.14), (3.15), (3.16) and the property of hyperbolic equation, we get Theorem 3.2.

Q.E.D.

§ 4. Reduction to symmetric hyperbolic system and the energy inequality.

Firstly, we treat the case where problems (I), (II) and (III) have complex constant coefficients. We set $D = \{z \in C \mid |z| \leqq 1, \text{Re}\, z \leqq 0, z \neq \pm i\}$. We consider the following condition:

(C.7) The quadratic equation

$$(4.1)\quad (c+1)z^2 + 2bz + (c-1) = 0$$

has two different roots in D or has the double root in $\dot{D}$.

We remark that the condition (C.7) is equivalent to (C.2)-(C.4) under (C.1) and $b=b_1$, b_2, $c=c_1$, c_2, d_1 and d_2 are complex constants.

We choose z_1 and z_2 in the following ways:

Case (I). The equation (4.1) has two roots in $\dot{D}$.

We determine z_1 and z_2 the solution of the equation

$$\sqrt{1-\varepsilon^2}(c+1)z^2+2bz+\sqrt{1-\varepsilon^2}(c-1)=0 \tag{4.2}$$

where ε is a sufficiently small positive constant. Then, we have

$$\begin{cases} z_1,\ z_2 \in \dot{D}\ , \quad z_1 \neq z_2 \\ z_1+z_2=-\dfrac{2b}{\sqrt{1-\varepsilon^2}(c+1)}\ , \quad z_1z_2=\dfrac{c-1}{c+1}\ . \end{cases} \tag{4.3}$$

Case (II). The condition (C.7) holds and the Case (I) does not hold.

We determine z_1 and z_2 the solution of the equation (4.1). Then, we have

$$z_1 \neq z_2\ , \quad z_1+z_2=-\frac{2b}{c+1}\ , \quad z_1z_2=\frac{c-1}{c+1}\ . \tag{4.4}$$

Also, z_1 and z_2 satisfy any one of the following cases:

$$\begin{cases} (\mathrm{i}) & |z_1|=|z_2|=1 \\ (\mathrm{ii}) & |z_1|=1\ , \quad z_2 \in \dot{D} \\ (\mathrm{iii}) & \mathrm{Re}\, z_1=0\ , \quad z_2 \in \dot{D} \\ (\mathrm{iv}) & \mathrm{Re}\, z_1=\mathrm{Re}\, z_2=0 \\ (\mathrm{v}) & |z_1|=1\ , \quad \mathrm{Re}\, z_2=0\ . \end{cases} \tag{4.5}$$

LEMMA 4.1. *Assume conditions* (C.1)-(C.4). *Then, the problem* (I) *is transformed into the following problem:*

$$\begin{cases} \dfrac{\partial U}{\partial t}=A_1\dfrac{\partial U}{\partial x}+B_1\dfrac{\partial U}{\partial y}+(D_1+E_1)U+F(t, x, y) \\ U(0, x, y)=U_0(x, y) \\ P_1U|_{x=0}=G_1(t, y) \\ Q_1U|_{y=0}=G_2(t, x) \quad (t, x, y) \in (R_+^1)^3 \end{cases} \tag{4.6}$$

where A_1, B_1 and D_1 *are* 6×6 *constant matrices of the following forms*

$$A_1=\begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & \frac{1-|z_1|^2}{1+|z_1|^2} & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & \frac{1-|z_2|^2}{1+|z_2|^2} \end{pmatrix}$$

$$B_1=\begin{pmatrix} 0 & \sqrt{1-\varepsilon^2} & \frac{\varepsilon}{\sqrt{1+|z_1|^2}} & & & \\ \sqrt{1-\varepsilon^2} & 0 & \frac{\varepsilon z_1}{\sqrt{1+|z_1|^2}} & & 0 & \\ \frac{\varepsilon}{\sqrt{1+|z_1|^2}} & \frac{\varepsilon \bar{z}_1}{\sqrt{1+|z_1|^2}} & -\frac{2\sqrt{1-\varepsilon^2}}{1+|z_1|^2}\operatorname{Re} z_1 & & & \\ & & & 0 & \sqrt{1-\varepsilon^2} & \frac{\varepsilon}{\sqrt{1+|z_2|^2}} \\ & 0 & & \sqrt{1-\varepsilon^2} & 0 & \frac{\varepsilon z_2}{\sqrt{1+|z_2|^2}} \\ & & & \frac{\varepsilon}{\sqrt{1+|z_2|^2}} & \frac{\varepsilon \bar{z}_2}{\sqrt{1+|z_2|^2}} & -\frac{2\sqrt{1-\varepsilon^2}}{1+|z_2|^2}\operatorname{Re} z_2 \end{pmatrix}$$

$$D_1=\begin{pmatrix} 0 & -\beta & -\frac{m}{\sqrt{1+|z_1|^2}} & & & \\ \beta & 0 & -\frac{m z_1}{\sqrt{1+|z_1|^2}} & & 0 & \\ \frac{m}{\sqrt{1+|z_1|^2}} & \frac{m \bar{z}_1}{\sqrt{1+|z_1|^2}} & \frac{2\beta}{1+|z_1|^2}\operatorname{Im} z_1\cdot i & & & \\ & & & 0 & \beta & -\frac{m}{\sqrt{1+|z_2|^2}} \\ & 0 & & -\beta & 0 & -\frac{m z_2}{\sqrt{1+|z_2|^2}} \\ & & & \frac{m}{\sqrt{1+|z_2|^2}} & \frac{m \bar{z}_2}{\sqrt{1+|z_2|^2}} & -\frac{2\beta}{1+|z_2|^2}\operatorname{Im} z_2\cdot i \end{pmatrix}$$

E_1 is a 6×6 constant matrix,

$$P_1=Q_1=\begin{pmatrix}1 & z_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & z_1 & 0\end{pmatrix}$$

$$F={}^t(f,\ z_1f,\ 0,\ f,\ z_2f,\ 0)$$

$$G_1=\begin{pmatrix}-\dfrac{2}{c_1+1} & g_1\\ -\dfrac{2}{c_1+1} & g_1\end{pmatrix},\quad G_2=\begin{pmatrix}-\dfrac{2b_1}{c_1+1} & g_2\\ -\dfrac{2b_1}{c_1+1} & g_2\end{pmatrix}$$

and $\varepsilon=0$ *for the Case* (II).

PROOF. We choose z_1 and z_2 as roots of the quadratic equation

$$\sqrt{1-\varepsilon^2}(c_1+1)z^2+2b_1z+\sqrt{1-\varepsilon^2}(c_1-1)=0 \tag{4.7}$$

or

$$(c_1+1)z^2+2b_1z+(c_1-1)=0 \tag{4.8}$$

for the Case (I) or Case (II) respectively. We set

$$U=\begin{pmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ U_6\end{pmatrix}=\begin{pmatrix}u_t-u_x+z_1(\sqrt{1-\varepsilon^2}u_y-\beta u)\\ z_1(u_t+u_x)+\sqrt{1-\varepsilon^2}u_y+\beta u\\ \sqrt{1+|z_1|^2}(\varepsilon u_y+mu)\\ u_t-u_x+z_2(\sqrt{1-\varepsilon^2}u_y+\beta u)\\ z_2(u_t+u_x)+\sqrt{1-\varepsilon^2}u_y-\beta u\\ \sqrt{1+|z_2|^2}(\varepsilon u_y+mu)\end{pmatrix} \tag{4.9}$$

where m is a complex constant and

$$\beta=\frac{2d_1}{(c_1+1)(z_1-z_2)}\,. \tag{4.10}$$

Then, we get easily

$$\frac{\partial U}{\partial t}=A_1\frac{\partial U}{\partial x}+B_1\frac{\partial U}{\partial y}+(D_1+E_1)U+F$$

and by (4.3), (4.4), (4.7), (4.8), (4.9) and (4.10), we obtain

$$P_1U=Q_1U=\frac{2}{c_1+1}(c_1u_t-u_x-b_1u_y-d_1u)\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{4.11}$$

Therefore, we have Lemma 4.1. Q.E.D.

REMARK 4. The equation $u_{tt}=u_{xx}+u_{yy}-(m^2+\beta^2)u$ is transformed into the equation $U_t=A_1U_x+B_1U_y+D_1U$.

LEMMA 4.2. *Assume conditions* (C.1)-(C.4).

(i) *Suppose that the Case* (I) *holds.*

Then, we have

$$\begin{cases}((A_1U,\ U))\geqq C((U,\ U)) & \textit{for all}\quad U\in \operatorname{Ker} P_1\\ ((B_1U,\ U))\geqq C((U,\ U)) & \textit{for all}\quad U\in \operatorname{Ker} Q_1\end{cases}\tag{4.12}$$

where C *is a positive constant.*

(ii) *Suppose that the Case* (II) *holds.*

Then, we obtain

$$\begin{cases}((A_1U,\ U))\geqq 0 & \textit{for all}\quad U\in \operatorname{Ker} P_1\\ ((B_1U,\ U))\geqq C((U,\ U)) & \textit{for all}\quad U\in \operatorname{Ker} Q_1\end{cases}\tag{4.13}$$

for the Case (II)-(i) *and* (ii),

$$\begin{cases}((A_1U,\ U))\geqq C((U,\ U)) & \textit{for all}\quad U\in \operatorname{Ker} P_1\\ ((B_1U,\ U))\geqq 0 & \textit{for all}\quad U\in \operatorname{Ker} Q_1\end{cases}\tag{4.14}$$

for the Case (II)-(iii) *and* (iv),

$$\begin{cases}((A_1U,\ U))\geqq C((V,\ V)) & \textit{for all}\quad U\in \operatorname{Ker} P_1\\ ((B_1U,\ U))\geqq C((W,\ W)) & \textit{for all}\quad U\in \operatorname{Ker} Q_1\end{cases}\tag{4.15}$$

for the Case (II)-(v) *respectively where* $V={}^t(U_1,\ U_2,\ 0,\ 0,\ 0,\ 0)$, $W={}^t(0,\ 0,\ 0,\ U_4,\ U_5,\ 0)$ *and* C *is a positive constant.*

PROOF. By direct calculations, we can easily obtain Lemma 4.2.

LEMMA 4.3. *Let* $u\in\mathscr{H}_{2,\mu}[(R^1_+)^3]$. *Then, we have*

$$\begin{cases}|||\mu u(t)|||^2_{0,\mu}+C_1\mu||\mu u||^2_{0,\mu,t}\leqq|||u(0)|||^2_{1,\mu}+C_2\mu||u_t||^2_{0,\mu,t}\\ \langle\mu u\rangle^2_{0,\mu,t}\leqq C_3\mu(||\mu u||^2_{0,\mu,t}+||u_x||^2_{0,\mu,t})\\ \langle\!\langle\mu u\rangle\!\rangle^2_{0,\mu,t}\leqq C_3\mu(||\mu u||^2_{0,\mu,t}+||u_y||^2_{0,\mu,t})\end{cases}\tag{4.16}$$

where C_1, C_2 *and* C_3 *are positive constants.*

PROOF. See [1:§5].

Now, we treat the problem (II) with constant coefficients.

LEMMA 4.4. *Assume the condition* (C.5). *Then, the problem* (II) *is transformed into the following problem:*

$$(4.17)\qquad \begin{cases} \dfrac{\partial V}{\partial t}+A_2\dfrac{\partial V}{\partial x}+B_2\dfrac{\partial V}{\partial y}+D_2 V+H(t, x, y) \\ V(0, x, y)=V_0(x, y) \\ P_2 V|_{x=0}=0 \\ Q_2 V|_{y=0}=0 \\ \quad (t, x, y)\in(R^1_+)^3 \end{cases}$$

where A_2, B_2 *and* D_2 *are* 4×4 *constant matrices,*

$$A_2=\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 1 \end{pmatrix}$$

$$B_2=\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H={}^t(h, h, 0, 0)$$

$$P_2=\left(1, \frac{c_3-1}{c_3+1}, 0, 0\right), \quad Q_2=(0, 0, 1, 0)$$

and

$$(4.18)\qquad \begin{cases} ((A_2 V, V))\geqq 0 & \textit{for all}\quad V\in \operatorname{Ker} P_2 \\ ((B_2 V, V))\geqq 0 & \textit{for all}\quad V\in \operatorname{Ker} Q_2\,. \end{cases}$$

PROOF. We set

$$(4.19)\qquad V=\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}=\begin{pmatrix} v_t-(v_x+d_3 v) \\ v_t+(v_x+d_3 v) \\ \sqrt{2}(v_y+d_4 v) \\ v \end{pmatrix}.$$

Then, by direct calculations, we have Lemma 4.4. Q.E.D.

We consider the problem (III) with constant coefficients.

LEMMA 4.5. *The problem* (III) *is transformed into the following problem*:

$$(4.20)\quad \begin{cases} \dfrac{\partial W}{\partial t}=A_8\dfrac{\partial W}{\partial x}+B_8\dfrac{\partial W}{\partial y}+C_8\dfrac{\partial W}{\partial z}+D_8W+K(t,x,y,z) \\ W(0,x,y,z)=W_0(x,y,z) \\ P_8W|_{x=0}=0 \\ Q_8W|_{y=0}=0 \\ R_8W|_{z=0}=0 \\ \quad (t,x,y,z)\in(R^1_+)^4 \end{cases}$$

where A_8, B_8, C_8 *and* D_8 *are* 7×7 *constant matrices*

$$A_8=\begin{pmatrix} -1 & & & & & & \\ & 1 & & & & 0 & \\ & & 0 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & 0 & & & & 0 & \\ & & & & & & 1 \end{pmatrix}$$

$$B_8=\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & & & & \\ 0 & 0 & \frac{1}{\sqrt{2}} & & & 0 & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & & & & \\ & & & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ & & & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ & 0 & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ & & & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$C_3=\begin{pmatrix} 0 & -i & 0 & & & & \\ i & 0 & 0 & & & 0 & \\ 0 & 0 & 0 & & & & \\ & & & 0 & i & 0 & 0 \\ & & & -i & 0 & 0 & 0 \\ & 0 & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$K={}^t(k, k, 0, k, k, 0, 0)$$

$$P_3=\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_3=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R_3=\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$\begin{cases} ((A_3W, W))\geqq 0 & \text{for all} \quad W\in \operatorname{Ker} P_3 \\ ((B_3W, W))\geqq 0 & \text{for all} \quad W\in \operatorname{Ker} Q_3 \\ ((C_3W, W))\geqq 0 & \text{for all} \quad W\in \operatorname{Ker} R_3 . \end{cases} \tag{4.21}$$

PROOF. We set

$$W=\begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \\ W_7 \end{pmatrix}=\begin{pmatrix} w_t-(w_x+\alpha w)-i(w_z+\gamma w) \\ w_t+(w_x+\alpha w)+i(w_z+\gamma w) \\ \sqrt{2}(w_y+\beta w) \\ w_t-(w_x+\alpha w)+i(w_z+\gamma w) \\ w_t+(w_x+\alpha w)-i(w_z+\gamma w) \\ \sqrt{2}(w_y+\beta w) \\ w \end{pmatrix}. \tag{4.22}$$

Then, by direct calculations, we have Lemma 4.5. Q.E.D.

Secondly, we treat the case where problems (I), (II) and (III) have variable coefficients.

PROOF OF THEOREM 3. Each element of D_2 in (4.17) is smooth and has bounded derivatives. By the integration and Lemma 4.4, we get Theorem 3. Q.E.D.

PROOF OF THEOREM 5. By Lemma 4.5 and the same as the above method, we obtain Theorem 5. Q.E.D.

PROOF OF THEOREM 1. We consider a partition of unity on $(\bar{R}^1_+)^3$ which is locally finite sum. It is sufficient to consider a neighborhood of $(t, 0, 0)$, because on other components of a partition of unity, we use results for Cauchy problem or the mixed problem in the half space to obtain the energy inequality. Therefore, from now on, we treat the problem (I) where (t, x, y) belongs to a neighborhood of $(t_0, 0, 0)$ for fixed $t_0(t_0 \geqq 0)$. We choose $\rho(z)$ as

$$\rho(z)=\begin{cases}1 & |z|\leqq \varepsilon_1\\ 0 & |z|\geqq 2\varepsilon_1\end{cases}$$

and $\rho(z)\in C^\infty$ where ε_1 is a positive constant. We set

$$\sigma(t, x, y)=\rho(4(t-t_0))\cdot\rho(4x)\cdot\rho(4y)$$

and

$$w(t, x, y)=\sigma(t, x, y)\cdot u(t, x, y)\ .$$

Then, we have for the problem (I)

$$\begin{cases}L[w]=\sigma\cdot f+[L, \sigma]u\\ w(0, x, y)=w_0(x, y),\ w_t(0, x, y)=w_1(x, y)\\ B_1[w]|_{x=0}=\sigma\cdot g_1+[B_1, \sigma]u|_{x=0}\\ B_2[w]|_{y=0}=\sigma\cdot g_2+[B_2, \sigma]u|_{y=0}\\ (t, x, y)\in(R^1_+)^3\ .\end{cases}\tag{4.23}$$

By conditions (C.1)–(C.4), we get

$$b_1(t, 0)\neq 0\ ,\quad b_2(t, 0)\neq 0\ ,\quad c_1(t, 0)\neq 0$$

and

$$d_1(t, 0)\neq 0 \quad\text{or}\quad d_1(t, 0)\equiv 0\ .$$

Then, we set

$$\begin{cases}b(t, x, y)=\dfrac{b_1(t, y)}{b_1(t, 0)b_2(t, x)}\\ c(t, x, y)=\dfrac{c_1(t, y)c_2(t, x)}{c_1(t, 0)b_2(t, x)}\\ d(t, x, y)=\dfrac{d_1(t, y)d_2(t, x)}{d_1(t, 0)b_2(t, x)} \quad\text{or}\quad \equiv 0\end{cases}\tag{4.24}$$

where (t, x, y) belongs to a neighborhood $U=\{(t, x, y)\,|\,|t-t_0|<3\varepsilon_1, |x|<3\varepsilon_1, |y|<3\varepsilon_1\}$. We have

$$
(4.25)\qquad \begin{cases} b(t, 0, y)=b_1(t, y) & b(t, x, 0)=\dfrac{1}{b_2(t, x)} \\ c(t, 0, y)=c_1(t, y) & c(t, x, 0)=\dfrac{c_2(t, x)}{b_2(t, x)} \\ d(t, 0, y)=d_1(t, y) & d(t, x, 0)=\dfrac{d_2(t, x)}{b_2(t, x)}\,. \end{cases}
$$

We choose $z_1(t, x, y)$ and $z_2(t, x, y)$ in following ways:
Case (I). The equation

$$(c(t_0, 0, 0)+1)z^2+2b(t_0, 0, 0)z+c(t_0, 0, 0)-1=0$$

has two roots in the interior of $D=\{z\in C\,|\,|z|\leqq 1, \operatorname{Re} z\leqq 0, z\neq \pm i)$.

Then, we determine $z_1(t, x, y)$ and $z_2(t, x, y)$ the solutions of the equation

$$(4.26)\qquad \sqrt{1-\varepsilon^2}(c(t, x, y)+1)z^2+2b(t, x, y)z+\sqrt{1-\varepsilon^2}(c(t, x, y)-1)=0$$

where ε is a sufficiently small positive constant. We have (4.3) for smooth functions $z_1(t, x, y)$ and $z_2(t, x, y)$ where $(t, x, y)\in U$.
Case (II). Case (I) does not hold.

Then, we determine $z_1(t, x, y)$ and $z_2(t, x, y)$ the solutions of the equation

$$(4.27)\qquad (c(t, x, y)+1)z^2+2b(t, x, y)z+c(t, x, y)-1=0\,.$$

We obtain (4.4) for smooth functions $z_1(t, x, y)$ and $z_2(t, x, y)$ where $(t, x, y)\in U$.

We can extend $z_1(t, x, y)$ and $z_2(t, x, y)$ to the region $\{(t, x, y)|t\geqq 0, (x, y)\in R^2\}$ which have the following property:

(1) They are constant outside a compact set in $\bar{R}^1_+\times R^2$.

(2) They are in $\mathring{D}=\{z\in C\,|\,|z|<1, \operatorname{Re} z<0\}$ for any

$$(t, x, y)\in\{(t, x, y)\,|\,|t-t_0|\geqq 2\varepsilon_1, |x|\geqq 2\varepsilon_1, |y|\geqq 2\varepsilon_1\}\,.$$

We set

$$(5.28)\qquad \beta=\frac{2d(t, x, y)\rho(2(t-t_0))\cdot\rho(2x)\cdot\rho(2y)}{[c(t, x, y)+1](z_1(t, x, y)-z_2(t, x, y))}\,.$$

Using the results in §3, Lemma 4.1, 4.2, 4.3, the fact that the principal part of the equation (4.6) for the Case (II) has a special form ($\varepsilon=0$), a partition of unity, (4.2) and the theory of pseudo-differential operators [2] and [3], we get Theorem 1. Q.E.D.

§ 5. The existence of the solution I.

In this chapter, we shall prove Theorem 2 and 4.

Firstly, we treat Theorem 2. We set

$$u=we^{(pt-qx-ry)/2} \tag{5.1}$$

where p, q and r are complex constant coefficients of the operator L in (I).

LEMMA 5.1. *The problem* (I) *is transformed into the following*

$$\begin{cases} L_1[w]=\dfrac{\partial^2 w}{\partial t^2}-\dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial^2 w}{\partial y^2}+aw=e^{-(pt-qx-ry)/2}f=f_1 \\ w(0, x, y)=e^{(qx+ry)/2}u_0(x, y)=W_0 \\ w_t(0, x, y)=e^{(qx+ry)/2}\left[u_1(x, y)-\dfrac{1}{2}p\cdot u_0(x, y)\right]=W_1 \\ B_1'[w]\Big|_{x=0}=\dfrac{\partial w}{\partial x}+b_1\dfrac{\partial w}{\partial y}-c_1\dfrac{\partial w}{\partial t}+ew\Big|_{x=0}=e^{-(pt-ry)/2}g_1=k_1 \\ B_2'[w]\Big|_{y=0}=\dfrac{\partial w}{\partial y}+\dfrac{1}{b_1}\dfrac{\partial w}{\partial x}-\dfrac{c_1}{b_1}\dfrac{\partial w}{\partial t}+\dfrac{e}{b_1}w\Big|_{y=0}=e^{-(pt-qx)/2}g_2=k_2 \\ \quad (t, x, y)\in(R^1_+)^3 \end{cases} \tag{5.2}$$

where $a=-p^2/4+q^2/4+r^2/4-s$ *and* $e=d_1-q/2-(b_1/2)r-(c_1/2)p$.

PROOF. By (5.1) and direct calculations, we can get Lemma 5.1. Q.E.D.

LEMMA 5.2. *Let* $p, q, r, s, b_1, b_2, c_1, c_2, d_1$ *and* d_2 *be complex constants and assume conditions* (C.1)-(C.4). *Let* w *be the solution of the problem* (5.2) *which belongs to* $\mathscr{H}_{6,\mu}[(R^1_+)^3]$.

Then, there exist positive constants C *and* μ_0 *such that the following inequality holds for any* $t\in R^1_+$ *and any* $\mu\geqq\mu_0$

$$|||w(t)|||^2_{5,\mu}+\mu||w||^2_{5,\mu,t} \tag{5.3}$$
$$+\mu\sum_{k=0}^{5}\left\{\left\langle \Lambda_{y,\mu}^{-1/2}\left(\frac{\partial}{\partial x}\right)^k w\right\rangle^2_{5-k,\mu,t}+\left\langle\left\langle \Lambda_{x,\mu}^{-1/2}\left(\frac{\partial}{\partial y}\right)^k w\right\rangle\right\rangle^2_{5-k,\mu,t}\right\}$$

$$\leqq C\Big\{|||w(0)|||^2_{5,\mu}+\frac{1}{\mu}\|f_1\|^2_{4,\mu,t}$$
$$+\frac{1}{\mu}\langle\Lambda^{1/2}_{y,\mu}k_1\rangle^2_{4,\mu,t}+\frac{1}{\mu}\langle\!\langle\Lambda^{1/2}_{x,\mu}k_2\rangle\!\rangle^2_{4,\mu,t}\Big\}\,.$$

PROOF. By the same method in [1: §5], we have Lemma 5.2.

Q.E.D.

PROOF OF THEOREM 2. By Lemma 5.1, we have only to prove Theorem 2 for the problem (5.2) under the condition (α) without loss of generality. We solve the problem

$$\begin{cases} L_1[w_1]=\tilde{f}_1 \\ w_1(0,x,y)=\tilde{W}_0\,, \quad w_{1t}(0,x,y)=\tilde{W}_1 \\ B'_1[w]=\tilde{k}_1 \\ (t,x,y)\in(R^1_+)^2\times R^1 \end{cases} \tag{5.4}$$

where $\tilde{l}$ is a extended function in the domain $\{(t,x,y)|t\geqq 0, x\geqq 0, y<0\}$ or $\{(x,y)|x\geqq 0, y<0\}$ by the assumption. Then, we have the solution w_1 of the problem (5.4) which belongs to $\mathscr{H}_{7,\mu}[(R^1_+)^2\times R^1]$. We set

$$k_2^*=e^{-(pt-qx)/2}g_2(t,x)-B_2[w_1]|_{y=0}\,. \tag{5.5}$$

Then, by the assumption, we have a extended function $\tilde{k}_2$ of k_2^* in the domain $\{(t,x)|t\geqq 0, x<0\}$ which is an odd function in x and belongs to $\mathscr{H}_{5,\mu}[(R^1_+)\times R^1]$. Let $\tilde{k}_{2\theta}$ be

$$\tilde{k}_{2\theta}(t,x)=(\rho_\theta*\tilde{k}_2)(t,x)$$

where

$$\rho(x)=\begin{cases} C\exp\left[-\dfrac{1}{1-|x|^2}\right] & (|x|<1) \\ 0 & (|x|\geqq 1) \end{cases}$$
$$\int_{-\infty}^{\infty}\rho(x)dx=1$$

C is a positive constant and $\rho_\theta(x)=(1/\theta)\rho(x/\theta)$.

We consider the problem

$$(5.6)\quad \begin{cases} L_1[w_{2\theta}]=0 \\ w_{2\theta}(0,x,y)=0\,, \quad w_{2\theta t}(0,x,y)=0 \\ B_2'[w_{2\theta}]|_{y=0}=\tilde{k}_{2\theta} \\ \quad (t,x,y)\in \boldsymbol{R}_+^1\times\boldsymbol{R}^1\times\boldsymbol{R}_+^1\,. \end{cases}$$

Then, we obtain the solution $w_{2\theta}$ of the problem (5.6) which belongs to $\mathscr{H}_{6,\mu}[\boldsymbol{R}_+^1\times\boldsymbol{R}^1\times\boldsymbol{R}_+^1]$ and $B_1'[w_{2\theta}]|_{x=0}=0$. Using Lemma 5.2 and $w=w_1+w_{2\theta}$, we have Theorem 2. Q.E.D.

Secondly, we treat Theorem 4. We set

$$(5.7)\quad v=e^{(pt-qx-ry)/2}w$$

where p, q and r are complex constant coefficients of the operator L in (II).

LEMMA 5.3. *The problem* (II) *is transformed into the following*

$$(5.8)\quad \begin{cases} L_1[w]=h_1=e^{-(pt-qx-ry)/2}h \\ w(0,x,y)=V_0=e^{(qx+ry)/2}v_0 \\ w_t(0,x,y)=V_1=e^{(qx+ry)/2}\left[v_1-\dfrac{1}{2}pv_0\right] \\ B_3'[w]\Big|_{x=0}=\dfrac{\partial w}{\partial x}-c_3\dfrac{\partial w}{\partial t}+\left(d_3-\dfrac{1}{2}q-\dfrac{c_3}{2}p\right)w\Big|_{x=0}=0 \\ B_4'[w]\Big|_{y=0}=\dfrac{\partial w}{\partial y}+\left(d_4-\dfrac{1}{2}r\right)w\Big|_{x=0}=0 \\ \quad (t,x,y)\in(\boldsymbol{R}_+^1)^3. \end{cases}$$

PROOF. By (5.7) and direct calculations, we can get Lemma 5.3. Q.E.D.

PROOF OF THEOREM 4. By the condition, we extend h_1, V_0 and V_1 to the regions $\{(t,x,y)|t\geqq 0,\ x<0,\ y\geqq 0\}$ and $\{(x,y)|x<0,\ y\geqq 0\}$ respectively by followings

$$\tilde{h}_1=\begin{cases} h(t,x,y) & (x\geqq 0) \\ h(t,-x,y) & (x<0) \end{cases}$$

$$\tilde{V}_0=\begin{cases} V_0(x,y) & (x\geqq 0) \\ V_0(-x,y) & (x<0) \end{cases}$$

and

$$\tilde{V}_1=\begin{cases} V_1(x,y) & (x\geqq 0) \\ V_1(-x,y) & (x<0)\,. \end{cases}$$

Now, we consider the problem

$$(5.9)\quad \begin{cases} L_1[w_3]=\tilde{h}_1 \\ w_3(0, x, y)=\tilde{V}_0\,, \quad w_{3t}(0, x, y)=\tilde{V}_1 \\ B_4'[w_3]|_{y=0}=\dfrac{\partial w_3}{\partial y}+\beta w_3\Big|_{y=0}=0 \\ \quad (t, x, y)\in R^1_+\times R^1\times R^1_+ \end{cases}$$

where $\beta=d_4-(1/2)r$. By the assumption, we have the solution $w_3\in C^\infty[(\bar{R}^1_+)\times R^1\times(\bar{R}^1_+)]$ of the problem (5.9) and w_3 has a compact support in the domain $R^1_x\times R^1_{+,y}$ for fixed $t(\geqq 0)$. And we obtain $w_3(t, x, y)=w_3(t, -x, y)$. Therefore, we have

$$(5.10)\qquad w_{3x}(t, 0, y)=0\,.$$

Next, we set

$$(5.11)\qquad \frac{\partial w_3}{\partial x}-c_3\frac{\partial w_3}{\partial t}+\alpha w_3\Big|_{x=0}=m(t, y)$$

where $\alpha=d_3-(1/2)q-(c_3/2)p$ and we define

$$(5.12)\qquad n(t, y)=-\beta\cdot m(t, y)-m_y(t, y)\,.$$

Then, by $w_{3x}(t, 0, y)$, we get

$$(5.13)\qquad \begin{cases} m(t, 0)=-c_3\cdot w_{3t}(t, 0, 0)-\alpha w_3(t, 0, 0) \\ m_y(t, 0)=\left[-c_3\dfrac{\partial}{\partial t}+\alpha\right]w_{3y}(t, 0, 0)\,. \end{cases}$$

Therefore, we have

$$\begin{aligned}(5.14)\qquad n(t, 0)&=-\beta[-c_3w_{3t}(t, 0, 0)+\alpha w_3(t, 0, 0)]\\ &\quad+\left[c_3\frac{\partial}{\partial t}-\alpha\right]w_{3y}(t, 0, 0)\\ &=c_3\frac{\partial}{\partial t}[w_{3y}(t, 0, 0)+\beta w_3(t, 0, 0)]\\ &\quad-\alpha[w_{3y}(t, 0, 0)+\beta w_3(t, 0, 0)]=0\end{aligned}$$

and

$$(5.15)\qquad n_{yy}(t, 0)=-\beta\left(\alpha-c_3\frac{\partial}{\partial t}\right)w_{3yy}-\left(\alpha-c_3\frac{\partial}{\partial t}\right)w_{3yyy}$$

$$=-\beta\left(\alpha-c_3\frac{\partial}{\partial t}\right)(w_{3tt}-w_{3xx}+aw_3)$$

$$-\left(\alpha-c_3\frac{\partial}{\partial t}\right)(w_{3ytt}-w_{3yxx}+aw_{3y})$$

$$=\left(c_3\frac{\partial}{\partial t}-\alpha\right)\left[\left(\frac{\partial}{\partial t}\right)^2-\left(\frac{\partial}{\partial x}\right)^2+a\right]\cdot(w_{3y}+\beta w_3)(t, x, 0)|_{x=0}=0$$

where $w_{3yy}|_{x=0(y=0)}=w_{3tt}-w_{3xx}+aw_3|_{x=0(y=0)}$ and $w_{3yyy}|_{x=0(y=0)}=w_{3ytt}-w_{3yxx}+aw_{3y}|_{x=0(y=0)}$. By the same method, we can get

$$n_{yyyy}(t, 0)=0 . \tag{5.16}$$

We extend $n(t, y)$ to the region $\{(t, y)|t\geqq 0, y<0\}$ by the following

$$\tilde{n}(t, y)=\begin{cases} n(t, y) & (y\geqq 0) \\ -n(t, -y) & (y<0) . \end{cases}$$

Then, we have $\tilde{n}\in C^5_{t,y}$ and $\tilde{n}$ has a compact support in R^1_y for fixed $t(\geqq 0)$. And we consider the problem

$$\begin{cases} L_1[w_4]=0 \\ w_4(0, x, y)=0 , \quad w_{4t}(0, x, y)=0 \\ B'_3[w_4]\Big|_{x=0}=\dfrac{\partial w_4}{\partial x}-c_3\dfrac{\partial w_4}{\partial t}+\alpha w_4\Big|_{x=0}=\tilde{n} \\ \quad (t, x, y)\in(R^1_+)^2\times R^1 . \end{cases} \tag{5.17}$$

Then, we have the solution w_4 of the problem (5.17) which belongs to $\mathscr{H}_{5,\mu}[(R^1_+)^3]$ and has a compact support in the region $R^1_{+x}\times R^1_y$ for fixed $t(\geqq 0)$. Also, we have $w_4(t, x, 0)=0$.

We solve the equation

$$\frac{\partial w_5}{\partial y}+\beta w_5=w_4 \tag{5.18}$$

for $L^2(R^1_{+y})$ space. Then, we have the solution

$$w_5=e^{-\beta y}\int_\infty^y e^{\beta s}w(t, x, s)ds . \tag{5.19}$$

We set

$$w=w_3+w_5 .$$

By the above construction, we obtain the solution of the problem (5.8) which satisfies Theorem 4. Q.E.D.

§6. The existence of the solution II.

In this chapter, we shall prove Theorem 6.
We set

$$w=v\cdot e^{(p_0t-p_1x-p_2y-p_3z)/2} \tag{6.1}$$

where p_0, p_1, p_2 and p_3 are constant coefficients of the operator M in (III).

Lemma 6.1. *The problem* (III) *is transformed into the following*

$$\begin{cases} M_1[v]=\dfrac{\partial^2 v}{\partial t^2}-\dfrac{\partial^2 v}{\partial x^2}-\dfrac{\partial^2 v}{\partial y^2}-\dfrac{\partial^2 v}{\partial z^2}+av=k_1(t, x, y, z) \\ v(0, x, y, z)=W_0,\ v_t(0, x, y, z)=W_1 \\ B_5'[v]\Big|_{x=0}=\dfrac{\partial v}{\partial x}+\alpha_1 v\Big|_{x=0}=0 \\ B_6'[v]\Big|_{y=0}=\dfrac{\partial v}{\partial y}+\beta_1 v\Big|_{y=0}=0 \\ B_7'[v]\Big|_{z=0}=\dfrac{\partial v}{\partial z}+\gamma_1 v\Big|_{z=0}=0 \\ (t, x, y, z)\in (R_+^1)^4 \end{cases} \tag{6.2}$$

where

$$a=-\frac{1}{4}p_0^2+\frac{1}{4}p_1^2+\frac{1}{4}p_2^2+\frac{1}{4}p_3^2-p_4\,,\quad k_1=e^{-(p_0t-p_1x-p_2y-p_3z)/2}k$$

$$W_0=e^{(p_1x+p_2y+p_3z)/2}w_0\,,\quad W_1=e^{(p_1x+p_2y+p_3z)/2}\left[w_1-\frac{1}{2}p_0w_0\right]$$

$$\alpha_1=\alpha-\frac{1}{2}p_1\,,\quad \beta_1=\beta-\frac{1}{2}p_2$$

and

$$\gamma_1=\gamma-\frac{1}{2}p_3\,.$$

Proof. By (6.1) and direct calculations, we can obtain Lemma 6.1. Q.E.D.

We consider the mixed problem

$$(6.3)\qquad \begin{cases} M_1[u]=\tilde{k}_1 \\ \quad u(0,x,y,z)=\tilde{W}_0\,, \quad u_t(0,x,y,z)=\tilde{W}_1 \\ B_5'[u]|_{x=0}=0 \\ B_6'[u]|_{y=0}=0 \\ \quad (t,x,y,z)\in(R_+^1)^3\times R^1\,. \end{cases}$$

where

$$\tilde{k}_1(t,x,y,z)=\begin{cases} k_1(t,x,y,z) & (z\geqq 0) \\ k_1(t,x,y,-z) & (z<0) \end{cases}$$

$$\tilde{W}_0(x,y,z)=\begin{cases} W_0(x,y,z) & (z\geqq 0) \\ W_0(x,y,-z) & (z<0) \end{cases}$$

and

$$\tilde{W}_1(x,y,z)=\begin{cases} W_1(x,y,z) & (z\geqq 0) \\ W_1(x,y,-z) & (z<0)\,. \end{cases}$$

THEOREM 6.2. *Let* (f, w_0, w_1) *be* $C_0^\infty[(R_+^1)^4]\times[C_0^\infty[(R_+^1)^3]]^2$.

Then, we have the smooth solution u *of the problem* (6.3) *which satisfies*

$$u(t,x,y,z)=u(t,x,y,-z)$$

and has a compact support in (x, y, z) *for fixed* $t(\geqq 0)$. *Also,* $u(t, x, y, z)$ *is zero in a neighborhood of* $(0, x, y, 0)$.

PROOF. By the same method in Proof of Theorem 4 in §5 and properties of dates, we have Theorem 6.2. Q.E.D.

Let u be the solution of the problem (6.3). We set

$$(6.4)\qquad u_z+\gamma_1 u|_{z=0}=m(t,x,y)$$

and define

$$(6.5)\qquad n(t,x,y)=-\Big(\frac{\partial}{\partial y}+\alpha_1\Big)\Big(\frac{\partial}{\partial y}+\beta_1\Big)m(t,x,y)\,.$$

By $u_z(t, x, y, 0)=0$, we get

$$(6.6)\qquad n(t,x,y)=-\gamma_1\Big(\frac{\partial}{\partial x}+\alpha_1\Big)\Big(\frac{\partial}{\partial y}+\beta_1\Big)u(t,x,y,0)\,.$$

Then, we have

(6.7)
$$n(t, 0, y) = -\gamma_1\Big(\frac{\partial}{\partial y}+\beta_1\Big)\cdot\Big[\Big(\frac{\partial}{\partial x}+\alpha_1\Big)u\Big]\Big|_{x=0} = 0$$

(6.8)
$$\begin{aligned} n_{xx}(t, 0, y) &= -\gamma_1\Big(\frac{\partial}{\partial y}+\beta_1\Big)\Big(\frac{\partial}{\partial x}+\alpha_1\Big)u_{xx} \\ &= -\gamma_1\Big(\frac{\partial}{\partial y}+\beta_1\Big)\Big(\frac{\partial}{\partial x}+\alpha_1\Big)(u_{tt}-u_{yy}-u_{zz}+au) \\ &= -\gamma_1\Big(\frac{\partial}{\partial y}+\beta_1\Big)\Big[\Big(\frac{\partial}{\partial t}\Big)^2-\Big(\frac{\partial}{\partial y}\Big)^2-\Big(\frac{\partial}{\partial z}\Big)^2+a\Big] \\ &\quad \cdot\Big\{\Big(\frac{\partial}{\partial x}+\alpha_1\Big)u\Big|_{x=0}\Big\}\Big|_{z=0} = 0 \end{aligned}$$

and by the same method

(6.9)
$$n_{xxxx}(t, 0, y) = 0 .$$

Also, we obtain

(6.10)
$$n(t, x, 0) = n_{yy}(t, x, 0) = n_{yyyy}(t, x, 0) = 0 .$$

Now, we treat the mixed problem

(6.11)
$$\begin{cases} M_1[v] = 0 \\ \quad v(0, x, y, z) = 0 , \quad v_t(0, x, y, z) = 0 \\ B_7'[v]|_{z=0} = \tilde{n}(t, x, y) \\ \quad (t, x, y) \in \boldsymbol{R}^1_+ \times (\boldsymbol{R}^1)^2 \times \boldsymbol{R}^1_+ \end{cases}$$

where

(6.12)
$$\tilde{n}(t, x, y) = \begin{cases} n(t, x, y) & (x \geqq 0, y \geqq 0) \\ -n(t, -x, y) & (x < 0, y \geqq 0) \\ -n(t, x, -y) & (x \geqq 0, y < 0) \\ n(t, -x, -y) & (x < 0, y < 0) . \end{cases}$$

Then, we have the solution $v \in \mathscr{H}_{5,\mu}[\boldsymbol{R}^1_+ \times (\boldsymbol{R}^1)^2 \times \boldsymbol{R}^1_+]$ of the problem (6.11) which satisfies

(6.13)
$$v(t, 0, y, z) = v(t, x, 0, z) = 0$$

and has a compact support in (x, y, z) for fixed $t(\geqq 0)$.

We solve the equation

(6.14)
$$\Big(\frac{\partial}{\partial x}+\alpha_1\Big)\Big(\frac{\partial}{\partial y}+\beta_1\Big)w_1 = v$$

for $L^2(\boldsymbol{R}^1_{+x}\times\boldsymbol{R}^1_{+y})$ space. Then, we get the solution

$$w_1=e^{-\alpha_1 x-\beta_1 y}\int_\infty^x\int_\infty^y e^{\alpha_1 r+\beta_1 s}v(t, r, s, z)drds\ . \tag{6.15}$$

The function w_1 satisfies

$$\begin{cases} w_1(0, x, y, z)=w_{1t}(0, x, y, z)=0 \\ \left(\dfrac{\partial}{\partial x}+\alpha_1\right)w_1\Big|_{x=0}=0 \\ \left(\dfrac{\partial}{\partial y}+\beta_1\right)w_1\Big|_{y=0}=0 \\ \left(\dfrac{\partial}{\partial z}+\gamma_1\right)w_1\Big|_{z=0}=-\gamma u(t, x, y, 0) \end{cases} \tag{6.16}$$

and

$$M_1[w_1]=0\ . \tag{6.17}$$

We set $w=u+w_1$. By the above construction, we obtain the solution of the problem (III). Q.E.D.

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