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A Topologically Mixing 2 Dimensional Infinite Particles System

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Introduction

Little are known about the ergodic properties of infinite systems of particles in spaces of dimension$\geqq 2$ except those of ideal gasses, in spite of their importance in statistical mechanics [1, 2].

In the case of one dimension, some ergodic properties are known for the systems of hard rods [3, 4, 5], also the orders of decay of the time correlation functions are known for a system [5].

We show in this note two dimensional infinite system introduced by Hardy et al in [7] is topologically mixing. Our argument depends essentially only on the dissipative character and the time reversibiliy combined with infiniteness of the system. So similar arguments may work also for other systems which have similar nature, in particular for the continuous systems with hard core.

§1. Description of the system $(\mathfrak{X}, T)$.

For the completeness we give the definition of the system we consider. Let Z^2 be 2 dimensional integral lattice. On each lattice site there are at most 4 particles, the velocity of a particle is one of the 4 unit vectors (1, 0), (0, 1), (−1, 0) and (0, −1). The configurations where there are at least 2 particles with the same velocity on the same lattice site are excluded.

More precisely, the phase space $\mathfrak{X}$ of allowed configurations of particles is

$$\mathfrak{X}=\{X \mid X: Z^2\times P \longrightarrow \{0, 1\}\} ,$$

where

$$P=\{v=(v^1, v^2)\in Z^2 \mid |v^1|+|v^2|=1\}$$

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Naturally, we have

$$\mathfrak{X}=\prod_{a\in Z^2}\mathfrak{X}_a$$

where

$$\mathfrak{X}_a=\{X_a\,|\,X_a\colon \{a\}\times P\cong P\longrightarrow\{0, 1\}\}$$

is the space of the configurations on the lattice site $a\in Z^2$. (The element X_a is naturally identified with the subset $\{v\in P\,|\,X_a(v)=1\}$ of P.) $\mathfrak{X}$ is a topological space with product topology.

The time evolution map T of the system is defined as follows. T is made of the free motion T_0 and the collision C, $T=T_0CT_0$. T_0 is merely a translation:

$$(T_0X)(a, v)=X(a-v, v)\,, \qquad (a, v)\in Z^2\times P\,.$$

Let us define the map $\varphi_a\colon \mathfrak{X}_a\to\mathfrak{X}_a$ by

$$\varphi_a(X_a)=\begin{cases} X_1\colon & X_a=X_0 \\ X_0\colon & X_a=X_1 \\ X_a\colon & \text{otherwise}\,, \end{cases}$$

where $X_0=\{(1, 0), (-1, 0)\}$, $X_1=\{(0, 1), (0, -1)\}$. The collision C is defined by

$$(CX)_a=\varphi_aX_a\,, \qquad a\in Z^2\,.$$

§2. Basic properties of the system ($\mathfrak{X}$, T).

T is dissipative in the following sence [6]. For any bounded subset K of Z^2, there exist a bounded subset $K_1\supset K$ and a positive integer n such that if $X(a, v)=0$ for $\forall a\in K_1-K$, then $(T^nX)\ (a, v)=0$ for $\forall a\in K$.

Next, let i be the direction reversing map of $\mathfrak{X}$, that is,

$$(iX)_a=-X_a\,, \qquad a\in Z^2$$

where $-X_a=\{-v_1, \cdots, -v_k\}\subset P$, when $X_a=\{v_1, \cdots, v_k\}\subset P$. T is time reversible, that is,

$$iT=T^{-1}i\,.$$

These properties can be easily checked by the definition of T.

§3. $(\mathfrak{X}, T)$ is topologically mixing.

Let us introduce some notations which will be used latter. For any positive integers $k_2 > k_1 > k > 0$,

$$Q(k) = [-k, k] \times [-k, k] \subset Z^2$$
$$Q^+(k; k_1, k_2) = [-k, k] \times [k_1, k_2]$$
$$Q^-(k; k_1, k_2) = [-k, k] \times [-k_2, -k_1]$$
$$Q_+(k; k_1, k_2) = [k_1, k_2] \times [-k, k]$$
$$Q_-(k; k_1, k_2) = [-k_2, -k_1] \times [-k, k]$$
$$\bar{Q}(k; k_1, k_2) = Q^+ \cup Q^- \cup Q_+ \cup Q_- ,$$

where $[a, b] = \{n \in Z; a \leqq n \leqq b\}$.

From dissipative property of the system, we can easily show that

(i) for any $k > 0$ and large $n > 0$, there exist positive integers $k_3 > k_2 > k_1 (> k)$ satisfying the following condition

$$\begin{aligned} &\text{if} \quad X_a = 0 \quad (\text{i.e. } X(a, v) = 0, \ \forall v \in P) \quad \text{for} \quad \forall a \in Q(k_3) - Q(k) \\ &\text{then} \quad (T^n X)_a = 0 \quad \text{for} \quad \forall a \in Q(k_2) - \bar{Q}(k; k_1, k_2) , \end{aligned}$$

and that all of the velocities of the particles of $T^n X$ on a $Q^+(k; k_1, k_2)$ (Q^-, Q_+, Q_-, respectively) are $(0, 1)$ ($(0, -1)$, $(1, 0)$ $(-1, 0)$, respectively).

Combining this property and the time reversibility of T, we can easily show that:

(ii) for any configuration $\{x_a\}_{a \in Q(k)}$ ($x_a \in \mathfrak{X}_a$) on $Q(k)$, there exist $n > 0$, $k_2 > k_1 (> k)$ and a configuration $\{x'_a\}_{a \in \bar{Q}(k; k_1, k_2)}$ such that

$$\text{if} \quad X_a = \begin{cases} x'_a & a \in \bar{Q}(k; k_1, k_2) \\ 0 & a \in Q(k_2) - \bar{Q}(k; k_1, k_2) \end{cases} \quad \text{then} \quad (T^n X)_a = x_a \quad \text{for} \quad a \in Q(k) .$$

Note that all of the velocities of the particles of X on $a \in Q^+(k; k_1, k_2)$ (Q^-, Q_+, Q_- respectively) are $(0, -1)$ ($(0, 1)$, $(-1, 0)$, $(1, 0)$ respectively), and k_1 can be taken arbitrarily large.

From (i) and the property of the collision C, it is easy to see that

(iii) for any $k > 0$ and for any configuration $\{x_a\}_{a \in Q(k)}$ on $Q(k)$, there exist $k_4 > k_3 > k$, and a configuration $\{x'_a\}_{a \in Q(k_4) - Q(k)}$ on $Q(k_4) - Q(k)$ such that

$$\begin{aligned} &\text{if} \quad X_a = \begin{cases} x'_a, & a \in Q(k_4) - Q(k) \\ x_a, & a \in Q(k) \end{cases} \\ &\text{then} \quad (T^m X)_a = 0 \quad \text{for} \quad \forall a \in \bar{Q}(k; k, k_3) \cup Q(k) \quad \text{for some} \quad m > 0 , \end{aligned}$$

As a matter of fact, the configuration $\{X'_a\}$ has following properties;

$$x'_a=0 \quad \text{for} \quad \forall a \in \bar{Q}(k; k, k_4)$$

and the velocities of the particles on $Q^+(k; k, k_4)$ (Q^-, Q_+, Q_- resp.) are $(0, -1)$ ($(0, 1)$, $(-1, 0)$, $(1, 0)$ resp.). For instance, the particles of $\{x_a\}_{a \in Q(k)}$ which go out from the right boundary of the $Q(k)$ are repelled upward or downward by the particles of $\{x'_a\}_{a \in Q_+(k;k,k_4)}$. In that case, we must arrange the particles on Q_+ appropriately so that the repelled particles do not interact mutually.

The velocities of the particles of $T^m X$ on a $Q(k_3)-(\bar{Q}(k; k, k_3) \cup Q(k))$ are

$$(1, 0) \quad \text{or} \quad (0, 1) \quad \text{if} \quad a \in [k+1, k_3] \times [k+1, k_3],$$

$$(-1, 0) \quad \text{or} \quad (0, 1) \quad \text{if} \quad a \in [-k_3, -k-1] \times [k+1, k_3],$$

$$(-1, 0) \quad \text{or} \quad (0, -1) \quad \text{if} \quad a \in [-k_3, -k-1] \times [-k_3, -k-1],$$

and

$$(1, 0) \quad \text{or} \quad (0, -1) \quad \text{if} \quad a \in [k+1, k_3] \times [-k_3, -k-1].$$

From (i), (ii) and (iii) it is not hard to see that

(iv) for any configurations $\{x'_a\}_{a \in Q(k)}$ and $\{x''_a\}_{a \in Q(k)}$ on $Q(k)$ and for large $n>0$, there exist a number $k_5>0$ and a configuration $\{x'''_a\}_{a \in Q(k_5)-Q(k)}$ such that

$$\text{if} \quad X_a=\begin{cases} x'_a, & a \in Q(k) \\ x'''_a, & a \in Q(k_5)-Q(k) \end{cases} \quad \text{then} \quad (T^n X)_a = x''_a \quad \text{for} \quad a \in Q(k).$$

This means that

$$T^n S(\{x'_a\}) \cap S(\{x''_a\}) \neq \varnothing,$$

where $S(\{x'_a\})=\{X \mid X_a=x'_a, a \in Q(k)\}$ is a cyclinder set of $\mathfrak{X}$.

Hence we have proved

THEOREM. *The system $(\mathfrak{X}, T)$ is topologically mixing, that is, for any open sets A and B, there exists a number $N>0$ such that for all $n>N$ we have $T^n(A) \cap B \neq \varnothing$.*

REMARK. The same result holds also in the case when $T=CT_0$.

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