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On Modules over a Serial Ring Whose Endomorphism Rings are Quasi-Frobenius

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Introduction

The purpose of this paper is to establish several necessary and sufficient conditions for a module over a serial ring to have a quasi-Frobenius endomorphism ring.

In the study of properties of modules, it is greatly important to investigate their endomorphism rings. By Schur's Lemma the endomorphism ring of a simple module is a division ring, and we have enough knowledge about the endomorphism rings of modules over a semi-simple ring. Here we shall investigate the following problem:

PROBLEM. *Find a necessary and sufficient condition for a module U over a ring R to have a quasi-Frobenius endomorphism ring.*

Quasi-Frobenius rings are one of the most important classes of rings which are not semi-simple; in fact, a group algebra KG of a finite group G over a field K such that $\mathrm{char}(K)\,|\,|G|$ is not semi-simple, but it is quasi-Frobenius. As for the problem in the case U being a faithful module over a quasi-Frobenius ring, C. W. Curtis [1] gave a sufficient condition and K. Morita [6] obtained a necessary and sufficient condition. Recently J. A. Green [4] and H. Sawada [11] showed that a certain nonfaithful module over a group algebra of a finite group with a split (B, N)-pair has a Frobenius endomorphism algebra. Stimulated with Sawada's result [10], Green [4] gave a necessary condition for our problem in the case of U being a module over a group algebra under a certain assumption, and again Sawada [12] extended Green's result. On the other hand, K. Morita gave a sufficient condition for the above problem in the case U being a module over an Artinian ring (cf. Remark 14). However, each of these conditions is not a necessary

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and sufficient condition for our problem. Indeed, even a special problem of finding a necessary and sufficient condition for a module to have a division ring as its endomorphism ring has not yet been settled. In this paper, as the first step to solve our problem, we shall restrict ourselves to the case of U being a module over a serial ring, and solve the problem in this case.

Serial rings were introduced by T. Nakayama [8] in 1940 (he called them "generalized uniserial rings") as a generalization of uniserial rings in the sense of G. Köthe. Since then they were studied by H. Kupisch [5], and later by I. Murase [7] and by K. R. Fuller [2, 3]. The class of serial rings seems to be the unique class of rings which is fairly studied, except the class of semi-simple rings and that of quasi-Frobenius rings.

The main theorem of this paper is stated as follows.

THEOREM. *Let R be an indecomposable serial ring with the radical J. Write* 1 *as a sum of mutually orthogonal primitive idempotents*

$$1=\sum_{i=1}^{n}\sum_{j=1}^{k_i} e_{ij}$$

where $Re_{ij}\cong Re_{rt}$ if and only if $i=r$. Let ${}_RU$ be a faithful module such that

$${}_RU=\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{p_i} U_{ij}$$

where each ${}_RU_{ij}$ is indecomposable and ${}_RU_{ij}\cong{}_RU_{rt}$ if and only if $i=r$. Put

$$\sigma=\{i \mid Re_{i1}/Je_{i1}\cong \mathrm{Top}(U_{k1}) \text{ for some } k\}$$

and

$$e=\sum_{i\in\sigma} e_{i1}\,.$$

Assume that $\mathrm{End}_R(U)$ *is an indecomposable ring. Then the following conditions are equivalent:*

(a) $\mathrm{End}_R(U)$ *is a quasi-Frobenius ring.*

(b) $\bigoplus_{i=1}^{s} U_{i1}$ *is a minimal faithful left R-module and*

$$\mathrm{Top}\,({}_{eRe}\bigoplus_{i=1}^{s} eU_{i1})\cong \mathrm{Soc}\,({}_{eRe}\bigoplus_{i=1}^{s} eU_{i1})\,.$$

(c) $c({}_{eRe}eU_{i1})=c({}_{eRe}eU_{j1})$ *for all i and j, and* $\mathrm{Top}\,(U_{i1})\cong\mathrm{Top}\,(U_{j1})$ *if and only if $i=j$, where $c(M)$ denotes the composition length of M.*

Although there is an additional assumption that $\mathrm{End}_R(U)$ is an indecomposable ring, it is not essential as we will point out in Remark 9 of section 3.

In the first section, we will define some terminology and recall the known results. In the second section, we shall prove our fundamental lemma. In sections 3 and 4, we establish the main theorem stated as above.

The author wishes to express his thanks to Professor Kiiti Morita for his helpful suggestions.

§ 1. Preliminaries.

Throughout this paper, R denotes an Artinian ring with unit element 1. Put $J=\mathrm{Rad}\,(R)$. Write 1 as a sum of mutually orthogonal primitive idempotents

$$1=\sum_{i=1}^{n}\sum_{j=1}^{k_i} e_{ij}$$

where $Re_{ij}\cong Re_{st}$ if and only if $i=s$. For a left R-module ${}_RM$, $c({}_RM)$ denotes the composition length of M and $\mathrm{Top}\,({}_RM)$ denotes the top of M, i.e., $\mathrm{Top}\,({}_RM)=M/JM$. If ${}_RM$ is a uniserial module with the composition factor modules

$$J^{k-1}M/J^kM\cong Re_{i_k1}/Je_{i_k1} \qquad \text{for} \quad 1\leqq k\leqq m=c(M)\ ,$$

then we say that the composition type of M is $(i_1, i_2, \cdots, i_m)$; in particular, the composition type of Re_{j1}/Je_{j1} is (j).

Homomorphisms between left R-modules will be written on the right, so that fg is first f, then g; similarly, the endomorphism ring of a left R-module will be act on the right.

For each integer j, $[j]$ denotes the least positive remainder of j *modulo* n. This notation is very convenient to consider a left Kupisch series of a serial ring.

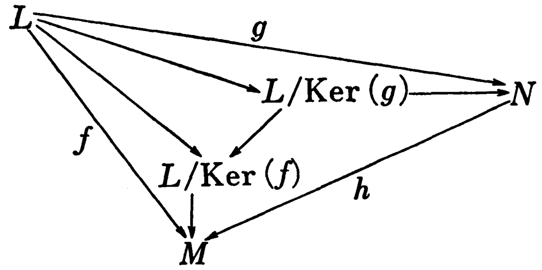
The terminology is the same as in K. R. Fuller [2]. The following lemmas are useful.

LEMMA 1. *Each indecomposable module over a serial ring is quasi-injective, quasi-projective and uniserial.*

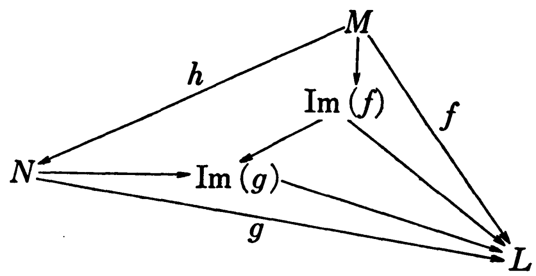
PROOF. cf. T. Nakayama [9] and K. R. Fuller [3].

LEMMA 2. *Let L, M and N be indecomposable left R-modules over a serial ring R.*

(i) *Let* $f: L \to M$ *and* $g: L \to N$ *be homomorphisms such that* $\mathrm{Ker}(f) \supseteqq \mathrm{Ker}(g)$. *If* $c(N)+c(\mathrm{Ker}(g)) \leqq c(M)+c(\mathrm{Ker}(f))$, *then there exists* $h: N \to M$ *such that* $gh=f$:



(ii) *Let* $f: M \to L$ *and* $g: N \to L$ *be homomorphisms such that* $\mathrm{Im}(g) \supseteqq \mathrm{Im}(f)$. *If* $c(\mathrm{Ker}(f)) \leqq c(\mathrm{Ker}(g))$, *then there exists* $h: M \to N$ *such that* $hg=f$:



PROOF. Obvious by Lemma 1.

COROLLARY 3. *Let* M *and* N *be indecomposable modules over a serial ring such that* $c(M) \leqq c(N)$. *Then*

(i) *If* $\mathrm{Top}(M) \cong \mathrm{Top}(N)$, *then there exists an epimorphism* $\pi: N \to M$.

(ii) *If* $\mathrm{Soc}(M) \cong \mathrm{Soc}(N)$, *then there exists a monomorphism* $\theta: M \to N$.

PROOF. Obvious from Lemma 2.

§ 2. Fundamental lemma.

First in this section, we shall prove the following lemma.

LEMMA 4. *Let* R *be a serial ring and* ${}_RU$ *be a faithful left* R-*module such that*

$$ {}_RU = {}_RU_1 \oplus \cdots \oplus {}_RU_s $$

where each ${}_RU_i$ *is indecomposable. If* $\mathrm{End}_R(U)$ *is a quasi-Frobenius ring, then*

(i) $\mathrm{Top}\,({}_RU_i)\cong\mathrm{Top}\,({}_RU_j)$ *if and only if* ${}_RU_i\cong{}_RU_j$.
(ii) $\mathrm{Soc}\,({}_RU_i)\cong\mathrm{Soc}\,({}_RU_j)$ *if and only if* ${}_RU_i\cong{}_RU_j$.

PROOF. Put $S=\mathrm{End}_R(U)$, $N=\mathrm{Rad}\,(S)$ and let $f_i\colon {}_RU\to{}_RU_i$ be the projection for all i. First notice that

$$(1)\qquad {}_SSf_i\cong{}_SSf_j \text{ if and only if } {}_RU_i\cong{}_RU_j\,.$$

Proof of (i). Since the 'if' part is trivial, we shall prove the 'only if' part.

Assume $\mathrm{Top}\,(U_i)\cong\mathrm{Top}\,(U_j)$. Without loss of generality, we can assume $c(U_i)\geqq c(U_j)$. Then, by Corollary 3, there exists $\pi\in S$ such that $\pi=f_i\pi f_j$ and it induces an epimorphism $\pi|_{U_i}\colon U_i\to U_j$. Since S is QF, there exists Sf_k such that $\mathrm{Soc}\,(Sf_k)\cong Sf_j/Nf_j$. Then $f_j\cdot\mathrm{Soc}\,(Sf_k)\neq 0$, hence there exists a nonzero element $\varphi\in f_j\cdot\mathrm{Soc}\,(Sf_k)$. Since $\mathrm{Soc}\,(Sf_k)$ is a left S-module, $\pi\varphi\in\mathrm{Soc}\,(Sf_k)$. On the other hand, $\pi\varphi\neq 0$ because $\varphi\neq 0$ and π is an epimorphism. Hence $f_i\cdot\mathrm{Soc}\,(Sf_k)\neq 0$, i.e., Sf_i/Nf_i is isomorphic to a direct summand of $\mathrm{Soc}\,(Sf_k)$. Therefore $Sf_i/Nf_i\cong Sf_j/Nf_j$, thus $Sf_i\cong Sf_j$. Hence $U_i\cong U_j$.

Proof of (ii). Assume $\mathrm{Soc}\,(Sf_i)\cong\mathrm{Soc}\,(Sf_j)$. Without loss of generality, we can assume $c(U_i)\leqq c(U_j)$. Then, by Corollary 3, there exists $\theta\in S$ such that $\theta=f_i\theta f_j$ and it induces a monomorphism $\theta|_{U_i}\colon U_i\to U_j$. Since S is QF, there exists Sf_k such that $\mathrm{Soc}\,(Sf_i)\cong Sf_k/Nf_k$. Then $f_k\cdot\mathrm{Soc}\,(Sf_i)\neq 0$, hence there exists a nonzero element $\psi\in f_k\cdot\mathrm{Soc}\,(Sf_i)$. Since $N(\psi\theta)=(N\psi)\theta=0$, we have $\psi\theta\in\mathrm{Soc}\,(Sf_j)$. On the other hand, $\psi\theta\neq 0$ because $\psi\neq 0$ and θ is a monomorphism. Hence $f_k\cdot\mathrm{Soc}\,(Sf_j)\neq 0$, i.e., Sf_k/Nf_k is isomorphic to a direct summand of $\mathrm{Soc}\,(Sf_j)$. Since S is QF, we have $Sf_i\cong Sf_j$, thus $U_i\cong U_j$ by (1).

LEMMA 5. *Let R be a serial ring and ${}_RU$ be a left R-module such that*

$${}_RU={}_RU_1\oplus\cdots\oplus{}_RU_s$$

where each ${}_RU_i$ is indecomposable. Assume that $\mathrm{Top}\,(U_i)\not\cong\mathrm{Top}\,(U_j)$ *and* $\mathrm{Soc}\,(U_i)\not\cong\mathrm{Soc}\,(U_j)$ *if $i\neq j$. If* $\mathrm{End}_R\,(U)$ *is an indecomposable ring, then each U_i is not simple.*

PROOF. If U_i is simple, then

$$\begin{aligned}\mathrm{Hom}_R\,(U_i, U_j)\neq 0 &\Longleftrightarrow U_i\cong\mathrm{Soc}\,(U_j)\Longleftrightarrow i=j\\ &\Longleftrightarrow \mathrm{Top}\,(U_j)\cong U_i\Longleftrightarrow \mathrm{Hom}_R\,(U_j, U_i)\neq 0\,.\end{aligned}$$

Thus $\mathrm{Hom}_R\,(U_i, U_j)=0=\mathrm{Hom}_R\,(U_j, U_i)$ if $i\neq j$. Therefore

$$\mathrm{End}_R(U)=\mathrm{End}_R(U_i)\oplus\mathrm{End}_R(\bigoplus_{k\neq i}U_k)$$

as rings. Thus $\mathrm{End}_R(U)$ decomposes.

Now, let us proceed to our fundamental lemma of this paper. Let R be an indecomposable self-basic serial ring with the radical J. Write 1 as a sum of mutually orthogonal primitive idempotents $1=e_1+\cdots+e_n$ such that $Re_1, Re_2, \cdots, Re_n$ is a left Kupisch series of R.

Let ${}_RU$ be a faithful left R-module such that

$${}_RU={}_RU_1\oplus\cdots\oplus{}_RU_n$$

where each ${}_RU_i$ is indecomposable. Assume further that

$${}_RU_i\cong Re_i/J^{m_i}e_i \qquad \text{for all} \quad i \ (m_i\neq 0)\ .$$

Put $S=\mathrm{End}_R(U)$, $N=\mathrm{Rad}(S)$ and let $f_i\colon U\to U_i$ be the projection. Then ${}_RU_i\cong{}_RU_j$ if and only if $i=j$, and S is a self-basic ring. Our fundamental lemma is stated as follows:

LEMMA 6. *Under the above assumptions, if $S=\mathrm{End}_R(U)$ is an indecomposable ring, then the following conditions are equivalent*:

(a) *S is a quasi-Frobenius ring.*

(b) *R is a quasi-Frobenius ring.*

(c) *${}_RU$ is a minimal faithful left R-module.*

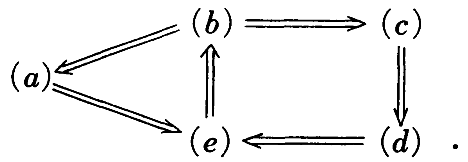
(d) *${}_RU$ is an injective left R-module.*

(e) *$c({}_RU_i)=c({}_RU_j)$ for all i and j.*

Moreover, if the above conditions are satisfied, then

$${}_RU\cong{}_RR \quad \textit{and} \quad R\cong\mathrm{End}_R(U)\ .$$

PROOF. We shall prove this lemma as in indicated by the following diagram;



$(b)\Rightarrow(c)$. Since R is self-basic, QF and serial, a minimal faithful left R-module is isomorphic to ${}_RR$. On the other hand, ${}_RU$ is a factor module of ${}_RR$ and faithful, and hence ${}_RU\cong{}_RR$. Thus ${}_RU$ is a minimal faithful left R-module.

$(c)\Rightarrow(d)$. A minimal faithful module is injective.

$(d)\Rightarrow(e)$. If $m_i\neq m_j$ for some i and j, then there exists k such that $m_k\lneqq m_{[k+1]}$. By K. R. Fuller [2],

$$\begin{aligned}{}_R U_k &\cong Re_k/J^{m_k}e_k\cong Je_{[k+1]}/J^{m_k+1}e_{[k+1]}\\ &\subsetneqq Re_{[k+1]}/J^{m_k+1}e_{[k+1]}\cong U_{[k+1]}/J^{m_k+1}U_{[k+1]}\ .\end{aligned}$$

Since $U_{[k+1]}$ is indecomposable, $U_{[k+1]}/J^{m_k+1}U_{[k+1]}$ is also indecomposable. On the other hand, U_k is injective. Thus, an indecomposable module $U_{[k+1]}/J^{m_k+1}U_{[k+1]}$ has a proper injective submodule U_k. This is a contradiction. Hence we have proved that $m_i=m_j$ for all i and j.

$(e)\Rightarrow(b)$. Let us put $c(Re_i)=\max\{c(Re_1),\cdots,c(Re_n)\}$. Then Re_i is a direct summand of a minimal faithful left R-module (cf. I. Murase [7]), and hence Re_i is also isomorphic to a direct summand of U since U is faithful. Since U_i is the unique direct summand of U whose top is isomorphic to $\mathrm{Top}(Re_i)$, we have $Re_i\cong U_i$. Then

$$\begin{aligned}c(Re_j)&\geqq c(U_j)=c(U_i)=c(Re_i)\\ &=\max\{c(Re_1),\cdots,c(Re_n)\}\quad\text{for all } j\ .\end{aligned}$$

Hence $c(Re_i)=c(Re_j)$ for all j. Therefore R is QF (cf. I. Murase [7]).

$(b)\Rightarrow(a)$. K. Morita [4], Theorem 16.6.

$(a)\Rightarrow(e)$. Assume (a). First, notice that S is self-basic, and hence $N=\{\varphi\in S|\ \mathrm{Im}(\varphi)\subseteqq JN\}$.

Now we distinguish two cases;

(i) $\mathrm{Soc}(U_i)\cong\mathrm{Soc}(U_j)$ for some i and j $(i\neq j)$,

(ii) $\mathrm{Soc}(U_i)\not\cong\mathrm{Soc}(U_j)$ for all i and j $(i\neq j)$.

Case (i). In this case, S is not QF from Lemma 4.

Case (ii). If $m_i\neq m_j$ for some $i\neq j$, then there exists k such that $m_k\lneqq m_{[k+1]}$. By K. R. Fuller [2],

$$\begin{aligned}U_k&\cong Re_k/J^{m_k}e_k\cong Je_{[k+1]}J^{m_k+1}e_{[k+1]}\\ &\cong JU_{[k+1]}/J^{m_k+1}U_{[k+1]}\ .\end{aligned}$$

If $m_k+1=m_{[k+1]}$, then $\mathrm{Soc}(U_k)\cong\mathrm{Soc}(JU_{[k+1]})=\mathrm{Soc}(U_{[k+1]})$. This contradicts the assumption (ii). Thus $m_k+1\lneqq m_{[k+1]}$.

We shall next prove

$$\mathrm{Im}(\varphi)\subsetneqq JU_{[k+1]}\qquad\text{for all}\quad \varphi\in Nf_{[k+1]}\ . \tag{2}$$

Since $N=\{\varphi\in S|\ \mathrm{Im}(\varphi)\subseteqq JU\}$, we have $\mathrm{Im}(\varphi)\subseteqq JU_{[k+1]}$ for all $\varphi\in Nf_{[k+1]}$. If $\mathrm{Im}(\varphi)=JU_{[k+1]}$ for some $\varphi\in Nf_{[k+1]}$, then $\mathrm{Im}(f_k\varphi)=JU_{[k+1]}$. Therefore

$$m_k = c(U_k) \geqq c(\mathrm{Im}\,(f_k\varphi)) = c(JU_{[k+1]}) \\ = c(U_{[k+1]}) - 1 = m_{[k+1]} - 1\ .$$

This contradicts $m_k + 1 \lneqq m_{[k+1]}$. Hence (2) is proved.

Since $\mathrm{Soc}\,(U_1), \cdots, \mathrm{Soc}\,(U_n)$ are mutually non-isomorphic, there exists i such that $\mathrm{Soc}\,(U_i) \cong Re_k/Je_k$. By Lemma 5, U_i is not simple, and hence the composition type of U_i is $(i, \cdots, \cdots, [k+1], k)$. Then there exists $\psi \in \mathrm{Hom}_R\,(U_{[k+1]}, U_i)$ such that $\mathrm{Im}\,(\psi) = J^{m_i-2}U_i$, i.e., $c(\mathrm{Im}\,(\psi)) = 2$. Then for all $\varphi \in Nf_{[k+1]}$, we have

$$c(\mathrm{Im}\,(\varphi\psi)) = c(\mathrm{Im}\,(\varphi)) - c(\mathrm{Ker}\,(\psi)) \\ \lneqq c(JU_{[k+1]}) - (c(U_{[k+1]}) - c(\mathrm{Im}\,(\psi))) \\ = (m_{[k+1]} - 1) - (m_{[k+1]} - 2) = 1\ .$$

Hence $N\psi = Nf_{[k+1]}\psi = 0$. This means $\psi(\neq 0) \in \mathrm{Soc}\,(Sf_i)$. Thus $f_{[k+1]} \cdot \mathrm{Soc}\,(Sf_i) \neq 0$. Therefore $Sf_{[k+1]}/Nf_{[k+1]}$ is isomorphic to a direct summand of $\mathrm{Soc}\,(Sf_i)$. By the same argument, Sf_k/Nf_k is isomorphic to a direct summand of $\mathrm{Soc}\,(Sf_i)$. Hence $\mathrm{Soc}\,(Sf_i)$ is not simple, and this contradicts that S is QF. Thus we have proved that $m_i = m_j$ for all i and j.

§ 3. Main theorem.

Let R be an indecomposable serial ring with the radical J, and write 1 as a sum of mutually orthogonal primitive idempotents

$$1 = \sum_{i=1}^{n} \sum_{j=1}^{k_i} e_{ij}$$

where $Re_{ij} \cong Re_{rt}$ if and only if $i = r$. Assume that $Re_{11}, Re_{21}, \cdots, Re_{n1}$ is a left Kupisch series of R.

Let ${}_RU$ be a faithful left R-module such that

$${}_RU = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{p_i} U_{ij}$$

where each U_{ij} is indecomposable and $U_{ij} \cong U_{rt}$ if and only if $i = r$. Put

$$\sigma = \{\, i \mid Re_{i1}/Je_{i1} \cong \mathrm{Top}\,(U_{k1}) \text{ for some } k\}\ ,$$

and

$$e = \sum_{i \in \sigma} e_{i1}\ .$$

The following theorem is the main result of this paper.

THEOREM 7. *The notations are as above. Assume that* $\mathrm{End}_R(U)$ *is an indecomposable ring. Then the following conditions are equivalent*:

(a) $\mathrm{End}_R(U)$ *is a quasi-Frobenius ring.*

(b) $\bigoplus_{i=1}^{s} U_{i1}$ *is a minimal faithful left R-module and*

$$\mathrm{Top}\,({}_{eRe}\bigoplus_{i=1}^{s} eU_{i1})\cong \mathrm{Soc}\,({}_{eRe}\bigoplus_{i=1}^{s} eU_{i1})\ .$$

(c) $c({}_{eRe}eU_{i1})=c({}_{eRe}eU_{j1})$ *for all* i *and* j, *and* $U_{ij}\cong U_{rt}$ *if and only if* $\mathrm{Top}\,(U_{ij})\cong \mathrm{Top}\,(U_{rt})$.

Moreover, if the above conditions are satisfied, then the rings $\mathrm{End}_R(U)$ *and eRe are Morita equivalent.*

Theorem 7 is derived from the next theorem.

THEOREM 8. *The notations are as above. Assume that* $\mathrm{End}_R(U)$ *is an indecomposable ring. Assume further that*

(3) $\qquad U_{ij}\cong U_{rt}$ *if and only if* $\mathrm{Top}\,(U_{ij})\cong \mathrm{Top}\,(U_{rt})$.

Then the following conditions are equivalent:

(a) $\mathrm{End}_R(U)$ *is a quasi-Frobenius ring.*

(b) $\mathrm{End}_{eRe}(eU)$ *is a quasi-Frobenius ring.*

(c) *eRe is a quasi-Frobenius ring.*

(d) $\bigoplus_{i=1}^{s} eU_{i1}$ *is a minimal faithful left eRe-module.*

(e) $\bigoplus_{i=1}^{s} eU_{i1}$ *is an injective left eRe-module.*

(f) $c({}_{eRe}eU_{i1})=c({}_{eRe}eU_{j1})$ *for all* i *and* j.

(g) $\bigoplus_{i=1}^{s} U_{i1}$ *is a minimal faithful left R-module and*

$$\mathrm{Top}\,({}_{eRe}\bigoplus_{i=1}^{s} eU_{i1})\cong \mathrm{Soc}\,({}_{eRe}\bigoplus_{i=1}^{s} eU_{i1})\ .$$

Moreover, if the above conditions are satisfied, then the rings $\mathrm{End}_R(U)$, $\mathrm{End}_{eRe}(eU)$ *and eRe are Morita equivalent.*

REMARK 9. Several assumptions in Theorem 7 such as

(i) R is an indecomposable ring,

(ii) ${}_RU$ is faithful,

(iii) $\mathrm{End}_R(U)$ is an indecomposable ring,

are not essential. Ad (i): If R decomposes into a direct sum of indecomposable two-sided ideals, then we can apply Theorem 7 over each indecomposable component. Ad (ii): If ${}_RU$ is not faithful, then we have only to consider $R/(0\colon U)$ instead of R. Ad (iii): If $\mathrm{End}_R(U)$ decomposes, then we have only to consider the direct summand of U which corresponds to an indecomposable component of $\mathrm{End}_R(U)$.

Here we shall prove Theorem 7 under the assumption that Theorem 8 is true.

PROOF OF THEOREM 7. To prove Theorem 7, it is sufficient to prove $(a)\Rightarrow(3)$ and $(b)\Rightarrow(3)$.

$(a)\Rightarrow(3)$. This is nothing else than Lemma 4.

$(b)\Rightarrow(3)$. Assume $\mathrm{Top}(U_{i1})\cong\mathrm{Top}(U_{j1})$. Without loss of generality, we can assume that $c(U_{i1})\geqq c(U_{j1})$. Then there exists an epimorphism $\pi\colon U_{i1}\to U_{j1}$. Since U_{j1} is a direct summand of a minimal faithful left R-module, U_{j1} is projective. If π is not an isomorphism, then U_{i1} is not indecomposable since U_{j1} is projective. This is a contradiction. Thus we have $U_{i1}\cong U_{j1}$.

§ 4. Proof of Theorem 8.

In this section, we shall prove Theorem 8. Throughout this section, the notations and the assumptions are as in Theorem 8.

Since 'QF' is Morita invariant, we have only to prove Theorem 8 for the case where $k_i=1$ for $1\leqq i\leqq n$ and $p_j=1$ for $1\leqq j\leqq s$. In this case, we write

$$e_i=e_{i1}(1\leqq i\leqq n),\ \ U_j=U_{j1}(1\leqq j\leqq s)\ .$$

Without loss of generality, we can assume that

$$\mathrm{Top}(U_i)\cong Re_{q(i)}/Je_{q(i)}\ \text{for all } i \text{ and } q(1)\lneqq q(2)\lneqq\cdots\lneqq q(s)\ .$$

Here, by the definition, we have

$$e=\sum_{j=1}^{s}e_{q(j)}\ .$$

Moreover, put

$$\begin{aligned}&m_i=c({}_RU_j)\ \text{for } 1\leqq j\leqq s\ ,\\ &S=\mathrm{End}_R(U),\ N=\mathrm{Rad}(S)\ ,\\ &S'=\mathrm{End}_{eRe}(eU),\ N'=\mathrm{Rad}(S')\ .\end{aligned}$$

Let $f_j\colon {}_RU\to{}_RU_j$ be the projection and $f_j'=f_j|_{eU}\colon {}_{eRe}eU\to{}_{eRe}eU_j$.

Then there exists a ring homomorphism

$$\Phi\colon S=\mathrm{End}_R(U)\ni\varphi\longmapsto\varphi|_{eU}\in\mathrm{End}_{eRe}(eU)=S'\ .$$

For any $\varphi\in S$, $\mathrm{Im}(\varphi)\neq0$ implies that $e(\mathrm{Im}(\varphi))\neq0$ by virtue of the definition of e. Hence we have

$$\varphi \in \operatorname{Ker}(\Phi) \Longleftrightarrow \varphi|_{eU} = \Phi(\varphi) = 0$$
$$\Longleftrightarrow e(\operatorname{Im}(\varphi)) = 0 \Longleftrightarrow \operatorname{Im}(\varphi) = 0 \Longleftrightarrow \varphi = 0\,,$$

and hence we see that Φ is one-to-one. Put $\Phi_{ij} = \Phi|_{f_i S f_j}$; then

$$\Phi_{ij}(f_i s f_j) = f_i s f_j|_{eU_i} \qquad \text{for all} \quad f_i s f_j \in f_i S f_j$$

and Φ_{ij} is a module monomorphism.

LEMMA 10. *The notations and the assumptions are as above. If* $m_i \geqq m_j - [q(j) - q(i)]$, *then* Φ_{ij} *is onto* $(i \neq j)$. (*For the notation* [ ], *cf. its definition in the first section*).

PROOF. In this proof, we will identify U_i with $Re_{q(i)}/J^{m_i}e_{q(i)}$. Let $h_i \colon Re_{q(i)} \to U_i$ be a canonical epimorphism and $h'_i = h_i|_{eRe_{q(i)}}$ for all i. Moreover, put

$$\Psi \colon \operatorname{Hom}_R(Re_{q(i)}, U_j) \ni \psi \longmapsto \psi|_{eRe_{q(i)}} \in \operatorname{Hom}_{eRe}(eRe_{q(i)}, eU_j)\,.$$

Then we get the following commutative diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Hom}_R(U_i, U_j) & \xrightarrow{\Phi_{ij}} & \operatorname{Hom}_{eRe}(eU_i, eU_j) \\
 & & \downarrow {\scriptstyle \operatorname{Hom}(h_i, 1)} & & \downarrow {\scriptstyle \operatorname{Hom}(h'_i, 1)} \\
0 & \longrightarrow & \operatorname{Hom}_R(Re_{q(i)}, U_j) & \xrightarrow{\Psi} & \operatorname{Hom}_{eRe}(eRe_{q(i)}, eU_j) \\
 & & \wr\downarrow & & \wr\downarrow \\
 & & e_{q(i)}U_j & \xrightarrow{=} & e_{q(i)}U_j\,.
\end{array}$$

Then Ψ is an isomorphism. Put $t = [q(j) - q(i)]$. Then $J^t U_j$ is the largest submodule of U_j whose top is isomorphic to $\operatorname{Top}(U_i)$.

For all $\varphi \in \operatorname{Hom}_{eRe}(eU_i, eU_j)$, let us put $\varphi^\# = [\Psi^{-1} \circ \operatorname{Hom}(h'_i, 1)](\varphi)$. If $\varphi \neq 0$, $\operatorname{Im}(\varphi^\#)$ is a submodule of $J^t U_j$ since $\operatorname{Top}(\operatorname{Im}(\varphi^\#)) \cong \operatorname{Top}(Re_{q(i)}) \cong \operatorname{Top}(U_i)$. Therefore

$$c(\operatorname{Im}(\varphi^\#)) \leqq c(J^t U_j) = m_j - t\,.$$

Since $m_i \geqq m_j - t$, we have $m_i \geqq c(\operatorname{Im}(\varphi^\#))$ for all $\varphi \in \operatorname{Hom}_{eRe}(eU_i, eU_j)$. On the other hand,

$$\varphi \in \operatorname{Im}(\Phi_{ij}) \Longleftrightarrow \varphi^\# \in \operatorname{Im}(\operatorname{Hom}(h_i, 1))$$
$$\Longleftrightarrow c(\operatorname{Ker}(\varphi^\#)) \geqq c(\operatorname{Ker}(h_i)) = c(Re_{q(i)}) - m_i$$
$$\Longleftrightarrow m_i \geqq c(Re_{q(i)}) - c(\operatorname{Ker}(\varphi^\#)) = c(\operatorname{Im}(\varphi^\#))\,.$$

Hence Lemma 10 holds.

For all integer j, $[j]^*$ denotes the least positive remainder of j *modulo* s; for example, we will use this notation in the case $U_{[s+1]^*}=U_1$. Notice that $[\ \ ]^*$ is different from $[\ \ ]$; $[j]$ is the least positive remainder of j *modulo* n.

COROLLARY 11. *The notations and assumptions are as above. Then*
(i) Φ_{ii} *is a ring isomorphism for all* i.
(ii) *If* $c({}_{eRe}eU_i)\geqq c({}_{eRe}eU_j)$, *then* Φ_{ij} *is onto.*
(iii) *If there exist* i *and* j *such that* Φ_{ij} *is not onto, then there exists* k *such that*

$$m_k \lneqq m_{[k+1]^*}-[q([k+1]^*)-q(k)]\ .$$

PROOF. (i) Lemma 10 is true in the case $i=j$ (use 0 instead of $[q(j)-q(i)]$).

(ii) If Φ_{ij} is not onto, then $m_i \lneqq m_j-[q(j)-q(i)]$ by Lemma 10. Put $t=[q(j)-q(i)]$. Since $c(U_i)\lneqq c(J^tU_j)$ and $\mathrm{Top}\,(U_i)\cong \mathrm{Top}\,(J^tU_j)$, there exists an epimorphism $\pi\colon J^tU_j \to U_i$ by Corollary 3. Then

$$\begin{aligned} c({}_{eRe}eU_j) &= c({}_{eRe}eU_j/eJ^tU_j)+c({}_{eRe}eJ^tU_j) \\ &\gneqq c({}_{eRe}eJ^tU_j) \geqq c({}_{eRe}e\,\mathrm{Im}\,(\pi)) = c({}_{eRe}eU_i)\ . \end{aligned}$$

(iii) If $m_k \geqq m_{[k+1]^*}-[q([k+1]^*)-q(k)]$ for all k, then

$$\begin{aligned} &m_i \geqq m_{[i+1]^*}-[q([i+1]^*)-q(i)]\ , \\ &0 \geqq -m_{[i+1]^*}+m_{[i+2]^*}-[q([i+2]^*)-q([i+1]^*)]\ , \\ &\quad\vdots \\ &0 \geqq -m_{[j-1]^*}+m_j-[q(j)-q([j-1]^*)]\ . \end{aligned}$$

Adding these inequalities, we have

$$m_i \geqq m_j-[q(j)-q(i)]\ .$$

Hence Φ_{ij} is onto for all i and j by Lemma 10.

LEMMA 12. *The notations and the assumptions are as above. Then*
(i) *If* $S'=\mathrm{End}_{eRe}(eU)$ *is QF, then* Φ *is onto.*
(ii) *If* $S=\mathrm{End}_R(U)$ *is QF, then* Φ *is onto.*

PROOF. If S' is QF, then Φ is onto by Lemma 6 and Corollary 11 (ii).

(ii) Assume that S is QF and that Φ is not onto. We will show that these assumptions lead to a contradiction.

Since Φ is not onto, there exists k such that

$$m_k \lneqq m_{[k+1]^*} - [q([k+1]^*) - q(k)] . \tag{4}$$

Put $t = [q([k+1]^*) - q(k)]$. We divide the proof into several steps.

Step 1. First we shall prove

$$\operatorname{Im}(\varphi) \subsetneqq J^t U_{[k+1]^*} \quad \text{for all} \quad \varphi \in Nf_{[k+1]^*} . \tag{5}$$

Since $\operatorname{Top}(J^t U_{[k+1]^*}) \cong \operatorname{Top}(U_k)$ and $q(1) \lneqq q(2) \lneqq \cdots \lneqq q(s)$, we have $\operatorname{Top}(J^j U_{[k+1]^*}) \not\cong \operatorname{Top}(U_i)$ for all $i = 1, \cdots, s$ and $j = 1, \cdots, t-1$. Therefore $\operatorname{Im}(\varphi) \subseteqq J^t U_{[k+1]^*}$ for all $\varphi \in Nf_{[k+1]^*}$. If $\varphi \in Nf_{[k+1]^*}$ satisfies the equality $\operatorname{Im}(\varphi) = J^t U_{[k+1]^*}$, then $\operatorname{Im}(f_k \varphi) = J^t U_{[k+1]^*}$ since U_k is the unique direct summand of U whose top is isomorphic to $\operatorname{Top}(J^t U_{[k+1]^*})$. Thus we have

$$c(U_k) \geqq c(\operatorname{Im}(f_k \varphi)) = c(U_{[k+1]^*}) - t .$$

This contradicts (4). Hence (5) is proved.

Step 2. Since S is QF, there exists i such that $\operatorname{Soc}(Sf_i) \cong Sf_k / Nf_k$. Let $\alpha (\neq 0) \in f_k \cdot \operatorname{Soc}(Sf_i)$, and fix α.

Step 3. We shall prove that

$$\operatorname{Im}(\alpha) \subsetneqq U_i . \tag{6}$$

If $\operatorname{Im}(\alpha) = U_i$, then $\operatorname{Top}(U_k) \cong \operatorname{Top}(\operatorname{Im}(\alpha)) = \operatorname{Top}(U_i)$, and hence $i = k$ from the assumption of Theorem 8. Hence α is an automorphism of U_i, and $\operatorname{Soc}(Sf_i) = S\alpha = Sf_i$. This implies that Sf_i is a simple, injective and projective left S-module. Then

$$f_i S f_j = \operatorname{Hom}_S(Sf_i, Sf_j) = 0 \quad \text{and}$$
$$f_j S f_i = \operatorname{Hom}_S(Sf_j, Sf_i) = 0$$

for $j \neq i$. This means that S decomposes as a ring, which contradicts the assumption that S is indecomposable as a ring. Thus we have proved (6).

Step 4. From (6), the composition type of U_i is $(q(i), \cdots, \cdots, q([k+1]^*), \cdots, q(k), \cdots)$. Since $c(\operatorname{Im}(\alpha)) + t \leqq c(U_k) + t \lneqq c(U_{[k+1]^*})$, there exists $\psi \in \operatorname{Hom}_R(U_{[k+1]^*}, U_i)$ such that $c(\operatorname{Im}(\psi)) = c(\operatorname{Im}(\alpha)) + t$ by Corollary 3; in this case, the composition types of $\operatorname{Im}(\alpha)$, $\operatorname{Im}(\psi)$ and U_i are respectively

$$\begin{aligned}
&\operatorname{Im}(\alpha): && (q(k), \cdots) , \\
&\operatorname{Im}(\psi): && (q([k+1]^*), \cdots, q(k), \cdots) , \\
&U_i \quad : && (q(i), \cdots, \cdots, q([k+1]^*), \cdots, q(k), \cdots) .
\end{aligned}$$

Step 5. Applying the functor $\mathrm{Hom}_R({}_RU_S, -)$, we get the following diagram;

$$(7)\qquad \begin{array}{ccc} & & Sf_k \\ & & \downarrow{\scriptstyle \mathrm{Hom}(1,\alpha)} \\ Sf_{[k+1]^*} & \xrightarrow{\mathrm{Hom}(1,\psi)} & Sf_i\,. \end{array}$$

We shall prove

$$(8)\qquad \mathrm{Im}(\mathrm{Hom}(1, \psi)) \supseteqq \mathrm{Im}(\mathrm{Hom}(1, \alpha))\,.$$

First, notice that $\mathrm{Im}(\mathrm{Hom}(1, \psi)) = Sf_{[k+1]^*}\psi$ and $\mathrm{Im}(\mathrm{Hom}(1, \alpha)) = Sf_k\alpha$. Assume $Sf_{[k+1]^*}\psi \not\supseteqq Sf_k\alpha$. Then the right annihilator ideal of $Sf_{[k+1]^*}$ is not contained in the right annihilator ideal of Sf_k since S is QF. Thus there exists $s \in S$ such that $Sf_{[k+1]^*}\psi s = 0$ and $Sf_k\alpha s \neq 0$. On the other hand, $USf_{[k+1]^*}\psi = \mathrm{Im}(\psi) \supseteqq \mathrm{Im}(\alpha) = USf_k\alpha$. Hence we have

$$0 = (USf_{[k+1]^*}\psi)s \supseteqq (USf_k\alpha)s \neq 0\,.$$

This is a contradiction. Thus we have proved (8).

Step 6. From (8) and the projectivity of ${}_SSf_k$, there exists a S-homomorphism $\bar{\varphi}$: ${}_SSf_k \to {}_SSf_{[k+1]^*}$ such that $\bar{\varphi}\,\mathrm{Hom}(1, \psi) = \mathrm{Hom}(1, \alpha)$. Put $\varphi = (f_k)\bar{\varphi} \in f_kSf_{[k+1]^*}$. Then $\varphi\psi = \alpha$. Therefore

$$\begin{aligned}[q([k+1]^*) - q(k)] &= c(\mathrm{Im}(\psi)) - c(\mathrm{Im}(\alpha)) \\ &= c(\mathrm{Coker}(\varphi)) = c(U_{[k+1]^*}) - c(\mathrm{Im}(\varphi)) \\ &\geqq m_{[k+1]^*} - m_k\,.\end{aligned}$$

This contradicts (4).

Therefore Φ is onto if S is QF. This completes the proof of Lemma 12.

Now, let us proceed to the proof of Theorem 8.

$(a) \Leftrightarrow (b)$. By Lemma 12, Φ is a ring isomorphism in the case S or S' is QF. Thus $(a) \Leftrightarrow (b)$ holds.

$(b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$. Since ${}_{eRe}eU$ is faithful and $\mathrm{Top}({}_{eRe}eU) \cong {}_{eRe}(eRe/eJe)$, Lemma 6 is applicable to the left eRe-module eU, and we have these equivalences.

$(b) \Rightarrow (g)$. Assume (b). Then by the equivalence of (b) and (f), we have ${}_{eRe}eRe \cong {}_{eRe}eU$ and $\mathrm{Top}({}_{eRe}eU) \cong \mathrm{Soc}({}_{eRe}eU)$.

We shall prove that U_i is isomorphic to a chain end for all i. Since $E(Re_{q(i)})$ is isomorphic to a direct summand of a minimal faithful left R-module and ${}_RU$ is faithful, it is also isomorphic to a direct summand of U. Here, there exists k such that $U_k \cong E(Re_{q(i)})$. Since U_i is a factor module of $Re_{q(i)}$ and $Re_{q(i)}$ is a submodule of U_k, we have

$$c({}_{eRe}eU_k) = c({}_{eRe}eU_i) \leqq c({}_{eRe}eRe_{q(i)}) \leqq c({}_{eRe}eU_k)$$

by (f). Thus $c({}_{eRe}eU_k) = c({}_{eRe}eRe_{q(i)})$, and hence $U_k = Re_{q(i)}$ because, if $U_k \supsetneqq Re_{q(i)}$ then the composition lengths are different by at least $c({}_{eRe}\operatorname{Top}(eU_k))$. Then $\operatorname{Top}(U_i) \cong \operatorname{Top}(Re_{q(i)}) \cong \operatorname{Top}(U_k)$, we have $U_i = U_k$ from the assumption (3) of Theorem 8. Hence U_i is isomorphic to a chain end of R for all i.

Then U_i is isomorphic to a direct summand of a minimal faithful left R-module. On the other hand, a minimal faithful left R-module is isomorphic to a direct summand of U. Hence U itself is a minimal faithful left R-module (notice that $U_i \not\cong U_j$ if $i \neq j$).

$(g) \Rightarrow (c)$. Assume (g). Then each U_i is projective and $U_i \cong Re_{q(i)}$. Since $e = \sum_i e_{q(i)}$, we have ${}_{eRe}eRe \cong {}_{eRe}eU$. Since eRe is a serial self-basic ring, the condition $\operatorname{Top}({}_{eRe}eRe) \cong \operatorname{Soc}({}_{eRe}eRe)$ in (g) implies that eRe is a QF ring. Hence (c) holds.

Thus the proof of Theorem 8 is completed.

EXAMPLE 13. In the general case, the ring homomorphism Φ defined in Theorem 8 is not onto.

Let R be an indecomposable self-basic serial ring with the radical J. We assume the admissible sequence of R is 3, 4, 5, i.e., $1 = e_1 + e_2 + e_3$ where Re_1, Re_2, Re_3 is a left Kupisch series of R and $c(Re_1) = 3$, $c(Re_2) = 4$, $c(Re_3) = 5$. Put

$$U_1 = Re_1/J^2e_1, \quad U_2 = Re_3 \quad \text{and} \quad U = U_1 \oplus U_2 .$$

The U is a faithful left R-module ($U_2 = Re_3$ is a minimal faithful left R-module). Put $e = e_1 + e_3$. Then it is easy to show that $\operatorname{Hom}_R(U_1, U_2) = 0$ and $\operatorname{Hom}_{eRe}(eU_1, eU_2) \neq 0$. Thus Φ is not onto. In this case, neither $\operatorname{End}_R(U)$ nor $\operatorname{End}_{eRe}(eU)$ is quasi-Frobenius.

REMARK 14. K. Morita had proved earlier the following theorem: *If U is a finitely generated projective and injective left module over a left Artinian ring R such that simple components of* $\operatorname{Top}({}_RU)$ *and those of* $\operatorname{Soc}({}_RU)$ *are coincident in disregard of multiplicity, then* $\operatorname{End}_R(U)$ *is a quasi-Frobenius ring.*

He pointed out further that the use of the idempotent e in this paper improves his result as follows:

Let U be a finitely generated projective (resp. injective) left module over an Artinian ring R. Let $e \in R$ (resp. $f \in R$) be an idempotent defined by $\mathrm{Top}({}_R U)$ *(resp.* $\mathrm{Soc}({}_R U)$*) as in Theorem 7. Then* $\mathrm{End}_R(U)$ *is quasi-Frobenius if and only if ${}_{eRe}eU$ is injective (resp. projective).*

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