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Standard Subgroups of Type $G_2(3)$

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Introduction

A quasisimple subgroup L of a finite group G is said to be *standard* if $|C_G(L)|$ is even, $|C_G(L)\cap C_G(L)^g|$ is odd for each $g\in G-N_G(L)$, and $[L, L^g]\neq 1$ for each $g\in G$. Let $G_2(3^n)$ denote the Chevalley group of type (G_2) over the finite field $GF(3^n)$. The objective of this paper is to prove the following theorem.

THEOREM. *Let G be a finite group which possesses a standard subgroup L such that $L/Z(L)\cong G_2(3)$. Assume that $C_G(L)$ has a cyclic Sylow 2-subgroup and that $LO(G)\ntriangleleft G$. Then one of the following holds.*

(1) $E(G)\cong G_2(9)$.

(2) $E(G)/Z(E(G))\cong G_2(3)\times G_2(3)$.

(3) $N_G(L)/C_G(L)\cong \mathrm{Aut}\,(G_2(3))$ *and for an involution z of L, $C_G(z)$ has a quasisimple subgroup K which satisfies the following conditions:*

(i) $z\in K$, *$O_2(K)$ is cyclic of order 4, and* $K/O(K)\cong SU_4(3)$.

(ii) $[K, O(C_G(z))]=1$.

(iii) *$K/\langle z\rangle$ is a standard subgroup of $C_G(z)/\langle z\rangle$ and $O_2(K)$ is a Sylow 2-subgroup of $C_G(K/\langle z\rangle)$.*

We remark that Case (3) does not occur in any known examples of G. Thus it is anticipated that once the classification of finite groups with a standard subgroup of type $PSU_4(3)$ is established, Case (3) will be eliminated. This paper represents a contribution to the program of classifying all finite groups having a standard subgroup of known type.

As usual the method used in the proof is essentially a detailed analysis of 2-local subgroups of G depending heavily on the structure of 2-local subgroups of $G_2(3)$. In this context the group $G_2(3)$ seems to be "small". There are two reasons. First, $G_2(3)$ is of characteristic 2-type (a group X is said to be of *characteristic 2-type* provided $F^*(Y)=$

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$O_2(Y)$ for every 2-local subgroup Y of X), although it is a Chevalley group of characteristic 3. Secondly, $G_2(3)$ is almost a N-group [27, section 8], that is to say, it has only one conjugacy class of nonsolvable 2-local subgroups and the remaining local subgroups are solvable. These properties cause some technical difficulties in the proof. One more unpleasant situation appears when $N_G(L)\neq C_G(L)L$.

Let t be an involution of $C_G(L)$. Note that $|Z(L)|=1$ or 3 by Griess [15]. A Sylow 2-subgroup of L has a unique maximal subgroup B whose center is a four-group. In section 3 we study the fusion of the involution t and show that $t^G\cap L=\varnothing$ and $N(B\langle t\rangle)$ acts transitively on $Z(B)t$. There is an elementary abelian subgroup F of order 8 in B with the property that $N_L(F)'$ is the nonsplit extension of E_8 by $GL_3(2)$. In section 4 we show that if $N_G(F\langle t\rangle)\leqq C_G(t)$, $N_G(B\langle t\rangle)$ contains a Sylow 2-subgroup of G. By using a transfer lemma we see that $t\notin O^2(G)$ and F is self-centralizing in a Sylow 2-subgroup of $O^2(G)$. Thus $E(G)\cong G_2(9)$ by [13] and [17]. If $N_G(F\langle t\rangle)$ acts transitively on Ft, then in section 5 we show that $N_G(F\langle t\rangle)$ has a normal subgroup M of order 2^6 such that $C_M(t)=F$ and either M is elementary abelian or homocyclic abelian of exponent 4. The case where M is homocyclic abelian is treated in section 6. It can be shown that Case (3) of the main theorem occurs. In the last stage of this argument we make use of [11, Lemma (1R)] and the classification of simple groups whose Sylow 2-subgroups are isomorphic to a Sylow 2-subgroup of $PSL_6(q)$, $q\equiv 3 \pmod 4$ by Foote [30]. Finally, in section 7 we handle the case where M is elementary abelian. After determining the structure of a Sylow 2-subgroup of $O^2(G)$, we can appeal to Shult's product fusion theorem [24] to conclude that $E(G)/Z(E(G))\cong G_2(3)\times G_2(3)$.

Our notation is fairly standard. Possible exceptions are as follows. For a group X, $m(X)$ and $r(X)$ denote respectively the 2-rank and the sectional 2-rank of X. $\mathscr{I}(D)$ denotes the set of involutions in a subset D of a group and $Y\hookrightarrow X$ implies that X has a subgroup isomorphic to Y. If Q is a 2-group, $\mathscr{E}^*(Q)$ is the set of maximal elementary abelian subgroups of Q, $J_0(Q)$ is the subgroup generated by all abelian subgroups of Q of maximal order, and $J_r(Q)$ is the subgroup generated by all abelian subgroups of Q of maximal rank. Moreover, A_n and Σ_n are respectively an alternating and a symmetric group of degree n and E_{p^n} is an elementary abelian group of order p^n. As is customary, for a gruop X a 2-group P is said to be of type X provided P is isomorphic to a Sylow 2-subgroup of X.

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§ 1. Properties of $G_2(3)$.

We enumerate some properties of the Chevalley group $G_2(3)$ of type (G_2) defined over $GF(3)$ and its automorphisms. An excellent description of $G_2(3^n)$ can be found in Ree [22]. Proofs will be omitted in the case where the assertions are consequences of direct computation.

Let Σ be a root system of type (G_2). In some fixed ordering the set of positive roots Σ^+ can be written as $\{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$ where a and b are the fundamental roots. The set Σ consists of the elements of Σ^+ and their negatives. For $r, s \in \Sigma$, define a rational integer $s(r)$ by $s(r)=2$ if $r=s$ and $s(r)=p-q$ if $r\neq s$ where

$$p=\max\{i\,|\,s-ir\in\Sigma\} \quad\text{and}\quad q=\max\{i\,|\,s+ir\in\Sigma\} .$$

The reflection w_r of Σ with respect to a root r is given by $w_r(s)=s-s(r)r$. For each root r there is an injective homomorphism φ_r: $SL_2(3)\rightarrow G_2(3)$. Set

$$x_r(\alpha)=\varphi_r\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}, \qquad n_r=\varphi_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad h_r(\beta)=\varphi_r\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}.$$

There is an isomorphism ψ from $J=\langle h_r(\beta)\,|\,r\in\Sigma,\ \beta\in GF(3)^\times\rangle$ onto the group of all $GF(3)$-characters of a free abelian group on the generators a, b. Denote by ψ_h the image of h under ψ. The isomorphism ψ is given by $\psi_{h_r(\beta)}(s)=\beta^{s(r)}$. The commutator formulas are taken from [22, (3.10)]. They are

$$\begin{aligned}
&[x_a(\alpha), x_b(\beta)]=x_{a+b}(-\alpha\beta)x_{2a+b}(-\alpha^2\beta)x_{3a+b}(\alpha^3\beta)x_{3a+2b}(\alpha^3\beta^2) ,\\
&[x_a(\alpha), x_{a+b}(\beta)]=x_{2a+b}(\alpha\beta) ,\\
&[x_b(\alpha), x_{3a+b}(\beta)]=x_{3a+2b}(\alpha\beta) ,\\
&[x_r(\alpha), x_s(\beta)]=1 \quad \text{for all other pairs} \quad r, s\in\Sigma^+ .
\end{aligned}$$

For $r, s\in\Sigma$ *and* $h\in J$ *we have* $hx_r(\alpha)h^{-1}=x_r(\psi_h(r)\alpha)$, $n_rx_s(\alpha)n_r^{-1}=x_{w_r(s)}(\eta_{r,s}\alpha)$, and $n_rhn_r^{-1}=h'$, where h' is the element of J satisfying $\psi_{h'}(t)=\psi_h(w_r(t))$ for $t\in\Sigma$ and the values $\eta_{r,s}=\pm1$ are given in [22, (3.4)]. The group $L=G_2(3)$ is generated by the elements $x_r(\alpha)$, $r\in\Sigma$, $\alpha\in GF(3)$ and $|L|=2^6\cdot3^6\cdot7\cdot13$. Let ρ be the permutation on Σ of order 2 defined by $\rho(\pm a)=\pm b$, $\rho(\pm(a+b))=\pm(3a+b)$, $\rho(\pm(2a+b))=\pm(3a+2b)$. Then the graph automorphism σ of L is given by $\sigma: x_r(\alpha) \mapsto x_{\rho(r)}(\alpha)$ for $r \in \Sigma$ and $\alpha \in GF(3)$. Hence $n_r^\sigma = n_{\rho(r)}$ and $h_r(\beta)^\sigma = h_{\rho(r)}(\beta)$. By a theorem of Steinberg [25], Aut $(L) = L\langle\sigma\rangle$. Now set

$$a_1 = x_{a+b}(1) n_{a+b}^3 x_{a+b}(-1) = \varphi_{a+b}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad b_1 = n_{a+b},$$

$$a_2 = x_{3a+b}(1) n_{3a+b}^3 x_{3a+b}(-1) = \varphi_{3a+b}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad b_2 = n_{3a+b},$$

$$h_0 = h_a(-1), \qquad z = h_{a+b}(-1) = h_{3a+b}(-1).$$

Note that the image of $h_{a+b}(-1)$ under ψ is identical with that of $h_{3a+b}(-1)$, so we have $h_{a+b}(-1) = h_{3a+b}(-1)$. Set $S = \langle a_1, b_1, a_2, b_2, h_0 \rangle$.

(1.1) $\langle S, \sigma \rangle$ is generated by $a_1, b_1, a_2, b_2, h_0, \sigma$ subject to the relations

$$z^2 = h_0^2 = \sigma^2 = 1, \qquad a_1^2 = b_1^2 = a_2^2 = b_2^2 = z;$$
$$[a_i, b_i] = z, \qquad [a_i, h_0] = b_i z, \qquad [b_i, h_0] = z,$$
$$[a_i, \sigma] = a_1 a_2 z, \qquad [b_i, \sigma] = b_1 b_2 z, \quad \text{for} \quad i = 1, 2;$$
$$[h_0, \sigma] = z.$$

All other commutators of pairs of generators are trivial. The subgroup S is a Sylow 2-subgroup of L and $S\langle\sigma\rangle$ is a Sylow 2-subgroup of Aut(L).

(1.2) S has seven conjugacy classes of involutions. They are

$$\{z\}, \qquad \{b_1b_2, b_1b_2z\},$$
$$\{a_1a_2, a_1a_2z, a_1b_1a_2b_2, a_1b_1a_2b_2z\},$$
$$\{a_1b_2, a_1b_2z, a_1b_1b_2, a_1b_1b_2z\},$$
$$\{b_1a_2, b_1a_2z, b_1a_2b_2, b_1a_2b_2z\},$$
$$\{a_1b_1a_2, a_1b_1a_2z, a_1a_2b_2, a_1a_2b_2z\},$$
$$\{h_0, h_0z, b_1h_0, b_1h_0z, b_2h_0, b_2h_0z, b_1b_2h_0, b_1b_2h_0z\}.$$

The centralizers of involutions in S are as follows:

$$C_S(z) = S, \qquad C_S(b_1b_2) = \langle a_1a_2, b_1, b_2, h_0 \rangle,$$
$$C_S(a_1a_2) = \langle a_1a_2 \rangle \times \langle a_1, b_1b_2 \rangle \cong Z_2 \times D_8,$$
$$C_S(a_1b_2) = \langle a_1b_2 \rangle \times \langle a_1, b_1a_2 \rangle \cong Z_2 \times D_8,$$
$$C_S(b_1a_2) = \langle b_1a_2 \rangle \times \langle b_1, a_1b_2 \rangle \cong Z_2 \times D_8,$$
$$C_S(a_1b_1a_2) = \langle a_1b_1a_2 \rangle \times \langle a_2, a_1b_2 \rangle \cong Z_2 \times D_8,$$
$$C_S(h_0) = \langle b_1b_2, h_0, z \rangle \cong E_8.$$

(1.3) We have $C_S(\sigma)=\langle a_1a_2, b_1b_2, z\rangle$ and $\mathscr{J}(S\sigma)=C_S(\sigma)\sigma=\sigma^S$. The group $S\langle\sigma\rangle$ has seven conjugacy classes of involutions. Their representatives are $z, b_1b_2, a_1a_2, a_1b_2, a_1b_1a_2, h_0, \sigma$. The centralizers of these involutions are $C_{S\langle\sigma\rangle}(z)=S\langle\sigma\rangle$, $C_{S\langle\sigma\rangle}(b_1b_2)=C_S(b_1b_2)\langle\sigma\rangle$, $C_{S\langle\sigma\rangle}(a_1a_2)=C_S(a_1a_2)\langle\sigma\rangle$, $C_{S\langle\sigma\rangle}(a_1b_2)=C_S(a_1b_2)$, $C_{S\langle\sigma\rangle}(a_1b_1a_2)=C_S(a_1b_1a_2)\langle h_0\sigma\rangle$, $C_{S\langle\sigma\rangle}(h_0)=C_S(h_0)\langle b_1\sigma\rangle$, $C_{S\langle\sigma\rangle}(\sigma)=C_S(\sigma)\langle\sigma\rangle$.

Set $A=\langle a_1, b_1, a_2, b_2\rangle$, $B=C_S(b_1b_2)$, $E=\langle b_1b_2, a_1b_1a_2, z\rangle$, and $F=\langle a_1a_2, b_1b_2, z\rangle$. These subgroups play an important role in later sections. The next two lemmas can be verified by straightforward computation.

(1.4) (1) $Z(S)=\langle z\rangle$, $Z_2(S)=S'=\langle b_1, b_2\rangle$, $S/Z_2(S)\cong E_8$.

(2) $A=\langle a_1, b_1\rangle * \langle a_2, b_2\rangle\cong Q_8 * Q_8$ and A is the unique extra-special subgroup of S of order 2^5.

(3) $Z(B)=\Omega_1(Z_2(S))=\langle b_1b_2, z\rangle$, B is the unique maximal subgroup of S whose center is noncyclic, $J_0(S)=J_0(B)=\langle b_1, a_1a_2h_0\rangle\cong Z_4\times Z_4$, $Z(B)=\Omega_1(J_0(B))$, and h_0 inverts $J_0(B)$.

(4) $EF=A\cap B=\langle b_1b_2\rangle\times\langle a_1a_2, b_1\rangle\cong Z_2\times D_8$, $\mathscr{E}^*(EF)=\{E, F\}$, and $E\cap F=Z(B)$.

(5) $\mathscr{E}^*(S/F)=\{A/F, B/F\}$, $\mathscr{E}^*(S/\langle z\rangle)=\{A/\langle z\rangle, Z_2(S)\langle h_0\rangle/\langle z\rangle\}$, and $Z_2(S)\langle h_0\rangle\cong Z_2\times D_8$.

(6) $|\mathscr{E}^*(S)|=8$ and each member of $\mathscr{E}^*(S)$ is conjugate in S to one of E, F, $Z(B)\langle h_0\rangle$, $\langle a_1b_2, b_1a_2, z\rangle$, or $\langle a_1b_2, a_1b_1a_2, z\rangle$.

(7) $m(S)=3$ and $r(S)=4$.

(1.5) (1) $Z(S\langle\sigma\rangle)=\langle z\rangle$, $Z_2(S\langle\sigma\rangle)=Z(B)$, $Z_3(S\langle\sigma\rangle)=EF$, $S\langle\sigma\rangle/EF\cong E_8$.

(2) $\mathscr{E}^*(S\langle\sigma\rangle/Z(B))=\{A/Z(B), B\langle\sigma\rangle/Z(B)\}$ and $B\langle\sigma\rangle/\langle z\rangle\cong Q_8 * Q_8$.

(3) $J_r(S\langle\sigma\rangle/\langle z\rangle)=A/\langle z\rangle$.

(4) $J_r(S\langle\sigma\rangle)=F\langle\sigma\rangle\cong E_{16}$ and $\mathscr{E}^*(S\langle\sigma\rangle)=\{F\langle\sigma\rangle\}\cup\mathscr{E}^*(S)-\{F\}$.

(5) Every abelian subgroup of $S\langle\sigma\rangle$ has order at most 16.

(6) $m(S\langle\sigma\rangle)=r(S\langle\sigma\rangle)=4$.

(1.6) The group L has only one conjugacy class of involutions by [7] or [27, Lemma 8.1(v)]. Since $\psi_z(a)=\psi_z(b)=-1$, by using Bruhat factorization we can verify that $C_L(z)=K_{a+b}K_{3a+b}\langle h_0\rangle$ where $K_r=\langle x_{\pm r}(\alpha)\,|\,\alpha\in GF(3)\rangle\cong SL_2(3)$, $r=a+b, 3a+b$. The subgroups K_{a+b} and K_{3a+b} are normal in $C_L(z)$. Note that $[K_{a+b}, K_{3a+b}]=1$ and $K_{a+b}\cap K_{3a+b}=\langle z\rangle$. Moreover, $O_2(K_{a+b})=\langle a_1, b_1\rangle$ and $O_2(K_{3a+b})=\langle a_2, b_2\rangle$. Let $X_r=\langle x_r(\alpha)\,|\,\alpha\in GF(3)\rangle$, $r=a+b, 3a+b$. Then X_r is a Sylow 3-subgroup of K_r and h_0 inverts X_r. Also, $K_{a+b}^\sigma=K_{3a+b}$.

(1.7) It is well-known that $C_L(\sigma)\cong \mathrm{Aut}\,(PSL_2(8))$. In particular, F is a Sylow 2-subgroup of $C_L(\sigma)$.

(1.8) (1) $N_L(S)=N_L(A)\cap N_L(B)=S$.

(2) $N_L(A)=C_L(z)$ and $N_L(A)/A$ is a Frobenius group of order 18. A Sylow 3-subgroup of $N_L(A)$ has three orbits on $(A/\langle z\rangle)^\#$, which are $(\langle a_i, b_i\rangle/\langle z\rangle)^\#$, $i=1, 2$ and the remaining elements.

(3) $N_L(B)=N_L(Z(B))\leqq N_L(F)$, $N_L(B)/B\cong\Sigma_3$, and a Sylow 3-subgroup of $N_L(B)$ acts regularly both on $Z(B)^\#$ and on $(J_0(B)/Z(B))^\#$. Moreover, $C_L(Z(B))=B$.

(4) $C_L(E)=E$, $N_L(E)\leqq N_L(A)$, $N_L(E)/A\cong\Sigma_3$, $N_L(E)$ acts transitively on $E-\langle z\rangle$, and a Sylow 3-subgroup of $N_L(E)$ acts regularly both on $(E/\langle z\rangle)^\#$ and on $(A/E)^\#$.

(5) $C_L(F)=F$, $N_L(F)/F\cong GL_3(2)$, and $N_L(F)$ does not split over F. Moreover, $N_L(A)\cap N_L(F)/A\cong\Sigma_3$ and a Sylow 3-subgroup of $N_L(A)\cap N_L(F)$ acts regularly both on $(F/\langle z\rangle)^\#$ and on $(A/F)^\#$.

PROOF. By (1.4), A and B are weakly closed in S with respect to L. As $S=AB$ and $Z(S)=\langle z\rangle$, (1) follows from the structure of $C_L(z)$. As $Z(A)=\langle z\rangle$ and $O_2(C_L(z))=A$, (2) is a consequence of (1.6). Also, $C_L(Z(B))=B$. We now proceed as in the proof of [27, Lemma 8.3(a)]. Since B is weakly closed in S, $N_L(B)$ controls the L-fusion of elements in $Z(B)$. All involutions of L are conjugate, so $N_L(B)$ acts transitively on $Z(B)^\#$ and thus $N_L(B)/B\cong\mathrm{Aut}\,(Z(B))\cong\Sigma_3$. Let $\langle k\rangle$ be a Sylow 3-subgroup of $N_L(B)$. Then k acts regularly on $Z(B)^\#$. Since $J_0(B)\cong Z_4\times Z_4$ and $\Omega_1(J_0(B))=Z(B)$, k acts regularly on $(J_0(B)/Z(B))^\#$ as well. Now $\mathscr{E}^*(B)=\{E, F, Z(B)\langle h_0\rangle, Z(B)\langle h_0\rangle^{a_1}\}$. Since E and F are normal in $S=N_L(S)$, they are not conjugate in L by Burnside's fusion lemma. If k normalizes both E and F, then $E=Z(B)C_E(k)$ and $F=Z(B)C_F(k)$ so that $|C_B(k)|\geqq 4$. But as k acts fixed-point-freely on $J_0(B)$, we have $|C_B(k)|=2$, a contradiction. Thus each member of $\mathscr{E}^*(B)$ is conjugate to E or F by an element of $\langle k\rangle$. The structure of $N_L(A)$ shows that $N_L(A)\cap N(F)/A\cong\Sigma_3$. By (1.7), $C_L(\sigma)\cap N(F)$ has order $2^3\cdot 3\cdot 7$. Hence $2^3\cdot 3\cdot 7$ divides $|N_L(F)/F|$ and as $\mathrm{Aut}\,(F)\cong GL_3(2)$ and $C_L(F)=F$, we conclude that $N_L(F)/F\cong GL_3(2)$. We can verify that $x^2\in Z(B)$ for every element x of S of order 4, so $N_L(F)$ does not split over F and (5) holds. Now B/F is a four-group and $N_L(B)/B\cong\Sigma_3$, so it follows from the structure of $N_L(F)/F$ that $N_L(B)\leqq N_L(F)$ and (3) holds. We see that $N_L(A)\cap N(E)$ acts transitively on $E-\langle z\rangle$ and $N_L(A)\cap N(E)/E$ is an extension of a four-group by Σ_3. Thus if $N_L(A)\ngeqq N_L(E)$, we have $N_L(E)/E\cong\mathrm{Aut}\,(E)\cong GL_3(2)$. But then $N_L(B)\leqq N(E)$, which is impossible since k does not normalize E. Therefore $N_L(E)\leqq N(A)$ and (4) holds.

(1.9) Every maximal elementary abelian 2-subgroup of L is conjugate to E or F in L.

PROOF. By the structure of $N_L(A)$, $\langle a_1b_2, b_1a_2, z\rangle$ and $\langle a_1b_2, a_1b_1a_2, z\rangle$ are conjugate to E and F in $N_L(A)$ respectively. We have shown in the proof of the above lemma that $Z(B)\langle h_0\rangle$ is conjugate to E in $N_L(B)$. Now the assertion follows from (1.4) (6).

(1.10) (1) $N_L(A)\langle\sigma\rangle/A\cong\Sigma_3\times\Sigma_3$ and $C_{L\langle\sigma\rangle}(A/\langle z\rangle)=C_{L\langle\sigma\rangle}(EF/\langle z\rangle)=A$.
(2) σ centralizes $N_L(F)/F$.

PROOF. Since σ centralizes S/F, (2) holds. (1) can be easily verified.

§ 2. Preliminaries.

In this section we collect some preliminary lemmas to be used in later sections. The following two lemmas are well-known.

(2.1) (1) $\mathrm{Aut}(PSL_2(9))=P\Gamma L_2(9)=PGL_2(9)\langle f\rangle$, *where f denotes the involutive field automorphism. If K is a subgroup of $P\Gamma L_2(9)$ of index 2 then $K\cong PGL_2(9)$ or Σ_6 or else K has a quasidihedral Sylow 2-subgroup of order 16.*

(2) *A Sylow 2-subgroup of $PGL_2(9)$ is dihedral, all involutions in $PGL_2(9)-PSL_2(9)$ are conjugate, and if a is an involution in $PGL_2(9)-PSL_2(9)$ then $PSL_2(9)\cap C(a)$ is a Frobenius group of order 10.*

(3) *$PSL_2(9)\langle f\rangle\cong\Sigma_6$ and a Sylow 2-subgroup of Σ_6 is isomorphic to $Z_2\times D_8$.*

(4) *There is no involution in $K-PSL_2(9)$ if K has a quasidihedral Sylow 2-subgroup.*

(5) *A Sylow 3-subgroup of $PSL_2(9)$ is self-centralizing in $P\Gamma L_2(9)$.*

(2.2) *Let $A\cong Q_8*Q_8$. Then A has a unique expression as the central product of two quaternion subgroups and* $\mathrm{Out}(A)\cong\Sigma_3$ wreath Z_2.

(2.3) *Let $D=\langle v_1, v_2\rangle\cong Z_4\times Z_4$. Then $X=\mathrm{Aut}(D)$ can be represented as a matrix group*

$$\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\middle| a, b, c, d\in Z/4Z \text{ with } ad-bc\not\equiv 0 \pmod 4\right\},$$

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}: v_1\longmapsto v_1^a v_2^b \quad\text{and}\quad v_2\longmapsto v_1^c v_2^d .$$

We have $|X|=2^5\cdot 3$, $O_2(X)=C_X(\Omega_1(D))\cong E_{16}$, and $X/O_2(X)\cong\mathrm{Aut}(\Omega_1(D))\cong\Sigma_3$. Let $\langle k\rangle$ be a Sylow 3-subgroup of X. Then $C_D(x)=\Omega_1(D)$ for all $1\neq x\in[O_2(X), k]$.

PROOF. See [13, Part II, Lemma 2.1] or [20, p. 364].

The next lemma is due to Harada and Yamaki [18].

(2.4) *A simple subgroup of* $GL_6(2)$ *is isomorphic to one of the following groups*: A_m, $5 \leqq m \leqq 7$; $GL_n(2)$, $3 \leqq n \leqq 6$; $SL_2(8)$, $Sp_6(2)$, *or* $SU_3(3)$.

(2.5) (1) $|GL_8(2)| = 2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 31 \cdot 127$.

(2) *A Sylow 3-subgroup of* $GL_8(2)$ *is isomorphic to* $(Z_3 \text{ wreath } Z_3) \times Z_3$ *and it has a unique elementary abelian subgroup of order* 3^4. *If* $GL_8(2) \geqq D \cong E_{3^4}$ *then the normalizer of* D *in* $GL_8(2)$ *is an extension of* $\Sigma_3 \times \Sigma_3 \times \Sigma_3 \times \Sigma_3$ *by* Σ_4.

(3) *The normalizer in* $GL_8(2)$ *of a Sylow 7-subgroup has order* $2^2 \cdot 3^3 \cdot 7^2$ *and it is isomorphic to* $(F_{21} \text{ wreath } Z_2) \times \Sigma_3$ *where* F_{21} *denotes a Frobenius group of order* 21.

(4) $PSL_2(q) \hookrightarrow GL_8(2)$ *if and only if* $q = 2, 2^2, 2^3, 2^4, 3, 3^2$, *or* 7.

PROOF. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Define $e_i \in GL_8(2)$ to be

$$e_1 = \begin{pmatrix} K & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix}, \quad e_2 = \begin{pmatrix} I & & & \\ & K & & \\ & & I & \\ & & & I \end{pmatrix}, \quad e_3 = \begin{pmatrix} I & & & \\ & I & & \\ & & K & \\ & & & I \end{pmatrix},$$

$$e_4 = \begin{pmatrix} I & & & \\ & I & & \\ & & I & \\ & & & K \end{pmatrix}, \quad e_0 = \begin{pmatrix} & I & & \\ & & I & \\ I & & & \\ & & & I \end{pmatrix}.$$

Then $e_1^{e_0} = e_2$, $e_2^{e_0} = e_3$, and $\langle e_i \mid 0 \leqq i \leqq 3 \rangle \cong Z_3$ wreath Z_3. Let $P = \langle e_i \mid 0 \leqq i \leqq 4 \rangle$. Then P is a Sylow 3-subgroup of $X = GL_8(2)$ and $D = \langle e_i \mid 1 \leqq i \leqq 4 \rangle$ is the unique elementary abelian subgroup of P of order 3^4. We can verify that $N_X(D)$ induces a permutation representation on the set $\{\langle e_i \rangle \mid 1 \leqq i \leqq 4\}$. The image of this representation is Σ_4 and the kernel consists of those elements which normalize each $\langle e_i \rangle$, $1 \leqq i \leqq 4$, so in fact it is isomorphic to $\Sigma_3 \times \Sigma_3 \times \Sigma_3 \times \Sigma_3$. Thus (2) holds. For the proof of (3), let

$$L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad f_1 = \begin{pmatrix} L & & \\ & J & \\ & & I \end{pmatrix}, \quad f_2 = \begin{pmatrix} J & & \\ & L & \\ & & I \end{pmatrix},$$

where J denotes the 3 dimensional unit matrix. Then $Q = \langle f_1, f_2 \rangle$ is a Sylow 7-subgroup of X and if $x \in N_X(Q)$, we have either $x \in N_X(\langle f_1 \rangle) \cap$

$N_X(\langle f_2\rangle)$ or $\langle f_1\rangle^x=\langle f_2\rangle$ and $\langle f_2\rangle^x=\langle f_1\rangle$. Let

$$u=\begin{pmatrix} & J & \\ J & & \\ & & I\end{pmatrix}.$$

Then u is an involution of X with $f_1^u=f_2$. Moreover $N_X(\langle f_1\rangle)\cap N_X(\langle f_2\rangle)\cong F_{21}\times F_{21}\times\Sigma_3$ since in $GL_3(2)$ the normalizer of a Sylow 7-subgroup is a Frobenius group of order 21. Thus (3) holds. The order of $PSL_2(q)$ divides $|X|$ only if q is one of the values listed in (4) or else $q\in\{7^2, 17, 31, 127\}$. It follows from [31, section 6] that $GL_8(2)$ does not have a subgroup isomorphic to $PSL_2(q)$ if $q=17, 31$, or 127. In $PSL_2(7^2)$ the normalizer of a Sylow 7-subgroup has order $2^3\cdot 3\cdot 7^2$, so $PSL_2(7^2)\not\hookrightarrow GL_8(2)$ by (3) and thus (4) holds.

(2.6) *Let X be a nontrivial extension of $Z_{2^n}\times Z_{2^n}\times Z_{2^n}$ by $GL_3(2)$ and set $M=O_2(X)$. Then for each value of $n\geqq 1$, the isomorphism class of X is determined by whether X does or does not split over M. Furthermore, if P is a Sylow 2-subgroup of X then P is generated by the elements u, v, w, r, s with $M=\langle u, v, w\rangle$ subject to the relations*

$$u^r=w^{-1},\quad v^r=v^{-1},\quad w^r=u^{-1},\quad s^r=s^{-1},$$
$$u^s=v,\quad v^s=w,\quad w^s=uv^{-1}w,$$
$$r^2=1,\ \text{and}\ s^4=1\ \text{or}\ s^4=uw.$$

The group X splits over M if and only if $s^4=1$. If $n=1$, P is of type $GL_4(2)$ or $G_2(3)$ according as $s^4=1$ or $s^4=uw$. If $n=2$, P is of type HS or OS according as $s^4=1$ or $s^4=uw$ where HS and OS denote respectively the Higman-Sims simple group and the O'Nan-Sims simple group.

PROOF. See [1] and [21].

In the next two lemmas we use the above notation. The assertions of these lemmas can be verified by direct computation.

(2.7) *If $n=2$ and $s^4=uw$, then $J_0(P)=M$ and $P/\Omega_1(M)$ is of type $G_2(3)$.*

(2.8) *If $n=2$ and $s^4=1$, then $J_0(P)=M$ and $P/\Omega_1(M)$ is of type $GL_4(2)$. Moreover the following conditions hold.*

(1) *$Z(P)=\langle u^2w^2\rangle$, $Z_2(P)=\langle u^2v^2, uw\rangle$, and $Z_3(P)=\langle u^2, v^2, uw\rangle$.*

(2) *If x is an involution of P such that $|C_P(x)|\geqq 2^6$, then x is conjugate in P to u^2w^2, u^2v^2, u^2, s^2, or vw^3s^2. We have $C_P(u^2v^2)=\langle rs, s^2\rangle M$, $C_P(u^2)=\langle rs^2\rangle M$, $C_P(s^2)=\langle r, s\rangle Z_2(P)$, and $C_P(vw^3s^2)=\langle vr, vs\rangle Z_2(P)$. Further-*

more $u^2w^2 \in P'$, $u^2v^2 \in C_P(u^2v^2)'$, $u^2 \in C_P(u^2)'$, $s^2 \in C_P(s^2)'$, *and* $vw^3s^2 \notin C_P(vw^3s^2)' = \langle (vs)^2 \rangle$. *Observe that* $(vs)^2 = uvs^2$ *and* $(vs)^4 = u^2w^2$.

(3) *Let* $N = \Omega_1(M)$ *and* $\bar{P} = P/N$. *Then* $\bar{P}$ *has four conjugacy classes of complements for* $\bar{M}$ *in* $\bar{P}$. *We can take* $\langle \bar{r}, \bar{s} \rangle$, $\langle \bar{u}\bar{w}\bar{r}, \bar{s} \rangle$, $\langle \bar{v}\bar{r}, \bar{v}\bar{s} \rangle$, *and* $\langle \bar{u}\bar{v}\bar{w}\bar{r}, \bar{v}\bar{s} \rangle$ *as their representatives. Furthermore both* $\langle r, s \rangle N$ *and* $\langle uwr, s \rangle N$ *split over* N *but neither* $\langle vr, vs \rangle N$ *nor* $\langle uvwr, vs \rangle N$ *splits over* N.

(2.9) *Let* X *be the nonsplit extension of* E_8 *by* $GL_3(2)$. *Then* $|\mathrm{Aut}(X): X| = 2$ *and* $\mathrm{Aut}(X) \cong (G_2(3) \cap N(F))\langle \sigma \rangle$ *in the notation of section* 1.

PROOF. By (2.6) such a group X is uniquely determined. Since $Z(X) = 1$, we can regard X as a subgroup of $H = \mathrm{Aut}(X)$. Set $M = O_2(X)$. Then X/M induces the automorphism group of M, so we have $H/M = X/M \times Y/M$ where $Y = C_H(M)$. Assume that Y is not a 2-group and let $1 \neq W \in \mathrm{Syl}_p(Y)$ with p an odd prime. For each Sylow 2-subgroup P of X, W stabilizes the series $P > M > 1$ and so $[W, P] = 1$. But then $W \leqq C_H(X) = 1$, a contradiction. Thus Y is a 2-group. If there exists an element v of order 4 in Y, then $|\mho^1(M\langle v \rangle)| = 2$. As $[X, Y] \leqq M$, X normalizes $M\langle v \rangle$ and so $\mho^1(M\langle v \rangle) \leqq C_H(X) = 1$, a contradiction. Hence Y is elementary abelian. Take a subgroup Q of X of order 21, so that $Y = M \times C_Y(Q)$ and QY is a maximal subgroup of X. Then for $y \in C_Y(Q)^\#$, we have $C_H(y) = QY$, $|y^H| = 8$, and $y^H = My$. For each $y \in Y - M$, $H \triangleright M\langle y \rangle$ and so $|y^H| = 8$. Let x be a 2-element of $X - M$ such that $x^2 \in M$. Then $|Y: C_Y(x)| \leqq |C_Y(x)| = |C_M(x)| = 4$. Thus $|Y| \leqq 16$ and the lemma holds.

§ 3. Fusion of the involution t.

Henceforth let G be a group which possesses a standard subgroup L with $L/Z(L) \cong G_2(3)$ such that $C(L)$ has a cyclic Sylow 2-subgroup and $LO(G)$ is not normal in G. Let S be a Sylow 2-subgroup of L. The Schur multiplier of $G_2(3)$ is of order 3 by Griess [15], so we can identify S with a Sylow 2-subgroup of $G_2(3)$. We shall use the same symbols as in section 1 for elements and subgroups of S for the rest of the paper. Let t be an involution of $C(L)$ and set $H = C(t)$. Then L is normal in H by our hypothesis and $|H: LC_H(L)| \leqq 2$. Let R be a Sylow 2-subgroup of $LC_H(L)$ and T a Sylow 2-subgroup of H with $S \leqq R \leqq T$. We begin by studying the fusion of the involution t.

(3.1) $\langle t \rangle \in \mathrm{Syl}_2(C(L))$, $C_H(L) = \langle t \rangle O(H)$, $t^G \cap L = \varnothing$, *and* $t^{N(Z(R))} = Z(S)t$.

PROOF. Since $LO(G)$ is not normal in G, $t \notin Z^*(G)$ and by the Z^*-

theorem [9] we can take $x \in G$ such that $t \neq t^x$ and $[t, t^x]=1$. Then $t \in H^x \triangleright L^x$. If $t \in L^x C(L^x)$, $t \neq t^{x^{-1}} \in LC_H(L)$. If $t \notin L^x C(L^x)$, each involution of tL^x is conjugate to t by an element of L^x and $SL_2(8) \cong C_{L^x}(t)^{(\infty)} \leqq H^{(\infty)} = L$ by (1.3) and (1.7). So $\mathscr{I}(tC_{L^x}(t)^{(\infty)}) \leqq t^G \cap tL$. Thus in either case $t^G \cap LC_H(L) \neq \{t\}$. Since L has exactly one conjugacy class of involutions, we conclude that $t^G \cap \langle z, t\rangle \neq \{t\}$. Now t is extremal in a Sylow 2-subgroup of G containing T with respect to G and $\langle z, t\rangle \leqq Z(T)$, so we have $t^G \cap \langle z, t\rangle = t^{N(T)} \cap \langle z, t\rangle$. Put $Q=C_R(L)$, so that $R=S\times Q$ and Q is cyclic with t the unique involution. Suppose $|Q|>2$. By (1.5) (1), $Z(T) \leqq Z(R) = \langle z\rangle \times Q$ and so $R \neq T$ and $Z(T)=\langle z, t\rangle$, for otherwise $N(T) \leqq H$. Now $J_r(T/Z(R)) = AQ/Z(R)$ by (1.5) (3), whence $J_r(T \bmod Z(T)) = AQ$. As $Z(AQ)=\langle z\rangle Q$, this yields $N(T) \leqq H$, a contradiction. Thus $\langle t\rangle \in \mathrm{Syl}_2(C(L))$ and $C_H(L)=\langle t\rangle O(H)$. Then there is no element v in G such that $v^4=t$. As $(a_1h_0)^4=z$, it follows that $t^G \cap L=\varnothing$. Finally we have $t^{N(Z(R))}=Z(S)t$.

DEFINITION. If $R\neq T$, $T/\langle t\rangle$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}(G_2(3))$ and there is an element $g \in T-R$ such that g acts on $L/Z(L)$ as the graph automorphism and $g^2=1$ or $g^2=t$. Let T_1 be a Sylow 2-subgroup of $N(Z(R))$ containing T, so that $|T_1: T|=2$ and $t^{T_1}=Z(S)t$. Let $R_1=T_1$ if $R=T$ and $R_1=N_{T_1}(\langle a_1, b_1\rangle)$ if $R\neq T$. Set

$$C_2=O_2([C_{N(B\langle t\rangle)}(Z(B)), N_H(B)])B\langle t\rangle \quad \text{and} \quad T_2=TC_2.$$

(3.2) *A and S are normal in T_1 and $R_1 \cap T=R$. If $R\neq T$, then* $|T_1: R_1|=2$.

PROOF. As $|T_1: T|=2$, $A\langle t\rangle = J_r(T \bmod Z(T))$ is normal in T_1. Suppose $A \ntriangleleft T_1$ and take $x \in T_1 - T$. Then $t^x=zt$ and $A^x \leqq A \cup At$. As A is generated by its involutions, $u^x \in At$ for some involution u of A. But then $u^{xy}=zt$ for some $y \in L$ and $u^{xyx}=t$, contrary to $t^G \cap L=\varnothing$. Thus $A \triangleleft T_1$. Similarly we have $S \triangleleft T_1$, for $S\langle t\rangle = R_1 \cap T \triangleleft T_1$ and S is generated by its involutions. If $R\neq T$, the element g interchanges $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$, so $|T_1: R_1|=2$ by (2.2).

(3.3) $N(B\langle t\rangle)=N(Z(B)\langle t\rangle)=N_H(B)C_2 \leqq N(B)$, $[O(H), C_2]=1$, $H\cap C_2 = B\langle t\rangle$, $t^{C_2}=Z(B)t$, *and* $C_2/B\langle t\rangle \cong Z(B)$ *as* $N_H(B)$*-modules. Moreover,* $T_1 \leqq T_2 \in \mathrm{Syl}_2(N(B\langle t\rangle))$ *and* $T_2/C_T(Z(B))$ *is dihedral of order* 8 *with* $C_{T_2}(Z(B))/C_T(Z(B))$ *and* $T_1/C_T(Z(B))$ *the only four-subgroups.*

PROOF. Let $X=N(B\langle t\rangle)$, $Y=C_X(Z(B))$, $Y_1=[Y, N_H(B)]BC_H(L)$, and $X_0=C_H(Z(B))$. Since $T_1 \triangleright R_1 \cap T = R$ and $B\langle t\rangle$ is the unique maximal subgroup of R whose center is elementary abelian of order 8 by (1.4), $T_1 \leqq X$. Thus $t^X=Z(B\langle t\rangle)-L=Z(B)t$, for $N_L(B)$ acts transitively on $Z(B)^\#$.

As $C_X(t)=N_H(B)=N_H(Z(B))$, this implies $X=N(Z(B)\langle t\rangle)$. We also have $\langle uv \mid u, v \in t^G \cap B\langle t\rangle\rangle = B$ since B is generated by its involutions and all involutions of B are conjugate to each other in L. Thus $X \leqq N(B)$. Now $LC_H(L) \cap C(Z(B)) = BC_H(L)$ and $|X_0: BC_H(L)| = |T: R|$. By (1.8) (3), $N_L(B)$ induces the automorphism group of $Z(B)$, hence $X = YN_H(B)$. Then the map defined by $X_0 y \mapsto [y, t] = t^y t$ for $y \in Y$ is an $N_H(B)$-isomorphism of Y/X_0 onto $Z(B)$ and $X/X_0 = Y/X_0 \cdot N_H(B)/X_0 \cong \Sigma_4$.

We wish to show that $Y = Y_1 X_0$ and $Y_1 \cap X_0 = BC_H(L)$. If $R=T$, then $Y = Y_1$ and these assertions hold. Assume that $R<T$. As $O(H)=O(X)$, $BC_H(L)$ is normal in X. Let $\bar{X} = X/BC_H(L)$. Then $\bar{Y}$ has order 8 and $\overline{N_H(B)} = \langle \bar{g} \rangle \times \overline{N_L(B)}$ with $\overline{N_L(B)} \cong \Sigma_3$. Let $\langle \bar{k} \rangle = O_3(\overline{N_L(B)})$. Then $\bar{k}$ acts nontrivially on $\bar{Y}/\bar{X}_0$, so $\bar{Y}$ is quaternion or abelian. The subgroup $T_1 \cap Y$ is of index 2 in T_1 and by (3.2) we have $\mho^1(T_1) \leqq R_1 \cap T \cap Y = B\langle t\rangle$, whence $\overline{T_1 \cap Y}$ is a four-group and $\bar{Y}$ is abelian. Therefore $\bar{Y}$ is a direct product of $[\bar{Y}, \bar{k}] = \bar{Y}_1$ and $C_{\bar{Y}}(\bar{k}) = \bar{X}_0$ as required.

Since $C_{Y_1}(O(H))O(H)$ contains $BC_H(L)$, the action of $\bar{k}$ shows that it is equal to Y_1 and so $Y_1 = O_2(Y_1)O(H)$. Note that $O_2(Y_1) = C_2$ by the definition. Since C_2 is normal in X, T_2 is a unique Sylow 2-subgroup of X containing T, so $T_1 \leqq T_2$. Finally, $C_T(Z(B)) = T_2 \cap X_0$ is normal in T_2 and $C_{T_2}(Z(B))/C_T(Z(B)) \cong Y/X_0$, hence the structure of $T_2/C_T(Z(B))$ is determined.

§ 4. The case $C_{T_1}(A) = \langle z, t\rangle$.

In this section we assume that $C_{T_1}(A) = \langle z, t\rangle$. Under this hypothesis we shall prove that $E(G) \cong G_2(9)$.

(4.1) $\mathscr{I}(R_1 - R) \neq \emptyset$. *If d is an involution in $R_1 - R$ and if bars denote images in $A/\langle z\rangle$, then one of the following holds.*

(i) $\bar{a}_1^d = \bar{a}_1 \bar{b}_1$, $\bar{a}_2^d = \bar{a}_2$, *and* $\bar{b}_i^d = \bar{b}_i$, $i=1, 2$.

(ii) $\bar{a}_1^d = \bar{a}_1$, $\bar{a}_2^d = \bar{a}_2 \bar{b}_2$, *and* $\bar{b}_i^d = \bar{b}_i$, $i=1, 2$.

(iii) $\langle \bar{a}_1, \bar{b}_1\rangle^d = \langle \bar{a}_2, \bar{b}_2\rangle$ *and* $\bar{b}_1^d = \bar{b}_2$.

If $R \neq T$, then (i) *or* (ii) *holds and $T_1/A\langle t\rangle$ is dihedral of order* 8 *with $R_1/A\langle t\rangle$ and $T/A\langle t\rangle$ the only four-subgroups.*

PROOF. If $\mathscr{I}(T_1) = \mathscr{I}(T)$, then $N(T_1) \leqq N(Z(R))$ since $\mathscr{I}(T)$ generates T or R and $Z(T) = Z(R)$. But $T_1 \lhd T_2$, so we get $\mathscr{I}(T_1 - T) \neq \emptyset$.

Suppose $R=T$. Then $R_1 = T_1$ by the definition. As d normalizes A by (3.2), it follows from (2.2) that $\langle a_1, b_1\rangle^d = \langle a_1, b_1\rangle$ or $\langle a_2, b_2\rangle$. Also, d centralizes $\bar{b}_1 \bar{b}_2$ since $Z(B)$ is normal in R_1 by (3.3). Thus if d interchanges $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$, (iii) holds. Assume that d normalizes $\langle a_i, b_i\rangle$, $i=1, 2$. Then d centralizes $\bar{b}_1$ and $\bar{b}_2$. Recall that S is normal in T_1 by

(3.2). We have $C_{R_1}(\bar{A})=C_{R_1}(A)A=A\langle t\rangle$ by the hypothesis of this section, so $\widetilde{S\langle d\rangle}=S\langle d\rangle/A$ acts faithfully on $\bar{A}$ and is isomorphic to a subgroup of Out (A). Moreover, $\widetilde{S\langle d\rangle}$ is a four-group generated by $\tilde{h}_0$ and $\tilde{d}$. Now $\bar{a}_i^{h_0}=\bar{a}_i\bar{b}_i$ and $\bar{b}_i^{h_0}=\bar{b}_i$ for $i=1, 2$ by (1.1). If d centralizes $\bar{a}_1$ and $\bar{a}_2$ then $\tilde{d}$ centralizes $\bar{A}$. If $\bar{a}_i^d=\bar{a}_i\bar{b}_i$ for $i=1, 2$ then $\tilde{h}_0\tilde{d}$ centralizes $\bar{A}$. Thus (i) or (ii) holds.

Next suppose $R\neq T$. Then $C_{T_1}(\bar{A})=C_{T_1}(A)A$ is equal to $A\langle t\rangle$ by our hypothesis and $T_1/A\langle t\rangle$ is isomorphic to a Sylow 2-subgroup of Out (A), which is dihedral of order 8. Certainly $T/A\langle t\rangle$ is a four-group. Let d be an involution in T_1-T. Then $R\langle d\rangle/A\langle t\rangle$ is the other four-subgroup of $T_1/A\langle t\rangle$. Since $g\in T-R$ interchanges $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$, (2.2) shows that d normalizes $\langle a_1, b_1\rangle$ and thus $R\langle d\rangle=R_1$ by the definition of R_1. Hence $\mathscr{I}(R_1-R)\neq\varnothing$. As before we see that d satisfies (i) or (ii).

(4.2) $T_2\in \mathrm{Syl}_2(G)$.

PROOF. In view of (3.3) it is enough to show that $Z(B)\langle t\rangle$ is a characteristic subgroup of T_2. For this purpose we distinguish two cases: $R=T$ and $R\neq T$. First assume that $R\neq T$. By our hypothesis $C_{T_1}(A)=\langle z, t\rangle$, so $Z(T_1)=\langle z\rangle$. As $C_{T_1}(A/\langle z\rangle)=C_{T_1}(A)A$, (1.5) (1) shows that $Z_2(T_1)\leqq Z(T \bmod \langle z\rangle)=Z(B)\langle t\rangle$. Since $Z(B)$ and $\langle z, t\rangle$ are normal in T_1, we have $Z_2(T_1)=Z(B)\langle t\rangle$. It follows from (4.1) that $Z(T_1/A\langle t\rangle)=R/A\langle t\rangle$, so $Z_3(T_1)\leqq R$. Let $\tilde{T}_1=T_1/Z_2(T_1)$. Then $Z(\tilde{T}_1)\leqq Z(\tilde{T})=\langle\tilde{a}_1\tilde{a}_2, \tilde{b}_1\rangle$ by (1.5) (1). Since an involution $d\in R_1-R$ does not centralize $\tilde{a}_1\tilde{a}_2$ by (4.1), we conclude that $Z_3(T_1)=\langle b_1, b_2, t\rangle$. Now $C_{T_2}(t)=T$, so $Z(T_2)=\langle z\rangle$. Then $N_{T_2}(\langle z, t\rangle)=T_1$ implies $Z_2(T_2)\leqq Z_2(T_1)$. Since $Z(B)$ is normal in T_2, it follows that $Z_2(T_2)=Z(B)$. Also, $N_{T_2}(R)=T_1$ gives that $Z_3(T_2)\leqq Z_3(T_1)$. Moreover, $Z(B)\langle t\rangle$ is normal in T_2. Hence we have $\Omega_1(Z_3(T_2))=Z(B)\langle t\rangle$, which is characteristic in T_2.

Next assume that $R=T$. Then $R_1=T_1$ by the definition and as above we have $Z(R_1)=\langle z\rangle$ and $Z(B)\langle t\rangle\leqq Z_2(R_1)\leqq Z(R \bmod \langle z\rangle)=\langle b_1, b_2, t\rangle$ since $C_{R_1}(A/\langle z\rangle)=A\langle t\rangle$. Now $Z(T_2)=\langle z\rangle$ and since $N_{T_2}(\langle z, t\rangle)=R_1$, it follows that $t\notin Z_2(T_2)\leqq Z_2(R_1)$. Since $Z(B)$ is normal in T_2, we conclude that $\Omega_1(Z_2(T_2))=Z(B)$, which is characteristic in T_2. As before $Z(T_2 \bmod Z(B))\leqq N_{T_2}(R)=R_1$. Let $\tilde{T}_2=T_2/Z(B)$ and let d be an involution in R_1-R. Suppose $\langle a_1, b_1\rangle\lhd R_1$. Then (i) or (ii) of (4.1) holds. Since $\tilde{b}_1=\tilde{b}_2$ and $\tilde{R}_1=\langle\tilde{t}\rangle\times\tilde{S}\langle\tilde{d}\rangle$ with $\tilde{S}=\tilde{A}\langle\tilde{h}_0\rangle$, $Z(\tilde{R}_1)=\langle\tilde{b}_1, \tilde{t}\rangle$. Thus $Z(T_2 \bmod Z(B))\leqq\langle b_1, b_2, t\rangle$. Since $Z(B)\langle t\rangle$ is normal in T_2, we conclude that $\Omega_1(Z(T_2 \bmod Z(B)))=Z(B)\langle t\rangle$. Suppose $\langle a_1, b_1\rangle\ntriangleleft R_1$. Then (iii) of (4.1) holds and $Z_2(R_1)=Z(B)\langle t\rangle$, whence $Z_2(T_2)=Z(B)$. In this case we have $\tilde{t}\in Z(\tilde{T}_2)\leqq Z(\tilde{R}_1)=\langle\tilde{a}_1\tilde{a}_2, \tilde{b}_1, \tilde{t}\rangle$. By

(3.3), B is normal in T_2 and by (1.4) (3), $\mathscr{E}^*(B)=\{D_1, D_2, E, F\}$ where $D_1=Z(B)\langle h_0\rangle$ and $D_2=Z(B)\langle b_1h_0\rangle$. The subgroups D_1 and D_2 are conjugate in S; so if one of E of F is normal in T_2 then the other is also normal in T_2. Thus if $\tilde{a}_1\tilde{a}_2\in Z(\tilde{T}_2)$ then E and F are normal in T_2 and we have $Z(\tilde{T}_2)=Z(\tilde{R}_1)$. Similarly, if $\tilde{a}_1\tilde{a}_2\tilde{b}_1\in Z(\tilde{T}_2)$ then $Z(\tilde{T}_2)=Z(\tilde{R}_1)$. Hence $Z(\tilde{T}_2)$ is equal to one of $\langle\tilde{t}\rangle$, $\langle\tilde{b}_1, \tilde{t}\rangle$, or $Z(\tilde{R}_1)$. If $Z(\tilde{T}_2)=\langle\tilde{t}\rangle$, $Z_3(T_2)=Z(B)\langle t\rangle$. If $Z(\tilde{T}_2)=\langle\tilde{b}_1, \tilde{t}\rangle$, $\Omega_1(Z_3(T_2))=Z(B)\langle t\rangle$. Finally if $Z(\tilde{T}_2)=Z(\tilde{R}_1)$, $Z(Z_3(T_2))=Z(B)\langle t\rangle$. In any case $Z(B)\langle t\rangle$ is characteristic in T_2 as required.

(4.3) $N(F\langle t\rangle)\leqq H$ *and* $N_{T_2}(F)=T$. *If* $R\neq T$, *the element* $g\in T-R$ *defined in section* 3 *is of order* 4, *that is* $g^2=t$.

PROOF. If $N(F\langle t\rangle)\not\leqq H$, we have $t^{N(F\langle t\rangle)}=Ft$ since $t^G\cap L=\emptyset$ and $N_L(F)$ acts transitively on $F^{\sharp}$. But then $|N(F\langle t\rangle)|_2=8|T|$, contrary to (4.2). Thus $N(F\langle t\rangle)\leqq H$. As $Z(B)\langle t\rangle$ is normal in T_2, $N_{T_2}(F)$ is contained in $N_{T_2}(F\langle t\rangle)=T$. Hence $N_{T_2}(F)=T$. Suppose $R\neq T$ and $|g|=2$. Then $F\langle g, t\rangle$ is the unique elementary abelian subgroup of T of order 2^5 by (1.5) (4). Since B is normal in T_1. it follows that $T_1\triangleright F\langle g, t\rangle\cap B=F$, which is a contradiction. Thus $g^2=t$.

(4.4) *A Sylow* 2-*subgroup of* $O^2(G)$ *is of order* 2^8 *and of sectional rank* 4.

PROOF. As S splits over B, there is a complement K for B in $N_L(B)$ by Gaschütz's theorem. By (1.8) (3), $K/Z(L)\cong\Sigma_3$. Hence setting $\langle e\rangle=S\cap K$ and $\langle k\rangle=[O_3(K), e]$, we have $K=\langle e, k\rangle\times Z(L)$.

We shall show that $N(B\langle t\rangle)$ has a normal subgroups M_2 with the property that $C_2=M_2\langle t\rangle\neq M_2\geqq B$. As $\mathscr{I}(R_1-R)\neq\emptyset$ and as S is normal in T_1 by (3.2), R_1/S is a four-group. Hence R_1/B is elementary abelian of order 8, for $|R_1/C_{R_1}(Z(B))|=2$ and $C_S(Z(B))=B$. Suppose $R\neq T$ and choose $y\in T_2-T_1$. Then as $N_{T_2}(S)=T_1$ has index 2 in T_2 and as $B\triangleleft T_2$, SS^y is a normal subgroup of T_2 contained in T_1 and $S\cap S^y=B$. Likewise, $Z(T)=Z(R)$ implies $N_{T_2}(T)=T_1$ and as $C_T(Z(B))=C_{T_2}(Z(B)\langle t\rangle)$ is normal in T_2, we have $T\cap T^y=C_T(Z(B))$. This shows that $SS^y\not\leqq T$ and thus $SS^y\cap T=S$. Since T_1/SS^y is of order 4, it follows that $T_1'\leqq SS^y\cap T\cap C(Z(B))=B$ and T_1/B is abelian of order 16. By (4.3), $g^2=t$, so we conclude that $\Omega_1(T_1/B)=R_1/B$. In particular, $R_1\triangleleft T_2$ and T_2/R_1 is abelian of order 4. Since $(T_2/C_T(Z(B)))'=C_{T_1}(Z(B))/C_T(Z(B))$ by (3.3) and since $C_T(Z(B))\cap R_1=B\langle t\rangle$, we get that $(T_2/B\langle t\rangle)'=C_{R_1}(Z(B))/B\langle t\rangle$. Then as T_2/C_2 has order 4, C_2/B contains $C_{R_1}(Z(B))/B$, which is a four-group. Suppose $R=T$. Then $|T_2: R_1|=2$ and $|C_2: C_2\cap R_1|=2$, whence $C_2\cap R_1/B$ is a four-group. Therefore in either case C_2/B has rank at least 2. Now the element k acts nontrivially on $C_2/B\langle t\rangle \cong Z(B)$ by (1.8) (3) and (3.3), so it follows that $C_2/B=[C_2/B, k]\times B\langle t\rangle/B$. Set $M_2=[C_2, k]B$. Then $C_2=M_2\langle t\rangle \neq M_2$. Moreover, $M_2=[C_2, N_L(B)]B$ by the structure of $N_L(B)$ so that $M_2 \lhd N(B\langle t\rangle)$.

Let x be an extremal conjugate of t in T_2 with respect to G and set $S_2=M_2S$. Note that $|C_{T_2}(x)|\geqq|C_{T_2}(t)|=|T|$ by the definition. Also, $C_{S_2}(t)=S$ and $T_2=S_2\langle t\rangle$ if $R=T$ and $T_2=S_2\langle g\rangle$ if $R\neq T$ by (4.3). Thus $|S_2\langle t\rangle \cap C(x)|\geqq 2^7$. We wish to show that x is not contained in S_2. Assume by way of contradiction that $x\in S_2$. We shall study the structure of the group $C_2\langle e, k\rangle$ and derive a contradiction. First of all we have $x\notin H$, for otherwise $x\in C_{S_2}(t)=S$, which conflicts with $t^G\cap L=\varnothing$. By (4.3) this implies that x normalizes neither F nor $F\langle t\rangle$.

We argue that $x\in M_2\cap T_1$. Since $M_2/B\cong Z(B)$ and S_2/B is a semidirect product of M_2/B and S/B, S_2/B is dihedral of order 8. As $|T_2\colon T_1|=2$, we have $|M_2\colon M_2\cap T_1|=2$ and $M_2\cap T_1=Z(S_2 \bmod B)$. Thus if $x\notin M_2\cap T_1$, the centralizer of x in $S_2\langle t\rangle/B\langle t\rangle$ has order 4. Consider the series $S_2\langle t\rangle \geqq B\langle t\rangle \geqq F\langle t\rangle \geqq Z(B)\langle t\rangle$ of subgroups of $S_2\langle t\rangle$. Since x does not normalize $F\langle t\rangle$, x centralizes at most four elements of $B\langle t\rangle/Z(B)\langle t\rangle$. Then as $|S_2\langle t\rangle\cap C(x)|\geqq 2^7$, x must centralize $Z(B)\langle t\rangle$. This contradicts $x\notin H$. Thus $x\in M_2\cap T_1$.

Let $W=[M_2, k]$. Since k acts nontrivially on $M_2/B\cong Z(B)$ and since x is an involution in M_2-B, it follows that $M_2/J_0(B)$ is abelian of order 8 and so $M_2/J_0(B)=W/J_0(B)\times B/J_0(B)$. If $W/Z(B)$ is nonabelian, then $(W/Z(B))'=J_0(B)/Z(B)$, for k acts transitively on the nonidentity elements of $J_0(B)/Z(B)$ by (1.8) (3). But then $W/Z(B)$ is dihedral, quasidihedral, or generalized quaternion of order 16 by [12, Theorem 5.4.5], which is impossible since $J_0(B)/Z(B)$ is a four-group. Thus $W'=Z(B)$. Suppose $x\notin W$ and let $\bar{T}_2=T_2/Z(B)\langle t\rangle$. Let u be the involution in $C_B(k)$, so that $B=J_0(B)\langle u\rangle$. As $x\in M_2-W$ and $M_2=WB$, we have $M_2=W\langle x\rangle=W\langle u\rangle$. The element x does not normalize $F\langle t\rangle$, so it does not centralize $\bar{B}$ and the group $\bar{M}_2$ is nonabelian. Thus $\bar{u}$ does not centralize $\bar{W}$. Since k normalizes $C_{\bar{W}}(\bar{u})$ and acts irreducibly on $\bar{W}/\overline{J_0(B)}$, we get that $C_{\bar{W}}(\bar{u})=\overline{J_0(B)}$. Now $\bar{x}\in\bar{W}\bar{u}$ and $\bar{W}$ is abelian by the above, so $C_{\bar{W}}(\bar{u})=C_{\bar{W}}(\bar{x})$. Since $x\notin H$, it follows that $C_{W\langle t\rangle}(x)$ is a proper subgroup of $J_0(B)\langle t\rangle$ and is of order at most 16. But $|S_2\langle t\rangle\cap C(x)|\geqq 2^7$ and $|S_2\langle t\rangle\colon W\langle t\rangle|=4$, a contradiction. Therefore, $x\in W$.

We argue that $C_{M_2}(J_0(B))=J_0(B)$. For this purpose let $V=J_0(B)$ and suppose $C_{M_2}(V)\neq V$. Since k is transitive on the nonidentity elements of W/V, the only $\langle k\rangle$-invariant proper subgroups of M_2/V are W/V and B/V. Hence we have $C_{M_2}(V)=W$. Let $\tilde{W}=W/Z(B)$. Then since $x\in W-B$ and $W'=Z(B)$, W is generated by V and $x^{\langle k\rangle}$ and so $\tilde{W}$ is elementary abelian. The element k has order 3 and acts fixed-point-freely on $\tilde{W}$, so $\langle \tilde{x}^{\langle k\rangle}\rangle$ is a four-group and $\tilde{W}=\langle \tilde{x}^{\langle k\rangle}\rangle \times \tilde{V}$. Set $U=\langle Z(B), x^{\langle k\rangle}\rangle$. Then U has order 16 and k is transitive on $(U/Z(B))^{\#}$, whence $U=Z(B)\cup Z(B)x\cup Z(B)x^k\cup Z(B)x^{k^2}$. This implies that U is elementary abelian. As $W=UV$, we get that W is abelian of order 2^6. Now, suppose there is an abelian subgroup D of T of order 2^6. Then as S has index 2 or 4 in T, $|S\cap D|\geqq 2^4$. Hence $S\cap D=V$ by (1.4) (3). But it follows from (1.1) that $C_T(V)=V\langle t\rangle$. Thus T does not have an abelian subgroup of order 2^6. Since $T\in \mathrm{Syl}_2(H)$ and x is a conjugate of t contained in W, W must be nonabelian. This contradiction shows that $C_{M_2}(J_0(B))=J_0(B)$.

Let $Y=M_2\langle e, k\rangle$ and $\bar{Y}=Y/J_0(B)$. As $C_Y(Z(B))=M_2$, $J_0(B)$ is self-centralizing in Y by the above and $\bar{Y}$ is isomorphic to a subgroup of the automorphism group of $J_0(B)\cong Z_4\times Z_4$. Now $\langle \bar{k}\rangle$ is a Sylow 3-subgroup of $\bar{Y}$ and $\bar{W}=[\bar{M}_2, \bar{k}]$, so $J_0(B)\cap C(\bar{w})=Z(B)$ for all $\bar{w}\in \bar{W}^{\#}$ by (2.3). In particular, $J_0(B)\cap C(x)=Z(B)$. As F lies between B and $Z(B)$ and x does not normalize F, it follows that $C_{B/Z(B)}(x)=J_0(B)/Z(B)$ and thus $C_B(x)=Z(B)$. But $|S_2\langle t\rangle\cap C(x)|\geqq 2^7$ and $|S_2\langle t\rangle/B|=2^4$, so $|C_B(x)|\geqq 8$, which is a contradiction. Therefore, $x\notin S_2$ as asserted.

We have shown that any extremal conjugate of t in T_2 with respect to G is not contained in S_2. Thus by [29, Corollary 5.3.2], $t\notin O^2(G)$. As $S\leqq L\leqq O^2(G)$, the action of $\langle e, k\rangle$ on M_2/B gives that $S_2\leqq O^2(G)$. Hence $S_2=T_2\cap O^2(G)$ is a Sylow 2-subgroup of $O^2(G)$. It follows from (4.3) that $N_{S_2}(F)=S$. Thus $C_{S_2}(F)=F$ and by Harada [17, Theorem 2], S_2 is of sectional rank 4. The proof is complete.

(4.5) $E(G)\cong G_2(9)$.

PROOF. Let $\bar{G}=G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$ and so $F^*(\bar{G})$ is simple. The preceding lemma together with [13] and [6] shows that $F^*(\bar{G})\cong G_2(9)$. Now the assertion follows from [28, (2.10)].

§ 5. The case $C_{T_1}(A)\neq\langle z, t\rangle$.

From now on we assume that $C_{T_1}(A)\neq\langle z, t\rangle$. Set $C_0=C_{T_1}(A)$. It is dihedral of order 8 since $C_T(A)=\langle z, t\rangle$. Also, $C_0=C_{R_1}(A)$. Let d be an involution in $C_0-\langle z, t\rangle$, so that $C_0=\langle t, d\rangle$, $\mathscr{E}^*(C_0)=\{\langle z, t\rangle, \langle z, d\rangle\}$, and $R_1=R\langle d\rangle$. Set $C_1=AC_0$.

(5.1) *If $R\neq T$, the element g defined in section 3 is an involution and furthermore we may assume that $d^g=d$.*

PROOF. The element g normalizes $\langle z, d\rangle$, so $d^g=d$ or zd. Replacing

g with tg if necessary, we may assume that $d^g=d$. By the definition $g^2=1$ or else $g^2=t$, so we get $g^2=1$.

DEFINITION. Let $M_2=[C_2, N_L(B)]B$ and $R_2=RM_2$.

(5.2) *$C_2=M_2\langle t\rangle\neq M_2$, $C_0\leqq C_2$, and $C_2/Z(B)$ is elementary abelian.*

PROOF. As $T_1<T_2$, d lies in $C_{T_2}(Z(B))=C_T(Z(B))C_2$. If $R=T$, $C_T(Z(B))=B\langle t\rangle$ and $d\in C_2$. Suppose $R\neq T$. Then $C_T(Z(B))=B\langle g, t\rangle$. Let bars denote images in $N(B\langle t\rangle)/J_0(B)C_H(L)$. Let k be an element of a Sylow 3-subgroup of $N_L(B)$ not contained in $Z(L)$, so that $N_L(B)=Z(L)B\langle k\rangle\langle a_1b_1b_2\rangle$ and $|\bar{k}|=3$ by (1.8) (3). As $\overline{C_{T_1}(Z(B))}=\bar{B}\langle\bar{d}, \bar{g}\rangle\cong E_8$, $\bar{C}_2\cap\bar{B}\langle\bar{d}, \bar{g}\rangle$ is a four-group. By (3.3), $\bar{C}_2/\bar{B}\cong Z(B)$, so $\bar{C}_2=[\bar{C}_2, \bar{k}]\times\bar{B}$. Then as $\bar{B}\langle\bar{k}\rangle\lhd\overline{N_H(B)}$, $N_H(B)$ normalizes $[\bar{C}_2, \bar{k}]=[\bar{C}_2, \bar{B}\langle\bar{k}\rangle]$ and $\overline{C_{T_2}(Z(B))}=[\bar{C}_2, \bar{k}]\times\bar{B}\langle\bar{g}\rangle$. Now $a_1b_1b_2\in A$ and $C_{\bar{B}\langle\bar{g}\rangle}(a_1b_1b_2)=\bar{B}$, for $[a_1b_1b_2, g]\notin J_0(B)$ by (1.1). Hence $\bar{d}\in\overline{C_{T_2}(Z(B))}\cap C(a_1b_1b_2)\leqq\bar{C}_2$, so $d\in C_2C_H(L)=C_2\times O(H)$ by (3.3). Thus $C_0\leqq C_2$.

As C_0 and $\langle z, t\rangle$ are normal in T_1, $\langle z, d\rangle$ is also normal in T_1 and $[d, B\langle t\rangle]=\langle z\rangle$. Now k is transitive on $(C_2/B\langle t\rangle)^{\#}$, whence $C_2=B\langle t\rangle\cup B\langle t\rangle d\cup B\langle t\rangle d^k\cup B\langle t\rangle d^{k^2}$. As $B\langle t\rangle/Z(B)$ is elementary abelian, $C_2/Z(B)$ is of exponent 2 and thus it is elementary abelian. Let tildes denote images in $N(B\langle t\rangle)/B$. Then $\tilde{C}_2=[\tilde{C}_2, \tilde{k}]\times\langle\tilde{t}\rangle$. As $Z(L)B\langle k\rangle\lhd N_L(B)$ and $[C_2, Z(L)]=1$, we have $[\tilde{C}_2, \tilde{k}]=\tilde{M}_2$ and $C_2=M_2\langle t\rangle\neq M_2$.

(5.3) *$N(A\langle t\rangle)=N_H(A)\langle d\rangle\leqq N(C_0)$ and $J_r(T_1/\langle z\rangle)=C_1/\langle z\rangle$.*

PROOF. As $Z(A\langle t\rangle)=Z(R)$, $N(A\langle t\rangle)\leqq N(Z(R))=N_H(Z(R))\langle d\rangle$ by (3.1). Then $N_H(A)=N_H(Z(R))$ implies $N(A\langle t\rangle)=N_H(A)\langle d\rangle$. Now $C_H(A)=O(H)Z(R)$ and $[O(H), C_0]=1$, so $N(A\langle t\rangle)\cap C(A)=O(H)\times C_0$ and $C_0\lhd N(A\langle t\rangle)$. As $C_1/\langle z\rangle$ is elementary abelian, the latter assertion follows from (1.5) (3).

DEFINITION. Let $C_3=O_2([C_{N(F\langle t\rangle)}(F), N_H(F)])F\langle t\rangle$, $M_3=[C_3, N_L(F)]F$, $S_3=SM_3$, and $R_3=RM_3$.

(5.4) (1) *$N(F\langle t\rangle)=N_H(F)M_3$, $H\cap M_3=F$, $t^{M_3}=Ft$, and $M_3/F\cong F$ as $N_H(F)$-modules.*

(2) *$C_3=M_3\langle t\rangle\geqq C_0$ and $[C_1, C_3]\leqq FC_0$.*

(3) *$M_3\geqq C_{M_2}(F)$ and $M_2=C_{M_2}(F)B$ with $|C_{M_2}(F)/F|=4$. Thus R_2 is a subgroup of R_3 of index 2.*

PROOF. Set $X=N(F\langle t\rangle)$ and $X_0=C_H(F)$. Then $C_3\leqq X$, for $C_3/Z(B)$ is abelian. As $N_L(F)\leqq X$, $t^X=Ft$ by (3.1) and $F\lhd X$. Set $Y=C_X(F)$. Then $d\in Y$, so the map defined by $y\mapsto[y, t]$ for $y\in Y$ is a $N_H(F)$-homomorphism of Y onto F. As $C_Y(t)=X_0$, Y/X_0 is $N_H(F)$-isomorphic to F. Also, $t^Y=Ft$ and $X=N_H(F)Y$. Now $X/Y\cong N_H(F)/X_0\cong GL_3(2)$ and $|X_0: C_H(L)F|=|T: R|$, so $O(X)=O(H)$. Let $\bar{X}=X/O(H)F$. A Sylow 7-subgroup Q of $N_L(F)$ acts fixed-point-freely on $Y/X_0\cong F$, so $C_{\bar{Y}}(Q)=\bar{X}_0$. Then $\bar{X}_0\leqq Z(\bar{Y})$ by [28, (2.4)] and $\bar{X}_0\langle\bar{d}\rangle$ is elementary abelian. Since $\bar{Y}$ is a union of the $N_L(F)$-conjugates of $\bar{X}_0\langle\bar{d}\rangle$, $\bar{Y}$ is elementary abelian.

Let $\tilde{X}=X/FC_H(L)$ and $Y_1=[Y, N_H(F)]FC_H(L)$. If $R=T$, $Y=Y_1$. Suppose $R\neq T$. Then $|\tilde{X}_0|=2$. Set $D=C_{C_2}(F)$. Then C_2/D acts faithfully on F and centralizes $Z(B)$, so its order is at most 4. As $C_B(F)=F$, we get $C_2=BD$ and $|C_2/D|=|D/F\langle t\rangle|=4$. Then $\tilde{Y}\geqq\tilde{D}\times\tilde{X}_0$, so $|C_{\tilde{Y}}(\tilde{S})|\geqq 4$. As $\widetilde{N_L(F)}\cong GL_3(2)$, [13, Part II, Lemma 3.7] shows that there is a subgroup $\tilde{Y}^*$ normalized by $N_L(F)$ such that $\tilde{Y}=\tilde{Y}^*\times\tilde{X}^0$. Then $\tilde{Y}^*=[\tilde{Y}, N_L(F)]=\tilde{Y}_1$. The element k in the first paragraph of the proof of (4.4) acts fixed-point-freely both on B/F and on $C_2/B\langle t\rangle\cong Z(B)$, so also on $\tilde{C}_2$. Hence $D=[D, k]F\langle t\rangle\leqq Y_1$. The action of $N_L(F)$ on Y_1 gives $Y_1=C_{Y_1}(O(H))O(H)$, for $[D, O(H)]=1$ by (3.3). Thus Y_1 is 2-closed with $O_2(Y_1)=C_3$ and $d\in D\leqq C_3$. Now $X=N_H(F)C_3$, $H\cap C_3=F\langle t\rangle$, and $C_3/F\langle t\rangle\cong F$ as $N_H(F)$-modules. Since $C_F(A)=\langle z\rangle$, $(C_3/F\langle t\rangle)\cap C(A)=FC_0/F\langle t\rangle$. Then $[A, F]=\langle z\rangle$ implies $[A, C_3]\leqq FC_0$. As $FC_0\lhd C_3$, $[C_1, C_3]\leqq FC_0$.

As $C_2=M_2\langle t\rangle$, we have $M_2=C_{M_2}(F)B$ and $|C_{M_2}(F)/F|=4$. Let $\bar{C}_3=C_3/F$, so that $\bar{C}_3\cong E_{16}$ by the first paragraph. As $\bar{C}_3\geqq\overline{C_{M_2}(F)}\times\langle\bar{t}\rangle$, $|C_{\bar{C}_3}(S)|\geqq 4$. By [13, Part II, Lemma 3.7] the action of $N_L(F)/Z(L)F$ on $\bar{C}_3$ is decomposable and so $\bar{C}_3=\bar{M}_3\times\langle\bar{t}\rangle$. The element k of the above paragraph acts fixed-point-freely on M_2/F', so $C_{M_2}(F)=[C_{M_2}(F), k]F\leqq M_3$. The proof is complete.

(5.5) *If* $R\neq T$, *then* $[g, M_3]=1$.

PROOF. By (1.10), g centralizes $N_L(F)/Z(L)F$, so $N_L(F)$ acts on $C_{M_3}(Z(L)F\langle g\rangle)=C_{M_3}(g)$. By (5.1), $C_{M_3}(g)\geqq F(C_0\cap M_3)$. As $C_0\cap M_3\not\leqq F$, the action of $N_L(F)$ yields the assertion.

DEFINITION. Let $C_4=O_2([C_{N(E\langle t\rangle)}(E), N_H(E)])E\langle t\rangle$.

(5.6) (1) $N(E\langle t\rangle)=N_H(E)C_4$, $H\cap C_4=E\langle t\rangle$, $t^{C_4}=Et$, *and* $C_4/E\langle t\rangle\cong E$ *as* $N_H(E)$*-modules.*

(2) $C_0\leqq C_4$ *and* $[C_1, C_4]\leqq EC_0$.

PROOF. Set $X=N(E\langle t\rangle)$, $X_0=C_H(E)$, and $Y=C_X(E)$. Then $C_2\leqq X$ and $C_0\leqq Y$ by (5.2). As $N_L(E)\leqq X$ and $t^G\cap L=\varnothing$, (1.8) (4) shows $t^X=Et$ and $E\lhd X$. If P is a Sylow 2-subgroup of X containing S, then $t^P=Et$ and as $|S/C_S(E)|=|\mathrm{Aut}(E)|_2$, $P=C_P(E)S$. Hence $t^Y=Et$ and $X=N_H(E)Y$.

The map defined by $X_0y \mapsto [y, t]$ for $y \in Y$ is an $N_H(E)$-isomorphism of Y/X_0 onto E. Now $N_L(E)/Z(L)E \cong \Sigma_4$, $|X_0: EC_H(L)| = |T: R|$, and an element f of a Sylow 3-subgroup of $N_L(E)$ not contained in $Z(L)$ acts transitively both on $(E/\langle z\rangle)^\#$ and on $(A/E)^\#$ by (1.8) (4). Note that $O(X) = O(H)$. Let $\bar{X} = X/EC_H(L)$. If $R = T$, $\bar{Y} \cong E$. Suppose $R \neq T$. Then $|\bar{Y}| = 16$ and $|\bar{X}_0| = 2$. As $N_H(E) \leqq N(A)$, $[f, C_0] = 1$ by (5.3). As $\bar{Y}/\bar{X}_0 \cong E$, $C_{\bar{Y}}(f) = \bar{C}_0 \times \bar{X}_0$ and $[\bar{Y}, f] \cap C(f) \leqq \bar{X}_0$. Moreover, $\bar{Y} = [\bar{Y}, f] * C_{\bar{Y}}(f)$ by [28, (2.4)]. The group $C_2/C_{C_2}(E)$ centralizes $Z(B)$, so its order is at most 4. As $C_B(E) = E$, we have $C_2 = C_{C_2}(E)B$ and $C_{C_2}(E)/E\langle t\rangle \cong C_2/B\langle t\rangle$. If $[\bar{Y}, f] \geqq \bar{X}_0$, $[\bar{Y}, f]$ is quaternion. But $\overline{C_{C_2}(E)} \times \bar{X}_0 \cong E_8$, a contradiction. Thus $\bar{Y} = [\bar{Y}, f] \times C_{\bar{Y}}(f) \cong E_{16}$.

As $Z(T_1) = \langle z\rangle$ and $|T_2: T_1| = 2$, $C_1 \lhd T_2$ by (5.3) and $N_{\bar{Y}}(\bar{C}_1) \geqq \overline{C_{C_2}(E)X_0}$. Thus $\bar{Y}$ normalizes $\bar{C}_1$ by the action of f, so $[\bar{C}_1, \bar{Y}] \leqq \bar{C}_1 \cap \bar{Y} = \bar{C}_0$.

Let $Y_1 = [Y, N_H(E)]EC_H(L)$ and $\tilde{Y} = \bar{Y}/\bar{C}_0$. If $R = T$, $Y_1 = Y$. If $R \neq T$, $\tilde{Y} = [\tilde{Y}, f] \times \tilde{X}_0$ with $|\tilde{X}_0| = 2$. As A centralizes $\tilde{Y}$ by the above and $\bar{A}\langle\bar{f}\rangle = O_{2,3}(\overline{N_L(E)})$, we get $[\tilde{Y}, f] = \tilde{Y}_1$. If $\bar{C}_0 \not\leqq \bar{Y}_1$, $\bar{Y} = \bar{Y}_1 \times \bar{C}_0 \times \bar{X}_0$. This is impossible since $N_L(E)$ is indecomposable on Y/X_0. Thus $\bar{C}_0 \leqq \bar{Y}_1$ and $\bar{Y} = \bar{Y}_1 \times \bar{X}_0$. Now $C_X(O(H))O(H)$ is a normal subgroup of X containing $N_L(E)C_H(L)$, so it contains Y_1. Hence $Y_1 = O_2(Y_1) \times O(H)$. As $O_2(Y_1) = C_4$, (1) holds. Moreover, $\bar{C}_0 \leqq \bar{Y}_1$ implies $C_0 \leqq C_4$. As $[\bar{C}_1, \bar{Y}] \leqq \bar{C}_0$, $[C_1, C_4] \leqq C_0EC_H(L) \cap C_4 = EC_0$ and (2) holds.

(5.7) *Let $V_1 = C_0 \cap M_3$. Then $V_1 \leqq M_2$ and one of the following holds.*

(1) *$V_1 = \langle z, d\rangle$ and M_3 is elementary abelian.*

(2) *$V_1 = \langle dt\rangle$ and M_3 is homocyclic abelian of exponent 4 and is inverted by t.*

PROOF. By (5.2), $C_0 \cap M_2 = \langle dt\rangle$ or $\langle z, d\rangle$. Also, $C_0 \cap M_2 = V_1$ since $C_{M_2}(F) \leqq M_3$. (5.4) shows that C_3 normalizes C_1 and $SM_2 = S_3 \cap R_2 \lhd R_3$. Then as $C_1 \cap SM_2 = AV_1$, $Z(AV_1) = V_1 \lhd M_3$ and hence $|M_3/C_{M_3}(V_1)| \leqq 2$. As M_3/F is $N_L(F)$-isomorphic to F and $N_L(A) \leqq N(C_0)$, $N_L(A) \cap N_L(F)$ acts transitively on $(M_3/FV_1)^\#$ by (1.8) (5). Thus $Z(M_3) \geqq FV_1$. Now the assertion follows from the action of $N_L(F)$.

§ 6. The case $V_1 = \langle dt\rangle$.

In this section we assume that $V_1 = \langle dt\rangle$. By (5.7), M_3 is homocyclic abelian of exponent 4 and t inverts it. We shall show that Case (3) of the main theorem occurs.

(6.1) *$N(C_0) \leqq N(C_1) \leqq N(AV_1) \leqq N(V_1)$, $|N(C_0): N(A(\langle t\rangle)| = 2$, $C(C_1/V_1) = O(H)C_1$, and $N(C_1)/O(H)C_1$ is isomorphic to a subgroup of* Aut (A_8) *con-*

taining A_6 *with* $TM_3 \in \mathrm{Syl}_2(N(C_1))$.

PROOF. Let $X=N(C_1)\cap N(V_1)$, $Y=N(A\langle t\rangle)$, and $\bar{X}=X/V_1$. By (5.3), $Y=N(\langle z, t\rangle)=N_H(A)\langle d\rangle \leqq N(C_0) \leqq N(V_1)$ and as $C_1=AC_0$, we have $Y\leqq X$. Also, $M_3\leqq X$ by (5.4) (2) and $\bar{t}^{M_3}=\bar{F}\bar{t}$. The only four-subgroups of C_0 are $\langle z, t\rangle$ and $\langle z, d\rangle$, so Y is a subgroup of $C_X(\bar{t})=N(C_0)$ of index at most 2. As $AV_1=C_{C_1}(V_1)\lhd X$, $\bar{t}^X\leqq\bar{C}_1-\bar{A}=\bar{A}\bar{t}$. (1.8) (2) shows that under the action of $N_L(A)$, $\bar{A}\cong A/\langle z\rangle$ is divided into four orbits of lengths 1, 3, 3, and 9 and that $\bar{b}_1\bar{b}_2\in\bar{F}$ belongs to the orbit of length 9. Thus $|\bar{t}^X|=10$, 13, or 16. In any case $\bar{t}^X$ generates $\bar{C}_1$. The order of $\mathrm{Aut}(\bar{C}_1)\cong GL_6(2)$ is not divisible by 13, so $|\bar{t}^X|\neq 13$. If $\bar{t}^X=\bar{A}\bar{t}$, there is an element $x\in X$ such that $\bar{t}^x=\bar{a}_1\bar{t}$. Then $t^x=va_1t$ for some $v\in V_1$. But t inverts V_1 and $a_1\in A$ centralizes C_0, so $(va_1t)^2=a_1^2(vt)^2=z$ and $|va_1t|=4$, a contradiction. Thus $|\bar{t}^X|=10$. Let $\tilde{X}=X/C_X(\bar{C}_1)$. Then $(\tilde{X}, \bar{t}^X)$ is a 2-transitive permutation group of degree 10. (1.10) (1) shows that $C_Y(\bar{C}_1)=C_H(A/\langle z\rangle)V_1=O(H)C_1$, so $Y/C_Y(\bar{C}_1)\cong N_H(A)/C_H(L)A$ is of order $2\cdot 3^2$ or $2^2\cdot 3^2$. Since $|C_X(\bar{t}): Y|\leqq 2$, we have $|\tilde{X}|=2^n\cdot 3^2\cdot 5$ where $2\leqq n\leqq 4$. Then a minimal normal subgroup $\tilde{N}$ of $\tilde{X}$ is simple and by Brauer [5] it is isomorphic to A_5 or A_6. If $\tilde{N}\cong A_5$, then $|C_{\tilde{X}}(\tilde{N})|$ is divisible by 3, a contradiction. Thus $\tilde{N}\cong A_6$ and $C_{\tilde{X}}(\tilde{N})=1$. If $C_X(\bar{t})=Y$ or $C_X(\bar{C}_1)\neq C_Y(\bar{C}_1)$, then $\tilde{X}\cong A_6$ and $R\neq T$ so that $\tilde{Y}\cong Y/O(H)C_1\cong\Sigma_3\times\Sigma_3$ by (1.10) (1). But the normalizer of a Sylow 3-subgroup of A_6 is a Frobenius group of order 36, a contradiction. Thus $|C_X(\bar{t}): Y|=2$ and $C_X(\bar{C}_1)=O(H)C_1$. Now $N_H(C_1)=N_H(A)$, for $H\cap C_1=A\langle t\rangle$. Hence $|t^{N(C_1)}|=|N(C_1): N_H(A)|=|N(C_1): X|\cdot|X: C_X(\bar{t})|\cdot|C_X(\bar{t}): N_H(A)|=40|N(C_1): X|$. By (3.1), $t^{N(C_1)}\leqq C_1-A$. Moreover, there are precisely $71-19=52$ involutions in C_1-A since $C_1\cong D_8*D_8*D_8$. Thus $N(C_1)\leqq N(V_1)$. As $Z(AV_1)=V_1$, (6.1) holds.

(6.2) S_3 *is isomorphic to a Sylow* 2*-subgroup of the Higman-Sims simple group.*

PROOF. The Schur multiplier of $GL_3(2)$ has order 2, so $N_L(F)=N_L(F)'\times Z(L)$ and $N_L(F)'M_3$ is an extension of $Z_4\times Z_4\times Z_4$ by $GL_3(2)$. Since $N_L(F)'\cap M_3=F=\Omega_1(M_3)$, the assertion follows from (2.6) and (2.7).

(6.3) $R\neq T$.

PROOF. By (2.8), $J_0(S_3)=M_3$ and $Z(S_3)$ has order 2. Thus $Z(S_3)=\langle z\rangle$ and so $C_{S_3}(V_1)=AM_3$, for AM_3 is a maximal subgroup of S_3. By [14, (2.11), (2.27), (2.28)], $S_3/\langle z\rangle$ has exactly two elementary abelian subgroups of order 32, whose preimages in S_3 are isomorphic to $Q_8*Q_8*Z_4$. Now $AV_1\cong Q_8*Q_8*Z_4$ and $(AV_1)'=\langle z\rangle$. Denote by $W/\langle z\rangle$ the other elementary abelian subgroup of $S_3/\langle z\rangle$ of order 32. As $Z(AV_1)=V_1$, [14, (2.25)] shows that $AM_3=C_{S_3}(V_1)=AW$ and AM_3/V_1 is of type $PSL_3(4)$. Hence $\mathscr{E}^*(AM_3/V_1)=\{AV_1/V_1, W/V_1\}$. It then follows that $\mathscr{E}^*(AM_3/\langle z\rangle)=\{AV_1/\langle z\rangle, W/\langle z\rangle\}$ and $Z(AM_3/\langle z\rangle)=AV_1\cap W/\langle z\rangle$. As $|AV_1\cap W|=16$, we also have $Z(AM_3/\langle z\rangle)=FV_1/\langle z\rangle$. As $N_L(F)'M_3/F$ is the split extension of an elementary abelian group of order 8 by $GL_3(2)$, S_3/F is of type $GL_4(2)$. Hence $Z(S_3/F)=FV_1/F$, AM_3/FV_1 is the unique elementary abelian subgroup of S_3/FV_1 of order 16, and $AM_3/F\cong Q_8*Q_8$. Thus $Z(AM_3\langle t\rangle/F)=FV_1\langle t\rangle/F$. As $[t, M_3]=F$, we get $Z(AM_3\langle t\rangle/\langle z\rangle)=FV_1/\langle z\rangle$. Suppose there exists an elementary abelian subgroup $U/\langle z\rangle$ of $R_3/\langle z\rangle$ of order 2^6 different from $C_1/\langle z\rangle$. Then since $R_3=S_3\langle t\rangle$, $S_3\cap U=W$ and so $AM_3\langle t\rangle=C_1U$ and $|C_1\cap U|=2^5$. But $C_1\cap U/\langle z\rangle\leqq Z(AM_3\langle t\rangle/\langle z\rangle)=FV_1/\langle z\rangle$, a contradiction. Therefore, $C_1/\langle z\rangle$ is a unique elementary abelian subgroup of $R_3/\langle z\rangle$ of order 2^6.

Assume that $R=T$. Then $R_3\in \mathrm{Syl}_2(N(C_1))$ by (6.1). As $Z(R_3)=\langle z\rangle$, the above shows $N(R_3)\leqq N(C_1)$. Hence $R_3\in\mathrm{Syl}_2(G)$. (2.6) gives a presentation of S_3, namely, S_3 is generated by the elements u, v, w, r, s subject to the relations listed in (2.6). We have $Z(S_3)=\langle u^2w^2\rangle$ and $Z_2(S_3)=\langle u^2v^2, uw\rangle$. As $V_1\lhd S_3$, it follows that $Z_2(S_3)=Z(B)V_1$. Since S_3/F is a split extension of M_3/F by S/F with $S/F\cong D_8$, $S^a=\langle vr, vs\rangle F$ or $\langle uvwr, vs\rangle F$ for some $a\in M_3$ by (2.8) (3). Replacing r and s with $r^{a^{-1}}$ and $s^{a^{-1}}$, we may assume that $S=\langle vr, vs\rangle F$ or $\langle uvwr, vs\rangle F$. We argue that every extremal conjugate of t in R_3 with respect to G lies in R_3-S_3. Suppose false and choose an extremal conjugate e of t in R_3 such that $e\in S_3$. Then as $C_{R_3}(t)=R\in\mathrm{Syl}_2(H)$, there is an element $y\in G$ with $R^y=C_{R_3}(e)$ and $t^y=e$. As $R_3=S_3\langle t\rangle$, $|C_{S_3}(e)|=2^6$ or 2^7. Thus (2.8) (2) implies that e is conjugate to vw^3s^2 in S_3, for $t\notin H'$. Hence we may assume that $e=vw^3s^2$. Now $R^y=C_{R_3}(e)$, $z=u^2w^2$, $Z(R)=\langle z, t\rangle$, and $Z(R)\cap R'=\langle z\rangle$. We see that $Z(C_{R_3}(e))=\langle z, e\rangle$ and $Z(C_{R_3}(e))\cap C_{R_3}(e)'=\langle z\rangle$ since $C_{S_3}(vw^3s^2)'=\langle (vs)^2\rangle$ contains u^2w^2. Therefore, $\langle z\rangle^y=(Z(R)\cap R')^y=\langle z\rangle$ and $y\in C(z)$. As $Z(B)V_1=Z_2(S_3)=\langle u^2v^2, uw\rangle$, we have $e=vw^3s^2=(uw)^3(vs)^2\in SV_1$. Moreover, $[e, Z_2(S_3)]=1$ and so $e\in SV_1\cap C(V_1)=AV_1$. Let $\overline{C(z)}=C(z)/\langle z\rangle$. Then $\bar{R}_3$ is a Sylow 2-subgroup of $\overline{C(z)}$ and $\bar{t}, \bar{e}\in\bar{C}_1$. Since $\bar{t}^{\bar{y}}=\bar{e}$ and $\bar{C}_1$ is weakly closed in $\bar{R}_3$ by the first paragraph, $\bar{t}$ and $\bar{e}$ are conjugate in $\overline{C(z)}\cap N(\bar{C}_1)$ and so $t^x\in\langle e, z\rangle$ for some $x\in N(C_1)$. By (6.1), $N(C_1)\leqq N(V_1)$, whereas $[t, V_1]\neq 1$ and $[t^x, V_1]=1$. This contradiction implies that every extremal conjugate of t in R_3 lies in R_3-S_3. Now [29, Corollary 5.3.2] gives that $t\notin O^2(G)$ and since $L\leqq O^2(G)$ and $M_3=[M_3, N_L(F)]\leqq O^2(G)$, we have $S_3=R_3\cap O^2(G)\in\mathrm{Syl}_2(O^2(G))$. Hence $E(G/O(G))$ is isomorphic to the Higman-Sims simple group by [14, Theorem A]. But in view of the centralizers of the involutions in the automorphism group of the Higman-Sims simple group [3, p. 441], we see that this is incompatible with the structure of H. The proof is complete.

(6.4) *Case* (3) *of the main theorem holds.*

PROOF. We apply the argument in the proof of [11, (6F)] to A, V_1, C_0, and C_1 in place of A_1, W, D_0, and D_1 respectively. For this purpose it is enough to prove the following four statements:

(1) $N(AV_1)/C(AV_1/V_1)\cong\Sigma_6$,

(2) $C_1\in \mathrm{Syl}_2(C(AV_1/V_1))$ and $N(AV_1)=N(C_1)C(AV_1/V_1)$,

(3) $N(C_0)C(AV_1/V_1)/C(AV_1/V_1)\cong\Sigma_3$ wreath Z_2,

(4) $|N(V_1): N(AV_1)|$ is even.

As $C_1=AV_1\langle t\rangle$, (6.1) gives $N(C_1)\cap C(AV_1/V_1)=O(H)C_1$. Moreover, $C(AV_1/V_1)\lhd N(AV_1)$, so we have $N(AV_1)=N(C_1)C(AV_1/V_1)$ by the Frattini argument. Thus (2) holds. By (5.5), g centralizes S_3/AV_1, so $TM_3/C_1=R_3/C_1\times\langle g\rangle C_1/C_1$. Hence it follows from (2.1) and (6.1) that $N(C_1)/O(H)C_1\cong\Sigma_6$. Thus (1) holds. Now $N(A\langle t\rangle)/O(H)C_1\cong N_H(A)/C_H(L)A\cong\Sigma_3\times\Sigma_3$ by (1.10) (1). Since $|N(C_0): N(A\langle t\rangle)|=2$ and $N(C_0)\leqq N(C_1)$, we have $N(C_0)/O(H)C_1\cong\Sigma_3$ wreath Z_2 which is the normalizer of a Sylow 3-subgroup of $N(C_1)/O(H)C_1\cong\Sigma_6$. Thus (3) holds.

We wish to show that $C(g)\cap N(\langle g, t\rangle)\not\leqq H$. Suppose false and set $C=C(g)$. Then $N_C(\langle g, t\rangle)=C_C(t)$. By (6.3), $H=LO(H)\langle g, t\rangle$ and so $C_C(t)=C_H(g)=C_L(g)C_{O(H)}(g)\langle g, t\rangle$. Set $J=C_L(g)'$. Then by (1.7), $J\cong SL_2(8)$, $F\in\mathrm{Syl}_2(C_L(g))$, and $C_C(t)\cap C(J)=C_{O(H)}(g)\langle g, t\rangle$. Let $\bar{C}=C/\langle g\rangle$. Then as $C_{\bar{C}}(\bar{t})=N_C(\langle g, t\rangle)/\langle g\rangle=\overline{C_C(t)}$, we have $C_{\bar{C}}(\bar{t})\cap C(\bar{J})=\overline{C_{O(H)}(g)}\langle\bar{t}\rangle$. Thus $\bar{J}$ is a standard subgroup of $\bar{C}$ isomorphic to $SL_2(8)$ and $\langle\bar{t}\rangle$ is a Sylow 2-subgroup of $C_{\bar{C}}(\bar{J})$. Moreover, $\bar{C}$ contains $\bar{M}_3\cong Z_4\times Z_4\times Z_4$ by (5.5). A theorem of Griess, Mason, and Seitz [16] together with [28, (2.10)] shows that $E(\bar{C})/Z(E(\bar{C}))\cong PSU_3(8)$ or $PSL_3(8)$ and $C_{\bar{C}}(E(\bar{C}))$ has odd order. In view of the Schur multipliers of these simple groups, we have $E(C \bmod \langle g\rangle)=C^*\times\langle g\rangle$ and $\langle g\rangle\in\mathrm{Syl}_2(C_C(C^*))$ where $C^*=E(C \bmod \langle g\rangle)'$. Thus C^* is a standard subgroup of G and $\langle g\rangle$ is a Sylow 2-subgroup of $C(C^*)$. As $C^*\cong PSU_3(8)$, $SU_3(8)$, or $PSL_3(8)$, [16] and [23] show that $E(G/O(G))$ is isomorphic to one of $PSU_3(8)\times PSU_3(8)$, $PSL_3(8)\times PSL_3(8)$, or $PSL_3(64)$. By using [4], we see that this is incompatible with the structure of H. Therefore, $N_C(\langle g, t\rangle)\not\leqq H$.

Put $X=N(F\langle g, t\rangle)$. Since T does not have an abelian subgroup of order 2^7 by (1.4) (3) and $M_3\langle g\rangle$ is abelian of order 2^7, $t^G\cap F\langle g\rangle=\varnothing$. By the above paragraph $|N_C(\langle g, t\rangle): C_H(g)|=2$ and $F\langle g, t\rangle\in\mathrm{Syl}_2(C_H(g))$, whence a Sylow 2-subgroup of $N_C(\langle g, t\rangle)$ containing $F\langle g, t\rangle$ lies in X and acts transitively on $\{t, gt\}$. As $M_3 \leqq X$ and $t^{M_3}=Ft$, we have that $t^X=F\langle g\rangle t$ and $X\leqq N(F\langle g\rangle)$. By (1.10) (2), g centralizes $N_L(F)'/F$ since $N_L(F)=N_L(F)'\times Z(L)$, so $T\in \mathrm{Syl}_2(C_X(t))$. Let T_4 be a Sylow 2-subgroup of X containing $T_3=TM_3$. Then $|T_4: T_3|=2$ and $t^{T_4}=t^X$. We argue that $V_1 \lhd T_4$. We have shown in the proof of (6.3) that $Z(S_3)=\langle z\rangle$, $Z_2(S_3)=Z(B)V_1$, and $Z(S_3/F)=FV_1/F$. As $C_{T_3}(t)=T$ and $Z(T)=\langle z, t\rangle$ by (1.5) (1), $Z(T_3)=\langle z\rangle$. Also, we get $Z(T_3/F)=FV_1\langle g, t\rangle/F$ and $FV_1\langle g, t\rangle \cap C(M_3/Z(B))=FV_1\langle g\rangle$, for $[M_3, t]\not\leqq Z(B)$. Furthermore, it follows from (1.1) that $FV_1\langle g\rangle \cap C(A/Z(B))=FV_1$. Thus $Z(T_3/Z(B))=FV_1/Z(B)$ and $Z_2(T_3)=Z(B)V_1$. As $\Omega_1(Z(B)V_1)=Z(B)$, we conclude that FV_1 and $F=\Omega_1(FV_1)$ are characteristic subgroups of T_3. In particular, they are normal in T_4. Now $C_X(t)=N_H(F\langle g\rangle)=N_{LO(H)}(F\langle g\rangle)\langle g, t\rangle \leqq N_L(F)C_{O(H)}(g)\langle g, t\rangle$, for $F\langle g\rangle \cap L=F$, and $N_L(F)'\leqq C_X(t)$. As $X=\langle T_4, C_X(t)\rangle$, this implies that $F\lhd X$. Let $Y=C_X(F)$ and $\bar{X}=X/C(F\langle g, t\rangle)$. As $C(F\langle g, t\rangle)=FC_{O(H)}(g)\langle g, t\rangle$, $\overline{C_X(t)}\cong N_L(F)'/F\cong GL_3(2)$. Hence X/Y is isomorphic to $\mathrm{Aut}(F)\cong GL_3(2)$ and $\bar{X}$ is a semidirect product of $\bar{Y}$ by $\overline{C_X(t)}$ with $|\bar{Y}|=16$. In particular, $T_4C_{O(H)}(g)=T_3Y$. Since $M_3/F\cong F$ as $N_H(F)$-modules by (5.4), $[\bar{Y}, \overline{C_X(t)}]=\bar{M}_3$. Take an element k of $N_L(F)'\cap N_L(A)$ of order 3. Then k centralizes V_1 and acts transitively on the nonidentity elements of $F/\langle z\rangle$ by (1.8) (5). We have $\bar{Y}=C_{\bar{Y}}(\bar{k})[\bar{Y}, \bar{k}]$ with $[\bar{Y}, \bar{k}]\leqq \bar{M}_3$. Denote by N the preimage of $C_{\bar{Y}}(\bar{k})$ in Y. Then $N\leqq Y\leqq T_4C_{O(H)}(g)\rhd FV_1$. Suppose V_1 is not normal in N and choose $a\in N-N_N(V_1)$. Then $(dt)^a=dtf$ for some $f\in F-\langle z\rangle$. As $[\bar{N}, \bar{k}]=1$, there is an element $b\in C(F\langle g, t\rangle)$ such that $ak=bka$. Note that a centralizes $\mho^1(FV_1)=\langle z\rangle$ and b normalizes V_1. Now $(dt)^{ak}=(dtf)^k=dtf^k$ and $(dt)^{bka}=(dt)^{ba}=dtf$ or $dtfz$, so that $f^k=f$ or zf, which conflicts with the action of k on $F/\langle z\rangle$. Hence V_1 is normal in N and so $Y=NM_3\rhd V_1$. Since $T_3\leqq N(C_1)\leqq N(V_1)$, we get $T_4\leqq T_3Y\rhd V_1$. Thus (4) holds, for T_3 is a Sylow 2-subgroup of $N(AV_1)$ by (2).

Now by Foote [30], $PSL_6(q)$, $q\equiv 3 \pmod 4$ and $PSU_6(q)$, $q\equiv 1 \pmod 4$ are the only simple groups whose Sylow 2-subgroups are isomorphic to a Sylow 2-subgroup of $PSL_6(q)$, $q\equiv 3 \pmod 4$. Since the Schur multipliers of these groups are subgroups of Z_6, arguing as in [11, (6F)] we see that Case (3) of the main theorem holds.

§ 7. The case $V_1=\langle z, d\rangle$.

In this section we assume that $V_1=\langle z, d\rangle$. Under this hypothesis we shall show that Case (2) of the main theorem occurs.

DEFINITION. Let $C_6=O_2(N(C_1))$, $M_6=C_{C_6}(V_1)$, $R_6=RM_6$, $M_4=C_{C_4}(V_1)$, and $V_2=M_3\cap M_4$.

(7.1) (1) $N(C_0)=N_H(A)\langle d\rangle \leqq N(V_1)$.

(2) $N(C_1)=N_H(A)M_6$, $H\cap M_6=A$, M_6 *acts transitively on* $C_1/V_1-AV_1/V_1$, $[M_6, AV_1]\leqq V_1$, *and* $M_6/AV_1\cong A/\langle z\rangle$ *as* $N_H(A)$*-modules.*

(3) $C_6=M_6\langle t\rangle$ *and* M_6 *contains* M_3 *and* M_4.

PROOF. Let $X=N(C_1)\cap N(V_1)$ and $\bar{X}=X/V_1$. As $\mathscr{E}^*(C_0)=\{\langle z, t\rangle, V_1\}$, (1) follows from (5.3) and (5.7). By (5.4), $C_3\leqq X$ and $\bar{t}^{M_3}=\bar{F}\bar{t}$. As in the proof of (6.1), $\bar{t}^X\leqq\bar{A}\bar{t}$ and $|\bar{t}^X|=10$ or 16. Let $\tilde{X}=X/C_X(\bar{C}_1)$. We have $C_X(\bar{t})=N(C_0)$ and $C_X(\bar{C}_1)=N(C_0)\cap C(\bar{A})=O(H)C_1$, so $C_{\tilde{X}}(\bar{t})=C_X(\bar{t})/C_X(\bar{C}_1)\cong N_H(A)/C_H(L)A$. If $|\bar{t}^X|=10$, $(\tilde{X}, \bar{t}^X)$ is a 2-transitive permutation group of order $2^2\cdot 3^2\cdot 5$ or $2^3\cdot 3^2\cdot 5$. Hence $\tilde{X}\cong A_6$ by [5]. But then $R\neq T$ and $C_{\tilde{X}}(\bar{t})\cong\Sigma_3\times\Sigma_3$, contrary to $\tilde{X}\cong A_6$, Thus $\bar{t}^X=\bar{A}\bar{t}$. As in the proof of (6.1), $|\mathscr{I}(C_1-A)|=52$ and $N_H(C_1)=N_H(A)$. We have $|t^{N(C_1)}|=|N(C_1): X|\cdot|X: C_X(\bar{t})|\cdot|C_X(\bar{t}): N_H(A)|=2^5|N(C_1): X|$, so $N(C_1)=X$.

Now $|\tilde{X}: C_{\tilde{X}}(\bar{t})|=16$ and $|\tilde{X}|=2^5\cdot 3^2$ or $2^6\cdot 3^2$, so $\tilde{X}$ is solvable with $O_3(\tilde{X})=1$ and $C_{\tilde{X}}(O_2(\tilde{X}))\leqq O_2(\tilde{X})$. This implies $|O_2(\tilde{X})|\geqq 16$, while $O_2(C_{\tilde{X}}(\bar{t}))=1$. Thus $O_2(\tilde{X})$ is a regular normal subgroup of $(\tilde{X}, \bar{A}\bar{t})$. As $O(H)=O(X)$, $N_L(A)\leqq C_X(O(H))\lhd X$ and $O_2(X \bmod C_X(\bar{C}_1))\leqq C_X(O(H))O(H)$. Thus $O_2(X \bmod C_X(\bar{C}_1))=O(H)\times C_6$. As $\bar{A}\lhd\bar{C}_6$, $Z(\bar{C}_6)$ contains $\langle\bar{a}_1, \bar{b}_1\rangle$ or $\langle\bar{a}_2, \bar{b}_2\rangle$ by (1.8) (2). If $Z(\bar{C}_6)\geqq\langle\bar{a}_1, \bar{b}_1\rangle$, then $AV_1\cap C(\langle a_1, b_1\rangle V_1)=\langle a_2, b_2\rangle V_1$ is normal in C_6 and $Z(\bar{C}_6)\geqq\langle\bar{a}_2, \bar{b}_2\rangle$. By symmetry, $Z(\bar{C}_6)\geqq\langle\bar{a}_2, \bar{b}_2\rangle$ implies $Z(\bar{C}_6)\geqq\langle\bar{a}_1, \bar{b}_1\rangle$. In any case we have $[C_6, AV_1]\leqq V_1$. As V_1 is a four-group, $C_6=M_6\langle t\rangle$. The map defined by $AV_1x\mapsto[x, \bar{t}]$ for $x\in M_6$ is an $N_H(A)$-isomorphism of M_6/AV_1 onto $\bar{A}$. Thus (2) holds.

Let $K=N_L(A)\cap N_L(F)$. Since $M_3/F\cong F$ as K-modules and $C_1\cap C_3=(A\cap C_3)C_0=FC_0$ by (5.4), it follows from (1.8) (5) that K acts irreducibly on C_1C_3/C_1 and either $C_6\geqq C_3$ or $C_6\cap C_1C_3=C_1$. Let $\hat{X}=X/O(H)C_6$, which is isomorphic to $N_H(A)/C_H(L)A$. If $C_6\cap C_1C_3=C_1$, $\hat{C}_3$ is a four-subgroup of $\hat{X}$ normalized by K. But this conflicts with the structure of $\hat{X}$. Thus $C_6\geqq C_3$, so $M_6\geqq M_3$ Similarly, $C_4/E\langle t\rangle\cong E$ as $N_L(E)$-modules and $C_1\cap C_4=EC_0$ by (5.6), so that $N_L(E)$ acts irreducibly on C_1C_4/C_1 by (1.8) (4). If $C_6\cap C_1C_4=C_1$, $\hat{C}_4$ is a four-subgroup of $\hat{X}$ normalized by $N_L(E)$, contrary to the structure of $\hat{X}$. Thus $C_6\geqq C_4$ and $M_6\geqq M_4$.

(7.2) (1) $M_2=B*V_2$, $B\cap V_2=Z(B)$, *and* $|V_2|=2^4$. *In particular,* $[S, V_1]=1$.

(2) M_4 *is elementary abelian.*

(3) $M_6'=V_1$ *and* M_6/V_1 *is elemetary abelan.*

(4) *If* $R\neq T$, *then* $[g, M_6]\leqq M_3$.

PROOF. Let $\bar{R}_6=R_6/V_1$, $R_4=RM_4$, and $R_5=N_{R_6}(EFC_0)$. By (7.1), $\bar{t}^{R_6}=$

$\overline{At}$, $C_{R_6}(\bar{t})=SC_0$, and C_8 centralizes $\bar{A}$. Thus $R_5=\{x \in R_6 \mid \bar{t}^x \in \overline{EFt}\}$, so $|R_6: R_5|=2$. We also have $R_3=\{x \in R_6 \mid \bar{t}^x \in \overline{Ft}\}$ and $R_4=\{x \in R_6 \mid \bar{t}^x \in \overline{Et}\}$. Similarly $R_2=\{x \in R_6 \mid \bar{t}^x \in \overline{Z(B)t}\}$, for $R_2 \leqq R_3$ and $t^{R_2}=Z(B)t$. Then $\overline{Ft} \cap \overline{Et}=\overline{Z(B)t}$ implies $R_3 \cap R_4=R_2$. As $|R_5: R_3|=|R_5: R_4|=2$, we conclude that $R_5/R_2=R_3/R_2 \times R_4/R_2$.

We argue that $\Omega_1(Z_2(R_5))=Z(B)V_1$. As $Z(R_5) \leqq C_{R_5}(t)=R$, $Z(R_5)=Z(R_1)=\langle z \rangle$. As $N_{R_5}(\langle z, t \rangle)=R_1$ by (3.1), $t \notin Z_2(R_5) \leqq Z_2(R_1)=Z_2(R)V_1=\langle b_1, b_2, d, t \rangle$. Now $R_5=\langle M_3, M_4, R \rangle$ and $[F, M_3]=[E, M_4]=1$. Thus $Z(B)V_1=\langle z, b_1b_2, d \rangle \leqq Z_2(R_5)$ and so $Z_2(R_5)$ is equal to one of $Z(B)V_1$, $\langle b_1, b_2, d \rangle$, or $\langle b_1b_2, d, b_1t \rangle$. As $\bar{t}^{R_5}=\overline{EFt}$, $\bar{R}_5 \ntriangleright \langle \bar{b}_1\bar{b}_2, \bar{b}_1\bar{t} \rangle$ and thus $Z_2(R_5)=Z(B)V_1$ or $\langle b_1, b_2, d \rangle$. In either case $\Omega_1(Z_2(R_5))=Z(B)V_1$.

As $C_{M_3}(t)=F$ and $C_{M_4}(t)=E$, $C_{V_2}(t)=Z(B)$. Hence $|V_2| \leqq 2^4$, for V_2 is elementary abelian and invariant under t. As $|M_3|=|M_4|=2^6$, this implies $|M_3M_4| \geqq 2^8$. Now $|M_6: M_6 \cap R_5|=2$ and $M_6 \cap R_5 \cap C(Z(B))$ has index 2 in $M_6 \cap R_5$, for $Z(B)$ is a normal subgroup of R_5 not centralized by A. Thus $M_6 \cap R_5 \cap C(Z(B))=M_3M_4$ and $|V_2|=2^4$. Then as $Z(B)V_1 \lhd M_6$ by the above, $M_6/C_{M_6}(Z(B)V_1)$ is abelian and $M_6' \leqq AV_1 \cap C(Z(B)V_1)=(A \cap B)V_1$ by (7.1) (2). Hence (1.8) (2) shows $M_6' \leqq V_1$. If $M_6' \neq V_1$, $M_6'=\langle z \rangle$ and $A \lhd M_6$. Then as $M_3 \leqq N_{M_6}(F\langle t \rangle)$ and $AF\langle t \rangle=A\langle t \rangle$, M_3 normalizes $Z(A\langle t \rangle)=\langle z, t \rangle$, a contradiction. Thus $M_6'=V_1$. As $EV_1 \leqq Z(M_4)$, EV_2 is elementary abelian of order 2^5 An element f of a Sylow 3-subgroup of $N_L(E)$ not contained in $Z(L)$ acts transitively on $(M_4/EV_1)^\#$ by (1.8) (4) and (5.6) (1). Set $W=(EV_2)^f$, so that $M_4=EV_2W$. If $C_{M_4}(x)=W$ for $x \in W-EV_1$, $\mathscr{E}^*(M_4)=\{EV_2, W\}$ by [28, (2.1)]. But then f normalizes EV_2, a contradiction. Hence M_4 is elementary abelian. By (1.8) (5), $N_L(A) \cap N_L(F)$ acts transitively on $(A/F)^\#$. The $N_H(A)$-isomorphism $M_6/AV_1 \to \bar{A}$ defined by $AV_1y \mapsto [y, \bar{t}]$ maps AM_3/AV_1 onto $\bar{F}$, so $M_6/AM_3 \cong A/F$. Now $M_4 \leqq M_6$ and M_6/M_3 is abelian, so the action of $N_L(A) \cap N_L(F)$ on M_6/AM_3 shows that M_6/M_3 is elementary abelian. Then $\mho^1(M_6) \leqq M_3 \cap AV_1=FV_1$ and so $\mho^1(M_6)=V_1$ by the action of $N_L(A)$, Thus (2) and (3) hold.

As $t^{V_2} \leqq t^{M_3} \cap t^{M_4}=Z(B)t$, V_2 normalizes $Z(B)\langle t \rangle$, As $t^{M_2}=Z(B)t$ and $S_3 \rhd Z(B)$, it follows from (5.4) that $N_{S_3}(Z(B)\langle t \rangle)=SM_2$. Let $I=C_{M_2}(F)$. Then $SM_2=SI$ and $V_2 \leqq SM_2 \cap C(F) \leqq I$, whence $M_2=BV_2$. An element k of a Sylow 3-subgroup of $N_L(B)$ not contained in $Z(L)$ acts transitively both on $(M_2/I)^\#$ and on $(I/F)^\#$ by (1.8) (3), for $I/F \cong M_2/B \cong Z(B)$ as $N_L(B)$-modules by (3.3). Then since EFV_2 is a maximal subgroup of M_2, $M_2=(EFV_2)(EFV_2)^k$ and $V_2 \cap V_2^k \leqq Z(M_2)$. Also, $|V_2 \cap V_2^k| \geqq 8$ since $|I: V_2|=2$. Now $M_2=BV_2$ implies $Z(M_2) \leqq C_I(E)=V_2$. Let tildes denote images in $N(Z(B))/Z(B)$. Then $\tilde{I}=[\tilde{I}, k] \times \tilde{F}$, so the only k-invariant proper subgroups of $\tilde{I}$ are $\tilde{F}$ and $[\tilde{I}, k]$. As $Z(M_2)\neq F$, $\widetilde{Z(M_2)}=[\tilde{I}, k]$ and $Z(M_2)=V_2$. Thus (1) holds.

Finally, suppose $R\neq T$. As g normalizes M_4, $[g, M_4]\leqq C_{M_4}(g)$. By (5.5), $C_{M_4}(g)\geqq V_2$. If $C_{M_4}(g)\neq V_2$, then as $[g, E]\neq 1$, $M_4=C_{M_4}(g)E$ and $[g, M_4]\leqq Z(B)$. In any case g centralizes $M_3M_4/M_3\cong M_4/V_2$. Since g centralizes AM_3/M_3 and since $[g, N_L(F)]\leqq FZ(L)$ by (1.10) and $Z(L)\leqq O(H)$ centalizes M_6, the action of $N_L(A)\cap N_L(F)$ on M_6/AM_3 yields $[g, M_6/M_3]=1$. Thus (4) holds.

(7.3) $N(C_6)=N_H(A)M_6$.

PROOF. Since $F=[M_3, t]\leqq C_6'$, the action of $N_L(A)$ gives $A\leqq C_6'$. Now $C_6'\leqq M_6\cap C_1=AV_1$. Hence $C_6'=AV_1$ by (7.2) (3) and $V_1=Z(AV_1)\lhd N(C_6)$. Let $\bar{C}_6=C_6/V_1$. Then $C_{M_6}(\bar{t})=N_{M_6}(C_0)=AV_1$, so $\mathscr{E}^*(\bar{C}_6)=\{\bar{M}_6, \bar{C}_1\}$. Thus (7.1) (2) shows $N(C_6)=N_H(A)M_6$.

(7.4) $C_1\in \mathrm{Syl}_2(C(A/\langle z\rangle))$ *and* $C_6\in \mathrm{Syl}_2(C(AV_1/V_1))$.

PROOF. Suppose $C_{M_6}(A)>V_1$. Then t centralizes some nonidentity element of $C_{M_6}(A)/V_1$. But $C_{M_6}(C_0/V_1)=AV_1$ by (7.1) and $AV_1\cap C(A)=V_1$, a contradiction. Thus $C_{M_6}(A)=V_1$. Set $D=N_{M_6}(A)$ and $Y=N_L(A)D$. Then $\tilde{Y}=Y/AC_Y(A)\hookrightarrow \mathrm{Out}(A)$. Let Q be a Sylow 3-subgroup of $N_L(A)$. Then $|\tilde{Q}|=9$ and as $\tilde{Q}$ normalizes $\tilde{D}$, (2.2) shows $\tilde{D}=1$. Then as $C_{M_6}(A)=V_1$, $D=AV_1$. By (7.1), $N(C_1)\cap N(A)=N_H(A)V_1$. Hence $N(C_1)\cap C(A/\langle z\rangle)=O(H)C_1$, so C_1 is a Sylow 2-subgroup of $C(A/\langle z\rangle)$. Since $AV_1/V_1\cong A/\langle z\rangle$, $N(C_6)\cap C(AV_1/V_1)=O(H)C_6$ by (7.3) and the assertion holds.

DEFINITION. Let $C_7=O_2([N(BC_3)\cap N(M_3)\cap C(BM_3/M_3), N_H(B)])BC_3$ and $M_7=[C_7, N_L(B)]BM_3$. Moreover, let $M_5=M_3M_4$.

(7.5) (1) $N(BC_3)\cap N(M_3)=N_H(B)M_7$, $H\cap M_7=B$, M_7 *acts transitively on* $BC_3/M_3-BM_3/M_3$, *and* $M_7/BM_3\cong B/F$ *as* $N_H(B)$-*modules.*

(2) $C_7=M_7\langle t\rangle$, $M_7\geqq M_4$, *and* $Z(M_7)\geqq V_2$.

(3) M_7/M_3 *is elementary abelian.*

(4) *If* $R\neq T$, *then* $[g, M_7]\leqq M_3$.

PROOF. Let $X=N(BC_3)\cap N(M_3)$, $\overline{N(M_3)}=N(M_3)/M_3$, $Y=C_X(\bar{B})$, and $X_0=C_X(\bar{B}\langle\bar{t}\rangle)$. As $\mathscr{E}^*(C_3)=\{M_3, F\langle t\rangle\}$, (5.4) gives $C_X(\bar{t})=N_X(C_3)=N_H(B)M_3$. Then as $\bar{B}\cong B/F$, $X_0=C_H(B/F)M_3$ with $C_H(B/F)=C_H(L)B$ if $R=T$ and $C_H(L)B\langle g\rangle$ if $R\neq T$.

We wish to show that $N_{M_6}(R_3)=AM_5\leqq X$, $\bar{t}^X=\overline{Bt}$, and $BM_3\lhd X$. Put $D=N_{M_6}(R_3)$. As $\bar{R}_3\cong R/F$, $\mathscr{E}^*(\bar{R}_3)=\{\bar{A}\langle\bar{t}\rangle, \bar{B}\langle\bar{t}\rangle\}$ by (1.4) (5). Moreover, M_6 normalizes $M_3C_1=M_3A\langle t\rangle$. Thus $D\leqq X$. As $R_3<R_6$, $D\nleqq R_3$. Hence

$\bar{t}^D \neq \{\bar{t}\}$. As $Z(\bar{R}_3)=\bar{E}\langle\bar{t}\rangle$ and $Z(\bar{R}_3)\cap\bar{M}_6=\bar{E}$, we have $\bar{t}^D=\overline{Et}$. By (1.8) (3), $N_L(B)$ acts transitively on $\bar{B}^\#$, so $\bar{t}^X=\overline{Bt}$ or $\bar{B}^\#\cup\overline{Bt}$. Now $TM_3\in \mathrm{Syl}_2(C_X(\bar{t}))$ and TD is a 2-subgroup of X properly containing TM_3, so $\bar{t}^X=\overline{Bt}$ and $BM_3 \lhd X$. Since $C_{M_6}(\bar{t})=AM_3$ and $\bar{t}^{M_6}=\overline{At}$ by (7.1) (2) and since $\bar{M}_6$ is abelian, $D=\{x\in M_6 \mid \bar{t}^x\in\overline{Et}\}$. This implies $D\geqq M_4$, for $\bar{t}^{M_4}=\overline{Et}$ by (5.6). Thus $D=AM_5$.

Since $|X: C_X(\bar{t})|=4$ and $\bar{X}\rhd\bar{B}$ and since $C_X(\bar{t})/X_0$ induces the automorphism group of $\bar{B}$, X/X_0 is a split extension of Y/X_0 by $C_X(\bar{t})/X_0$. Hence $X=\langle C_X(\bar{t}), D\rangle\rhd Z(B)$. Then $Y=C_X(Z(B))$ and $M_5\leqq Y$. The map defined by $X_0y\mapsto[y,\bar{t}]$ is an $N_H(B)$-isomorphism of Y/X_0 onto $\bar{B}$. Also, $O(X)=O(X_0)=O(H)$.

Let $I=BC_3O(H)$ and $Y_1=[Y, N_H(B)]I$. If $R=T$, $X_0=I$ and $Y=Y_1$. If $R\neq T$, $|Y/I|=8$ and the image of $M_4\langle g\rangle$ in Y/I is a four-group. Let k be an element of a Sylow 3-subgroup of $N_L(B)$ with $k\notin Z(L)$. Then $\langle k\rangle X_0=O_3(C_X(\bar{t}) \bmod X_0)$ and k acts transitively on $(Y/X_0)^\#$, so $Y/I=[Y/I, k]\times X_0/I$. As $\langle k\rangle X_0\lhd C_X(\bar{t})$, the preimage of $[Y/I, k]$ in Y is Y_1. Hence $Y=Y_1X_0$ and $Y_1\cap X_0=I$. As $X\rhd C_X(O(H))\geqq N_L(B)C_3$, $Y_1\leqq C_X(O(H))O(H)$. Thus $Y_1=C_7\times O(H)$.

Let $\tilde{X}=X/BM_3$. If $R=T$, $Y=Y_1$ and so $C_7\geqq M_4$ and $\tilde{C}_4$ is a four-subgroup of $\tilde{C}_7$. If $R\neq T$, $Y\geqq M_4\langle g\rangle$ and as $|Y: Y_1|=2$, $C_7\cap M_4\langle g\rangle$ has index 2 in $M_4\langle g\rangle$ and $\tilde{C}_7\cap\tilde{C}_4\langle\tilde{g}\rangle$ is a four-subgroup. In any case, as $|\tilde{C}_7|=8$, $\tilde{C}_7$ is abelian by the action of k and $\tilde{C}_7=\langle\tilde{t}\rangle\times[\tilde{C}_7, k]$. As $\langle k\rangle BZ(L)\lhd N_L(B)$ and $BZ(L)$ centralizes $\tilde{C}_7$, $[\tilde{C}_7, k]=[\tilde{C}_7, N_L(B)]=\tilde{M}_7$. Hence $C_7=M_7\langle t\rangle\neq M_7$ and (1) holds. By (7.2) (1), $V_2=Z(M_2)$, so $N_L(B)$ normalizes V_2. Now $C_{C_7}(V_2)\geqq BM_3$. If $R=T$, $C_{C_7}(V_2)\geqq M_4$ by (7.2) (2). If $R\neq T$, $C_{C_7}(V_2)\geqq C_7\cap M_4\langle g\rangle\not\leqq BM_3$ by (5.5). The only k-invariant proper subgroups of $\tilde{C}_7$ are $\langle\tilde{t}\rangle$ and $\tilde{M}_7$ and as $t\notin C(V_2)$, we get $C_{C_7}(V_2)=M_7$.

Suppose $R=T$. Then $|M_7: BM_5|=2$ and (7.2) (3) shows $M_5=BM_5\cap C(M_3/V_1)\lhd M_7$, for $C_B(F/\langle z\rangle)=EF$. Since $\tilde{M}_7\cong B/F$ as $N_L(B)$-modules, $\tilde{M}_7=\tilde{M}_4\times\tilde{M}_4^x$ for some involution $x\in N_L(B)-N_L(S)$. Now $EE^xF=B$ and so $M_4M_4^xM_3\geqq BM_3$. Comparing orders, we have $\bar{M}_7=\bar{M}_4\times\bar{M}_4^x$, which is elementary abelian. Thus (2) and (3) hold in this case.

Suppose $R\neq T$. Then $M_7\langle g\rangle=C_7\langle g\rangle\cap C(V_2)\geqq M_4\langle g\rangle$. As $g\in X_0$, $[M_7, g]\leqq M_7\cap X_0=BM_3$ and $\tilde{M}_7\langle\tilde{g}\rangle\cong E_8$. Hence $BM_5\lhd M_7\langle g\rangle$ and $M_5=BM_5\cap C(M_3/V_1)\lhd M_7\langle g\rangle$. Note that $N_L(F)=N_L(F)'\times Z(L)$, g centralizes $N_L(F)'/F$, and $N_L(B)=(N_L(B)\cap N_L(F)')\times Z(L)$. Let x be an involution of $N_L(B)\cap N_L(F)'$ not contained in $N_L(S)$, Then $[g, x]\leqq F$, so $\bar{M}_4$ and $\bar{M}_4^x$ are normal in $\bar{M}_7\langle\bar{g}\rangle$. Now $C_{\bar{M}_4}(\bar{t})=\bar{E}$ by the first paragraph, so $\bar{M}_4\cap\bar{M}_4^x\cap C(\bar{t})=1$ and thus $\bar{M}_4\cap\bar{M}_4^x=1$. Tnen $\bar{M}_4\bar{M}_4^x\cap C(\bar{t})=\bar{B}$ and as $\bar{g}\notin\bar{B}$, $\bar{M}_7\langle\bar{g}\rangle=\bar{M}_4\bar{M}_4^x\langle\bar{g}\rangle$. By (7.2) (4), $[\bar{g}, \bar{M}_4]=1$, so $\bar{M}_7\langle\bar{g}\rangle$ is elementary abelian. Thus (3) and (4) hold. It remains to show that $M_7 \geqq M_4$. As $C_{\bar{M}_7}(\bar{t})=\bar{B}$ and $C_B(M_3/V_1)=EF$, $C_{\bar{M}_7}(M_3/V_1)\cap C(\bar{t})=\bar{E}$. As $|\bar{E}|=2$, this implies $|C_{\bar{M}_7}(M_3/V_1)|\leqq 4$. Hence comparing orders, we get $M_7\langle g\rangle \cap C(M_3/V_1)=M_5\langle g\rangle$. By (7.2)(3), $C_S(M_3/V_1)=A$, so setting $U=SM_7\langle g\rangle$, we have $C_U(M_3/V_1)=AM_5\langle g\rangle$. By (1.1), g does not centralize $E/\langle z\rangle$, whence $U\triangleright C_U(M_5/V_1)=AM_5$. Thus $SM_5=ABM_5$ is normal in U. Now $|U/SM_5|=|U/M_7|=4$ and $|U/C_U(Z(B))|=2$. As $SM_5\cap C(Z(B))=BM_5$, it follows that $\tilde{U}'\leqq\tilde{M}_5\cap\tilde{M}_7$. Moreover, $\tilde{U}\geqq\tilde{S}\tilde{M}_7$ and as $\tilde{M}_7\cong B/F$, $\tilde{U}'\neq 1$. As $|BM_5/BM_3|=2$, we get $M_7\geqq BM_5$. The proof is complete.

(7.6) *Let $P=M_6M_7$. Then P is a subgroup of order 2^{12} with $C_P(t)=S$ and $P\triangleright M_6$.*

PROOF. Let $R_5=RM_5$. Note that $C_{M_5}(t)=EF$, for $M_5\leqq M_6\cap M_7$. As $C_{V_2}(t)=Z(B)$ and $R_2=RV_2$ by (7.2)(1), $R_2/V_2\cong R/Z(B)$. As $R_1=RV_1$ is of index 2 in R_2, we have $S'\leqq R_2'\leqq R_1\cap R'V_2=S'V_1$. If $R_2'=S'$, then $R_2\triangleright R$, which conflicts with $t^{R_2}\not\leqq Z(R)$. Thus $R_2'=S'V_1$. Now R_2 is a normal subgroup of R_5 of index 4, $(M_3\langle t\rangle)'=F$, and $R_5/M_5\cong R/EF$. Hence $EFV_1\leqq R_5'\leqq R_2\cap M_5=EFV_2$. If $R_5'=EFV_1$, $R_5\triangleright R_1$. But $R_1=S*C_0$ and $Z_2(R_1)=Z_2(S)C_0$, whence $t^{M_3}\not\leqq Z_2(R_1)$. Thus $R_5'=EFV_2$ and $Z(R_5')=V_2$. If $M_5',=\langle z\rangle$ then $M_5\triangleright E$ and as $M_3\leqq N(F\langle t\rangle)$, M_3 normalizes $Z(EF\langle t\rangle)$. But $t^{M_3}=Ft$. Thus $M_5'=V_1$ by (7.2)(3). Now $C_{R_5}(Z(B))=BM_5\langle t\rangle$, so $C_{R_5}(V_2)=BM_5$. As $FV_1/V_1\cong F/\langle z\rangle C_{R_5}(FV_1/V_1)=AM_5\langle t\rangle$ and $C_{R_5}(M_5/V_1)=AM_5$.

Let $X=N(R_5)\cap N(M_5)$. We have shown that $X\leqq N(V_1)\cap N(V_2)\cap N(AM_5)\cap N(BM_5)$. Suppose there is an abelian subgroup D of $EFV_2\langle t\rangle$ of order 2^5 not contained in EFV_2. Then $EFV_2\langle t\rangle=EFV_2D$ and $V_2\cap D\leqq Z(EFV_2\langle t\rangle)=Z(B)$. Now $(EFV_2\cap D)V_2$ is an abelian subgroup of EFV_2, so its order is at most 2^5. Then $|EFV_2\cap D|\leqq 8$ and $|D|\leqq 16$, a contradiction. Thus $J_0(EFV_2\langle t\rangle)=EFV_2$. By (7.2)(1), $Z(M_2)=V_2$, so V_2 is invariant under $N_H(B)$. As $\mathscr{E}^*(V_2\langle t\rangle)=\{V_2, Z(B)\langle t\rangle\}$, $N(V_2\langle t\rangle)=N_H(B)V_2$ by (3.3). This shows $\mathscr{E}^*(M_5\langle t\rangle/V_2)=\{M_5/V_2, EFV_2\langle t\rangle/V_2\}$. Thus $N_X(M_5\langle t\rangle)\leqq N_X(EFV_2\langle t\rangle)\leqq N_X(EFV_2)$. Let tildes denote images in $N(V_2)/V_2$. Then $\tilde{t}^{M_5}=\tilde{E}\tilde{F}\tilde{t}$. Hence $N_X(M_5\langle t\rangle)$ acts transitively on $\tilde{E}\tilde{F}\tilde{t}$ and as $N_X(M_5\langle t\rangle)\cap C(\tilde{t})=N_X(M_5\langle t\rangle)\cap N(V_2\langle t\rangle)=N_H(S)V_2$, we have $N_X(M_5\langle t\rangle)=N_H(S)M_5$.

Let $\bar{X}=X/M_5$. Then $C_X(\bar{t})=N_H(S)M_5$ by the above. As $|R_6: R_5|=|RM_7: R_5|=2$, $\langle R_6, M_7\rangle\leqq X$. As $\bar{A}$ and $\bar{B}$ are normal subgroups of $\bar{X}$ of order 2, $X=C_X(\bar{S})$. Now $\bar{t}^X\leqq\overline{St}$ and the map $X\to\bar{S}$; $x\mapsto[x,\bar{t}]$ is a $N_H(S)$-homomorphism with kernel $C_X(\bar{t})$. As $\bar{t}^{M_6}=\overline{At}$ and $\bar{t}^{M_7}=\overline{Bt}$, this homomorphism is surjective. Thus $|X: N_H(S)M_6|=|X: N_H(S)M_7|=2$ and $X=N_H(S)P$. As $O_2(N_H(S)M_i)=RM_i$ or $RM_i\langle g\rangle$ for $i=6, 7$, TP is a group. Set $Q=TP$ and $U=C_Q(M_5/V_1)$. If $U>M_6$, t centralizes some nonidentity element of U/M_6, so $N_U(C_6)>M_6$. But (7.3) shows $N_U(C_6)=C_T(M_5/V_1)M_6=$

M_6, for $Q\cap H=T$ and $C_T(EF/\langle z\rangle)=A$. Hence $M_6=C_Q(M_5/V_1)\lhd Q$. Now $SM_6\cap SM_7=SM_5$, so $|P|=2^{12}$. As $|P\colon C_P(\bar t)|=4$, $C_P(\bar t)=SM_5$ and we get $C_P(t)=S$.

(7.7) *Let K be the normal closure of $N_L(F)'M_3$ in $N(M_3)$. Then $P\in \mathrm{Syl}_2(K)$ and either $K/M_3\cong GL_3(2)\times GL_3(2)$ and t interchanges its components or $K/M_3\cong SL_3(4)$.*

PROOF. Let $J=N_L(F)'$, $X=N(M_3)$, and $\bar X=X/M_3$. Then $N_L(F)=J\times Z(L)$ and if $R\neq T$, g centralizes J/F. As $\mathscr{E}^*(M_3\langle t\rangle)=\{M_3, F\langle t\rangle\}$, (5.4) gives $C_X(\bar t)=N_H(F)M_3$. Thus $C_{\bar X}(\bar t)=\langle \bar t\rangle\overline{JO(H)}$ or $\langle \bar g, \bar t\rangle\overline{JO(H)}$ with $\bar J\cong GL_3(2)$. As in the proof of (7.2), $M_6/AM_3\cong A/F$. Thus $N_J(A)$ acts irreducibly on M_6/AM_3, so $\bar M_6=[\overline{N_J(A)}, \bar M_6]\leqq \bar K$. Similarly, $N_J(B)$ acts irreducibly on M_7/BM_3 by (7.5) and $\bar M_7=[\overline{N_J(B)}, \bar M_7]\leqq \bar K$. Hence $P\leqq K$.

Let $\tilde X=X/O(X \bmod M_3)$. Then $C_{\tilde X}(\tilde t)\cong C_{\bar X}(\bar t)/C_{O(\bar X)}(\bar t)$, for $\tilde X\cong \bar X/O(\bar X)$. As $O_2(\tilde X)\langle\tilde t\rangle\cap C(\tilde t)\leqq O_2(C_{\tilde X}(\tilde t))$ and $|O_2(C_{\tilde X}(\tilde t))|\leqq 4$, $m(O_2(\tilde X))\leqq 2$ by Suzuki [26, Lemma 4] and so $[\tilde J, O_2(\tilde X)]=1$. Then there is a component of $\tilde X$ not centralized by $\tilde J$, for $F^*(\tilde X)=E(\tilde X)O_2(\tilde X)$ and $C_{\tilde X}(F^*(\tilde X))\leqq O_2(\tilde X)$. As $\tilde J$ is a component of $C_{\tilde X}(\tilde t)$, [2, Lemma 2.7] shows $\tilde J\leqq E(\tilde X)$. Thus $\tilde K=E(\tilde K)$, for $\tilde K=\langle \tilde J^{\tilde X}\rangle$ is a perfect normal subgroup. Now $[\bar M_6, \bar t]=\bar A$ by (7.1) (2). As $\overline{O(X)}\cap C(\bar t)\leqq O(C_{\bar X}(\bar t))=\overline{O(H)}$ and $[O(H), A]=1$, $[O(\bar X), \bar A]=1$ by [10, (1J)]. Thus $\bar K$ centralizes $O(\bar X)$ and $\bar K=E(\bar K)$. Applying [2, Lemma 2.7] to $\bar K\langle\bar t\rangle$, we can choose a component $\bar Y$ of $\bar K$ such that $\bar J\leqq \bar Y$ or $\bar Y\neq \bar Y^t$ and $\bar J=(\overline{YY^t}\cap C(\bar t))'$ with $\bar Y/Z(\bar Y)\cong \bar J$. In any case $\bar X$ acts transitively on the set of components of $\bar K$ since $\bar K=\langle \bar J^{\bar X}\rangle$. If $\overline{C_K(M_3)}\not\leqq Z(\bar K)$, then $\overline{C_K(M_3)}$ contains some component of $\bar K$ and so $\bar K=\overline{C_K(M_3)}$. But then $J\leqq K\leqq C_K(M_3)$, a contradiction. Thus $\overline{C_K(M_3)}\leqq Z(\bar K)$. Now $K/C_K(M_3)\hookrightarrow \mathrm{Aut}\,(M_3)\cong GL_6(2)$. Since 7 divides $|\bar Y/Z(\bar Y)|$ and each component of $\bar K$ is conjugate to $\bar Y$, $\bar K$ has at most two components.

Suppose $\bar K$ has two components. If $\bar J\leqq \bar Y$, t normalizes $\bar Y$ and the other component of $\bar K$, say $\bar Y_1$. Let $\bar U$ be a t-invariant Sylow 2-subgroup of $\bar Y_1$. Then $C_{\bar U}(\bar t)\langle\bar t\rangle$ lies in $C_{\bar X}(\bar t)\cap C(\bar J)$, so $|C_{\bar U}(\bar t)\langle \bar t\rangle|\leqq 4$ and by Suzuki's lemma, $m(\bar U)\leqq 2$. But $P\leqq K$ and $\bar M_6=[\overline{N_J(A)}, \bar M_6]\leqq \bar Y$. As $\bar Y$ and $\bar Y_1$ are conjugate and $\bar M_6\cong E_{16}$, this is impossible. Thus $\bar K=\overline{YY^t}$ and $\bar Y/Z(\bar Y)\cong GL_3(2)$. Suppose $Z(\bar Y)\neq 1$. Then $\bar Y\cong SL_2(7)$. If $\bar Y\cup \bar Y^t=1$, then $C_{\bar X}(\bar t)\rhd C_{\bar K}(\bar t)\cong SL_2(7)$, contrary to the structure of $C_{\bar X}(\bar t)$. If $|\bar Y\cap \bar Y^t|=2$, then $m(\bar K)=3$, a contradiction. Thus $\bar Y\cong GL_3(2)$.

Suppose $\bar K$ is quasisimple. By (7.6), $C_{\bar P}(\bar t)=\bar S\leqq \bar J$ and $\bar S\cap Z(\bar K)=1$. Since $\bar P\cap Z(\bar K)$ is a $\bar t$-invariant 2-subgroup, this implies $\bar P\cap Z(\bar K)=1$. If $\overline{C_K(M_3)}=Z(\bar K)$, then as $|\bar P|=2^6$, $\bar K/Z(\bar K)\cong GL_6(2)$, $GL_5(2)$, $GL_4(2)$, or

$Sp_6(2)$ by (2.4). A Sylow 2-subgroup of $GL_4(2)$ possesses a unique elementary abelian subgroup of order 16, while $\bar{M}_6 \cong \bar{M}_7 \cong E_{16}$. The remaining three groups have trivial Schur multipliers. As $C_{\bar{K}}(\bar{t}) \triangleright \bar{J} \cong GL_3(2)$, [4, section 19] shows that this is not the case. If $\overline{C_K(M_3)} \neq Z(\bar{K})$, $\bar{K}/\overline{C_K(M_3)}$ is quasisimple with nontrivial center. Looking at the local subgroups of $GL_6(2)$, we have $\bar{K}/\overline{C_K(M_3)} \cong SL_3(4)$. Then $\bar{P}O_2(\bar{K}) \in \mathrm{Syl}_2(\bar{K})$ and as $\bar{P} \cap Z(\bar{K}) = 1$, $O_2(\bar{K})$ has a complement in $\bar{K}$ by Gaschütz's theorem [19, p. 121]. Since $\bar{K}$ is perfect, this implies $O_2(\bar{K}) = 1$. In view of the Schur multiplier of $PSL_3(4)$, we get $\bar{K} \cong SL_3(4)$.

(7.8) $K/M_3 \not\cong SL_3(4)$.

PROOF. Suppose $K/M_3 \cong SL_3(4)$ and let Q be a Sylow 3-subgroup of $Z(K \bmod M_3)$. Then $C_{M_3}(Q) = 1$ and $K = M_3 C_K(Q)$. As $|N_{M_3\langle t\rangle}(Q)| = 2$ and $\mathscr{F}(M_3 t) = t^{M_3}$, we may assume that $Q^t = Q$. Then $N_L(F)' = C_K(t)$ is a split extension of F by $C_K(Q) \cap C(t)$, a contradiction.

DEFINITION. By (7.7) and (7.8) we can write $K/M_3 = L_1/M_3 \times L_1^t/M_3$ with $L_1/M_3 \cong GL_3(2)$.

(7.9) *$K = L_1' \times L_1'^t$ and $L_1' \cong C_K(t) = N_L(F)'$.*

PROOF. Suppose $C_{M_3}(L_1) = 1$ and let $D = L_1 \cap P$ and $W = C_{M_3}(D)$. Then $1 \neq W \triangleleft L_1^t$ and as $C_{M_3}(L_1^t) = 1$, $|W| \geqq 8$. Now $L_1 = \langle D, D^x \rangle$ for some involution $x \in L_1$, so $W \cap W^x \leqq C_{M_3}(L_1) = 1$. As $|M_3| = 2^6$, we get $|W| = 8$ and L_1^t/M_3 acts irreducibly on W. As $L_1^t = \langle D^t, D^{txt} \rangle$, it follows that $|C_W(D^t)| = 2$. Now $V_1 \leqq Z(P)$ by (7.5) (2). But then $C_W(D^t) \geqq V_1$, a contradiction. Thus $U = C_{M_3}(L_1) \neq 1$. By (5.4) (1) a Sylow 7-subgroup Q of $N_L(F)'$ acts fixed-point-freely on M_3. Thus $U \cap U^t \leqq C_{M_3}(K) \leqq C_{M_3}(Q) = 1$, so $|U| = 8$ and L_1/M_3 is irreducible on U^t. As $Z(L_1/U^t) = M_3/U^t$ and the Schur multiplier of $GL_3(2)$ has order 2, $|L_1' \cap M_3 : U^t| \leqq 2$. As $C_{M_3}(Q) = 1$, we have $L_1' \cap M_3 = U^t$ and $K = L_1' \times L_1'^t$. As $C_{K/M_3}(t) = N_L(F)' M_3/M_3$, the assertion holds.

DEFINITION. Let $U = L_1' \cap P$. Thus $P = U \times U^t$ and $U \cong C_P(t) = S$. Note that $Z(P) = V_1$.

(7.10) *$t^G \cap P = \varnothing$ and $N(P\langle t\rangle) = N_H(S)P$.*

PROOF. In S the centralizer of every involution has rank 3, so in P the centralizer of every involution has rank 6. As $m(H) \leqq 5$, $t^G \cap P = \varnothing$. Now $N_H(P\langle t\rangle) = N_H(S)$, for $L \cap P\langle t\rangle = S$. As $\mathscr{F}(P\langle t\rangle - P) = t^P$, we conclude that $N(P\langle t\rangle) = N_H(S)P$.

(7.11) *If $R = T$, then $P \in \mathrm{Syl}_2(O^2(G))$.*

PROOF. If $R=T$, the preceding lemma shows $P\langle t\rangle \in \mathrm{Syl}_2(G)$ and moreover, $t\notin O^2(G)$ by the Thompson fusion lemma.

(7.12) *Every involution of U is conjugate to the involution of $Z(U)$ in $\langle L_1', N_L(E)\rangle$ and every involution of $P-U$ is conjugate to an involution of $V_1-Z(U)$ in $\langle K, N_L(E)\rangle$.*

PROOF. Let $F_0=O_2(L_1')$, so that $M_3=F_0\times F_0^t$. Let $E_0=M_4\cap U$, $A_0=J_r(U \bmod Z(U))$, and $B_0=M_7\cap U$. As $M_6/V_1\cong E_{2^8}$ and $J_r(S/Z(S))=A/Z(S)$, we have $J_r(P/V_1)=M_6/V_1$ and $M_6=A_0\times A_0^t$. By (7.1) (1), $[N_L(A), V_1]=1$, so $N_L(A)$ normalizes $Z(M_6 \bmod Z(U))=A_0Z(U)^t$. Take a Sylow 3-subgroup Q of $N_L(F)'\cap N_L(A)$. Then $[A, Q]=A$. As $Q\leqq K$, Q normalizes $L_1'\cap M_6=A_0$ and so A_0 is Q-isomorphic to $C_{M_6}(t)=A$. Thus $[A_0, Q]=A_0$ and we have $[A_0Z(U)^t, Z(L)Q]=A_0$, for $Z(L)$ centralizes P. If Q_1 is a Sylow 3-subgroup of $N_L(A)$ containing Q, then $Z(L)Q\lhd Q_1$. Thus $N_L(A)=\langle Q_1, S\rangle\leqq N(A_0)$. By (7.5) (2), $Z_2(P)\geqq V_2$ and as $\Omega_1(Z_2(S))=Z(B)$, $\Omega_1(Z_2(P))=V_2$. Hence $V_2=(V_2\cap U)\times(V_2\cap U)^t$ and $V_2\cap U\cong C_{V_2}(t)=Z(B)$. Then as $C_S(Z(B))=B$, $C_P(V_2)=M_7$. Thus $B_0=C_U(V_2\cap U)$, $Z(B_0)=V_2\cap U$, and $M_7=B_0\times B_0^t$. Now $M_6\cap M_7=C_{M_6}(V_2)=C_{A_0}(Z(B_0))\times C_{A_0}(Z(B_0))^t$ and $C_{A_0}(Z(B_0))\cong M_6\cap M_7\cap C(t)=A\cap B$. By (1.4) (4), $C_{A_0}(Z(B_0))=A_0\cap B_0$ has precisely two maximal elementary abelian subgroups, one of which is F_0. Denote by E_1 the other maximal elementary abelian subgroup of $A_0\cap B_0$. Then the only t-invariant members of $\mathscr{E}^*(M_6\cap M_7)$ are $E_1E_1^t$ and $F_0F_0^t$. Thus $M_4=E_1E_1^t$, so $E_0=E_1$. Hence $M_4=E_0\times E_0^t$, $\mathscr{E}^*(E_0F_0)=\{E_0, F_0\}$, and $Z(B_0)=E_0\cap F_0$. As $N_L(E)$ normalizes M_4 and $E_0=M_4\cap A_0$, E_0 is $N_L(E)$-isomorphic to $C_{M_4}(t)=E$. Then by (1.8) (4), an element e of a Sylow 3-subgroup of $N_L(E)$ with $e\notin Z(L)$ acts transitively on $(E_0/Z(U))^\#$.

Set $\langle z_0\rangle=Z(U)$. Let y be an involution in U. If $y\notin F_0$, then as $L_1'/F_0\cong GL_3(2)$ and $\mathscr{E}^*(E_0F_0)=\{E_0, F_0\}$, y is fused into E_0 by an element of L_1'. Thus y is fused into $Z(B_0)$ by an element of $\langle L_1', N_L(E)\rangle$. Now all involutions of F_0 are conjugate in L_1'. Thus y and z_0 are conjugate in $\langle L_1', N_L(E)\rangle$.

Let x be an involution in $P-U$. If $x\in U^t$, x and z_0^t are conjugate in $\langle K, N_L(E)\rangle$ by the above. Suppose $x\notin U^t$ and choose $x_1\in U$ and $x_2\in U^t$ so that $x=x_1x_2$. As L_1'/F_0 has only one conjugacy class of involutions and $\mathscr{E}^*(E_0F_0)=\{E_0, F_0\}$, replacing x with a suitable K-conjugate of x if necessary, we may assume that $x_1\in E_0\cup F_0$ and $x_2\in E_0^t\cup F_0^t$. If $x_1\in F_0$ and $x_2\in F_0^t$, x is conjugate to $z_0z_0^t$ in K. If $x_1\in F_0$ and $x_2\in E_0^t-F_0^t$, then $x_1^a=z_0$ for some $a\in L_1'$ and $x^a=z_0x_2$. Now $x_2^b\in Z(B_0)^t$ for some $b\in\langle e\rangle$ and $x^{ab}=z_0x_2^b$. Thus x is conjugate to $z_0z_0^t$ in $\langle K, N_L(E)\rangle$. If $x_1\in E_0-F_0$ and $x_2\in F_0^t$, then by symmetry x and $z_0z_0^t$ are conjugate in $\langle K, N_L(E)\rangle$. Finally, assume that $x_1 \in E_0 - F_0$ and $x_2 \in E_0^t - F_0^t$. There is an involution f such that $U = A_0\langle f\rangle$ and $ff^t = h_0$, for $S = A\langle h_0\rangle$. As $Z_2(U) \cap E_0 = Z(B_0)$, f has two orbits on $(E_0/Z(U))^{\#}$. Now $E_0 = Z(B_0) \cup Z(B_0)^e \cup Z(B_0)^{e^2}$ and $Z(B_0)^{ef} = Z(B_0)^{e^2}$. As $[e, t] = 1$, we also have $E_0^t = Z(B_0)^t \cup Z(B_0)^{te} \cup Z(B_0)^{te^2}$ and $Z(B_0)^{tef^t} = Z(B_0)^{te^2}$. Note that $[f, x_2] = [f^t, x_1] = 1$. As $x_1 \in E_0 - Z(B_0)$ and $x_2 \in E_0^t - Z(B_0)^t$, we can choose $a \in \langle f, f^t\rangle$ so that $x_1^a \in Z(B_0)^e$ and $x_2^a \in Z(B_0)^{te}$. Then $x^{ae^{-1}}$ and $z_0 z_0^t$ are conjuate in K. The proof is complete.

(7.13) *If $R \neq T$, then $P \in \mathrm{Syl}_2(O^2(G))$.*

PROOF. Suppose $R \neq T$ and let P_1 be a Sylow 2-subgroup of $N(K)$ containing $P\langle g, t\rangle$. Set $P_0 = N_{P_1}(L_1')$ and $U_0 = C_{P_1}(L_1')^t$. We have shown in the proof of (7.12) that $M_3 = F_0 \times F_0^t$, $M_4 = E_0 \times E_0^t$, $M_6 = A_0 \times A_0^t$, and $\mathscr{E}^*(E_0F_0) = \{E_0, F_0\}$ where $F_0 = O_2(L_1')$, $E_0 = M_4 \cap U$, and $A_0 = J_r(U \bmod Z(U))$. By (7.4), M_6 is a Sylow 2-subgroup of $C(AV_1/V_1) \cap C(V_1)$, so it is a Sylow 2-subgroup of $C(M_6/V_1) \cap C(V_1)$. Now $A_0 \cong C_{M_6}(t) = A$, so $Z(M_6) = V_1$. As $C(M_6) \lhd C(M_6/V_1) \cap C(V_1)$, it follows that $C(M_6) = V_1 O(C(M_6))$ by [12, Theorem 7.4.3]. Thus $C(K)$ has odd order and $U_0 \cap U_0^t = 1$. By the Krull-Schmidt theorem $N(K)$ acts on the set $\{L_1', L_1'^t\}$, so $P_1 = P_0\langle t\rangle$. (2.9) implies $|U_0 : U| \leqq 2$. Similarly, P_0/U_0^t is isomorphic to a subgroup of $\mathrm{Aut}(L_1')$ and $|P_0 : UU_0^t| \leqq 2$. If $|U_0, U| = 2$, then $P_0 = U_0 \times U_0^t$ and $U_0 \cong C_{P_0}(t) = S\langle g\rangle$, so there is an involution $g_0 \in U_0$ such that $g_0 g_0^t = g$. Thus one of the following two cases occurs:

Case 1. $U_0 = U\langle g_0\rangle \cong S\langle g\rangle$ with $g_0 g_0^t = g$ and $P_0 = U_0 \times U_0^t$.

Case 2. $U_0 = U$ and $P_0 = P\langle g\rangle$.

As $C_{P_1}(t) = T$, $Z(P_1) = \langle z\rangle$. As $P_1 \rhd V_1$, $Z_2(P_1)$ contains V_1. On the other hand, $Z_2(P_1) \leqq N_{P_1}(\langle z, t\rangle) = TC_0$ and as $TC_0/\langle z\rangle = S\langle g\rangle/\langle z\rangle \times C_0/\langle z\rangle$, $Z(TC_0 \bmod \langle z\rangle) = Z(B)C_0$ by (1.5) (1). Hence $Z_2(P_1) \leqq Z(B)C_0 \cap C(M_3/\langle z\rangle) = Z(B)V_1$. As $C_{C_6}(t) = A\langle t\rangle$ and $Z(A\langle t\rangle) = \langle z, t\rangle$, $Z(C_6) = \langle z\rangle$. Moreover, $Z(C_6/V_1) = AV_1/V_1$ since $N_{C_6}(C_0) = C_1$ by (7.1) (1). If $Z_2(P_1) = Z(B)V_1$, then $Z(B)V_1 \leqq Z_2(C_6) \leqq AV_1$, so we have $Z_2(C_6) = AV_1$ by (1.8) (2) and $M_6 \rhd A$. But $M_3 \leqq N_{M_6}(F\langle t\rangle)$ and $AF\langle t\rangle = A\langle t\rangle$, whereas M_3 does not normalize $Z(A\langle t\rangle)$, a contradiction. Thus $Z_2(P_1) = V_1$. As $P\langle g\rangle \cap C(L_1') = U^t$, $U\langle g\rangle \cong P\langle g\rangle/U^t$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Aut}(L_1')$. Hence by (5.5) and (1.5) (4), $J_r(U\langle g\rangle) = F_0\langle g\rangle$ and $J_r(P\langle g\rangle/U^t) = F_0U^t\langle g\rangle/U^t$, so $J_r(P\langle g\rangle) = M_3\langle g\rangle$. (1.5) (4) also shows $\mathscr{J}(Ug) = F_0 g$ and $\mathscr{J}(Pg) = M_3 g$.

We argue that in Case 1, $J_r(P_1) = M_3\langle g_0, g_0^t\rangle$ and $J_r(P_1/V_1) = M_6/V_1$. As $P_1 = P_0\langle t\rangle$ and $C_{P_1}(t) = T$, every involution of $P_1 - P_0$ is conjugate to t in P_1 and the centralizer of which in P_1 has rank 5. Now $J_r(U_0) = F_0\langle g_0\rangle$, so $M_3\langle g_0, g_0^t\rangle$ is a unique elementary abelian subgroup of P_1 of order 2^8 and is self-centralizing in P_1. Thus $J_r(P_1) = M_3\langle g_0, g_0^t\rangle$. Let $\bar{P}_1 = P_1/V_1$.

Then as $V_1=Z(U)\times Z(U)^t$, $\bar{P}_0=\bar{U}_0\times\bar{U}_0^t$ and $\bar{U}_0\cong C_{\bar{P}_0}(\bar{t})=\bar{S}\langle\bar{g}\rangle\cong S\langle g\rangle/\langle z\rangle$. (1.5) (3) shows $J_r(\bar{U}_0)=\bar{A}_0$ and $C_{\bar{P}_1}(\bar{M}_6)=\bar{M}_6$, so $J_r(P_1/V_1)=M_6/V_1$.

Next we argue that in Case 2, $J_r(P_1)=M_8\langle g\rangle$ and $J_r(P_1/V_1)=M_6/V_1$. As $P=U\times U^t=U\times U^{gt}$, $\mathscr{I}(Pt)=t^P$ and $\mathscr{I}(Pgt)=(gt)^P$. As $C_{P_1}(t)=S\langle g, t\rangle$ and $C_{P_1}(gt)=C_P(gt)\langle g, t\rangle$ with $C_P(gt)\cong U$ and as $P_1=P\langle g, t\rangle$, the centralizer in P_1 of every involution of $P_1-P\langle g\rangle$ has rank at most 5. As $J_r(P\langle g\rangle)=M_8\langle g\rangle$ is self-centralizing in P_1, we get $J_r(P_1)=M_8\langle g\rangle$. Let $\bar{P}_1=P_1/V_1$. Then $\bar{P}=\bar{U}\times\bar{U}^t=\bar{U}\times\bar{U}^{gt}$ and the rank of $\bar{U}$ is 4, so the centralizer in $\bar{P}_1$ of every involution of $\bar{P}_1-\bar{P}\langle\bar{g}\rangle$ has rank at most 6. As $P\langle g\rangle/U^t\cong S\langle g\rangle$, (1.5) (3) gives $J_r(P\langle g\rangle/U^tV_1)=A_0U^tV_1/U^tV_1$. Also, $U^tV_1/V_1\cong U^t/Z(U^t)$ and $J_r(U^tV_1/V_1)=A_0^tV_1/V_1$. Thus $J_r(P_1/V_1)=M_6/V_1$.

We have shown that $M_6/Z_2(P_1)=J_r(P_1/Z_2(P_1))$ and $M_8=M_6\cap J_r(P_1)$. Hence $N(P_1)\leq N(M_8)=N(K)$, which implies $P_1\in\mathrm{Syl}_2(G)$. As shown before $\mathscr{I}(Pg)=M_8g$, whence the centralizer in P of every involution of P_0 has rank 6. Since $m(H)=5$, we conclude that $t^G\cap P_0=\varnothing$.

We wish to show that $M_8\langle g_0, g_0^t\rangle\langle t\rangle\in\mathrm{Syl}_2(C(g))$ in Case 1 and $M_8\langle g, t\rangle\in\mathrm{Syl}_2(C(g))$ in Case 2. For this purpose let $X=C(g)$. Assume that Case 1 holds. As $M_8\langle g_0, g_0^t\rangle=F_0\langle g_0\rangle\times(F_0\langle g_0\rangle)^t$ and $M_8\langle g_0, g_0^t\rangle\cap H=F\langle g\rangle$, we have $\mathscr{E}^*(M_8\langle g_0, g_0^t\rangle\langle t\rangle)=\{M_8\langle g_0, g_0^t\rangle, F\langle g, t\rangle\}$ and $N(M_8\langle g_0, g_0^t\rangle\langle t\rangle)\leq N(F\langle g, t\rangle)$. Now $t^{M_8}=Ft$ and $t^{g_0}=gt$, so $M_8\langle g_0, g_0^t\rangle$ acts transitively on $Ft\cup Fgt$. As $t^G\cap P_0=\varnothing$, this implies $t^{N(F\langle g,t\rangle)}=Ft\cup Fgt$. Then as $F\langle g, t\rangle\in\mathrm{Syl}_2(C_X(t))$, $|N_X(F\langle g, t\rangle)|_2=2^9$. Hence $M_8\langle g_0, g_0^t\rangle\langle t\rangle\in\mathrm{Syl}_2(C(g))$.

Set $I=C_L(g)'$. By (1.7), $I\cong SL_2(8)$ and $F\in\mathrm{Syl}_2(C_L(g))$. Moreover, $C_X(t)=C_H(g)=C_L(g)C_{O(H)}(g)\langle g, t\rangle$. Assume that t and gt are not conjugate in $N_X(\langle g, t\rangle)$ and let $\tilde{X}=X/\langle g\rangle$. Then $C_{\tilde{X}}(\tilde{t})=C_X(t)/\langle g\rangle$, so $\tilde{I}$ is a standard subgroup of $\tilde{X}$ and $\langle\tilde{t}\rangle\in\mathrm{Syl}_2(C_{\tilde{X}}(\tilde{I}))$. As $\tilde{M}_8\cong E_{2^6}$, [16] and [28, (2.10)] show that $E(\tilde{X})\cong SL_2(2^6)$, $SL_3(8)$, or $SL_2(8)\times SL_2(8)$ and $C_{\tilde{X}}(E(\tilde{X}))$ is of odd order. If $E(\tilde{X})\cong SL_2(2^6)$ or $SL_3(8)$, then as the Schur multipliers of these groups are trivial, $E(X \bmod \langle g\rangle)=X^*\times\langle g\rangle$ where $X^*=E(X \bmod \langle g\rangle)'$. As $\langle g\rangle\in\mathrm{Syl}_2(C_X(X^*))$, X^* is a standard subgroup of G isomorphic to $SL_2(2^6)$ or $SL_3(8)$. Hence by [16] and [23], $E(G/O(G))$ is determined. But then [4] and the structure of H yield a contradiction. Thus $E(\tilde{X})\cong SL_2(8)\times SL_2(8)$ and $M_8\langle g, t\rangle\in\mathrm{Syl}_2(C(g))$.

Assume that Case 2 holds and that t and gt are conjugate in $N_X(\langle g, t\rangle)$. Let $Y=N(M_8\langle g\rangle)$ and $\bar{Y}=Y/M_8\langle g\rangle$. Let D be a Sylow 2-subgroup of $N_X(F\langle g, t\rangle)$ containing $M_8\langle g, t\rangle$. By our hypothesis $N_X(\langle g, t\rangle)$ is transitive on $\{t, gt\}$ and as $F\langle g, t\rangle\in\mathrm{Syl}_2(C_X(t))$, a Sylow 2-subgroup of $N_X(\langle g, t\rangle)$ containing $F\langle g, t\rangle$ lies in $N_X(F\langle g, t\rangle)$ and is transitive on $\{t, gt\}$. Thus $N_X(F\langle g, t\rangle)$ is transitive on $Ft\cup Fgt$ and $|N_X(F\langle g, t\rangle)|_2=2^9$. Then $|D\colon M_8\langle g, t\rangle|=2$, $t^D=F\langle g\rangle t$, and $C_D(t)=F\langle g, t\rangle$. As $\mathscr{E}^*(M_8\langle g, t\rangle)=\{M_8\langle g\rangle,$

$F\langle g, t\rangle\}$, $D \leqq Y$. By (7.2) (4), g centralizes K/M_3, so $K \leqq Y$. As $P_1 \in \mathrm{Syl}_2(G)$ and $t^G \cap P_0 = \varnothing$, $\bar{P}_1 = \bar{P}\langle \bar{t}\rangle \in \mathrm{Syl}_2(\bar{Y})$ and $\bar{t}^{\bar{Y}} \cap \bar{P} = \varnothing$. By the Thompson fusion lemma, $\bar{Y} = O^2(\bar{Y})\langle \bar{t}\rangle$ and $O^2(\bar{Y}) \geqq \bar{K}$ with $\bar{P} \in \mathrm{Syl}_2(\bar{K})$. Let $J = N_L(F)'$. Then $N_{O(H)L}(F\langle g, t\rangle) \leqq N_{O(H)L}(F) = O(H) \times J$ and as $[g, J] \leqq F$, $N_H(F\langle g, t\rangle) = JC_{O(H)}(g)\langle g, t\rangle$. Now $t^{N(F\langle g, t\rangle)} = t^D$, whence $N(F\langle g, t\rangle) = N_H(F\langle g, t\rangle)D = N_Y(M_3\langle g, t\rangle)$. As $N_Y(M_3\langle g, t\rangle)$ is the preimage of $C_{\bar{Y}}(\bar{t})$ in Y, $\overline{N_H(F\langle g, t\rangle)} = \bar{J} \times \overline{C_{O(H)}(g)} \times \langle \bar{t}\rangle$ is a subgroup of $C_{\bar{Y}}(\bar{t})$ of index 2 with $\bar{J} \cong GL_3(2)$. As $\bar{K} = \bar{L}_1 \times \bar{L}_1^t$ and $\bar{L}_1 \cong GL_3(2)$, $O_2(\bar{Y}/O(\bar{Y})) = 1$ and so $E(\bar{Y}/O(\bar{Y})) = F^*(\bar{Y}/O(\bar{Y})) \neq 1$. There are no proper t-invariant normal subgroups of $\bar{K}$, so $\bar{K} \leqq E(\bar{Y} \bmod O(\bar{Y}))$. Let X_1 be a component of $E(\bar{Y}/O(\bar{Y}))$. Then $1 \neq X_1 \cap \bar{P}O(\bar{Y})/O(\bar{Y}) \in \mathrm{Syl}_2(X_1)$, so X_1 contains $\bar{L}_1O(\bar{Y})/O(\bar{Y})$ or $\bar{L}_1^tO(\bar{Y})/O(\bar{Y})$, for they are the only proper normal subgroups of $\bar{K}O(\bar{Y})/O(\bar{Y})$. There are no simple groups whose Sylow 2-subgroups are isomorphic to $\bar{P}$, so we have $E(\bar{Y}/O(\bar{Y})) = X_1 \times X_1^t$ and $X_1 \cong E(\bar{Y}/O(\bar{Y})) \cap C(\bar{t}) \lhd C_{\bar{Y}}(\bar{t})O(\bar{Y})/O(\bar{Y})$. Thus $X_1 \cong GL_3(2)$ and $E(\bar{Y}/O(\bar{Y})) = \bar{K}O(\bar{Y})/O(\bar{Y})$. By (7.5) (1), $[\bar{M}_7, \bar{t}] = \bar{B}$ and $[C_{O(\bar{Y})}(\bar{t}), \bar{B}] \leqq [\overline{C_{O(H)}(g)}, \bar{B}] = 1$. Thus $[O(\bar{Y}), \bar{B}] = 1$ by [10, (1J)] and we conclude that $E(\bar{Y}) = \bar{K}$. Now $\bar{D} = (O^2(\bar{Y}) \cap \bar{D})\langle \bar{t}\rangle$. As $|O^2(\bar{Y})/\bar{K}|$ is odd, $O^2(\bar{Y}) \cap \bar{D} \leqq \bar{K}$ and $\bar{D} \leqq \bar{K}\langle \bar{t}\rangle \cap C(\bar{t}) = \bar{J}\langle \bar{t}\rangle$. But then $D \leqq M_3\langle g, t\rangle J \leqq N(F\langle t\rangle)$, contrary to $t^D = F\langle g\rangle t$. Thus in Case 2, t and gt are not conjugate in $N_X(\langle g, t\rangle)$ and by the preceding paragraph $M_3\langle g, t\rangle \in \mathrm{Syl}_2(C(g))$.

Next we wish to show that $|C(gt)|_2 = 2^8$. For this purpose let $C = C(gt)$ and $\bar{C} = C/\langle gt\rangle$. As g and t are not conjugate in G, $N_C(\langle t, gt\rangle) = C_C(t)$, and $C_{\bar{C}}(\bar{t}) = C_C(t)/\langle gt\rangle$. Recall that $C_C(t) = C_H(g) = C_L(g)C_{O(H)}(g)\langle g, t\rangle$ and $I = C_L(g)' \cong SL_2(8)$. Then $C_{\bar{C}}(\bar{t}) \cap C(\bar{I}) = \overline{C_{O(H)}(g)}\langle \bar{t}\rangle$, so $\bar{I}$ is a standard subgroup of $\bar{C}$ and $\langle \bar{t}\rangle \in \mathrm{Syl}_2(C_{\bar{C}}(\bar{I}))$. Moreover, $C \geqq C_P(gt)\langle g, t\rangle$ with $C_P(gt) \cong U$. By [16] and [28, (2.10)], $E(\bar{C})/Z(E(\bar{C}))$ is isomorphic to one of $SL_2(2^6)$, $SL_3(8)$, $PSU_3(8)$, $G_2(3)$, or $SL_2(8) \times SL_2(8)$ and $C_{\bar{C}}(E(\bar{C}))$ is of odd order. In the first three cases, setting $C^* = E(C \bmod \langle gt\rangle)'$ we have $E(C \bmod \langle gt\rangle) = C^* \times \langle gt\rangle$ and C^* is a standard subgroup of G with $\langle gt\rangle \in \mathrm{Syl}_2(C(C^*))$. Hence $E(G/O(G))$ is determined by [16] and [23]. But each possibility of $E(G/O(G))$ is incompatible with the structure of H. If $E(\bar{C}) \cong SL_2(8) \times SL_2(8)$, $\overline{C_P(gt)}\langle \bar{t}\rangle$ is a Sylow 2-subgroup of $\bar{C}$. But $\overline{C_P(gt)}\langle \bar{t}\rangle$ has rank 4, a contradiction. Thus $E(\bar{C})/Z(E(\bar{C})) \cong G_2(3)$ and $C_P(gt)\langle g, t\rangle$ is a Sylow 2-subgroup of $C(gt)$ as required.

Since $|C(g)|_2 \leqq 2^9$ and $|C(gt)|_2 = 2^8$, (7.12) gives $g^G \cap P = (gt)^G \cap P = \varnothing$. As shown before $P_1 \in \mathrm{Syl}_2(G)$ and $t^G \cap P_0 = \varnothing$, whence $t \notin O^2(G)$ by the Thompson fusion lemma. Thus in Case 2, $P = P_1 \cap O^2(G) \in \mathrm{Syl}_2(O^2(G))$ by Thompson's lemma. Suppose Case 1 holds. Then $gt = t^{g_0} \notin O^2(G)$. Hence if $g \notin O^2(G)$, R is a Sylow 2-subgroup of $O^2(G)\langle t\rangle \cap H$, so that P is a Sylow 2-subgroup of $O^2(O^2(G)\langle t\rangle)=O^2(G)$ by (7.11). If $g\in O^2(G)$, $P_1\cap O^2(G)$ is equal to $P\langle g\rangle$, $P\langle g_0t\rangle$, or $P\langle g_0, g_0^t\rangle$. As $g^G\cap P=\emptyset$ and $(g_0t)^2=g$, $P_1\cap O^2(G)\neq P\langle g\rangle$ or $P\langle g_0t\rangle$ by [28, (2.3)]. Assume that $P_1\cap O^2(G)=P\langle g_0, g_0^t\rangle$. Then there is an involution $x\in g^G\cap P\langle g_0\rangle$. Choose $x_1\in U\langle g_0\rangle$ and $x_2\in U^t$ such that $x=x_1x_2$. Then $x_1\in Ug_0$. By (1.3), $x_1^a=g_0$ for some $a\in U$. If $x_2=1$, then $C(x)\geqq F_0\langle g_0\rangle U^t$, which contradicts $|C(g)|_2=2^9$. Thus x_2 is an involution of U^t and as $L_1'/F_0\cong GL_3(2)$ and $\mathscr{E}^*(E_0F_0)=\{E_0, F_0\}$, we have $x_2^b\in E_0^t\cup F_0^t$ for some $b\in L_1'^t$. Then $C_{U^t}(x_2^b)$ is nonabelian by (1.2) and as $x^{ab}=g_0x_2^b$, $C_P(x^{ab})=C_U(g_0)C_{U^t}(x_2^b)$. But $M_3\langle g_0, g_0^t\rangle\langle t\rangle\in \mathrm{Syl}_2(C(g))$, so $M_3\langle g_0, g_0^t\rangle$ is an abelian Sylow 2-subgroup of $O^2(G)\cap C(g)$, a contradiction. Therefore $P=P_1\cap O^2(G)\in \mathrm{Syl}_2(O^2(G))$. The proof is complete.

(7.14) $E(G)/Z(E(G))\cong G_2(3)\times G_2(3)$.

PROOF. By (7.11) and (7.13), P is a Sylow 2-subgroup of $O^2(G)$. We argue that U is strongly involution closed in P with respect to $O^2(G)$. Suppose false and choose an involution $a\in U$ and an element $x\in O^2(G)$ such that $a^x\in P-U$. As $\langle K, L\rangle\leqq O^2(G)$, we may assume that $a\in Z(U)$ and $a^x\in V_1$ by (7.12). Then as $V_1=Z(P)$, a and a^x are conjugate in $O^2(G)\cap N(P)$. Now $P=U\times U^t$, so $N(P)$ acts on the set $\{U', U'^t\}$ by the Krull-Schmidt theorem. As $U'\lhd P$, this implies $U'\lhd O^2(G)\cap N(P)$. But then as $Z(U)\leqq U'$, we have $a^x\in U'$, a contradiction. Thus U is strongly involution closed in P with respect to $O^2(G)$. Let bars denote images in $G/O(G)$. Then $\bar{L}$ is a standard subgroup of $\bar{G}$, so [24, Corollary 2] and a property of groups with a standard subgroup show that $E(\bar{G})\cong G_2(3)\times G_2(3)$. Now [28, (2.10)] establishes the assertion.

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