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Double Normals of a Compact Submanifold

Dedicated to Professor Tosiya Saito on his sixtieth birthday

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Introduction

Let M be a smooth (C^∞) submanifold of $\boldsymbol{R}^n$. A *double normal* is a line segment combining two points p and q of M ($p\neq q$) at which the segment is perpendicular to the tangent space T_pM and T_qM. N. H. Kuiper [4] showed that there are at least n double normals for convex bodies in $\boldsymbol{R}^n$. (For generic immersions, see [10].) More generally, let L be a Riemannian manifold and M a submanifold of L. A non-constant geodesic $\gamma: [0, 1]\to L$ is called an *(M, M)-geodesic* if $\gamma(i)\in M$ and $\dot{\gamma}(i)\perp T_{\gamma(i)}M$ ($i=0, 1$). The problem of this type is treated in A. Riede [7]. We have

THEOREM 1. *Let $(\boldsymbol{R}^n, g)$ be a complete C^∞ Riemannian manifold and M a compact C^∞ submanifold of $\boldsymbol{R}^n$ without boundary.*
Then there exist at least $\dim M+1$ *(M, M)-geodesics in $\boldsymbol{R}^n$.*

In counting the number, we identify γ and its inverse $\gamma^{-1}(t)=\gamma(1-t)$.

§1. Path space and involution.

Let $(\boldsymbol{R}^n, g)$ and M be as in Theorem 1. We define

$$\Omega=\{\omega: [0, 1]\longrightarrow \boldsymbol{R}^n\,;\ \omega:\ \text{piecewise smooth and } \omega(0)\in M,\ \omega(1)\in M\}\,.$$

Ω is endowed with the compact open topology. We consider the involution $\xi: \Omega\to\Omega$ defined by $\xi\omega=\omega^{-1}$ and denote by Δ' the fixed point set of ξ.

In general, for a topological space X with continuous involution $\xi: X\to X$, we denote by $H^{\Pi}_*(X)$ the equivariant homology group with $\boldsymbol{Z}_2$ coefficient, that is $H_*(X)=H_*(X_\Pi)$, where X_Π is the orbit space of $S^\infty\times X$

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by the involution $(\zeta, x)\mapsto(-\zeta, \xi x)$. The equivariant cohomology group is similarly defined [5].

Satz (5.5) in [7] asserts that

PROPOSITION. *If there exist* $b\in H^{\Pi}_*(\Omega, \Delta')$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_s\in H^*_{\Pi}(\Omega)$ *with* $\deg\alpha_i>0$ $(i=1, 2, \cdots, s)$, *such that*

$$(\alpha_1\cup\cdots\cup\alpha_s)\cap b\neq 0 .$$

Then there are at least $s+1$ (M, M)*-geodesics.*

In fact, by the naturality (for example, 5.6.16 in [8]), $\alpha_{i|(\Omega-\Delta')_\Pi}$ plays the role of u_i in Definition (5.4) in [7]. Remark that one of u_i is 1 and take $h\in H^*_{\Pi}(\Omega, \Delta')$ with $\langle h, (\alpha_1\cup\cdots\cup\alpha_s)\cap b\rangle\neq 0$.

Instead of Ω, the Hilbert manifold of H^1-curves can be used [3], since the condition (C) of Palais-Smale for our problem is also proved in [1]. So the following theorem yields Theorem 1.

THEOREM 2. *Let* M *be a compact connected manifold without boundary with* $\dim M=m$. *We put* $M^2=M\times M$ *and define* $\xi: M^2\to M^2$ *by*

$$\xi(x, y)=(y, x) .$$

We denote by Δ *the diagonal set* $\{(x, x); x\in M\}$. *Then there exist* $b\in H^{\Pi}_{2m}(M^2, \Delta)$ *and* $\theta\in H^1_{\Pi}(M^2)$ *such that* $\theta^m\cap b\neq 0$ *in* $H^{\Pi}_m(M^2, \Delta)$.

The proof is given in §2.

PROOF OF THEOREM 1. Let $\Phi: M^2\to\Omega$ be the function defined by $\Phi(x, y)(t)=x+t(y-x)$. Clearly Φ is a homotopy equivalence and we can take the homotopy to be equivariant. Also $\Phi(\Delta)$ is a deformation retract of Δ'. Thus naturally we have

$$H^{\Pi}_*(M^2, \Delta)\cong H^{\Pi}_*(\Omega, \Phi(\Delta))\cong H^{\Pi}_*(\Omega, \Delta')$$

and

$$H^*_{\Pi}(M^2)\cong H^*_{\Pi}(\Omega) .$$

Therefore, by the naturality, Theorem 2 gives the assumption of the proposition, putting $s=m$. This proves Theorem 1. Q.E.D.

§2. Proof of Theorem 2.

Let M, Δ and $\xi: M^2\to M^2$ be as in Theorem 2 and $\pi: S^\infty\times M^2\to M^2_\Pi$ the covering projection. By the Künneth formula, we identify

(1) $$H_*(M^2)=H_*(S^\infty\times M^2) \quad\text{and}\quad H^*(M^2)=H^*(S^\infty\times M^2) .$$

Let $\pi_1: H^*(M^2)\to H^*_\Pi(M^2)$ be the transfer. The following lemma is easily proved.

LEMMA 1.

(i) $$\pi_*(\beta\cap\pi^!(b))=\pi_1(\beta)\cap b$$

where $\pi_*: H_*(M^2, \Delta)\to H^\Pi_*(M^2, \Delta)$, $\pi^!: H^\Pi_*(M^2, \Delta)\to H_*(M^2, \Delta)$ *and* $\beta\in H^*(M^2)$, $b\in H^\Pi_*(M^2, \Delta)$.

(ii) $$\pi^*\pi_1(\alpha\times\beta)=\alpha\times\beta+\beta\times\alpha \quad \textit{for} \quad \alpha, \beta\in H^*(M)\,.$$

Let $p: M^2_\Pi\to P^\infty=\boldsymbol{R}P^\infty$ be the mapping derived from the projection $p': S^\infty\times M^2\to S^\infty$, and $\theta\in H^1_\Pi(M^2)$ the characteristic class $p^*(\omega)$, where $\omega\in H^1(P^\infty)$ is the generator. Let $P: H^k(M)\to H^{2k}_\Pi(M^2)$ be the external square operation (§4 in [6] or [5]) and $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ a basis for $H^*(M)$. Then Steenrod Isomorphism Theorem (Theorem 4 in [6] or Theorem 20.2 in [5]) gives

LEMMA 2. *$\theta^i\cup P(\alpha_j)$ $(i=0, 1, \cdots; j=1, 2, \cdots, s)$ and $\pi_1(\alpha_i\times\alpha_j)$ $(1\leqq i<j\leqq s)$ form a basis for $H^*_\Pi(M^2)$.*

Let $d: M\to M^2$ be the diagonal map, that is, $d(x)=(x, x)$. We identify by d

(2) $$M=\Delta \quad\text{and}\quad \Delta_\Pi=P^\infty\times M\,,$$

and denote by d^* the restriction $H^*_\Pi(M^2)\to H^*_\Pi(\Delta)=H^*(P^\infty\times M)$. We have

LEMMA 3.
(i) $d^*(\theta)=\omega\times 1$,
(ii) $d^*\pi_1=0: H^*(M^2)\to H^*_\Pi(\Delta)$,
(iii) $d^*P(\alpha)=\sum_{i+j=g}\omega^i\times Sq^j\alpha$ $(\alpha\in H^g(M))$.

PROOF. (i) is easy. (ii) is given by Lemma 4.1 in Chapter VII of [9] (or the proof of (ii) of Theorem 20.3 in [5]). (iii) is Theorem 5 in [6]. Q.E.D.

We put $m=\dim M$, $g_i=\deg\alpha_i$ and assume $g_i\leqq g_j$ if $i<j$, so that $\alpha_1=1$ and $\alpha_s=\sigma\in H^m(M)$, the generator.

LEMMA 4. *$d^*(\theta^k)$, $d^*(\theta^{k-2g_2}\cup P(\alpha_2))$, $\cdots$, $d^*(\theta^{k-2g_r}\cup P(\alpha_r))$ are linearly independent in $H^k_\Pi(\Delta)$ $(k-2g_r\geqq 0)$.*

PROOF. $d^*(\theta^{k-2g_q}\cup P(\alpha_q))=\sum_{i+j=k-g_q}\omega^i\times Sq^j\alpha_q$ $(1\leqq q\leqq r)$ contains the term $\omega^{k-g_q}\times\alpha_q$ $(j=0)$, and for other terms $(j>0)$, $\deg Sq^j\alpha_q>g_q$. This yields the lemma. Q.E.D.

COROLLARY 5. *$d^*: H_\Pi^k(M^2)\cong H_\Pi^k(\Delta)$ if $k\geqq 2m$.*
Hence $H_\Pi^k(M^2, \Delta)=0$ if $k>2m$.

LEMMA 6. *In the exact sequence*

$$H_\Pi^{2m-1}(M^2)\xrightarrow{d^*} H_\Pi^{2m-1}(\Delta)\xrightarrow{\delta^*} H_\Pi^{2m}(M^2, \Delta)\longrightarrow 0\ ,$$

$H_\Pi^{2m}(M^2, \Delta)\cong Z_2$ and $\delta^*(\omega^{m-1}\times\sigma)$ is the generator.

PROOF. $\dim H_\Pi^{2m-1}(\Delta)=s$, $\dim(\operatorname{Im} d^*)=s-1$ by Lemma 4 and $\omega^{m-1}\times\sigma\notin\operatorname{Im} d^*$. Q.E.D.

LEMMA 7. $\pi^!: H^\Pi_{2m}(M^2, \Delta)\cong H_{2m}(M^2, \Delta)\cong Z_2$.

PROOF. For the inclusion $j: M^2\subset(M^2, \Delta)$, it is easily seen $j^*: H^{2m}(M^2, \Delta)\cong H^{2m}(M^2)$.

In the commutative diagram

$$\begin{array}{ccccc}
H_\Pi^{2m-1}(\Delta) & \longrightarrow & H^{2m-1}(\Delta) & & \\
\Big\downarrow{\scriptstyle\delta^*} & & \Big\downarrow{\scriptstyle 0} & & \\
H_\Pi^{2m}(M^2, \Delta) & \xrightarrow{\pi^*} & H^{2m}(M^2, \Delta) & \xrightarrow{\pi_!} & H_\Pi^{2m}(M^2, \Delta) \\
 & & \Big\downarrow{\scriptstyle j^*} & & \\
 & & H^{2m}(M^2)\ , & &
\end{array}$$

we have $\pi^*\delta^*(\omega^{m-1}\times\sigma)=0$. By Lemma 6, $\operatorname{Im}\pi^*=0$, hence $\pi_!$ is one-to-one. Since $\dim H^{2m}(M^2, \Delta)=\dim H_\Pi^{2m}(M^2, \Delta)=1$, the lemma is proved. Q.E.D.

Let $b\in H^\Pi_{2m}(M^2, \Delta)$ be the generator. This b and θ shall give Theorem 2.

Let $V_i\in H^i(M)$ be the Wu class, that is

$$(3)\qquad \alpha\cup V_i=Sq^i\alpha \quad\text{for any}\quad \alpha\in H^{m-i}(M)\ .$$

$$V_0=1 \quad\text{and}\quad V_i=0 \quad\text{for}\quad i>m/2\ .$$

We shall prove Theorem 2 by different ways corresponding to

the first case: $V_i=0$ for any $i>0$,

and

the second case: $V_i\neq 0$ for some $i>0$.

In the exact sequence

$$(4) \qquad H^m_\Pi(M^2, M^2-\Delta) \xrightarrow{j_1^*} H^m_\Pi(M^2) \xrightarrow{i_1^*} H^m_\Pi(M^2-\Delta) ,$$

where i_1: $M^2-\Delta \subset M^2$ and j_1: $M^2 \subset (M^2, M^2-\Delta)$, let $U_\Pi \in H^m_\Pi(M^2, M^2-\Delta) \cong Z_2$ be the generator and $U'_\Pi = j_1^*(U_\Pi) \in H^m_\Pi(M^2)$ the equivariant diagonal cohomology class for the trivial action on M.

Let $C=(c_{ij})$ be the inverse matrix of $Y=(y_{ij})$, where $y_{ij}=\langle \alpha_i \cup \alpha_j, [M]\rangle \in Z_2$ ($[M] \in H_m(M)$ is the fundamental homology class). It is known that [2 or Theorem 22.1 in 5]

$$U'_\Pi = \sum_{i=0}^{[m/2]} \theta^{m-2i} P(V_i) + \sum_{i<j} (c_{ij} + c_{ii}c_{jj}) \pi_1(\alpha_i \times \alpha_j) .$$

So for the first case, we have

$$(5) \qquad U'_\Pi = \theta^m + \pi_1(\beta)$$

where

$$(6) \qquad \beta = \sum_{i<j} (c_{ij} + c_{ii}c_{jj}) \alpha_i \times \alpha_j \in H^m(M^2) .$$

Then we have

LEMMA 8. *$\pi_1(\beta) \cap b \neq 0$ in $H^\Pi_m(M^2, \Delta)$.*

PROOF. Assume that

$$(7) \qquad \pi_*(\beta \cap \pi^!(b)) = \pi_1(\beta) \cap b = 0$$

(see (i) in Lemma 1). Consider the commutative diagram

$$\begin{array}{ccccc} & & \beta \cap [M^2] \in H_m(M^2) & \xrightarrow{j_*} & H_m(M^2, \Delta) \\ & & \downarrow \pi_* & & \downarrow \pi_* \\ H^\Pi_m(\Delta) & \xrightarrow{d_*} & H^\Pi_m(M^2) & \xrightarrow{\bar{j}_*} & H^\Pi_m(M^2, \Delta) . \end{array}$$

Lemma 7 gives $\pi^!(b) = j_*[M^2]$, so $\beta \cap \pi^!(b) = \beta \cap j_*[M^2] = j_*(\beta \cap [M^2])$. From (7), $\bar{j}_* \pi_*(\beta \cap [M^2]) = 0$, hence

$$(8) \qquad \pi_*(\beta \cap [M^2]) = d_*(e) \quad \text{for some} \quad e \in H^\Pi_m(\Delta) .$$

Now, by the definition, $c_{1s}=1$ and $c_{1j}=0$ $(1 \leqq j < s)$, hence

$$(9) \qquad (\sigma \times 1) \cup \beta = (\sigma \times 1) \cup (1 \times \sigma) = \sigma \times \sigma \in H^{2m}(M^2) .$$

And, since $\deg \alpha_j > 0$ if $j > 1$, we have

$$(10) \qquad (1 \times \sigma) \cup \beta = 0 .$$

Therefore

$$\begin{aligned}
&\langle \pi_1(1\times\sigma),\ \pi_*(\beta\cap[M^2])\rangle \\
&\quad =\langle \pi^*\pi_1(1\times\sigma),\ \beta\cap[M^2]\rangle \\
&\quad =\langle 1\times\sigma+\sigma\times 1,\ \beta\cap[M^2]\rangle && \text{(by (ii) in Lemma 1)} \\
&\quad =\langle (1\times\sigma+\sigma\times 1)\cup\beta,\ [M^2]\rangle \\
&\quad =\langle \sigma\times\sigma,\ [M^2]\rangle && \text{(by (9) and (10))} \\
&\quad =1\,.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\langle \pi_1(1\times\sigma),\ \pi_*(\beta\cap[M^2])\rangle \\
&\quad =\langle \pi_1(1\times\sigma),\ d_*(e)\rangle && \text{(by (8))} \\
&\quad =\langle d^*\pi_1(1\times\sigma),\ e\rangle \\
&\quad =0 && \text{(by (ii) in Lemma 3)}\,.
\end{aligned}$$

This is a contradiction. Q.E.D.

PROOF OF THEOREM 2.

The first case: Let N be a tubular neighborhood of Δ in M^2 which is invariant under ξ. Then

$$H^{\Pi}_*(M^2,\Delta)\cong H^{\Pi}_*(M^2,N)\cong H^{\Pi}_*(M^2-\Delta,\ N-\Delta)\,.$$

If $b'\in H^{\Pi}_*(M^2-\Delta,\ N-\Delta)$ is corresponding to b, then $U'_{\Pi\,|\,(M^2-\Delta)_\Pi}\cap b'$ corresponds to $U'_\Pi\cap b$, which is 0, for $U'_{\Pi\,|\,(M^2-\Delta)_\Pi}=0$. Therefore

$$\begin{aligned}
\theta^m\cap b&=-\pi_1(\beta)\cap b && \text{(by (5))} \\
&\neq 0 && \text{(by Lemma 8)}\,.
\end{aligned}$$

The second case: In this case,

$$Sq^i\alpha=\alpha\cup V_i=\sigma \quad \text{for some} \quad \alpha\in H^{m-i}(M) \qquad (0<i\leqq m/2)\,.$$

We consider the exact sequence

$$H^{2m-1}_\Pi(M^2)\xrightarrow{d^*}H^{2m-1}_\Pi(\Delta)\xrightarrow{\delta^*}H^{2m}_\Pi(M^2,\Delta)\longrightarrow 0\,.$$

For $\theta^{2i-1}\cup P(\alpha)\in H^{2m-1}_\Pi(M^2)$, we have

$$\begin{aligned}
&d^*(\theta^{2i-1}\cup P(\alpha)) \\
&\quad =\omega^{m+i-1}\times Sq^0\alpha+\omega^{m+i-2}\times Sq^1\alpha+\cdots+\omega^{m-1}\times Sq^i\alpha \\
&\quad =(\omega^m\times 1)\cup(\omega^{i-1}\times\alpha+\omega^{i-2}\times Sq^1\alpha+\cdots+1\times Sq^{i-1}\alpha)+\omega^{m-1}\times\sigma\,.
\end{aligned}$$

Put

$$\beta'=\omega^{i-1}\times\alpha+\cdots+1\times Sq^{i-1}\alpha\in H_{\Pi}^{m-1}(\Delta)=H^{m-1}(P^{\infty}\times M)\ .$$

Since we have $\delta^*((\omega^m\times 1)\cup\beta'+\omega^{m-1}\times\sigma)=\delta^*d^*(\theta^{2i-1}\cup P(\alpha))=0$, the element $\delta^*((\omega^m\times 1)\cup\beta')=\delta^*(\omega^{m-1}\times\sigma)\in H_{\Pi}^{2m}(M^2, \Delta)$ is the generator by Lemma 6. Therefore

$$\begin{aligned}1&=\langle\delta^*((\omega^m\times 1)\cup\beta'), b\rangle\\&=\langle(\omega^m\times 1)\cup\beta', \partial_* b\rangle\\&=\langle\beta', (\omega^m\times 1)\cap\partial_* b\rangle\\&=\langle\beta', d^*\theta^m\cap\partial_* b\rangle \qquad \text{(by (i) in Lemma 3)}\\&=\langle\beta', \partial_*(\theta^m\cap b)\rangle\ .\end{aligned}$$

In particular, $\theta^m\cap b\neq 0$. This completes the proof of Theorem 2. Q.E.D.

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