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Eta-Function on S^{2n-1}

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Let Y be a compact oriented riemannian manifold of dimension $2n-1$, $\Omega^q(Y)$ be the space of all differential q-forms on Y and put $\Omega^{ev}(Y)=\bigoplus_{p=0}^{n-1}\Omega^{2p}(Y)$. Let $A: \Omega^{ev}(Y)\to\Omega^{ev}(Y)$ be a first order differential operator defined by

$$(1)\qquad A\phi=i^n(-1)^{p+1}(*d-d*)\phi \qquad (\phi\in\Omega^{2p}(Y))$$

where $i=\sqrt{-1}$, d is the exterior differential and $*$ is the Hodge duality operator. Then A is formally self-adjoint, elliptic and the squre A^2 is the Laplace operator $\Delta=d\delta+\delta d$, where δ is the formal adjoint of d. Therefore A is diagonalizable with real eigenvalues and, of course, the eigenvalues of A can be either positive or negative—they are square roots of the eigenvalues of Δ.

Now let G be a compact group of orientation preserving isometries on Y and suppose that A commutes with the action of G, then the λ-eigenspace E_λ of A is a finite dimensional G-module. In this situation, Atiyah-Patodi-Singer [4] defined the so-called "eta-function"

$$(2)\qquad \eta_A(g, s)=\sum_{\lambda\neq 0}(\mathrm{sign}\,\lambda)\,\mathrm{Tr}\,(g|E_\lambda)\cdot|\lambda|^{-s}$$

for any $g\in G$, where the summation is taken over all distinct eigenvalues of A and $g|E_\lambda$ is the transformation induced by g on E_λ.

For example, when Y is the circle S^1 and g is rotation through an angle θ, we have already known that

$$\eta_A(g, s)=-2i\cdot\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^s},$$

(see [4, p. 413]), and when Y is the 3-sphere S^3 and g is represented by the matrix $\begin{pmatrix}D(\theta_1) & 0\\ 0 & D(\theta_2)\end{pmatrix}$, where $D(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ is rotation of $\boldsymbol{R}^2$ by an angle θ, K. Katase calculated directly this η-function by determining the basis for the eigenspace of A (see [12]). On the other hand, J. J.

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Millson [13] has found a formula to compute directly the η-invariant on homogeneous spaces in terms of a Selberg-like zeta function.

In §2 and §3 we show the result of K. Katase (Theorem 3 in §3) by a different way and in §4 we extend to the case where Y is the $(2n-1)$-sphere S^{2n-1} and g is represented by the matrix $\begin{pmatrix} D(\theta_1) & & 0 \\ & \ddots & \\ 0 & & D(\theta_n) \end{pmatrix}$, where $0<\theta_j<\pi$ $(1\leqq j\leqq n)$. In case $e^{i\theta_1}, e^{-i\theta_1}, \cdots, e^{i\theta_n}, e^{-i\theta_n}$ are the distinct eigenvalues of g, our result is the following equation:

$$\eta_A(g, s) = -2i^n \sum_{j=1}^{n} \left\{ \left(\prod_{\substack{l=1 \\ l\neq j}}^{n} \frac{\sin\theta_l}{\cos\theta_l - \cos\theta_j} \right) \cdot \sum_{k=0}^{\infty} \frac{\sin(k+n)\theta_j}{(k+n)^s} \right\} .$$

Lastly the author wishes to express his gratitude to Mr. K. Katase for his kind and helpful suggestions.

§1. We start by recalling some facts about the spectrum of the Laplace operator Δ on the standard sphere S^{2n-1} in $\boldsymbol{R}^{2n}$. For further details we refer to [14, p. 118] and [6, p. 2104].

Throughout this paper, we will denote by $*$, d, δ, Δ and $\tilde{*}$, $\tilde{d}$, $\tilde{\delta}$, $\tilde{\Delta}$ the intrinsic operators on S^{2n-1} and $\boldsymbol{R}^{2n}$, respectively.

Let $(y_1, \cdots, y_{2n})$ be the standard coordinate system on $\boldsymbol{R}^{2n}$ and set $\rho^2=\sum_{j=1}^{2n} y_j^2$. Let ψ be a q-form on $\boldsymbol{R}^{2n}$ which can be written as

$$\psi = g(\rho)\psi_1 + f(\rho)\tilde{d}\rho\wedge\psi_2 ,$$

where ψ_1 and ψ_2 are q and $q-1$ forms, respectively, on S^{2n-1} and f and g are powers of ρ. Then we have

$$\tilde{*}\psi = (-1)^q \rho^{2n-1-2q} g\tilde{d}\rho\wedge *\psi_1 + \rho^{2n-1-2(q-1)} f * \psi_2 .$$

Since $\tilde{\delta}$ and δ are expressed as

$$\tilde{\delta} = -\tilde{*}\tilde{d}\tilde{*} \quad \text{and} \quad \delta = (-1)^q * d *$$

on q-forms, respectively, a straightforward calculation shows that, for $\psi=\rho^{k+q}\psi_1+\rho^{k+q-1}\tilde{d}\rho\wedge\psi_2$,

$$\tilde{d}\psi = (k+q)\rho^{k+q-1}\tilde{d}\rho\wedge\psi_1 + \rho^{k+q}d\psi_1 - \rho^{k+q-1}\tilde{d}\rho\wedge d\psi_2$$

and

$$\tilde{\delta}\psi = \rho^{k+q-2}\delta\psi_1 - (k-q+2n)\rho^{k+q-2}\psi_2 - \rho^{k+q-3}\tilde{d}\rho\wedge\delta\psi_2 .$$

Further, we see that, for ψ to be harmonic, ψ_1 and ψ_2 must satisfy

$$\Delta\psi_1 - (k+q)(k-q+2n-2)\psi_1 = 2d\psi_2$$

and

$$\Delta\psi_2-(k+q-2)(k-q+2n)\psi_2=2\delta\psi_1\,.$$

Now, let $V_\lambda^q(S^{2n-1})$ be the subspace of $\Omega^q(S^{2n-1})$ consisting of eigenforms associated to each eigenvalue λ of Δ. For $\lambda\neq 0$, $V_\lambda^q(S^{2n-1})$ decomposes into the closed eigenspace $V_\lambda^q(S^{2n-1})\cap\operatorname{Ker} d$ and the coclosed eigenspace $V_\lambda^q(S^{2n-1})\cap\operatorname{Ker}\delta$ and the map

$$d\colon V_\lambda^q(S^{2n-1})\cap\operatorname{Ker}\delta\longrightarrow V_\lambda^{q+1}(S^{2n-1})\cap\operatorname{Ker} d$$

is an isomorphism. Therefore, it is enough to consider the closed eigenspace of S^{2n-1}. Thus we deduce the following result by a standard argument.

PROPOSITION 1. *Let $\phi_{k,q}$ be a closed eigenform of Δ on S^{2n-1} with eigenvalue $\lambda_{k,q}=(k+q)(k-q+2n)$, where $q\neq 0$ and $k\geqq 0$. Let $H_k^q(\boldsymbol{R}^{2n})$ denote the vector space consisting of all q-forms on $\boldsymbol{R}^{2n}$ of which coefficients are harmonic homogeneous polynomials of degree k on $\boldsymbol{R}^{2n}$ and put $\hat{H}_k^q(\boldsymbol{R}^{2n})=H_k^q(\boldsymbol{R}^{2n})\cap\operatorname{Ker}\tilde{d}\cap\operatorname{Ker}\tilde{\delta}$. Then the map*

$$\varphi\colon V_{\lambda_{k,q}}^q(S^{2n-1})\cap\operatorname{Ker} d\longrightarrow\hat{H}_k^q(\boldsymbol{R}^{2n})$$

given by $\phi_{k,q}\mapsto(1/\lambda_{k,q})\tilde{d}(\rho^{k+q}\delta\phi_{k,q})$ is an isomorphism and commutes with the action of g.

From now on we have only to consider forms on S^{2n-1} of even degree. Since $\lambda_{k,q}=\lambda_{k,2n-q}$, it follows from this proposition that the diagram

$$\begin{array}{ccc}V_{\lambda_k}^{2p}(S^{2n-1})\cap\operatorname{Ker} d & \xrightarrow{\varphi} & \hat{H}_k^{2p}(\boldsymbol{R}^{2n})\\ \big\downarrow d* & & \big\downarrow (k-2p+2n)\tilde{*}\\ V_{\lambda_k}^{2n-2p}(S^{2n-1})\cap\operatorname{Ker} d & \xrightarrow{\varphi} & \hat{H}_k^{2n-2p}(\boldsymbol{R}^{2n})\end{array}\tag{3}$$

commutes, where $\lambda_k=\lambda_{k,2p}=(k+2p)(k-2p+2n)$, $p\geqq 1$ and $k\geqq 0$. Moreover, by making use of the isomorphism d, we have the following commutative diagram:

$$\begin{array}{ccccc}V_{\mu_k}^{2p}(S^{2n-1})\cap\operatorname{Ker}\delta & \xrightarrow{d} & V_{\lambda_k}^{2p+1}(S^{2n-1})\cap\operatorname{Ker} d & \xrightarrow{\varphi} & \hat{H}_k^{2p+1}(\boldsymbol{R}^{2n})\\ \big\downarrow -*d & & \big\downarrow -d* & & \big\downarrow (k-2p+2n-1)\tilde{*}\\ V_{\mu_k}^{2n-2p-2}(S^{2n-1})\cap\operatorname{Ker}\delta & \xrightarrow{d} & V_{\lambda_k}^{2n-2p-1}(S^{2n-1})\cap\operatorname{Ker} d & \xrightarrow{\varphi} & \hat{H}_k^{2n-2p-1}(\boldsymbol{R}^{2n})\end{array}\tag{4}$$

where $\mu_k=\lambda_{k,2p+1}=(k+2p+1)(k-2p+2n-1)$, $p\geqq 0$ and $k\geqq 0$.

Now, let us return to our basic operator A and decompose A into the following form:

$$A\phi=\begin{cases}A_1\phi=i^n(-1)^p d*\phi & \text{on} \quad V_\lambda^{2p}(S^{2n-1})\cap \operatorname{Ker} d\\ A_2\phi=i^n(-1)^{p+1}*d\phi & \text{on} \quad V_\lambda^{2p}(S^{2n-1})\cap \operatorname{Ker}\delta\,.\end{cases}$$

Also, let τ be an involution defined by

$$\tau(\alpha)=i^{q(q-1)+n}\tilde{*}\alpha \qquad (\alpha\in H_k^q(\boldsymbol{R}^{2n}))$$

(see [5, p. 575]). Then, from the diagrams (3) and (4), we see that the operator A_1 corresponds to $(k-2p+2n)\tau$ and the operator A_2 corresponds to $(k-2p+2n-1)\tau$. In particular, if n is even and put $2p=n$, then $A_1=d*$ corresponds to $(k+n)\tilde{*}$ and if n is odd and put $2p+1=n$, then $A_2=-i*d$ corresponds to $(k+n)i\tilde{*}$. Hence, by a standard argument (see for example [5, pp. 579–580]), if we denote by $(\ \ )_\pm$ the ± 1-eigenspaces of τ, we can write the η-function (2) as follows:

$$\eta_A(g,s)=\sum_{k=0}^{\infty}\frac{1}{(k+n)^s}\{\operatorname{Tr}(g|\hat{H}_k^n(\boldsymbol{R}^{2n})_+)-\operatorname{Tr}(g|\hat{H}_k^n(\boldsymbol{R}^{2n})_-)\}\,. \tag{5}$$

Therefore we need to calculate $\operatorname{Tr}(g|\hat{H}_k^n(\boldsymbol{R}^{2n})_+)-\operatorname{Tr}(g|\hat{H}_k^n(\boldsymbol{R}^{2n})_-)$. (Cf. also [1].)

§2. In §2 and §3, let us work out the case when $n=2$, i.e., when Y is the 3-sphere S^3 in $\boldsymbol{R}^4$ and $g=\begin{pmatrix}D(\theta_1) & 0\\ 0 & D(\theta_2)\end{pmatrix}$, as a simple illustration of our methods.

Now we put

$$H_k^{\mathrm{ev}}=H_k^0(\boldsymbol{R}^4)\oplus H_k^2(\boldsymbol{R}^4)\oplus H_k^4(\boldsymbol{R}^4) \quad\text{and}\quad H_k^{\mathrm{odd}}=H_k^1(\boldsymbol{R}^4)\oplus H_k^3(\boldsymbol{R}^4)\,.$$

Then there is an exact sequence:

$$\cdots\longrightarrow H_{k+1}^{\mathrm{odd}}\xrightarrow{\tilde{d}+\tilde{\delta}}H_k^{\mathrm{ev}}\xrightarrow{\tilde{d}+\tilde{\delta}}H_{k-1}^{\mathrm{odd}}\longrightarrow\cdots\,. \tag{6}$$

Furthermore, if we put

$$K_k^{\mathrm{ev}}=H_k^{\mathrm{ev}}\cap\operatorname{Ker}(\tilde{d}+\tilde{\delta}) \quad\text{and}\quad K_k^{\mathrm{odd}}=H_k^{\mathrm{odd}}\cap\operatorname{Ker}(\tilde{d}+\tilde{\delta})\,,$$

then the sequence

$$\cdots\longrightarrow K_{k+1}^{\mathrm{odd}}\xrightarrow{\tilde{d}-\tilde{\delta}}K_k^{\mathrm{ev}}\xrightarrow{\tilde{d}-\tilde{\delta}}K_{k-1}^{\mathrm{odd}}\longrightarrow\cdots \tag{7}$$

is also exact.

The exactness of these sequences is proved by making use of the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^2_{k+2}(\boldsymbol{R}^4) & \xrightarrow{\tilde{d}} & H^3_{k+1}(\boldsymbol{R}^4) & \xrightarrow{\tilde{d}} & H^4_k(\boldsymbol{R}^4) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\tilde{\delta}} & & \downarrow{\scriptstyle \tilde{\delta}} & & \downarrow{\scriptstyle -\tilde{\delta}} & & \\
\cdots & \longrightarrow & H^1_{k+1}(\boldsymbol{R}^4) & \xrightarrow{\tilde{d}} & H^2_k(\boldsymbol{R}^4) & \xrightarrow{\tilde{d}} & H^3_{k-1}(\boldsymbol{R}^4) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle -\tilde{\delta}} & & \downarrow{\scriptstyle \tilde{\delta}} & & \downarrow{\scriptstyle -\tilde{\delta}} & & \\
0 & \longrightarrow & H^0_k(\boldsymbol{R}^4) & \xrightarrow{\tilde{d}} & H^1_{k-1}(\boldsymbol{R}^4) & \xrightarrow{\tilde{d}} & H^2_{k-2}(\boldsymbol{R}^4) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \vdots & & \vdots & &
\end{array}$$

where columns and rows are both exact (see [11]).

Next, a simple computation shows that

$$\tau\tilde{d} = -\tilde{\delta}\tau \quad \text{and} \quad \tau\tilde{\delta} = -\tilde{d}\tau \,.$$

That is, the involution τ anti-commutes with $\tilde{d}+\tilde{\delta}$ and commutes with $\tilde{d}-\tilde{\delta}$. Therefore, from the sequences (6) and (7), we have the following induced exact sequences:

$$(8) \qquad \cdots \longrightarrow H^{\text{odd}}_{k+1\mp} \xrightarrow{\tilde{d}+\tilde{\delta}} H^{\text{ev}}_{k\pm} \xrightarrow{\tilde{d}+\tilde{\delta}} H^{\text{odd}}_{k-1\mp} \longrightarrow \cdots$$

$$(9) \qquad \cdots \longrightarrow K^{\text{odd}}_{k+1\pm} \xrightarrow{\tilde{d}-\tilde{\delta}} K^{\text{ev}}_{k\pm} \xrightarrow{\tilde{d}-\tilde{\delta}} K^{\text{odd}}_{k-1\pm} \longrightarrow \cdots \,.$$

We now proceed to the calculation of the trace of g. From (8) it follows that

$$\operatorname{Tr}(g|H^{\text{ev}}_{k\pm}) = \operatorname{Tr}(g|K^{\text{ev}}_{k\pm}) + \operatorname{Tr}(g|K^{\text{odd}}_{k-1\mp}) \,.$$

Similarly, since

$$K^{\text{ev}}_k \cap \operatorname{Ker}(\tilde{d}-\tilde{\delta}) = \hat{H}^2_k(\boldsymbol{R}^4) \quad \text{and} \quad K^{\text{odd}}_k \cap \operatorname{Ker}(\tilde{d}-\tilde{\delta}) = \hat{H}^1_k(\boldsymbol{R}^4) \oplus \hat{H}^3_k(\boldsymbol{R}^4) \,,$$

it follows from (9) that

$$\operatorname{Tr}(g|K^{\text{ev}}_{k\pm}) = \operatorname{Tr}(g|\hat{H}^2_k(\boldsymbol{R}^4)_\pm) + \operatorname{Tr}(g|(\hat{H}^1_{k-1}(\boldsymbol{R}^4) \oplus \hat{H}^3_{k-1}(\boldsymbol{R}^4))_\pm)$$

and

$$\operatorname{Tr}(g|K^{\text{odd}}_{k-1\mp}) = \operatorname{Tr}(g|(\hat{H}^1_{k-1}(\boldsymbol{R}^4) \oplus \hat{H}^3_{k-1}(\boldsymbol{R}^4))_\mp) + \operatorname{Tr}(g|\hat{H}^2_{k-2}(\boldsymbol{R}^4)_\mp) \,.$$

Hence, we have

$$\mathrm{Tr}\,(g|H_{k\pm}^{\mathrm{ev}})=\mathrm{Tr}\,(g|\hat{H}_k^2(\boldsymbol{R}^4)_\pm)+\mathrm{Tr}\,(g|\hat{H}_{k-1}^1(\boldsymbol{R}^4)\oplus\hat{H}_{k-1}^3(\boldsymbol{R}^4))+\mathrm{Tr}\,(g|\hat{H}_{k-2}^2(\boldsymbol{R}^4)_\mp)\,.$$

Combined with

$$\mathrm{Tr}\,(g|H_{k+}^{\mathrm{ev}})-\mathrm{Tr}\,(g|H_{k-}^{\mathrm{ev}})=\mathrm{Tr}\,(g|H_k^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|H_k^2(\boldsymbol{R}^4)_-)\,,$$

this leads to the following formula:

$$(10)\quad \mathrm{Tr}\,(g|H_k^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|H_k^2(\boldsymbol{R}^4)_-)=\{\mathrm{Tr}\,(g|\hat{H}_k^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|\hat{H}_k^2(\boldsymbol{R}^4)_-)\}$$
$$-\{\mathrm{Tr}\,(g|\hat{H}_{k-2}^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|\hat{H}_{k-2}^2(\boldsymbol{R}^4)_-)\}\,.$$

Finally, we put

$$a_k=\mathrm{Tr}\,(g|\hat{H}_k^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|\hat{H}_k^2(\boldsymbol{R}^4)_-)$$

and

$$b_k=\mathrm{Tr}\,(g|H_k^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|H_k^2(\boldsymbol{R}^4)_-)$$

and, with this notation, rewrite the formula (10) as

$$(11)\qquad b_k=a_k-a_{k-2}\,.$$

§3. In this section we determine at first a_k and then the η-function and lastly the η-invariant.

First note that g acts on the j-th factor $\boldsymbol{R}^2=\boldsymbol{C}$ as multiplication by $e^{i\theta_j}$ ($j=1, 2$). We now regard as $H_k^q(\boldsymbol{R}^4)_\pm=H_k^0(\boldsymbol{R}^4)\otimes\Lambda^q(\boldsymbol{R}^4)_\pm$, where $\Lambda^q(\boldsymbol{R}^4)_\pm$ denote the ±1-eigenspaces of τ. Then it is well-known that

$$\mathrm{Tr}\,(g|\Lambda^2(\boldsymbol{R}^4)_+)-\mathrm{Tr}\,(g|\Lambda^2(\boldsymbol{R}^4)_-)=(e^{-i\theta_1}-e^{i\theta_1})(e^{-i\theta_2}-e^{i\theta_2})$$

(see for example [5, pp. 576–577], [1, p. 473] or [12]). On the other hand, it is convenient to express $\mathrm{Tr}\,(g|H_k^0(\boldsymbol{R}^4))$ by means of the generating function

$$\sum_{k=0}^{\infty}\mathrm{Tr}\,(g|H_k^0(\boldsymbol{R}^4))\cdot t^k=\frac{1-t^2}{(e^{i\theta_1}-t)(e^{-i\theta_1}-t)(e^{i\theta_2}-t)(e^{-i\theta_2}-t)}\qquad(|t|<1)$$

(see [9, pp. 80–81]). Therefore, using the multiplicative property of traces, we obtain

$$\sum_{k=0}^{\infty}b_kt^k=\frac{(1-t^2)(e^{-i\theta_1}-e^{i\theta_1})(e^{-i\theta_2}-e^{i\theta_2})}{(e^{i\theta_1}-t)(e^{-i\theta_1}-t)(e^{i\theta_2}-t)(e^{-i\theta_2}-t)}\qquad(|t|<1)\,.$$

Hence, together with (11), we have the following

PROPOSITION 2. *The generating function of a_k is given by*

(12)
$$\sum_{k=0}^{\infty} a_k t^k = \frac{(e^{-i\theta_1}-e^{i\theta_1})(e^{-i\theta_2}-e^{i\theta_2})}{(e^{i\theta_1}-t)(e^{-i\theta_1}-t)(e^{i\theta_2}-t)(e^{-i\theta_2}-t)}$$
$$\left(=\frac{-4\sin\theta_1\cdot\sin\theta_2}{(1-2t\cdot\cos\theta_1+t^2)(1-2t\cdot\cos\theta_2+t^2)}\right).$$

Next, denote by $f(t)$ the right hand side of (12). Then

$$a_k=\frac{1}{k!}f^{(k)}(0)\,,$$

where $f^{(k)}(t)$ is the k-th derivative of $f(t)$. We now write

$$f(t)=\left(\frac{1}{e^{i\theta_1}-t}-\frac{1}{e^{-i\theta_1}-t}\right)\cdot\left(\frac{1}{e^{i\theta_2}-t}-\frac{1}{e^{-i\theta_2}-t}\right)$$

and applying Leibniz' formula to this we obtain

(13)
$$a_k=-4\sum_{l=0}^{k}\sin(l+1)\theta_1\cdot\sin(k-l+1)\theta_2\,.$$

But there is another expression of a_k. To obtain it we need to consider two cases. At first consider the case $\theta_1\neq\theta_2$. Then $f(t)$ can be written as

$$f(t)=\frac{2\sin\theta_1\sin\theta_2}{\cos\theta_2-\cos\theta_1}\left(\frac{2\cos\theta_1-t}{1-2t\cdot\cos\theta_1+t^2}-\frac{2\cos\theta_2-t}{1-2t\cdot\cos\theta_2+t^2}\right).$$

Hence, by making use of

$$\frac{2\cos\theta-t}{(e^{i\theta}-t)(e^{-i\theta}-t)}=\frac{1}{-2i\cdot\sin\theta}\left(\frac{e^{-i\theta}}{e^{i\theta}-t}-\frac{e^{i\theta}}{e^{-i\theta}-t}\right),$$

we can express the formula (13) as

$(14)_1$
$$a_k=\frac{2\sin\theta_1\sin\theta_2}{\cos\theta_2-\cos\theta_1}\left(\frac{\sin(k+2)\theta_1}{\sin\theta_1}-\frac{\sin(k+2)\theta_2}{\sin\theta_2}\right).$$

Next consider the case $\theta_1=\theta_2(=\theta)$. Then $f(t)$ is equal to

$$\left(\frac{1}{e^{i\theta}-t}\right)^2+\left(\frac{1}{e^{-i\theta}-t}\right)^2+\frac{1}{i\cdot\sin\theta}\left(\frac{1}{e^{i\theta}-t}-\frac{1}{e^{-i\theta}-t}\right).$$

Hence, a simple computation shows that a_k is expressed as

$(14)_2$
$$a_k=2\sin\theta\cdot\frac{d}{d\theta}\left(\frac{\sin(k+2)\theta}{\sin\theta}\right).$$

Thus we deduce the following result:

THEOREM 3. (*See* [12].)

$$\eta_A(g,s)=\begin{cases}\dfrac{2\sin\theta_2}{\cos\theta_2-\cos\theta_1}\displaystyle\sum_{k=0}^{\infty}\frac{\sin(k+2)\theta_1}{(k+2)^s}-\frac{2\sin\theta_1}{\cos\theta_2-\cos\theta_1}\sum_{k=0}^{\infty}\frac{\sin(k+2)\theta_2}{(k+2)^s} & \text{for } \theta_1\neq\theta_2\,, \\ 2\sin\theta\cdot\dfrac{d}{d\theta}\left\{\dfrac{1}{\sin\theta}\displaystyle\sum_{k=0}^{\infty}\frac{\sin(k+2)\theta}{(k+2)^s}\right\} & \text{for } \theta=\theta_1=\theta_2\,.\end{cases}$$

Finally recall that

$$\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^s}\longrightarrow\frac{1}{2}\cot\frac{\theta}{2}\qquad\text{as}\quad s\longrightarrow 0 \tag{15}$$

(cf. [8, p. 106]). Then we obtain the result of Atiyah-Patodi-Singer [4, p. 413]:

COROLLARY 4 (Atiyah-Patodi-Singer).

$$\eta_A(g,0)=-\cot\frac{\theta_1}{2}\cdot\cot\frac{\theta_2}{2}\,.$$

REMARK 1. Since the series $\sum_{k=1}^{\infty}(\sin k\theta/k^s)$ is uniformly convergent, it can be written as

$$\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^s}=-\frac{d}{d\theta}\left(\sum_{k=1}^{\infty}\frac{\cos k\theta}{k^{s+1}}\right)\,.$$

Therefore, using a well-known fact that

$$\sum_{k=1}^{\infty}\frac{\cos k\theta}{k}=-\log\left(2\sin\frac{\theta}{2}\right)\,,$$

we can get the fact (15).

REMARK 2. By (12), we have

$$\sum_{k=0}^{\infty}a_k e^{-(k+2)t}=\frac{-\sin\theta_1\sin\theta_2}{(\cosh t-\cos\theta_1)(\cosh t-\cos\theta_2)}\,.$$

So consider $\Gamma(s)\cdot\eta(g,s)$, where $\Gamma(s)$ is the gamma function, and calculate its residue at $s=0$. Then we can obtain Corollary 4 not using Theorem 3. (See [2, p. 299].)

§4. The computation of the case of S^{2n-1} is essentially identical with that of S^3 in §2 and §3. So, without repeating the same treatment let us comment on a few facts.

First, we put

$$H_k^{\mathrm{ev}}(\boldsymbol{R}^{2n})=\bigoplus_{p=0}^{n} H_k^{2p}(\boldsymbol{R}^{2n}) \quad \text{and} \quad H_k^{\mathrm{odd}}(\boldsymbol{R}^{2n})=\bigoplus_{p=0}^{n-1} H_k^{2p+1}(\boldsymbol{R}^{2n}) .$$

Then it follows that

$$H_k^{\mathrm{ev}}(\boldsymbol{R}^{2n})\cap \operatorname{Ker}(\tilde{d}+\tilde{\delta})\cap \operatorname{Ker}(\tilde{d}-\tilde{\delta})=K_k^{\mathrm{ev}}(\boldsymbol{R}^{2n})\cap \operatorname{Ker}(\tilde{d}-\tilde{\delta})=\bigoplus_{p=1}^{n-1} \hat{H}_k^{2p}(\boldsymbol{R}^{2n})$$

and

$$H_k^{\mathrm{odd}}(\boldsymbol{R}^{2n})\cap \operatorname{Ker}(\tilde{d}+\tilde{\delta})\cap \operatorname{Ker}(\tilde{d}-\tilde{\delta})=K_k^{\mathrm{odd}}(\boldsymbol{R}^{2n})\cap \operatorname{Ker}(\tilde{d}-\tilde{\delta})=\bigoplus_{p=0}^{n-1} \hat{H}_k^{2p+1}(\boldsymbol{R}^{2n}) .$$

Furthermore, we see as before that the space $\hat{H}_*^q(\boldsymbol{R}^{2n})$ for $q\neq n$ contribute nothing to the trace formula. So if we put

$$a_k^\circ=\operatorname{Tr}(g|\hat{H}_k^n(\boldsymbol{R}^{2n})_+)-\operatorname{Tr}(g|\hat{H}_k^n(\boldsymbol{R}^{2n})_-)$$

and

$$b_k^\circ=\operatorname{Tr}(g|H_k^n(\boldsymbol{R}^{2n})_+)-\operatorname{Tr}(g|H_k^n(\boldsymbol{R}^{2n})_-) ,$$

then it follows that

$$b_k^\circ=a_k^\circ-a_{k-2}^\circ \tag{11)°$$

as in §2. Thus we obtain:

PROPOSITION 2°. *The generating function of a_k° is given by*

$$\sum_{k=0}^{\infty} a_k^\circ t^k=\prod_{l=1}^{n}\frac{e^{-i\theta_l}-e^{i\theta_l}}{(e^{i\theta_l}-t)(e^{-i\theta_l}-t)} \quad \left(=\frac{(-2i)^n\prod_{l=1}^{n}\sin\theta_l}{\prod_{l=1}^{n}(1-2t\cdot\cos\theta_l+t^2)}\right). \tag{12)°$$

Now, using Leibniz' formula we have

$$a_k^\circ=(-2i)^n\prod_{l=1}^{n}\sin\theta_l\cdot\sum_{k_1+\cdots+k_n=k}\prod_{l=1}^{n}\frac{\sin(k_l+1)\theta_l}{\sin\theta_l} . \tag{13)°$$

On the other hand, we want to get another expression of a_k° and then compute the η-invariant. For this purpose, at first we suppose that $e^{i\theta_1}, e^{-i\theta_1}, \cdots, e^{i\theta_n}, e^{-i\theta_n}$ are the distinct eigenvalues of g and put

$$g(t)=\prod_{l=1}^{n}(t-e^{i\theta_l})(t-e^{-i\theta_l}) .$$

Using the Lagrange's interpolation formula, it follows that

$$\frac{1}{k!}\left(\frac{1}{g(t)}\right)^{(k)}(0)=\sum_{j=1}^{n}\left\{\frac{-e^{-i(k+1)\theta_j}}{g'(e^{i\theta_j})}+\frac{-e^{i(k+1)\theta_j}}{g'(e^{-i\theta_j})}\right\},$$

where $g'(t)$ denotes the first derivative $g^{(1)}(t)$ of $g(t)$. Hence we obtain

$$(14)_1^\circ \qquad a_k^\circ=(-2i)^n\prod_{l=1}^{n}\sin\theta_l\left\{\frac{1}{(-2)^{n-1}}\sum_{j=1}^{n}\frac{\sin(k+n)\theta_j}{\sin\theta_j\prod_{l\neq j}(\cos\theta_l-\cos\theta_j)}\right\}.$$

Thus we have the following expression of η-function:

THEOREM 3°. $\eta_A(g, s)=-2i^n\sum_{j=1}^{n}\left\{\left(\prod_{\substack{l=1\\ l\neq j}}^{n}\frac{\sin\theta_l}{\cos\theta_l-\cos\theta_j}\right)\cdot\sum_{k=0}^{\infty}\frac{\sin(k+n)\theta_j}{(k+n)^s}\right\}.$

Now, letting s tend to 0, we have

$$(16) \qquad \eta_A(g, 0)=-2i^n\prod_{l=1}^{n}\sin\theta_l\left\{\sum_{j=1}^{n}\frac{(1/2)\cot(\theta_j/2)}{\sin\theta_j\prod_{l\neq j}(\cos\theta_l-\cos\theta_j)}\right.$$
$$\left.-\sum_{j=1}^{n}\frac{\sum_{r=1}^{n-1}\sin r\theta_j}{\sin\theta_j\prod_{l\neq j}(\cos\theta_l-\cos\theta_j)}\right\}.$$

Here if we put $h(t)=(\cos\theta_1-t)\cdots(\cos\theta_n-t)$, then

$$\frac{1}{h(1)}=\sum_{j=1}^{n}\frac{1}{h'(\cos\theta_j)\cdot 2\sin^2(\theta_j/2)}.$$

Therefore the first term in { } of (16) is equal to

$$-\frac{1}{2}\cdot\frac{1}{h(1)}=\frac{1}{(-2)^{n+1}\prod_{l=1}^{n}\sin^2(\theta_l/2)}$$

and also using the fact that

$$\sum_{j=1}^{n}\frac{\cos^r\theta_j}{\prod_{l\neq j}(\cos\theta_l-\cos\theta_j)}=0 \qquad \text{for } r\leqq n-2,$$

it is seen that the second term in { } of (16) vanishes. Thus we conclude that the η-invariant is given by

$$\eta_A(g, 0)=(-i)^n\prod_{l=1}^{n}\cot\frac{\theta_l}{2}.$$

Next, we show how to treat the general case where g has eigenvalues $e^{\pm i\theta_{r_1}}, \cdots, e^{\pm i\theta_{r_m}}$ with multiplicities $\nu_1, \cdots, \nu_m$, respectively, where

$\nu_1+\cdots+\nu_m=n$. Since Gegenbauer's polynomials $C_k^\nu(x)$ are defined by the generating function:

$$(1-2t\cdot x+t^2)^{-\nu}\equiv\sum_{k=0}^{\infty}C_k^\nu(x)t^k$$

and the formula

$$C_k^\nu(\cos\theta)=\frac{1}{2^{\nu-1}(\nu-1)!}\Big(\frac{d}{d\cos\theta}\Big)^{\nu-1}\Big(\frac{\sin(\nu+k)\theta}{\sin\theta}\Big)$$

holds (see for example [7]), we have

$$\begin{aligned}&\sum_{k_1+\cdots+k_m=k}\prod_{j=1}^{m}C_{k_j}^{\nu_j}(\cos\theta_{r_j})\\&=\frac{1}{2^{n-m}}\prod_{j=1}^{m}\frac{1}{(\nu_j-1)!}\Big(\frac{d}{d\cos\theta_{r_j}}\Big)^{\nu_j-1}\Big\{\sum_{k_1+\cdots+k_m=k}\prod_{j=1}^{m}\frac{\sin(\nu_j+k_j)\theta_{r_j}}{\sin\theta_{r_j}}\Big\}\\&=\frac{1}{2^{n-m}}\prod_{j=1}^{m}\frac{1}{(\nu_j-1)!}\Big(\frac{d}{d\cos\theta_{r_j}}\Big)^{\nu_j-1}\Big\{\frac{1}{(-2)^{m-1}}\sum_{j=1}^{m}\frac{\sin(k+n)\theta_{r_j}}{\sin\theta_{r_j}\prod_{l\neq j}(\cos\theta_{r_l}-\cos\theta_{r_j})}\Big\}\end{aligned}$$

(by $(14)_1^\circ$). Hence, it follows that

$$\begin{aligned}\eta_A(g,0)&=(-2i)^n\prod_{j=1}^{m}\sin^{\nu_j}\theta_{r_j}\cdot\frac{1}{2^n}\prod_{j=1}^{m}\frac{1}{(\nu_j-1)!}\Big(\frac{d}{d\cos\theta_{r_j}}\Big)^{\nu_j-1}\Big(\frac{1}{1-\cos\theta_{r_j}}\Big)\\&=(-1)^n\prod_{j=1}^{m}\Big(\frac{\sin\theta_{r_j}}{1-\cos\theta_{r_j}}\Big)^{\nu_j}=(-i)^n\prod_{j=1}^{m}\cot^{\nu_j}\frac{\theta_{r_j}}{2}.\end{aligned}$$

Thus at any rate we have the following

COROLLARY 4° (*Atiyah-Patodi-Singer*).

$$\eta_A(g,0)=(-i)^n\prod_{l=1}^{n}\cot\frac{\theta_l}{2}.$$

REMARK. By $(12)^\circ$, we have

$$\sum_{k=0}^{\infty}a_k^\circ e^{-(k+n)t}=\frac{(-i)^n\prod_{l=1}^{n}\sin\theta_l}{\prod_{l=1}^{n}(\cosh t-\cos\theta_l)}.$$

So we can obtain Corollary 4° as in Remark 2 in §3.

§5. Finally we shall consider about the η-function of the Dirac operator of S^{2n-1}. For the definition of the Dirac operator and formal properties we refer to [2, §6], [1] and [5].

Let ξ be a spin bundle of R^{2n} associated to the spin representation of Spin $(2n)$. This is the direct sum of two bundles ξ^+ and ξ^- associated to the two half-spin representation of Spin $(2n)$ and the restriction of ξ^+ to S^{2n-1} is identified with the spin bundle ζ of S^{2n-1} associated to the spin representation of Spin $(2n-1)$. So the Dirac operator D of S^{2n-1} acting on the sections of ζ is defined as usual. Furthermore, tensoring ζ with ξ we have a generalized Dirac operator D_ξ on S^{2n-1} and this operator D_ξ coincides with the operator $i^{q(q-1)+n}(d*-(-1)^q*d)$ on $\Omega^q(S^{2n-1})$. (Cf. [2, p. 316] and [3].) Since our basic operator A is the restriction of this operator to the even forms (see (1)), it follows that

$$\eta_{D_\xi}(\hat{g}, s)=2\eta_A(\hat{g}, s)=2\varepsilon(\hat{g})\eta_A(g, s) ,$$

where $\hat{g}$ is the lifting of g and $\varepsilon(\hat{g})=\pm 1$. On the other hand, using the formula for the character of the spin representation (see [5, p. 569]), we have

$$\eta_{D_\xi}(\hat{g}, s)=\prod_{l=1}^{n}(e^{-i\theta_l/2}+e^{i\theta_l/2})\eta_D(\hat{g}, s) .$$

Hence we can obtain the formula for $\eta_D(\hat{g}, s)$ from the result for $\eta_A(g, s)$. Consequently, as $s\rightarrow 0$, we conclude that

$$\eta_D(\hat{g}, 0)=2\varepsilon(\hat{g})\left(\frac{i}{2}\right)^n\prod_{l=1}^{n}\operatorname{cosec}\frac{\theta_l}{2} .$$

(Cf. [1, p. 485].)

References

[1] M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: II, Ann. of Math., **88** (1968), 451–491.

[2] M. F. Atiyah, R. Bott and V. K. Patodi, On the heat equation and the index theorem, Invent. Math., **19** (1973), 279–330. Errata, ibid **28** (1975), 277–280.

[3] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry: I, Math. Proc. Cambridge Philos. Soc., **77** (1975), 43–69.

[4] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry: II, Math. Proc. Cambridge Philos. Soc., **78** (1975), 405–432.

[5] M. F. Atiyah and I. M. Singer, The index of elliptic operators: III, Ann. of Math., **87** (1968), 546–604.

[6] J. Cheeger, On the spectral geometry of spaces with cone-like singularities, Proc. Nat. Acad. Sci. U.S.A., **76** (1979), 2103–2106.

[7] A. Erdélyi, Bateman Manuscript Project, "Higher Transcendental Functions", McGraw-Hill, New York, 1953.

[8] S. Igari, "Fourier Kyusu," Iwanami Shoten, Tokyo, 1975 (in Japanese).

[9] A. Ikeda, On the spectrum of a compact riemannian manifold of positive constant curvature, Osaka J. Math., **17** (1980), 75–93.

[10] I. IWASAKI and K. KATASE, On the spectra of Laplace operator on $\Lambda^*(S^n)$, Proc. Japan Acad., **55** (1979), 141–145.
[11] I. IWASAKI and K. KATASE, On the dimension of the (co-)closed harmonic forms on $\boldsymbol{R}^{n+1}$, to appear.
[12] K. KATASE, Eta-function on S^3, Proc. Japan Acad., **57** (1981), 233–237.
[13] J. J. MILLSON, Closed geodesics and the η-invariant, Ann. of Math., **108** (1978), 1–39.
[14] D. B. RAY, Reidemeister torsion and the Laplacian on lens spaces, Advances in Math., **4** (1970), 109–126.

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