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On the Number of Apparent Singularities of a Linear Differential Equation

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Introduction

Let M be a compact Riemann Surface of genus g, and let S be a finite subset of M. When a representation ρ of the fundamental group $\pi_1(M-S)$ to the general linear group $GL(n, C)$ is given, we have the so-called Riemann-Hilbert problem: Find a linear differential equation on M having ρ as its monodromy group. This problem has been solved by many mathematicians in various fashions.

In this note a linear differential equation on M means a collection of locally defined linear differential equations on M

$$\frac{d^n y}{dz^n}+A_1(z)\frac{d^{n-1}y}{dz^{n-1}}+\cdots+A_n(z)y=0 ,$$

where z is a local coordinate on M and $A_i(z)$ are meromorphic functions. They are compatible in the sense that any two of them have the same solutions on their common domain of definition.

Then a solution of the Riemann-Hilbert problem in this form has, necessarily, apparent singularities besides the given singularities S. The purpose of this note is to count the number of such apparent singularities. Our result is:

THEOREM. *If the representation ρ is irreducible and if the local representation at some point of S induced by ρ is semi-simple, then there exists a Fuchsian linear differential equation on M which has the given representation ρ as its monodromy group and has at most*

$$1-n(1-g)+\frac{n(n-1)}{2}(m+2g-2) \quad (m=\#S)$$

apparent singularities.

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Here the local representation induced by ρ at a point $p \in S$ is defined as follows: Let U be a neighborhood of p biholomorphic to the unit disc such that $U \cap S = \{p\}$. Then the injection $U - p \to M - S$ induces a representation of $\pi_1(U-p)$ to $GL(n, C)$. This is the local representation at $p \in S$ induced by ρ by definition.

This theorem gives answers to a problem in [2]. The totality of representations of $\pi_1(M-S)$ to $GL(n, C)$ and the totality of the corresponding Fuchsian differential equations form complex manifolds of dimension $n^2(m+2g-2)+1$, and $(n^2(m+2g-2)/2)+(nm/2)$, respectively. The difference of these dimensions is equal to the above mentioned number. So we might expect conversely that general solutions of the Riemann-Hilbert problem have *at least* $1-n(1-g)+(n(n-1)/2)(m+2g-2)$ apparent singularities.

To prove the theorem we use a solution of the Riemann-Hilbert problem given by Deligne in [1]. In §§ 1, 2 and 3 we will resume its essential points and in § 4, introducing Wronskians we will prove the theorem.

§ 1. Outline of Deligne's solution of the Riemann-Hilbert problem.

Let M be a compact Riemann surface of genus g and let S be a set of m points in M. When a representation $\rho: \pi_1(M-S) \to GL(n, C)$ is given, we can find a linear differential equation on M with the monodromy group isomorphic to ρ. Deligne's results are as follows:

(1) Take a local system V' of n-dimensional vector spaces on $M-S$ associated with the representation ρ.

(2) The local system V' determines canonically a holomorphic vector bundle $\mathscr{V}'$ on $M-S$ with a holomorphic connection ∇' such that $V' = \{\xi \in \mathscr{V}' \mid \nabla'\xi = 0\}$.

(3) Extend the pair $(\mathscr{V}', \nabla')$ onto the whole space M as a pair $(\mathscr{V}, \nabla)$, where $\mathscr{V}$ is a holomorphic vector bundle on M and ∇ is a meromorphic connection of $\mathscr{V}$ with simple poles on S. Here an extention of the pair $(\mathscr{V}', \nabla')$ means that the restriction $(\mathscr{V}|_{M-S}, \nabla|_{M-S})$ is isomorphic to the pair $(\mathscr{V}', \nabla')$.

(4) When we want to have an ordinary linear differential equation in the usual sense, take a holomorphic section φ of the dual bundle $\mathscr{V}^*$, and consider the local system $\varphi(V')$ as a subsheaf of $\mathscr{O}_{M-S}$. If $\varphi(V')$ is isomorphic to V' as local systems, then the differential equation with solution sheaf $\varphi(V')$ is the desired one.

§ 2. The Chern class of an extended bundle.

Now we analyze more closely the step (3). Let p be a point of S and let z be a local coordinate on a neighborhood U of p ($z(p)=0$). We assume that U is biholomorphic to the unit disc and that $U\cap S=\{p\}$. The representation $\rho\colon \pi_1(M-S)\to GL(n, \boldsymbol{C})$ induces a representation ρ_U: $\pi_1(U-p)\to GL(n, \boldsymbol{C})$, and the local system V'_U associated with the representation ρ_U is isomorphic to the restriction $V'|_{U-p}$ of V'.

On the other hand $\pi_1(U-p)$ is isomorphic to the infinite cyclic group $\boldsymbol{Z}$. Let γ be the generator of $\pi_1(U-p)$ represented by a loop in $U-p$ rounding p once counter-clockwise. Put $A=\rho_U(\gamma)\in GL(n, \boldsymbol{C})$ and choose a matrix B satisfying $A=\exp(-2\pi iB)$.

Consider the trivial vector bundle $\mathcal{O}_U^n$ over U and consider its meromorphic connection ∇_U with the connection matrix $(B/z)dz$ with respect to the natural frame of $\mathcal{O}_U^n$. Then the pair $(\mathcal{O}_U^n, \nabla_U)$ determines a local system V'' on $U-p$. This consists of solution vectors of the equation

$$\nabla_U\xi=d\xi+\frac{B}{z}dz\xi=0 .$$

By the condition $A=\exp(-2\pi iB)$, this local system V'' is isomorphic to $V'_U=V'|_{U-p}$.

Thus we can patch together $\mathscr{V}'$ and $\mathcal{O}_U^n$ identifying V'_U and V'', and we get an extention of the pair $(\mathscr{V}', \nabla')$ to the point $p\in S$. Let $(\mathscr{V}, \nabla)$ be an extention on M thus obtained.

PROPOSITION. *The Chern class $c(\mathscr{V})$ of $\mathscr{V}$ is equal to*

$$-\sum_{p\in S}\operatorname{tr}(B)$$

($H^2(M, \boldsymbol{Z})$ being identified with $\boldsymbol{Z}$).

The proof is easy. We recall that the trace of the connection ∇ is a connection of the determinant bundle $\det(\mathscr{V})$ of $\mathscr{V}$, and that $c(\mathscr{V})$ is equal to the Chern class $c(\det\mathscr{V})$ by definition. In the case of a line bundle, the sum of residues of a meromorphic connection is equal to the Chern class of the bundle.

The matrix B is arbitrary except that it satisfies the equation $\exp(-2\pi iB)=A$. Hereafter we assume that for a point $p\in S$ the local monodromy matrix A around p is semi-simple. If $A=\operatorname{diag}(a_1, \cdots, a_n)$, $-2\pi iB=\operatorname{diag}(\log a_1, \cdots, \log a_n)$ and we can take the values of $\log a_i$ arbitrarily. Taking into account of the above proposition, this enables us to give any integral value to the Chern class of the extended bundle.

§ 3. Local systems realized in $\mathcal{O}_{M-S}$ and its Wronskian.

Let V' be a subsheaf of $\mathcal{O}_{M-S}$, and assume that the stalk V'_q of V' at any $q \in M-S$ is an n-dimensional vector space. We call such V' a local system realized in $\mathcal{O}_{M-S}$. Clearly V' itself is a local system of vector spaces.

The construction of a linear differential equation on $M-S$ having V' as its solution sheaf is classical. Let $\varphi_1, \cdots, \varphi_n$ be a basis for V' on an open set U of $M-S$, and let z be a local coordinate on U. Then a holomorphic function $y=y(z)$ on U is contained in V' if and only if

$$\begin{vmatrix} D^n y & D^{n-1}y & \cdots & Dy & y \\ D^n\varphi_1 & D^{n-1}\varphi_1 & \cdots & D\varphi_1 & \varphi_1 \\ & & \cdots\cdots\cdots & & \\ D^n\varphi_n & D^{n-1}\varphi_n & & D\varphi_n & \varphi_n \end{vmatrix} = 0 ,$$

where D denotes the differential operator d/dz on U.

Expanding it, we have

$$A_0(z)D^n y + A_1(z)D^{n-1}y + \cdots + A_n(z)y = 0 ,$$

where

$$\begin{aligned} A_0(z) &= \begin{vmatrix} \varphi_1 & D\varphi_1 & \cdots & D^{n-1}\varphi_1 \\ \varphi_2 & D\varphi_2 & \cdots & D^{n-1}\varphi_2 \\ & \cdots\cdots\cdots & & \\ \varphi_n & D\varphi_n & & D^{n-1}\varphi_n \end{vmatrix} \\ &= \varphi \wedge D\varphi \wedge \cdots \wedge D^{n-1}\varphi , \qquad \varphi = {}^t(\varphi_1, \cdots, \varphi_n) , \\ A_1(z) &= -\varphi \wedge D\varphi \wedge \cdots \wedge D^{n-2}\varphi \wedge D^n\varphi , \\ & \cdots\cdots\cdots . \end{aligned}$$

Generally for a vector $\varphi = {}^t(\varphi_1, \cdots, \varphi_n)$ and for a differential operator D, we define $W(\varphi, D) = \varphi \wedge D\varphi \wedge \cdots \wedge D^{n-1}\varphi$ and call it the Wronskian of φ with respect to the operator D.

§ 4. The number of apparent singularities.

Let the pair $(\mathcal{V}, \nabla)$ be a solution of the Riemann-Hilbert problem explained in § 1. If the dual bundle $\mathcal{V}^*$ of $\mathcal{V}$ have a holomorphic section $\varphi \in \Gamma(M, \mathcal{V}^*)$, then the local system $\varphi(V')$ is realized in $\mathcal{O}_{M-S}$ and we have the exact sequence

$$V' \xrightarrow{\varphi} \varphi(V') \longrightarrow 0 .$$

The kernel of φ is a local subsystem of V' and it corresponds to a subrepresentation of ρ. Therefore if ρ is irreducible and if $\varphi(V')$ is not zero, then the local system V' and $\varphi(V')$ are isomorphic. The latter condition is satisfied when φ is not the zero section of $\mathscr{V}^*$. Because $\mathscr{V}'=\mathscr{O}_{M-S}\otimes_C V'$, and $\varphi(V')=0$ implies $\varphi(\mathscr{V}')=0$ and $\varphi(\mathscr{V})=0$ (φ is $\mathscr{O}_M$-linear). Thus we have

PROPOSITION. *If the representation ρ is irreducible, the local system $\varphi(V')$ is isomorphic to V' for any non-zero holomorphic section φ of $\mathscr{V}^*$.*

Now $\mathscr{V}^*$ has a connection dual to ∇. We also denote it by ∇. For a section φ of $\mathscr{V}^*$, we define the Wronskian $W(\varphi, \nabla)$ of φ with respect to ∇ as follows: Let U be an open set of M and let z be a local coordinate on U. Then $\nabla_D\varphi=\langle d/dz, \nabla\varphi\rangle$ is a section of $\mathscr{V}^*$ over U, and we define $W(\varphi, \nabla_D)=\varphi\wedge\nabla_D\varphi\wedge\cdots\wedge(\nabla_D)^{n-1}\varphi$. This is a section of $\det(\mathscr{V}^*)$ over U. For another coordinate z', put $D=KD'$, where $D'=d/dz'$ and $K=dz'/dz$. We have

$$\begin{aligned}\nabla_D\varphi&=K\nabla_{D'}\varphi\ ,\\ (\nabla_D)^2\varphi&=\nabla_D(K\nabla_{D'}\varphi)\\ &=D(K)\nabla_D\varphi+K^2(\nabla_{D'})^2\varphi\ .\end{aligned}$$

Thus

$$\varphi\wedge\nabla_D\varphi\wedge(\nabla_D)^2\varphi=K^3\varphi\wedge\nabla_{D'}\varphi\wedge(\nabla_{D'})^2\varphi\ .$$

Repeating this procedure, we have

$$W(\varphi, \nabla_D)=K^{n(n-1)/2}W(\varphi, \nabla_{D'})\ .$$

Thus $W(\varphi, \nabla_D)$ defines a section of $\det(\mathscr{V}^*)\otimes\Omega^{n(n-1)/2}$. We call it the Wronskian of φ with respect to ∇ and denote it by $W(\varphi, \nabla)$. Here Ω denote the canonical sheaf of M and Ω^k is the k-times tensor product.

Let $\xi_1, \cdots, \xi_n$ be a C-basis for $V'\subset\mathscr{V}'$ over U. Then we have $\nabla\xi_i=0$ for $i=1, \cdots, n$. For a global holomorphic section φ of $\mathscr{V}^*$, $\langle\varphi, \xi_1\rangle, \cdots, \langle\varphi, \xi_n\rangle\in\Gamma(U, \mathscr{O})$ is a C-basis for $\varphi(V')$ over U. The differential equation with the solution sheaf $\varphi(V')$ is

$$\begin{vmatrix}\langle\varphi, \xi_1\rangle & \cdots & \langle\varphi, \xi_n\rangle & y\\ D\langle\varphi, \xi_1\rangle & \cdots & D\langle\varphi, \xi_n\rangle & Dy\\ & \cdots\cdots\cdots & & \\ D^{n-1}\langle\varphi, \xi_1\rangle & \cdots & D^{n-1}\langle\varphi, \xi_n\rangle & D^{n-1}y\\ D^n\langle\varphi, \xi_1\rangle & \cdots & D^n\langle\varphi, \xi_n\rangle & D^ny\end{vmatrix}=0\ .$$

But $D^k\langle\varphi, \xi_i\rangle=\langle\nabla_D^k\varphi, \xi_i\rangle$ for any k because of the general identity $D\langle\varphi, \xi\rangle=\langle\nabla_D\varphi, \xi\rangle+\langle\varphi, \nabla_D\xi\rangle$. Therefore

$$\begin{vmatrix} \langle\varphi, \xi_1\rangle & \cdots & \langle\varphi, \xi_n\rangle & y \\ \langle\nabla\varphi, \xi_1\rangle & \cdots & \langle\nabla\varphi, \xi_n\rangle & Dy \\ & \cdots\cdots\cdots & & \\ \langle\nabla^n\varphi, \xi_1\rangle & \cdots & \langle\nabla^n\varphi, \xi_n\rangle & D^n y \end{vmatrix}=0 .$$

Expanding this, we have

$$A_0D^ny+A_1D^{n-1}y+\cdots+A_ny=0 ,$$

where

$$A_0=\langle W(\varphi, \nabla), \xi_1\wedge\cdots\wedge\xi_n\rangle ,$$
$$A_1=-\langle\varphi\wedge\nabla\varphi\wedge\cdots\wedge\nabla^{n-2}\varphi\wedge\nabla^n\varphi, \xi_1\wedge\cdots\wedge\xi_n\rangle$$
$$\cdots\cdots\cdots .$$

Dividing by $\xi_1\wedge\cdots\wedge\xi_n\neq0$, we see that our differential equation has singularities only at the zeros and poles of the Wronskian $W(\varphi, \nabla)$.

At any point $q\in M-S$, $W(\varphi, \nabla)$ is holomorphic. At $p\in S$, with respect to the natural frame of $\mathcal{O}_U^n$ let φ be represented by ${}^t(\varphi_1, \cdots, \varphi_n)$. Then

$$\nabla_D\varphi=D\varphi+\frac{B}{z}\varphi$$

$$(\nabla_D)^2\varphi=D^2\varphi+2\frac{B}{z}D\varphi-\frac{B}{z^2}\varphi+\frac{B^2}{z^2}\varphi$$

$$\varphi\wedge\nabla_D\varphi\wedge(\nabla_D)^2\varphi=\varphi\wedge\frac{B}{z}\varphi\wedge\left(-\frac{B}{z^2}+\frac{B^2}{z^2}\right)\varphi+\cdots$$
$$=\varphi\wedge\frac{B}{z}\varphi\wedge\frac{B^2}{z^2}\varphi+\cdots$$

where $\cdots$ are terms of higher power of z. Repeating this procedure, we have at last

$$\varphi\wedge\nabla_D\varphi\wedge\cdots\wedge(\nabla_D)^{n-1}\varphi=\varphi\wedge\frac{B}{z}\varphi\wedge\cdots\wedge\frac{B^{n-1}}{z^{n-1}}\varphi+\cdots ,$$

that is,

$$W(\varphi, \nabla_D)=z^{-n(n-1)/2}(\varphi\wedge B\varphi\wedge\cdots\wedge B^{n-1}\varphi+\cdots) .$$

Thus $W(\varphi, \nabla)$ has a pole at $p\in S$ of order at most $n(n-1)/2$, and the zeros are apparent singularities of our differential equation.

The Wronskian $W=W(\varphi, \nabla)$ being a meromorphic section of $\mathscr{V}^*\otimes\Omega^{n(n-1)/2}$,

$$\sharp(\text{Zeros of } W)-\sharp(\text{Poles of } W)=c(\mathscr{V}^*)+\frac{n(n-1)}{2}c(\Omega)$$
$$=c(\mathscr{V}^*)+\frac{n(n-1)}{2}(2g-2)\ .$$
$$\sharp(\text{Zeros of } W)\leqq c(\mathscr{V}^*)+\frac{n(n-1)}{2}(m+2g-2)\quad (m=\sharp S)\ .$$

On the other hand, by the Riemann-Roch theorem for vector bundles, we have

$$\dim \Gamma(M, \mathscr{V}^*)\geqq c(\mathscr{V}^*)+n(1-g)\ .$$

If we choose an extention $\mathscr{V}$ of $\mathscr{V}'$ with $c(\mathscr{V}^*)=1-n(1-g)$, the number of zeros of $W(\varphi, \nabla)$ does not exceed

$$1-n(1-g)+\frac{n(n-1)}{2}(m+2g-2)\ .$$

THEOREM. *Let M be a compact Riemann surface of genus g and let S be a set of m points on M. Assume that an irreducible representation $\rho\colon \pi_1(M-S)\to GL(n, \boldsymbol{C})$ is given and that the induced local representation at some point of S is semi-simple. Then there exists a Fuchsian linear differential equation on M having the given representation ρ as its monodromy group and at most*

$$1-n(1-g)+\frac{n(n-1)}{2}(m+2g-2)$$

apparent singularities.

References

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