TOKYO J. MATH.
VOL. 6, No. 1, 1983

The Centralizers of Semisimple Elements of the Chevalley Groups E_7 and E_8

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(Communicated by Y. Kawada)

The purpose of this paper is to give in detailed tables all the centralizers and their orders of semisimple elements of the finite Chevalley groups E_7 and E_8. These tables are very useful since they give also the character degrees of the semisimple irreducible complex representations constructed by Deligne and Lusztig [7] for the finite Chevalley groups of adjoint type. For, these degrees can be obtained if we know what subgroups of the finite Chevalley groups of universal type are centralizers of semisimple elements (see [7]). Similar tables giving these centralizers for the classical groups and for the groups G_2, F_4 and E_6 have been obtained in [5], [6], [12] and [11] respectively.

A considerable amount of detailed work was involved in the compilation of our tables which has not been included in the paper. However we outline below the general results on which we relied heavily for our calculations.

Let G be a simple linear algebraic Chevalley group of rank l defined over the algebraic closure K of the prime field F_p of p elements. Let Φ be a root system of G with respect to a maximal torus T_0 of G which splits over F_p. Consider the highest root r_0 in Φ and let $\tilde{\Delta}=\Delta\cup\{-r_0\}$ where $\Delta=\{r_i;\ i=1, \cdots, l\}$ is a fixed fundamental basis of Φ. Also we put $I_0=\{0, 1, 2, \cdots, l\}$.

We have shown [8] that, except for the bad primes of G (see [1, p. 178]), a connected reductive subgroup G_1 of maximal rank in G is the connected centralizer of a semisimple element if and only if some proper subset of the roots in $\tilde{\Delta}$ is equivalent under the Weyl group $W(=W(\Phi))$ to a fundamental basis of the root system of G_1. Thus every connected centralizer of a semisimple element in G is in some C_J, $J\subsetneqq I_0$, where by C_J we denote the set of all connected centralizers of semisimple elements

Received April 16, 1981
Revised December 14, 1982

of G which are G-conjugate to the connected centralizers whose root system is Φ_J, the root system generated by $\Delta_J=\{r_j;\ j\in J\}$. All the centralizers belonging in a given C_J, $J\subsetneqq I_0$, have the same Dynkin diagram the type of which we denote E_J. Notice (See [10].) that if two proper subsets Δ_J, $\Delta_{J'}$ of $\tilde{\Delta}$ have the same Dynkin diagrams then Δ_J and $\Delta_{J'}$ are W-conjugate, apart from a few exceptions when $\tilde{\Delta}$ is of type E_7 or E_8.

We consider now the Frobenius endomorphism σ of G which raises every matrix coefficient to its q^{th} power where $q=p^m$. Then the group G_σ of the fixed points under σ is a Chevalley group over the field F_q of q elements. To work within G_σ we have to consider first the connected centralizers of σ-stable semisimple elements (which are, of course, σ-stable subgroups) and the question is if each set $C_J(J\subsetneqq I_0)$ contains such a centralizer. Thus for a given $J\subsetneqq I_0$ we consider the set $\mathscr{C}_J$ of all σ-stable centralizers in C_J. Then the group G_σ acts on $\mathscr{C}_J$ and let $\mathscr{C}_J/G_\sigma$ be the set of G_σ-orbits in $\mathscr{C}_J$. Finally we denote by Ω_J the normalizer in W of the set Δ_J of the simple roots r_j, $j\in J$. Now the structure of the group of the fixed points of a centralizer in $\mathscr{C}_J$ under σ has been determined by Carter [4] as follows: Let $G_J\in\mathscr{C}_J$. Then

(a) Each conjugacy class $[w]$ of Ω_J gives rise to the orbit $\overline{G_J^g}$ in $\mathscr{C}_J/G_\sigma$ represented by the conjugate G_J^g of G_J, where $\pi(g^{-1}\sigma(g))=w$, π being the natural homomorphism of the normalizer $N_G(T_0)$ of T_0 onto W. The map $[w]\to\overline{G_J^g}$ is a bijection.

(b) If M is the semisimple part of G_J, then the group $(M^g)_\sigma$ is isomorphic to the subgroup of the finite Chevalley group of type M obtained by combining the graph automorphism τ of the Dynkin diagram of Δ_J induced by w with σ and taking the fixed points of the product $\sigma\tau$.

(c) If S is the identity component of the centre of G_J, then the group $(S^g)_\sigma$ is isomorphic to the group $X/\bar{P}_J/(qw-1)(X/\bar{P}_J)$. Here X denotes the group (considered as an additive group) of the K-rational characters of T_0 and $\bar{P}_J$ is the subgroup of X consisting of all rational linear combinations of roots in Φ_J.

Let G_J be as above. Then $G_J=MS$ and $M\cap S=A$ is finite. Also M and S are σ-stable, being characteristic subgroups of G_J and G_J is F_q-isogenous to the direct product $M\times S$ (since both the connected groups G_J and $M\times S$ are F_q-isogenous to $M/A\times S/A$ which is isomorphic to $(M\times S)/(A\times A)$. In general, it is known that if H_1, H_2 are two connected algebraic groups defined over k, a finite subfield of K, and H_1 is k-isogenous to H_2 then the groups of the k-rational points of H_1 and H_2

respectively have the same order. Therefore $|(G_J)_\sigma|=|M_\sigma|\,|S_\sigma|$. In particular, if $w=\pi(g^{-1}\sigma(g))$, then $|(G_J^g)_\sigma|=|(M^g)_\sigma|\,|(S^g)_\sigma|$, where the order $|(M^g)_\sigma|=|M_{\sigma\tau}|$ is well known (See [2, ch. 12].) and the order $|(S^g)_\sigma|$ is $f(q)/g(q)$, $f(t)$ and $g(t)$ being the characteristic polynomials of w on $X\otimes\boldsymbol{R}$ and on $\bar{P}_J\otimes\boldsymbol{R}$ respectively.

From the above discussion we see that each orbit in $\mathscr{C}_J/G_\sigma$ is characterized by the type E_J of the Dynkin diagram of the centralizers in $\mathscr{C}_J$ and by some conjugacy class $[w]$ in Ω_J. If we are given such an orbit we can now ask whether this orbit contains a centralizer of some σ-stable semisimple element. Carter [4, Cor. 20] has shown that this depends on whether the group $X/P_J+(qw-1)X$ has a character which does not annihilate any root in $\bar{\Phi}_J-\Phi_J$ for sufficiently large values of q, where $\bar{\Phi}_J=\Phi\cap\bar{P}_J$. We note that instead of this we have given in [8] a practical method (using the Brauer complex of G, [9]) to determine the conditions which have to be imposed on q for the occurrence of such a centralizer in a given orbit in $\mathscr{C}_J/G_\sigma$.

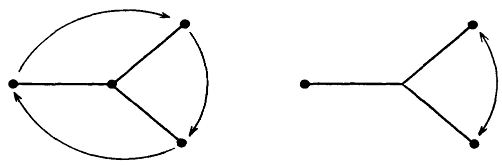
To determine, for each proper subset J of I_0, the structure and its conjugacy classes of the group Ω_J we made great use of the material of Carter's paper [3].

In the tables which follow one row corresponds to each orbit in $\mathscr{C}_J/G_\sigma$. The column headed with Δ_J gives the type E_J of the Dynkin diagram of the centralizers in $\mathscr{C}_J$. The column headed with Ω_J gives the abstruct type of group which is isomorphic to Ω_J. The columns headed with $|(M^g)_\sigma|$ and $|(S^g)_\sigma|$ give respectively the order of the semisimple and toral parts of the fixed points of the centralizers under σ belonging to a given orbit in $\mathscr{C}_J/G_\sigma$. In particular from the semisimple part we can deduce what kind of graph automorphism is induced by the elements of Ω_J. The last column gives the conditions which have to be imposed on q for the occurrence in the G_σ-orbits in $\mathscr{C}_J$ of centralizers of semisimple elements of G_σ. In this last column whenever there is no indication of condition for occurrence this will mean that in the corresponding G_σ-orbit they do occur as centralizers of semisimple elements of G_σ for all q sufficiently large. For the group E_7 we shall distinguish the simply-connected case from the adjoint one by putting "*sc*" for the former and "*ad*" for the latter.

When the group Ω_J is not too small, in the tables there is a column headed with $[w]$. In this column we give a representative element w for each conjugacy class in Ω_J which corresponds to the G_σ-orbit in $\mathscr{C}_J$ parametrized by $[w]$ and Δ_J so that one can distinguish the rows which have the same Δ_J, Ω_J, $|(M^g)_\sigma|$ and $|(S^g)_\sigma|$. For these cases, we indicate in the 2nd, 3rd and 4th row of the first column the chosen Δ_J, the type of the root subsystem $\Phi_J^\perp$ which is orthogonal to Φ_J and a fundamental basis $\Delta_J^\perp$ of $\Phi_J^\perp$ respectively. We give also in the 2nd and 3rd row of the second column the abstruct type of the group $\mathrm{Aut}_W(\Delta_J)$ and generators of this group respectively. These generators are given by their action on a suitable bases of Φ so that the reader can easily see the symmetries of Δ_J induced by them.

Our notation for the types of the Dynkin diagrams will be as that of Dynkin's paper [10]. The root system of type E_8 is considered to be embedded in the real vector space $\boldsymbol{R}^8$ with orthonormal basis $\{\varepsilon_i\}_{1\leq i\leq 8}$. The fundamental roots are chosen to be the vectors $1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$, $\varepsilon_2+\varepsilon_1$, $\varepsilon_2-\varepsilon_1$, $\varepsilon_3-\varepsilon_2$, $\varepsilon_4-\varepsilon_3$, $\varepsilon_5-\varepsilon_4$, $\varepsilon_6-\varepsilon_5$ and $\varepsilon_7-\varepsilon_6$ with respect to which the positive roots are the vectors $\pm\varepsilon_i+\varepsilon_j$, $i<j$ and the vectors $1/2(\varepsilon_8+\sum_{i=1}^{7}(-1)^{v_i}\varepsilon_i)$ such that $\sum_{i=1}^{7}v_i$ is even where $v_i=0, 1$. The root system of type E_7 is the root subsystem of E_8 consisting of the roots $\pm(\pm\varepsilon_i+\varepsilon_j)$, $1\leqq i<j\leqq 6$, $\pm(\varepsilon_8-\varepsilon_7)$ and $\pm 1/2(\varepsilon_8-\varepsilon_7+\sum_{i=1}^{6}(-1)^{v_i}\varepsilon_i)$ such that $\sum_{i=1}^{6}v_i$ is odd. In the tables below, the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22 and 23 will denote respectively the roots $1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$, $\varepsilon_1+\varepsilon_2$, $\varepsilon_2-\varepsilon_1$, $\varepsilon_3-\varepsilon_2$, $\varepsilon_4-\varepsilon_3$, $\varepsilon_5-\varepsilon_4$, $\varepsilon_6-\varepsilon_5$, $\varepsilon_7-\varepsilon_6$, $\varepsilon_7-\varepsilon_8$, $-\varepsilon_7-\varepsilon_8$, $1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$, $\varepsilon_3+\varepsilon_2$, $\varepsilon_4+\varepsilon_3$, $\varepsilon_6+\varepsilon_5$, $\varepsilon_4+\varepsilon_1$, $1/2(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7+\varepsilon_8)$, $1/2(-\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7+\varepsilon_8)$, $\varepsilon_7+\varepsilon_6$, $\varepsilon_8-\varepsilon_5$, $\varepsilon_8+\varepsilon_5$, $1/2(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$, $1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$ and $\varepsilon_4-\varepsilon_1$. Also the letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \pi, \rho, \tau, \varphi, x, y, z, \omega, \upsilon$ and i will denote respectively the reflections in the hyperplanes orthogonal to the roots 1 up to 23.

We shall denote by H_1 and H_2 the following groups. Let us write the symmetric group S_4 as the semi-direct product of $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ by S_3. Then H_1 denotes the semi-direct product of $\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ by S_4, where S_4 acts on the normal part such that its Klein subgroup $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ acts trivially and S_3 purmutes the components in all possible ways. H_2 denotes the semi-direct product of the Weyl group $W(D_4)$ of type D_4 by S_4, where here the Klein subgroup acts trivially on $W(D_4)$ and S_3 acts as in the figure:



Notice that if $J=\varnothing$ then the groups of the σ-fixed points of the centralizers in $\mathscr{C}_\phi$ are the maximal tori $(T_w)_\sigma$, $w\in\Omega_\phi$, in G_σ determined from

TABLE 1

The structure and the orders of connected centralizers of semisimple elements in E_7.

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc. ad.
A_1	$W(D_6)$	1	$\lvert A_1(q)\rvert$	$(q-1)^6$	
$\{7\}$	1	α		$(q^2-1)(q-1)^4$	
D_6		$\alpha\gamma$		$(q^3-1)(q-1)^3$	
$\{1,2,3,4,5,9\}$		$\alpha\beta$		$(q-1)^2(q^2-1)^2$	
		$\alpha\delta\gamma$		$(q-1)^2(q^4-1)$	
		$\alpha\gamma\beta$		$(q-1)(q^2-1)\times(q^3-1)$	
		$\alpha\delta\gamma\varepsilon$		$(q-1)(q^5-1)$	
		$\kappa\gamma\varepsilon$		$(q^2-1)^3$	
		$\kappa\gamma\beta$		$(q^2-1)^3$	
		$\kappa\gamma\alpha\varepsilon$		$(q^2-1)(q^4-1)$	
		$\kappa\gamma\alpha\beta$		$(q^2-1)(q^4-1)$	
		$\kappa\delta\alpha\varepsilon$		$(q^3-1)^2$	
		$\kappa\gamma\varepsilon\alpha\delta$		q^6-1	
		$\kappa\gamma\beta\alpha\delta$		q^6-1	
		$\beta\varepsilon$		$(q+1)^2(q-1)^4$	
		$\beta\varepsilon\pi\kappa$		$(q+1)^2(q^2-1)^2$	
		$\beta\varepsilon\pi\kappa\xi\gamma$		$(q+1)^6$	
		$\beta\varepsilon\gamma$		$(q^2-1)^3$	
		$\beta\varepsilon\pi\kappa\gamma$		$(q-1)(q+1)^5$	
		$\beta\varepsilon\gamma\xi\alpha$		$(q+1)^2(q^4-1)$	
		$\gamma\xi\alpha$		$(q-1)^2(q^4-1)$	
		$\beta\varepsilon\alpha\gamma$		$(q+1)(q^2-1)\times(q^3-1)$	
		$\beta\varepsilon\alpha\kappa\pi\gamma$		$(q+1)^3(q^3+1)$	
		$\kappa\gamma\pi\alpha$		$(q-1)(q^2-1)\times(q^3+1)$	
		$\beta\varepsilon\kappa\gamma$		$(q+1)(q^2-1)^2$	
		$\gamma\xi\alpha\varepsilon$		$(q^2+1)(q^2-1)^2$	
		$\kappa\pi\alpha\tau$		$(q+1)^2(q^4-1)$	
		$\kappa\pi\alpha\tau\beta\varepsilon$		$(q-1)^2(q^2+1)^2$	
		$\alpha\delta\gamma\beta\varepsilon$		$(q+1)^2(q^2+1)^2$	
		$\beta\varepsilon\kappa\alpha\delta$		$(q^2-1)(q^4+1)$	
		$\kappa\gamma\pi\alpha\varepsilon$		$(q^2+q+1)(q^4-1)$	
		$\kappa\pi\delta\alpha\tau$		$(q+1)(q^2-1)\times(q^3+1)$	
		$\beta\gamma\kappa\alpha\delta\varepsilon$		$(q-1)(q^2+1)\times(q^3+1)$	
		$\kappa\pi\alpha\tau\varepsilon$		$(q+1)(q^5+1)$	
		$\kappa\pi\delta\alpha\tau\varepsilon$		$(q^2+1)(q^4-1)$	
		$\kappa\pi\delta\iota\alpha\tau$		$(q^2+1)(q^4+1)$	
		$\beta\varepsilon\alpha\kappa\gamma$		$(q^3+1)^2$	
$2A_1$	$W(B_4)\times Z_2$	1	$\lvert A_1(q)\rvert^2$	$(q-1)^5$	
$\{7,9\}$	Z_2	β		$(q^2-1)(q-1)^3$	
D_4+A_1	s: $1\leftrightarrow 6$	$\gamma\varepsilon$		$(q-1)(q^2-1)^2$	
$\{2,3,4,5,14\}$	$2\to 2$	$\beta\gamma$		$(q-1)(q^2-1)^2$	
	$3\leftrightarrow 5$	$\gamma\delta$		$(q-1)^2(q^3-1)$	
	$4\to 4$	$\gamma\varepsilon\delta$		$(q-1)(q^4-1)$	
	$7\leftrightarrow 9$	$\beta\gamma\delta$		$(q-1)(q^4-1)$	
		$\beta\gamma\varepsilon\xi$		$(q^2-1)(q+1)^3$	
		$\beta\gamma\varepsilon$		$(q+1)(q^2-1)^2$	
		$\beta\gamma\varepsilon\delta$		$(q^2-1)(q^3+1)$	
		$\beta\gamma\delta\nu$		$(q-1)(q^2+1)^2$	
		π		$(q^2-1)(q-1)^3$	
		$\beta\pi$		$(q-1)(q^2-1)^2$	
		$\gamma\varepsilon\pi$		$(q+1)(q^2-1)^2$	
		$\beta\gamma\pi$		$(q+1)(q^2-1)^2$	
		$\gamma\delta\pi$		$(q^2-1)(q^3-1)$	
		$\gamma\varepsilon\delta\pi$		$(q+1)(q^4-1)$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc. ad.
		$\beta\gamma\delta\pi$		$(q+1)(q^4-1)$	
		$\beta\gamma\varepsilon\xi\pi$		$(q+1)^5$	
		$\beta\gamma\varepsilon\pi$		$(q-1)(q+1)^4$	
		$\beta\gamma\varepsilon\delta\pi$		$(q+1)^2(q^3+1)$	
		$\beta\gamma\delta\nu\pi$		$(q+1)(q^2+1)^2$	
		s	$\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)^2$	
		γs		$(q-1)(q^4-1)$	
		ιs		$(q+1)(q^2-1)^2$	
		$\delta\iota s$		$(q^2-1)(q+1)^3$	
		$\delta\gamma s$		$(q^2-1)(q^3+1)$	
		$\beta\delta s$		$(q+1)^2(q^3-1)$	
		$\beta\gamma s$		$(q+1)(q^4-1)$	
		$\beta\nu\varepsilon s$		$(q+1)(q^4+1)$	
		$\beta\xi\varepsilon s$		$(q^2+1)(q+1)^3$	
		πs		$(q^2-1)(q-1)^3$	
		$\gamma\pi s$		$(q^2+1)(q-1)^3$	
		$\iota\pi s$		$(q-1)(q^2-1)^2$	
		$\delta\iota\pi s$		$(q+1)(q^2-1)^2$	
		$\delta\gamma\pi s$		$(q-1)^2(q^3+1)$	
		$\beta\delta\pi s$		$(q^2-1)(q^3-1)$	
		$\beta\gamma\pi s$		$(q-1)(q^4-1)$	
		$\beta\nu\varepsilon\pi s$		$(q-1)(q^4+1)$	
		$\beta\xi\varepsilon\pi s$		$(q+1)(q^4-1)$	
A_2	$S_6\times Z_2$	1	$\lvert A_2(q)\rvert$	$(q-1)^5$	
$\{6, 7\}$	Z_2	β		$(q+1)(q-1)^4$	
A_5	s: $1\to-1$	$\kappa\gamma$		$(q-1)(q^2-1)^2$	
$\{1, 2, 3, 4, 9\}$	$2\to-2$	$\kappa\alpha$		$(q-1)^2(q^3-1)$	
	$3\to-3$	$\kappa\gamma\beta$		$(q-1)^2(q+1)^3$	
	$4\to-4$	$\kappa\gamma\delta$		$(q^2-1)(q^3-1)$	
	$6\leftrightarrow7$	$\beta\gamma\delta$		$(q-1)(q^4-1)$	

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc. ad.
	$9\to-9$	$\beta\kappa\alpha\delta$		$(q^2+q+1)(q^3-1)$	
		$\beta\gamma\kappa\delta$		$(q+1)(q^4-1)$	
		$\kappa\gamma\alpha\delta$		(q^5-1)	
		$\kappa\gamma\beta\alpha\delta$		$(q^2+q+1)(q^3+1)$	
		s	$\lvert{}^2A_2(q)\rvert^2$	$(q+1)^5$	
		βs		$(q^2-1)(q+1)^3$	
		$\kappa\gamma s$		$(q+1)(q^2-1)^2$	
		$\kappa\alpha s$		$(q+1)^2(q^3+1)$	
		$\kappa\gamma\beta s$		$(q-1)(q^2-1)^2$	
		$\kappa\gamma\delta s$		$(q^2-1)(q^3+1)$	
		$\beta\gamma\delta s$		$(q+1)(q^4-1)$	
		$\beta\kappa\alpha\delta s$		$(q^2-q+1)(q^3+1)$	
		$\beta\gamma\kappa\delta s$		$(q-1)(q^4-1)$	
		$\kappa\gamma\alpha\delta s$		(q^5+1)	
		$\kappa\gamma\beta\alpha\delta s$		$(q^2-q+1)(q^3-1)$	
$\lvert 3A_1\rvert'$	$W(B_3)\times Z_2$	1	$\lvert A_1(q)\rvert^3$	$(q-1)^4$	
$\{3, 5, 7\}$	S_3	β		$(q^2-1)(q-1)^2$	
$4A_1$	s_1: $2\leftrightarrow13$	κ		$(q^2-1)(q-1)^2$	
$\{2, 9, 13, 14\}$	$3\leftrightarrow5$	$\beta\xi$		$(q^2-1)^2$	
	$7\leftrightarrow7$	$\beta\kappa$		$(q^2-1)^2$	
	$9\leftrightarrow9$	$\beta\xi\kappa$		$(q^2-1)(q+1)^2$	
	$14\leftrightarrow14$	$\beta\xi\pi$		$(q^2-1)(q+1)^2$	
	s_2: $2\to14\to13$	$\beta\xi\pi\kappa$		$(q+1)^4$	
	$\to2$	s_1	$\lvert A_1(q)\rvert \times\lvert A_1(q^2)\rvert$	$(q^2-1)(q-1)^2$	
	$3\to7\to5\to3$	ξs_1		$(q^2-1)^2$	
	$9\to9$	πs_1		$(q^2-1)^2$	
		κs_1		$(q^2-1)^2$	
		$\pi\kappa s_1$		$(q^2-1)(q+1)^2$	
		$\xi\pi s_1$		$(q^2-1)(q+1)^2$	
		$\xi\kappa s_1$		$(q^2-1)(q+1)^2$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc.	ad.
		$\kappa\xi\pi s_1$		$(q^2+1)(q+1)^2$		
		s_2	$\lvert A_1(q^3)\rvert$	$(q-1)(q^3-1)$		
		κs_2		$(q+1)(q^3-1)$		
		ξs_2		$(q-1)(q^3+1)$		
		$\pi\kappa s_2$		$(q+1)(q^3+1)$		
$\lvert 3A_1\rvert''$	$W(F_4)$	1	$\lvert A_1(q)\rvert^3$	$(q-1)^4$		
$\{2,5,7\}$	S_3	α		$(q+1)(q-1)^3$		
D_4	s_1: $1\to1$ $3\to3$	$\tau\varphi$		$(q^2-1)^2$		
$\{1,3,16,$ $-17\}$	$2\leftrightarrow5$ $7\to7$	$\tau\gamma$		$(q-1)(q^3-1)$		
	$16\leftrightarrow-17$	$\tau\varphi\gamma$		q^4-1		
	s_2: $2\to7\to5$	$\alpha\tau\varphi\omega$		$(q+1)^4$		
	$\to2$	$\tau\alpha\varphi$		$(q-1)(q+1)^3$		
	$3\to3$	$\gamma\varphi\tau\alpha$		$(q+1)(q^3+1)$		
	$1\to16\to$	$\tau\varphi\gamma\kappa$		$(q^2+1)^2$		
	$-17\to1$	s_1	$\lvert A_1(q)\rvert$ $\times\lvert A_1(q^2)\rvert$	$(q^2-1)(q-1)^2$		
		τs_1		$(q^2+1)(q-1)^2$		
		ξs_1		$(q^2-1)^2$		
		$\gamma\xi s_1$		$(q^2-1)(q+1)^2$		
		$\gamma\tau s_1$		$(q-1)(q^3+1)$		
		$\alpha\gamma s_1$		$(q+1)(q^3-1)$		
		$\alpha\tau s_1$		q^4-1		
		$\alpha\kappa\varphi s_1$		q^4+1		
		$\alpha\omega\varphi s_1$		$(q^2+1)(q+1)^2$		
		s_2	$\lvert A_1(q^3)\rvert$	$(q-1)(q^3-1)$		
		αs_2		$(q-1)(q^3+1)$		
		$\alpha\omega s_2$		$(q+1)(q^3-1)$		
		$\alpha\tau\omega\varphi s_2$		$(q+1)(q^3+1)$		
		$\gamma\tau s_2$		(q^4-q^2+1)		

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc.	ad.
		$\xi\varphi s_2$		$(q^2-q+1)^2$		
		$\pi\omega s_2$		$(q^2+q+1)^2$		
A_2+A_1	$S_4\times Z_2$	1	$\lvert A_1(q)\rvert$ $\times\lvert A_2(q)\rvert$	$(q-1)^4$		
$\{1,2,3\}$	Z_2	ε		$(q+1)(q-1)^3$		
A_3	s: $1\leftrightarrow3$	$\varepsilon\zeta$		$(q-1)(q^3-1)$		
$\{5,6,7\}$	$2\to2$	$\varepsilon\eta$		$(q^2-1)^2$		
	$4\to-4$	$\varepsilon\eta\zeta$		q^4-1		
	$5\to-5$	s	$\lvert A_1(q)\rvert$ $\times\lvert{}^2A_2(q^2)\rvert$	$(q+1)^4$		
	$6\to-6$	εs		$(q+1)^2(q^2-1)$		
	$7\to-7$	$\varepsilon\zeta s$		$(q+1)(q^3+1)$		
		$\varepsilon\eta s$		$(q^2-1)^2$		
		$\varepsilon\eta\zeta s$		q^4-1		
A_3	$S_4\times(Z_2)^2$ Z_2	1	$\lvert A_3(q)\rvert$	$(q-1)^4$		
$\{3,4,5\}$	s: $3\leftrightarrow5$	η		$(q+1)(q-1)^3$		
A_3+A_1	$4\to4$	$\eta\mu$		$(q-1)(q^3-1)$		
$\{7,9,11,14\}$	$7\to-7$	$\eta\kappa$		$(q^2-1)^2$		
	$9\to-9$	$\eta\mu\kappa$		q^4-1		
	$11\to-11$	π		$(q+1)(q-1)^3$		
	$14\to14$	$\eta\pi$		$(q^2-1)^2$		
		$\eta\mu\pi$		$(q+1)(q^3-1)$		
		$\eta\kappa\pi$		$(q^2-1)(q+1)^2$		
		$\eta\mu\kappa\pi$		$(q^2+1)(q+1)^2$		
		s	$\lvert{}^2A_3(q^2)\rvert$	$(q+1)^2(q^2-1)$		
		ηs		$(q^2-1)^2$		
		$\eta\mu s$		$(q^3+1)(q-1)$		
		$\eta\kappa s$		$(q-1)^2(q^2-1)$		
		$\eta\mu\kappa s$		$(q-1)^2(q^2+1)$		
		πs		$(q+1)^4$		

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc.	ad.
		$\pi\eta s$		$(q+1)^2(q^2-1)$		
		$\pi\eta\mu s$		$(q+1)(q^3+1)$		
		$\pi\eta\kappa s$		$(q^2-1)^2$		
		$\pi\eta\mu\kappa s$		q^4-1		
$[4A_1]'$	H_1	1	$\lvert A_1(q)\rvert^4$	$(q-1)^3$	$2\lvert q-1$	$2\lvert q-1$
$\{3, 5, 7, 9\}$	S_4	β		$(q-1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
$3A_1$	$s_1: 2\to 2$	$\beta\xi$		$(q+1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
$\{2, 13, 14\}$	$3\leftrightarrow 9$	$\beta\xi\pi$		$(q+1)^3$	$2\lvert q-1$	$2\lvert q-1$
	$5\to 5$	s_1	$\lvert A_1(q)\rvert^2$ $\times\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
	$7\to 7$	βs_1		$(q+1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
	$13\leftrightarrow 14$	ξs_1		$(q-1)(q^2+1)$	$2\lvert q-1$	$2\lvert q-1$
	$s_2: 2\leftrightarrow 2$	$\beta\pi s_1$		$(q+1)(q^2+1)$	$2\lvert q-1$	$2\lvert q-1$
	$3\to 3$	$s_1 s_3$	$\lvert A_1(q)\rvert$ $\times\lvert A_1(q^3)\rvert$	q^3-1	$2\lvert q-1$	$2\lvert q-1$
	$5\leftrightarrow 7$	$\beta s_1 s_3$		q^3+1	$2\lvert q-1$	$2\lvert q-1$
	$9\to 9$	$s_1 s_3 s_2$	$\lvert A_1(q^4)\rvert$	$(q-1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
	$13\leftrightarrow 14$	$\beta s_1 s_3 s_2$		$(q-1)(q^2+1)$	$2\lvert q-1$	$2\lvert q-1$
	$s_3: 2\leftrightarrow 13$	$\xi s_1 s_3 s_2$		$(q+1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
	$3\leftrightarrow 5$	$\beta\xi s_1 s_3 s_2$		$(q+1)(q^2+1)$	$2\lvert q-1$	$2\lvert q-1$
	$7\to 7$	$s_1 s_2$	$\lvert A_1(q^2)\rvert^2$	$(q-1)^3$	$2\lvert q-1$	$2\lvert q-1$
	$9\to 9$	$\beta s_1 s_2$		$(q-1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
	$14\to 14$	$\xi s_1 s_2$		$(q-1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
		$\xi\pi s_1 s_2$		$(q+1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
		$\beta\xi s_1 s_2$		$(q+1)(q^2-1)$	$2\lvert q-1$	$2\lvert q-1$
		$\beta\xi\pi s_1 s_2$		$(q+1)^3$	$2\lvert q-1$	$2\lvert q-1$
$[4A_1]''$	$W(B_3)$	1	$\lvert A_1(q)\rvert^4$	$(q-1)^3$		
$\{2, 3, 5, 7\}$	S_3	κ		$(q+1)(q-1)^2$		
$3A_1$	$s_1: 2\to 5\to 7$	$\kappa\xi$		$(q+1)(q^2-1)$		
$9, 13, 14\}$	$\to 2$	$\kappa\xi\pi$		$(q+1)^3$		

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence sc.	ad.
	$3\to 3$ $9\to 14$	s_1	$\lvert A_1(q)\rvert$ $\times\lvert A_1(q^3)\rvert$	q^3-1		
	$\to 13\to 9$	κs_1		q^3+1		
	$s_2: 2\to 2$ $3\to 3$	s_2	$\lvert A_1(q)\rvert^2$ $\times\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)$		
	$5\leftrightarrow 7$	κs_2		$(q+1)(q^2-1)$		
	$9\to 9$	ξs_2		$(q-1)(q^2+1)$		
	$13\leftrightarrow 14$	$\kappa\xi s_2$		$(q+1)(q^2+1)$		
A_2+2A_1	$(\boldsymbol{Z}_2)^3$	1	$\lvert A_2(q)\rvert$ $\times\lvert A_1(q)^2\rvert$	$(q-1)^3$		
$\{2, 3, 5, 6\}$	$(\boldsymbol{Z}_2)^2$	κ		$(q-1)(q^2-1)$		
A_1	$s_1: 2\leftrightarrow 3$ $4\leftrightarrow 4$	s_1	$\lvert A_2(q)\rvert$ $\times\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)$		
$\{9\}$	$5\leftrightarrow 6$	κs_1		$(q+1)(q^2-1)$		
	$7\to -14$ $9\to 9$	s_2	$\lvert {}^2A_2(q^2)\rvert$ $\times\lvert A_1(q)\rvert^2$	$(q+1)(q^2-1)$		
	$s_2: 2\to 2$ $3\to 3$	κs_2		$(q+1)^3$		
	$4\to -(\varepsilon_2+\varepsilon_5)$ $5\leftrightarrow 6$ $7\leftrightarrow \varepsilon_3-\varepsilon_6$	$s_1 s_2$	$\lvert {}^2A_2(q^2)\rvert$ $\times\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)$		
	$9\to 9$	$\kappa s_1 s_2$		$(q+1)(q^2-1)$		
$2A_2$	$S_3\times(\boldsymbol{Z}_2)^2$	1	$\lvert A_2(q)\rvert^2$	$(q-1)^3$		
$\{2, 4, 6, 7\}$	$(\boldsymbol{Z}_2)^2$	α		$(q+1)(q-1)^2$		
A_2	$s_1: 1\to -1$ $2\leftrightarrow 4$	$\alpha\kappa$		q^3-1		
$\{1, 9\}$	$5\to -(\varepsilon_1+\varepsilon_6)$ $6\leftrightarrow 7$ $9\to 9$	s_1	$\lvert {}^2A_2(q^2)\rvert^2$	$(q+1)(q^2-1)$		
	$s_2: 1\to -1$ $2\leftrightarrow 7$	αs_1		(q^3+1)		
	$4\leftrightarrow 6$	$\alpha\kappa s_1$		$(q+1)(q^2-1)$		

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Δ_J	Ω_J	$[w]$	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence sc.	ad.
	$5\to5$ $9\to-9$	s_2	$\lvert A_2(q^2)\rvert$	$(q^2-1)(q-1)$		
		αs_2		$(q-1)(q^2-q+1)$		
		$\alpha\kappa s_2$		$(q^2-1)(q-1)$		
		s_1s_2		$(q+1)(q-1)^2$		
		αs_1s_2		$(q+1)(q^2-1)$		
		$\alpha\kappa s_1s_2$		$(q^2+q+1)(q+1)$		
$[A_3+A_1]'$	$(\boldsymbol{Z}_2)^3$	1	$\lvert A_3(q)\rvert \times\lvert A_1(q)\rvert$	$(q-1)^3$		
$\{3, 5, 6, 7\}$	$\boldsymbol{Z}_2$	κ		$(q+1)(q-1)^2$		
$2A_1$	s: $2\to2$	β		$(q+1)(q-1)^2$		
$\{2, 9\}$	$3\to3$	$\kappa\beta$		$(q-1)(q+1)^2$		
	$4\to-(\varepsilon_2+\varepsilon_6)$	s	$\lvert {}^2A_3(q^2)\rvert \times\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$		
	$5\leftrightarrow7$	κs		$(q+1)(q^2-1)$		
	$6\to6$	βs		$(q+1)(q^2-1)$		
	$9\to9$	$\kappa\beta s$		(q^3+1)		
$[A_3+A_1]''$	$S_4\times\boldsymbol{Z}_2$	1	$\lvert A_3(q)\rvert \times\lvert A_1(q)\rvert$	$(q-1)^3$		
$\{2, 5, 6, 7\}$	$\boldsymbol{Z}_2$	α		$(q+1)(q-1)^2$		
A_3	s: $1\to-1$	$\gamma\kappa$		$(q+1)(q^2-1)$		
$\{1, 3, 9\}$	$2\to2$	$\alpha\gamma\kappa$		$(q+1)(q^2+1)$		
	$3\to-3$	$\alpha\gamma$		q^3-1		
	$5\leftrightarrow7$	s	$\lvert {}^2A_3(q^2)\rvert \times\lvert A_1(q)\rvert$	$(q+1)^3$		
	$6\to6$	αs		$(q+1)(q^2-1)$		
	$9\to-9$	$\gamma\kappa s$		$(q-1)(q^2-1)$		
		$\alpha\gamma\kappa s$		$(q-1)(q^2+1)$		
		$\alpha\gamma s$		q^3+1		
A_4	$S_3\times\boldsymbol{Z}_2$	1	$\lvert A_4(q)\rvert$	$(q-1)^3$		
$\{4, 5, 6, 7\}$	$\boldsymbol{Z}_2$	α		$(q+1)(q-1)^2$		

Δ_J	Ω_J	$[w]$	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence sc.	ad.
A_2	s: $1\to-1$	$\alpha\kappa$		q^3-1		
$\{1, 9\}$	$2\to-(\varepsilon_1+\varepsilon_6)$ $4\leftrightarrow7$	s	$\lvert {}^2A_4(q^2)\rvert$	$(q+1)^3$		
	$5\leftrightarrow6$	αs		$(q+1)(q^2-1)$		
	$9\to-9$	$\alpha\kappa s$		q^3+1		
D_4	$W(B_3)$	1	$]D_4(q)\rvert$	$(q-1)^3$		
$\{2, 3, 4, 5\}$	S_3	η		$(q-1)(q^2-1)$		
$3A_1$	s_1: $2\to2$	$\kappa\eta$		$(q+1)(q^2-1)$		
$\{7, 9, 14\}$	$3\leftrightarrow5$	$\eta\kappa\pi$		$(q+1)^3$		
	$4\to4$	s_1	$\lvert {}^2D_4(q^2)\rvert$	$(q+1)(q^2-1)$		
	$7\leftrightarrow9$	ηs_1		$(q+1)(q^2+1)$		
	$14\to-14$	$\eta\pi s_1$		$(q-1)(q^2+1)$		
	s_2: $2\to3\to5$	πs_1		$(q-1)(q^2-1)$		
	$\to2$ $4\to4$	s_2	$\lvert {}^3D_4(q^3)\rvert$	q^3-1		
	$7\to-9\to14$ $\to7$	ηs_2		q^3+1		
$5A_1$	$(\boldsymbol{Z}_2)^2 \wr \boldsymbol{Z}_2$	1	$\lvert A_1(q)\rvert^5$	$(q-1)^2$	$2\mid q-1$	$2\mid q-1$
$\{2, 3, 5, 7, 9\}$	$W(B_2)$	ξ		q^2-1	$2\mid q-1$	$2\mid q-1$
$2A_1$	s_1: $2\to2$	$\xi\pi$		$(q+1)^2$	$2\mid q-1$	$2\mid q-1$
$\{13, 14\}$	$3\to3$	s_1	$\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert^3$	q^2-1	$2\mid q-1$	$2\mid q-1$
	$5\leftrightarrow7$	ξs_1		q^2+1	$2\mid q-1$	$2\mid q-1$
	$9\to9$ $13\leftrightarrow14$	s_1s_2	$\lvert A_1(q^4)\rvert \times\lvert A_1(q)\rvert$	q^2+1	never occurs	$2\mid q-1$
	s_2: $2\to2$ $3\leftrightarrow5$ $7\leftrightarrow9$	ξs_1s_2		q^2-1	never occurs	$2\mid q-1$
	$13\to13$ $14\to-14$	s_2	$\lvert A_1(q^2)\rvert^2 \times\lvert A_1(q)\rvert$	q^2-1	never occurs	$2\mid q-1$
		ξs_2		$(q+1)^2$	never occurs	$2\mid q-1$

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Δ_J	Ω_J	$[w]$	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence sc.	ad.
		πs_2		$(q-1)^2$	never occurs	$2\mid q-1$
		$\xi\pi s_2$		q^2-1	never occurs	$2\mid q-1$
		$(s_1s_2)^2$		$(q+1)^2$	$2\mid q-1$	$2\mid q-1$
		$\xi(s_1s_2)^2$		q^2-1	$2\mid q-1$	$2\mid q-1$
		$\xi\pi(s_1s_2)^2$		$(q-1)^2$	$2\mid q-1$	$2\mid q-1$
A_2+3A_1	$S_3\times \boldsymbol{Z}_2$		$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^3$	$(q-1)^2$		
			$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert$	q^2-1		
			$\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert$	q^2-1		
			$\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q^3)\rvert$	q^2-1		
			$\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q)\rvert^3$	$(q+1)^2$		
			$\lvert A_2(q)\rvert \times\lvert A_1(q^3)\rvert$	q^2+q+1		
$2A_2+A_1$	$(\boldsymbol{Z}_2)^2$		$\lvert A_2(q)\rvert^2\lvert A_1(q)\rvert$	$(q-1)^2$		
			$\lvert A_2(q^2)\rvert\lvert A_1(q)\rvert$	q^2-1		
			$\lvert A_2(q^2)\rvert\lvert A_1(q)\rvert$	q^2-1		
			$\lvert {}^2A_2(q^2)\rvert^2\lvert A_1(q)\rvert$	$(q+1)^2$		
$[A_3+2A_1]'$	$(\boldsymbol{Z}_2)^3$		$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$	$2\mid q-1$	$2\mid q-1$
				q^2-1	$2\mid q-1$	$2\mid q-1$
			$\lvert A_3(q)\rvert\lvert A_1(q^2)\rvert$	$(q+1)^2$	$2\mid q-1$	$2\mid q-1$
				q^2-1	$2\mid q-1$	$2\mid q-1$
			$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q-1)^2$	$2\mid q-1$	$2\mid q-1$
				q^2-1	$2\mid q-1$	$2\mid q-1$
			$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2$	$2\mid q-1$	$2\mid q-1$
				q^2-1	$2\mid q-1$	$2\mid q-1$
$[A_3+2A_1]''$	$(\boldsymbol{Z}_2)^2$		$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$		

Δ_J	Ω_J	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence sc.	ad.
			q^2-1		
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2$		
			q^2-1		
A_3+A_2	$(\boldsymbol{Z}_2)^2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert$	$(q-1)^2$		
			q^2-1		
		$\lvert {}^2A_3(q^2)\rvert\lvert {}^2A_2(q)\rvert$	$(q+1)^2$		
			q^2-1		
A_4+A_1	$\boldsymbol{Z}_2$	$\lvert A_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$		
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2$		
$[A_5]'$	$(\boldsymbol{Z}_2)^2$	$\lvert A_5(q)\rvert$	$(q-1)^2$		
			$(q+1)^2$		
		$\lvert {}^2A_5(q^2)\rvert$	$(q+1)^2$		
			q^2-1		
$[A_5]''$	$S_3\times \boldsymbol{Z}_2$	$\lvert A_5(q)\rvert$	$(q-1)^2$		
			(q^2-1)		
			q^2+q+1		
		$\lvert {}^2A_5(q^2)\rvert$	$(q+1)^2$		
			q^2-1		
			q^2-q+1		
D_4+A_1	$W(B_2)$	$\lvert D_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$		
			q^2-1		
			$(q+1)^2$		
		$\lvert {}^2D_4(q^2)\rvert\lvert A_1(q)\rvert$	q^2-1		
			q^2+1		
D_5	$(\boldsymbol{Z}_2)^2$	$\lvert D_5(q)\rvert$	$(q-1)^2$		
			q^2-1		
		$\lvert {}^2D_5(q^2)\rvert$	$(q+1)^2$		
			q^2-1		
$3A_2$	$S_3\times \boldsymbol{Z}_2$	$\lvert A_2(q)\rvert^3$	$q-1$	$3\mid q-1$	$3\mid q-1$
		$\lvert {}^2A_2(q^2)\rvert^3$	$q+1$	$3\mid q+1$	$3\mid q+1$

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Δ_J	Ω_J	$\vert(M^\sigma)_\sigma\vert$	$\vert(S^\sigma)_\sigma\vert$	Condition for occurrence sc.	ad.
		$\vert A_2(q^2)\vert\vert A_2(q)\vert$	$q+1$	$3\mid q-1$	$3\mid q-1$
		$\vert A_2(q^3)\vert$	$q-1$	$3\mid q-1$	$3\mid q-1$
		$\vert {}^2A_2(q^2)\vert\vert A_2(q^2)\vert$	$q-1$	$3\mid q+1$	$3\mid q+1$
		$\vert {}^2A_2(q^6)\vert$	$q+1$	$3\mid q+1$	$3\mid q+1$
A_3+3A_1	$(Z_2)^2$	$\vert A_3(q)\vert\vert A_1(q)\vert^3$	$q-1$	$2\mid q-1$	$2\mid q-1$
		$\vert {}^2A_3(q^2)\vert\vert A_1(q^2)\vert \times\vert A_1(q)\vert$	$q-1$	$2\mid q-1$	$2\mid q-1$
		$\vert {}^2A_3(q^2)\vert\vert A_1(q)\vert^3$	$q+1$	$2\mid q-1$	$2\mid q-1$
		$\vert A_3(q)\vert\vert A_1(q^2)\vert \times\vert A_1(q)\vert$	$q+1$	$2\mid q-1$	$2\mid q-1$
$A_3+A_2+A_1$	Z_2	$\vert A_3(q)\vert\vert A_2(q)\vert \times\vert A_1(q)\vert$	$q-1$		
		$\vert {}^2A_3(q^2)\vert\vert {}^2A_2(q^2)\vert \times\vert A_1(q)\vert$	$q+1$		
$2A_3$	$(Z_2)^3$	$\vert A_3(q)\vert^2$	$q-1$	$4\mid q-1$	$2\mid q-1$
			$q+1$	$4\mid q-1$	$2\mid q-1$
		$\vert A_3(q^2)\vert$	$q-1$	$4\mid q+1$	$2\mid q-1$
			$q+1$	$4\mid q+1$	$2\mid q-1$
			$q-1$	$4\mid q-1$	$2\mid q-1$
			$q+1$	$4\mid q-1$	$2\mid q-1$
		$\vert {}^2A_3(q^2)\vert^2$	$q-1$	$4\mid q+1$	$2\mid q-1$
			$q+1$	$4\mid q+1$	$2\mid q-1$
A_4+A_2	Z_2	$\vert A_4(q)\vert\vert A_2(q)\vert$	$q-1$		
		$\vert {}^2A_4(q^2)\vert\vert {}^2A_2(q^2)\vert$	$q+1$		
$[A_5+A_1]'$	Z_2	$\vert A_5(q)\vert\vert A_1(q)\vert$	$q-1$	$2\mid q-1$	$2\mid q-1$
		$\vert {}^2A_5(q^2)\vert\vert A_1(q)\vert$	$q+1$	$2\mid q-1$	$2\mid q-1$
$[A_5+A_1]''$	Z_2	$\vert A_5(q)\vert\vert A_1(q)\vert$	$q-1$		

Δ_J	Ω_J	$\vert(M^\sigma)_\sigma\vert$	$\vert(S^\sigma)_\sigma\vert$	Condition for occurrence sc.	ad.
		$\vert {}^2A_5(q^2)\vert\vert A_1(q)\vert$	$q+1$		
A_6	Z_2	$\vert A_6(q)\vert$	$q-1$		
		$\vert {}^2A_6(q^2)\vert$	$q+1$		
D_4+2A_1	$(Z_2)^2$	$\vert D_4(q)\vert\vert A_1(q)\vert^2$	$q-1$	$2\mid q-1$	$2\mid q-1$
			$q+1$	$2\mid q-1$	$2\mid q-1$
		$\vert {}^2D_4(q^2)\vert\vert A_1(q^2)\vert$	$q-1$	never occurs	$2\mid q-1$
			$q+1$	never occurs	$2\mid q-1$
D_5+A_1	Z_2	$\vert D_5(q)\vert\vert A_1(q)\vert$	$q-1$		
		$\vert {}^2D_5(q^2)\vert\vert A_1(q)\vert$	$q+1$		
D_6	Z_2	$\vert D_6(q)\vert$	$q-1$		
			$q+1$		
E_6	Z_2	$\vert E_6(q)\vert$	$q-1$		
		$\vert {}^2E_6(q^2)\vert$	$q+1$		
$2A_3+A_1$	$(Z_2)^2$	$\vert A_3(q)\vert^2\vert A_1(q)\vert$	1	$4\mid q-1$	$4\mid q-1$
		$\vert A_3(q^2)\vert\vert A_1(q)\vert$	1	never occurs	$4\mid q-1$
			1	never occurs	$4\mid q-1$
		$\vert {}^2A_3(q^2)\vert^2\vert A_1(q)\vert$	1	$4\mid q+1$	$4\mid q+1$
A_5+A_2	Z_2	$\vert A_5(q)\vert\vert A_2(q)\vert$	1	$3\mid q-1$	$3\mid q-1$
		$\vert {}^2A_5(q^2)\vert\vert {}^2A_2(q^2)\vert$	1	$3\mid q+1$	$3\mid q+1$
A_7	Z_2	$\vert A_7(q)\vert$	1	$4\mid q-1$	$4\mid q-1$
		$\vert {}^2A_7(q^2)\vert$	1	$4\mid q+1$	$4\mid q+1$
D_6+A_1	1	$\vert D_6(q)\vert\vert A_1(q)\vert$	1	$2\mid q-1$	$2\mid q-1$
E_7	1	$\vert E_7(q)\vert$	1		

TABLE 2

The structure and the orders of the centralizers of semisimple elements in E_8.

Δ_J	Ω_J	$[w]$	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence
A_1	$W(E_7)$ 1	1	$\lvert A_1(q)\rvert$	$(q-1)^7$	
$\{10\}$		α		$(q+1)(q-1)^6$	
E_7		$\alpha\beta$		$(q+1)^2(q-1)^5$	
$\{1, 2, 3, 4,$		$\alpha\gamma$		$(q^3-1)(q-1)^4$	
$5, 6, 7\}$		$\kappa\gamma\varepsilon$		$(q-1)(q^2-1)^3$	
		$\kappa\gamma\beta$		$(q-1)(q^2-1)^3$	
		$\alpha\gamma\beta$		$(q-1)^2(q^2-1)$ $\times(q^3-1)$	
		$\alpha\delta\gamma$		$(q-1)^3(q^4-1)$	
		$\kappa\gamma\varepsilon\eta$		$(q+1)(q^2-1)^3$	
		$\kappa\gamma\beta\eta$		$(q+1)(q^2-1)^3$	
		$\alpha\beta\gamma\varepsilon$		$(q^3-1)(q^2-1)^2$	
		$\kappa\delta\alpha\varepsilon$		$(q-1)(q^3-1)^2$	
		$\kappa\gamma\alpha\varepsilon$		$(q-1)(q^2-1)$ $\times(q^4-1)$	
		$\kappa\gamma\alpha\beta$		$(q-1)(q^2-1)$ $\times(q^4-1)$	
		$\alpha\delta\gamma\varepsilon$		$(q-1)^2(q^5-1)$	
		$\kappa\gamma\pi\alpha$		$(q+1)(q^3+1)$ $\times(q-1)^3$	
		$\kappa\pi\alpha\tau$		$(q-1)^3(q^2+1)^2$	
		$\kappa\gamma\beta\varepsilon\eta$		$(q-1)^2(q+1)^5$	
		$\alpha\beta\gamma\varepsilon\eta$		$(q+1)^2(q^2-1)$ $\times(q^3-1)$	
		$\kappa\delta\alpha\varepsilon\eta$		$(q+1)(q^3-1)^2$	
		$\kappa\gamma\alpha\varepsilon\eta$		$(q+1)(q^2-1)$ $\times(q^4-1)$	
		$\kappa\gamma\alpha\beta\eta$		$(q+1)(q^2-1)$ $\times(q^4-1)$	
		$\kappa\gamma\zeta\alpha\varepsilon$		$(q^3-1)(q^4-1)$	
		$\kappa\gamma\alpha\delta\zeta$		$(q^2-1)(q^5-1)$	
		$\kappa\gamma\varepsilon\alpha\delta$		$(q-1)(q^6-1)$	
		$\kappa\gamma\beta\alpha\delta$		$(q-1)(q^6-1)$	
		$\kappa\gamma\pi\alpha\varepsilon$		$(q^3+1)(q^2-1)^2$	
		$\kappa\pi\alpha\tau\varepsilon$		$(q-1)(q^2+1)$ $\times(q^4-1)$	
		$\alpha\delta\gamma\beta\varepsilon$		$(q+1)(q-1)^2$ $\times(q^4+1)$	
		$\kappa\pi\delta\alpha\tau$		$(q-1)^2(q^2+1)$ $\times(q^3+1)$	
		$\kappa\gamma\beta\varepsilon\eta\xi$		$(q^2-1)(q+1)^5$	
		$\kappa\zeta\beta\alpha\eta\delta$		$(q-1)(q^2+q+1)^3$	
		$\kappa\gamma\beta\alpha\varepsilon\eta$		$(q-1)(q^2+1)$ $\times(q+1)^4$	
		$\kappa\gamma\beta\zeta\alpha\eta$		$(q+1)^2(q^2+1)$ $\times(q^3-1)$	
		$\kappa\gamma\zeta\alpha\varepsilon\eta$		$(q+1)(q^2+1)$ $\times(q^4-1)$	
		$\kappa\gamma\zeta\alpha\delta\eta$		$(q^2+q+1)(q^5-1)$	
		$\kappa\gamma\varepsilon\alpha\delta\eta$		$(q+1)(q^6-1)$	
		$\kappa\gamma\beta\alpha\delta\eta$		$(q+1)(q^6-1)$	
		$\kappa\gamma\varepsilon\alpha\delta\zeta$		q^7-1	
		$\kappa\gamma\pi\alpha\varepsilon\eta$		$(q^2-1)(q+1)^2$ $\times(q^3+1)$	
		$\kappa\gamma\pi\alpha\delta\eta$		$(q+1)(q^2-1)$ $\times(q^4+1)$	
		$\kappa\pi\delta\alpha\tau\eta$		$(q^3+1)(q^4-1)$	
		$\kappa\gamma\beta\varepsilon\alpha\delta$		$(q^2-1)(q^5+1)$	
		$\kappa\pi\delta\alpha\tau\varepsilon$		$(q-1)(q^2+1)$ $\times(q^4+1)$	
		$\kappa\pi\delta\iota\alpha\tau$		$(q-1)(q^3+1)^2$	
		$\alpha\delta\zeta\beta\gamma\varepsilon$		$(q^3-1)(q^4-q^2+1)$	
		$\kappa\pi\delta\alpha\tau\zeta$		$(q-1)(q^6+q^3+1)$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrenec
		$\kappa\pi\rho\alpha\zeta\tau$		$(q^3-1)(q^2-q+1)^2$	
		$\kappa\pi\gamma\beta\varepsilon\eta\xi$		$(q+1)^7$	
		$\kappa\gamma\zeta\alpha\varepsilon\eta\beta$		$(q+1)^3(q^2+1)^2$	
		$\kappa\gamma\beta\zeta\alpha\delta\eta$		$(q^3+1)(q^2+q+1)^2$	
		$\kappa\gamma\varepsilon\eta\alpha\delta\zeta$		$(q+1)(q^2+1)$ $\times(q^4+1)$	
		$\kappa\gamma\pi\alpha\varepsilon\eta\beta$		$(q^3+1)(q+1)^4$	
		$\kappa\gamma\pi\beta\alpha\delta\eta$		$(q+1)^2(q^5+1)$	
		$\kappa\pi\rho\alpha\tau\zeta\eta$		$(q+1)(q^3+1)^2$	
		$\kappa\gamma\beta\varepsilon\zeta\alpha\delta$		$(q+1)(q^6-q^3+1)$	
		$\kappa\pi\delta\alpha\beta\zeta\tau$		q^7+1	
		$\kappa\pi\nu\delta\alpha\zeta\tau$		$(q^3+1)(q^4-q^2+1)$	
		$\kappa\pi\rho\alpha\tau\zeta\beta$		$(q^2-q+1)(q^5+1)$	
		$\kappa\pi\iota\rho\alpha\tau\zeta$		$(q+1)(q^2-q+1)^3$	
$2A_1$	$W(B_6)$	1	$\lvert A_1(q)\rvert^2$	$(q-1)^6$	
$\{7, 10\}$	Z_2	α		$(q+1)(q-1)^5$	
D_6	s: $1\to1$	$\alpha\beta$		$(q+1)^2(q-1)^4$	
$\{1, 2, 3, 4,$	$2\to2$	$\beta\varepsilon$		$(q+1)^2(q-1)^4$	
$5, 9\}$	$3\to3$	$\alpha\gamma$		$(q^3-1)(q-1)^3$	
	$5\to5$	$\kappa\gamma\varepsilon$		$(q^2-1)^3$	
	$7\leftrightarrow10$	$\beta\varepsilon\gamma$		$(q^2-1)^3$	
	$9\leftrightarrow14$	$\alpha\gamma\beta$		$(q-1)(q^2-1)$ $\times(q^3-1)$	
		$\alpha\delta\gamma$		$(q-1)^2(q^4-1)$	
		$\gamma\xi\alpha$		$(q-1)^2(q^4-1)$	
		$\beta\varepsilon\alpha\gamma$		$(q+1)^2(q^2-1)^2$	
		$\beta\varepsilon\pi\kappa$		$(q+1)^2(q^2-1)^2$	
		$\beta\varepsilon\alpha\gamma$		$(q-1)(q+1)^2$ $\times(q^3-1)$	
		$\kappa\delta\alpha\varepsilon$		$(q^3-1)^2$	
		$\kappa\gamma\alpha\varepsilon$		$(q^2-1)(q^4-1)$	

Δ_J	Ω_J	$[w]$	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrence
		$\gamma\xi\alpha\varepsilon$		$(q^2-1)(q^4-1)$	
		$\alpha\delta\gamma\varepsilon$		$(q-1)(q^5-1)$	
		$\kappa\gamma\pi\alpha$		$(q-1)(q^2-1)$ $\times(q^3+1)$	
		$\kappa\pi\alpha\tau$		$(q-1)^2(q^2+1)^2$	
		$\beta\varepsilon\pi\kappa\gamma$		$(q-1)(q+1)^5$	
		$\beta\varepsilon\alpha\kappa\gamma$		$(q+1)^2(q^4-1)$	
		$\beta\varepsilon\gamma\xi\alpha$		$(q+1)^2(q^4-1)$	
		$\beta\varepsilon\kappa\alpha\delta$		$(q+1)(q^2+1)$ $\times(q^3-1)$	
		$\kappa\gamma\varepsilon\alpha\delta$		q^6-1	
		$\kappa\gamma\pi\alpha\varepsilon$		$(q-1)(q+1)^2$ $\times(q^3+1)$	
		$\kappa\pi\alpha\tau\varepsilon$		$(q^2-1)(q^2+1)^2$	
		$\alpha\delta\gamma\beta\varepsilon$		$(q^2-1)(q^4+1)$	
		$\kappa\pi\delta\alpha\tau$		$(q-1)(q^2+1)$ $\times(q^3+1)$	
		$\beta\varepsilon\pi\kappa\xi\gamma$		$(q+1)^6$	
		$\beta\varepsilon\kappa\pi\alpha\tau$		$(q+1)^2(q^2+1)^2$	
		$\beta\varepsilon\alpha\kappa\pi\gamma$		$(q+1)^3(q^3+1)$	
		$\kappa\pi\delta\iota\alpha\tau$		$(q^3+1)^2$	
		$\kappa\pi\delta\alpha\tau\varepsilon$		$(q^2+1)(q^4+1)$	
		$\beta\gamma\kappa\delta\alpha\varepsilon$		$(q+1)(q^5+1)$	
		s	$\lvert A_1(q^2)\rvert$	$(q+1)(q-1)^5$	
		αs		$(q+1)^2(q-1)^4$	
		κs		$(q^2+1)(q-1)^4$	
		$\delta\alpha s$		$(q^2-1)^3$	
		$\varepsilon\beta s$		$(q^2-1)^3$	
		$\delta\gamma s$		$(q-1)(q^2-1)$ $\times(q^3-1)$	
		$\gamma\kappa s$		$(q-1)^2(q^4-1)$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
		$\kappa\alpha s$		$(q-1)^3(q^3+1)$	
		$\gamma\varepsilon\beta s$		$(q-1)^2(q+1)^4$	
		$\varepsilon\beta\kappa s$		$(q^2-1)(q^4-1)$	
		$\varepsilon\gamma\kappa s$		$(q^2-1)(q^4-1)$	
		$\varepsilon\gamma\alpha s$		$(q+1)(q^2-1) \times(q^3-1)$	
		$\delta\gamma\kappa s$		$(q-1)(q^2+1) \times(q^3-1)$	
		$\varepsilon\gamma\delta s$		$(q^2-1)(q^4-1)$	
		$\varepsilon\kappa\alpha s$		$(q-1)(q^2-1) \times(q^3+1)$	
		$\gamma\kappa\alpha s$		$(q-1)^2(q^4+1)$	
		$\alpha\varepsilon\gamma\xi s$		$(q^2-1)(q+1)^4$	
		$\gamma\alpha\varepsilon\beta s$		$(q^3-1)(q+1)^3$	
		$\varepsilon\beta\gamma\kappa s$		$(q+1)^2(q^4-1)$	
		$\varepsilon\beta\delta\kappa s$		$(q^2-1)(q^2+1)^2$	
		$\varepsilon\beta\kappa\alpha s$		$(q+1)(q^2-1) \times(q^3+1)$	
		$\varepsilon\kappa\alpha\delta s$		q^6-1	
		$\delta\varepsilon\gamma\kappa s$		$(q^2+1)(q^4-1)$	
		$\varepsilon\gamma\delta\alpha s$		$(q+1)(q^5-1)$	
		$\varepsilon\gamma\kappa\alpha s$		$(q^2-1)(q^4+1)$	
		$\kappa\gamma\alpha\delta s$		$(q-1)(q^5+1)$	
		$\varepsilon\beta\gamma\xi\kappa s$		$(q^2+1)(q+1)^4$	
		$\kappa\iota\delta\varepsilon\beta s$		$(q^2+1)^3$	
		$\varepsilon\beta\gamma\delta\kappa s$		$(q+1)(q^2+1) \times(q^3+1)$	
		$\varepsilon\beta\gamma\kappa\alpha s$		$(q+1)^2(q^4+1)$	
		$\varepsilon\gamma\kappa\alpha\delta s$		q^6+1	
A_2	$W(E_6)\times Z_2$	1	$\lvert A_2(q)\rvert$	$(q-1)^6$	
{8, 10}	Z_2	γ		$(q+1)(q-1)^5$	
E_6	s: $1\to-1$	$\gamma\varepsilon$		$(q+1)^2(q-1)^4$	

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
{1, 2, 3, 4, 5, 6}	$2\to-2$	$\gamma\delta$		$(q^3-1)(q-1)^3$	
	$3\to-3$	$\gamma\varepsilon\beta$		$(q^2-1)^3$	
	$4\to-4$	$\gamma\varepsilon\zeta$		$(q-1)(q^2-1) \times(q^3-1)$	
	$5\to-5$	$\gamma\varepsilon\delta$		$(q-1)^2(q^4-1)$	
	$6\to-6$	$\alpha\delta\zeta\mu$		$(q+1)^2(q^2-1)^2$	
	$8\leftrightarrow10$	$\alpha\beta\delta\zeta$		$(q+1)(q^2-1) \times(q^3-1)$	
		$\alpha\varepsilon\gamma\zeta$		$(q^3-1)^2$	
		$\gamma\beta\delta\zeta$		$(q^2-1)(q^4-1)$	
		$\gamma\varepsilon\delta\zeta$		$(q-1)(q^5-1)$	
		$\gamma\varepsilon\beta\delta$		$(q-1)(q^2-1) \times(q^3+1)$	
		$\gamma\varepsilon\delta\nu$		$(q-1)^2(q^2+1)^2$	
		$\alpha\beta\varepsilon\gamma\zeta$		$(q+1)(q^2+q+1) \times(q^3-1)$	
		$\alpha\delta\gamma\zeta\mu$		$(q+1)^2(q^4-1)$	
		$\gamma\varepsilon\delta\zeta\mu$		$(q+1)(q^5-1)$	
		$\alpha\delta\zeta\gamma\varepsilon$		q^6-1	
		$\beta\gamma\varepsilon\delta\zeta$		$(q^2-1)(q^4+1)$	
		$\gamma\varepsilon\delta\zeta\nu$		$(q-1)(q^2+1) \times(q^3+1)$	
		$\alpha\beta\varepsilon\gamma\mu\zeta$		$(q^2+q+1)^3$	
		$\alpha\delta\zeta\mu\gamma\varepsilon$		$(q+1)(q^2+q+1) \times(q^3+1)$	
		$\alpha\delta\zeta\gamma\varepsilon\beta$		$(q^2+q+1) \times(q^4-q^2+1)$	
		$\alpha\delta\rho\beta\gamma\zeta$		q^6+q^3+1	
		$\beta\varepsilon\omega\delta\nu\zeta$		$(q^2+q+1) \times(q^2-q+1)^2$	
		s	$\lvert {}^2A_2(q^2)\rvert$	$(q+1)^6$	
		γs		$(q^2-1)(q+1)^4$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
		$\gamma\varepsilon s$		$(q^2-1)^2(q+1)^2$	
		$\gamma\delta s$		$(q^3+1)(q+1)^3$	
		$\gamma\varepsilon\beta s$		$(q^2-1)^3$	
		$\gamma\varepsilon\zeta s$		$(q+1)(q^2-1)$ $\times(q^3+1)$	
		$\gamma\varepsilon\delta s$		$(q+1)^2(q^4-1)$	
		$\alpha\delta\zeta\mu s$		$(q-1)^2(q^2-1)^2$	
		$\alpha\beta\delta\zeta s$		$(q-1)(q^2-1)$ $\times(q^3+1)$	
		$\alpha\varepsilon\gamma\zeta s$		$(q^3+1)^2$	
		$\gamma\beta\delta\zeta s$		$(q^2-1)(q^4-1)$	
		$\gamma\varepsilon\delta\zeta s$		$(q+1)(q^5+1)$	
		$\gamma\varepsilon\beta\delta s$		$(q+1)(q^2-1)$ $\times(q^3-1)$	
		$\gamma\varepsilon\delta\nu s$		$(q+1)^2(q^2+1)^2$	
		$\alpha\beta\varepsilon\gamma\zeta s$		$(q^2-1)(q^2-q+1)^2$	
		$\alpha\delta\gamma\zeta\mu s$		$(q-1)^2(q^4-1)$	
		$\gamma\varepsilon\delta\zeta\mu s$		$(q-1)(q^5+1)$	
		$\alpha\delta\zeta\gamma\varepsilon s$		q^6-1	
		$\beta\gamma\varepsilon\delta\zeta s$		$(q^2-1)(q^4+1)$	
		$\gamma\varepsilon\delta\zeta\nu s$		$(q^2+q+1)(q^4-1)$	
		$\alpha\beta\varepsilon\gamma\mu\zeta s$		$(q^2-q+1)^3$	
		$\alpha\delta\zeta\mu\gamma\varepsilon s$		$(q-1)(q^2-q+1)$ $\times(q^3-1)$	
		$\alpha\delta\zeta\gamma\varepsilon\beta s$		(q^2-q+1) $\times(q^4-q^2+1)$	
		$\alpha\delta\rho\beta\gamma\zeta s$		q^6-q^3+1	
		$\beta\varepsilon\omega\delta\nu\zeta s$		(q^2+q+1) $\times(q^4+q^2+1)$	
$3A_1$	$W(F_4)\times \boldsymbol{Z}_2$	1	$\lvert A_1(q)\rvert^3$	$(q-1)^5$	
$\{2,5,7\}$	S_3	α		$(q+1)(q-1)^4$	
D_4+A_1	$s_1: 1\to 1$	$\tau\varphi$		$(q-1)(q^2-1)^2$	

Δ_J	Ω_J	$[w]$	$\lvert(M^g)\rvert_\sigma$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
$\{1,3,10,$	$2\leftrightarrow 5$	$\tau\gamma$		$(q-1)^2(q^3-1)$	
$16,-17\}$	$3\to 3$	$\tau\varphi\gamma$		$(q-1)(q^4-1)$	
	$7\to 7$	$\alpha\tau\varphi\omega$		$(q^2-1)(q+1)^3$	
	$10\to 10$	$\tau\alpha\varphi$		$(q+1)(q^2-1)^2$	
	$16\leftrightarrow$	$\gamma\varphi\tau\alpha$		$(q^2-1)(q^3+1)$	
	-17	$\tau\varphi\gamma\kappa$		$(q-1)(q^2+1)^2$	
	$s_2: 1\to 16\to$	λ		$(q+1)(q-1)^4$	
	$-17\to 1$	$\alpha\lambda$		$(q-1)(q^2-1)^2$	
	$3\to 3$	$\tau\varphi\lambda$		$(q+1)(q^2-1)^2$	
	$2\to 7\to$	$\tau\gamma\lambda$		$(q^2-1)(q^3-1)$	
	$5\to 2$	$\tau\varphi\gamma\lambda$		$(q+1)(q^4-1)$	
	$10\to 10$	$\alpha\tau\varphi\omega\lambda$		$(q+1)^5$	
		$\tau\alpha\varphi\lambda$		$(q^2-1)(q+1)^3$	
		$\gamma\varphi\tau\alpha\lambda$		$(q+1)(q^3+1)$	
		$\tau\varphi\gamma\kappa\lambda$		$(q+1)(q^2+1)^2$	
		s_1	$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)(q-1)^4$	
		τs_1		$(q^2+1)(q-1)^3$	
		ξs_1		$(q-1)(q^2-1)^2$	
		$\gamma\xi s_1$		$(q+1)(q^2-1)^2$	
		$\gamma\tau s_1$		$(q-1)^2(q^3+1)$	
		$\alpha\gamma s_1$		$(q^2-1)(q^3-1)$	
		$\alpha\tau s_1$		$(q-1)(q^4-1)$	
		$\alpha\kappa\varphi s_1$		$(q-1)(q^4+1)$	
		$\alpha\omega\varphi s_1$		$(q+1)(q^4-1)$	
		λs_1		$(q-1)(q^2-1)^2$	
		$\tau\lambda s_1$		$(q-1)(q^4-1)$	
		$\xi\lambda s_1$		$(q+1)(q^2-1)^2$	
		$\gamma\xi\lambda s_1$		$(q^2-1)(q+1)^3$	
		$\gamma\tau\lambda s_1$		$(q^2-1)(q^3+1)$	
		$\alpha\gamma\lambda s_1$		$(q+1)^2(q^3-1)$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrence
		$\alpha\tau\lambda s_1$		$(q+1)(q^4-1)$	
		$\alpha\kappa\varphi\lambda s_1$		$(q+1)(q^4+1)$	
		$\alpha\omega\varphi\lambda s_1$		$(q^2+1)(q+1)^3$	
		s_2	$\lvert A_1(q^3)\rvert$	$(q-1)^2(q^3-1)$	
		αs_2		$(q-1)^2(q^3+1)$	
		$\alpha\omega s_2$		$(q^2-1)(q^3-1)$	
		$\alpha\tau\omega\varphi s_2$		$(q^2-1)(q^3+1)$	
		$\gamma\tau s_2$		$(q-1)(q^4-q^2+1)$	
		$\xi\varphi s_2$		$(q-1)(q^2-q+1)^2$	
		$\pi\omega s_2$		$(q^2+q+1)(q^3-1)$	
		λs_2		$(q^2-1)(q^3-1)$	
		$\alpha\lambda s_2$		$(q^2-1)(q^3+1)$	
		$\alpha\omega\lambda s_2$		$(q+1)^2(q^3-1)$	
		$\alpha\tau\omega\varphi\lambda s_2$		$(q+1)^2(q^3+1)$	
		$\gamma\tau\lambda s_2$		$(q+1)(q^4-q^2+1)$	
		$\xi\varphi\lambda s_2$		$(q+1)(q^2-q+1)^2$	
		$\pi\omega\lambda s_2$		$(q+1)(q^2+q+1)^2$	
A_2+A_1	$S_6\times Z_2$	1	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^5$	
$\{1, 2, 3\}$	Z_2	λ		$(q+1)(q-1)^4$	
A_5	s: $1\leftrightarrow 3$	$\lambda\eta$		$(q-1)(q^2-1)^2$	
$\{5, 6, 7, 8,$	$2\rightarrow 2$	$\lambda\vartheta$		$(q-1)^2(q^3-1)$	
$10\}$	$5\rightarrow -5$	$\lambda\eta\varepsilon$		$(q-1)^2(q+1)^3$	
	$6\rightarrow -6$	$\lambda\varepsilon\zeta$		$(q^2-1)(q^3-1)$	
	$7\rightarrow -7$	$\lambda\eta\vartheta$		$(q-1)(q^4-1)$	
	$8\rightarrow -8$	$\lambda\varepsilon\zeta\vartheta$		$(q-1)(q^2+q+1)^2$	
	$10\rightarrow$	$\lambda\eta\varepsilon\vartheta$		$(q+1)(q^4-1)$	
	-10	$\lambda\eta\zeta\vartheta$		q^5-1	
		$\lambda\eta\varepsilon\zeta\vartheta$		$(q^2+q+1)(q^3+1)$	
		s	$\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^5$	
		λs		$(q^2-1)(q+1)^3$	

Δ_J	Ω_J	$[w]$	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrence
		$\lambda\eta s$		$(q+1)(q^2-1)^2$	
		$\lambda\vartheta s$		$(q+1)^2(q^3+1)$	
		$\lambda\eta\varepsilon s$		$(q-1)(q^2-1)^2$	
		$\lambda\varepsilon\zeta s$		$(q^2-1)(q^3+1)$	
		$\lambda\eta\vartheta s$		$(q+1)(q^4-1)$	
		$\lambda\varepsilon\zeta\vartheta s$		$(q^2-q+1)(q^3+1)$	
		$\lambda\eta\varepsilon\vartheta s$		$(q-1)(q^4-1)$	
		$\lambda\eta\zeta\vartheta s$		q^5+1	
		$\lambda\eta\varepsilon\zeta\vartheta s$		$(q-1)(q^4+q^2+1)$	
A_3	$W(B_5)$	1	$\lvert A_3(q)\rvert$	$(q-1)^5$	
$\{7, 8, 10\}$	Z_2	α		$(q+1)(q-1)^4$	
D_5	s: $1\rightarrow 21$	$\alpha\delta$		$(q+1)^2(q-1)^3$	
$\{1, 2, 3, 4, 5\}$	$2\rightarrow 2$	$\beta\varepsilon$		$(q+1)^2(q-1)^3$	
	$3\rightarrow 3$	$\alpha\gamma$		$(q-1)^2(q^3-1)$	
	$4\rightarrow 4$	$\beta\varepsilon\gamma$		$(q-1)^2(q+1)^3$	
	$5\rightarrow 5$	$\alpha\gamma\varepsilon$		$(q^2-1)(q^3-1)$	
	$7\leftrightarrow 10$	$\beta\varepsilon\delta$		$(q-1)(q^4-1)$	
	$8\rightarrow 8$	$\alpha\delta\gamma$		$(q-1)(q^4-1)$	
		$\alpha\omega\beta\varepsilon$		$(q-1)(q+1)^4$	
		$\beta\varepsilon\alpha\gamma$		$(q+1)^2(q^3-1)$	
		$\beta\varepsilon\delta\gamma$		$(q+1)(q^4-1)$	
		$\beta\varepsilon\delta\upsilon$		$(q-1)(q^2+1)^2$	
		$\gamma\beta\varepsilon\delta$		$(q^2-1)(q^3+1)$	
		$\alpha\delta\gamma\varepsilon$		q^5-1	
		$\gamma\beta\varepsilon\delta\upsilon$		$(q^2+1)(q^3+1)$	
		$\beta\varepsilon\delta\alpha\omega$		$(q^2+1)(q+1)^3$	
		$\alpha\delta\beta\varepsilon\gamma$		$(q+1)(q^4+1)$	
		s	$\lvert {}^2A_3(q^2)\rvert$	$(q^2-1)(q-1)^3$	
		γs		$(q-1)(q^2-1)^2$	
		αs		$(q^2+1)(q-1)^3$	

(Continued)

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
		$\varepsilon\gamma s$		$(q+1)(q^2-1)^2$	
		$\varepsilon\beta s$		$(q+1)(q^2-1)^2$	
		$\delta\gamma s$		$(q^2-1)(q^3-1)$	
		$\varepsilon\alpha s$		$(q-1)(q^4-1)$	
		$\alpha\gamma s$		$(q-1)^2(q^3+1)$	
		$\gamma\varepsilon\beta s$		$(q^2-1)(q+1)^3$	
		$\varepsilon\gamma\delta s$		$(q+1)(q^4-1)$	
		$\varepsilon\delta\alpha s$		$(q^2+1)(q^3-1)$	
		$\varepsilon\beta\alpha s$		$(q+1)(q^4-1)$	
		$\varepsilon\alpha\gamma s$		$(q^2-1)(q^3+1)$	
		$\gamma\alpha\delta s$		$(q-1)(q^4+1)$	
		$\varepsilon\beta\alpha\gamma s$		$(q+1)^2(q^3+1)$	
		$\gamma\xi\varepsilon\beta s$		$(q+1)^5$	
		$\varepsilon\beta\alpha\delta s$		$(q+1)(q^2+1)^2$	
		$\alpha\delta\varepsilon\gamma s$		q^5+1	
$[4A_1]'$	$W(B_4)$	1	$\lvert A_1(q)\rvert^4$	$(q-1)^4$	
$\{3, 5, 7, 10\}$	S_4	β		$(q+1)(q-1)^3$	
$4A_1$	s_1: $2\leftrightarrow 13$	$\beta\xi$		$(q^2-1)^2$	
$\{2, 9, 13, 14\}$	$3\leftrightarrow 5$	$\beta\xi\pi$		$(q-1)(q+1)^3$	
	$7\to 7$	$\beta\xi\pi\kappa$		$(q+1)^4$	
	$9\to 9$	s_1	$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2(q^2-1)$	
	$10\to 10$	βs_1		$(q-1)^2(q^2+1)$	
	$14\to 14$	πs_1		$(q^2-1)^2$	
	s_2: $2\to 2$	$\beta\pi s_1$		q^4-1	
	$3\to 3$	$\pi\kappa s_1$		$(q+1)^2(q^2-1)$	
	$5\to 5$				
	$7\leftrightarrow 10$	$\beta\pi\kappa s_1$		$(q+1)^2(q^2+1)$	
	$9\leftrightarrow 14$	$s_2 s_1$	$\lvert A_1(q^2)\rvert^2$	$(q^2-1)^2$	
	$13\to 13$	$\beta s_2 s_1$		q^4-1	
	s_3: $2\to 2$	$\beta\pi s_2 s_1$		$(q^2+1)^2$	

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
	$3\to 3$	$s_3 s_2$	$\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^3-1)$	
	$5\leftrightarrow 7$	$\beta s_3 s_1$		$(q-1)(q^3+1)$	
	$9\to 9$	$\kappa s_3 s_1$		$(q+1)(q^3-1)$	
	$10\to 10$	$\beta\kappa s_3 s_1$		$(q+1)(q^3+1)$	
	$13\leftrightarrow 14$	$s_2 s_3 s_1$	$\lvert A_1(q^4)\rvert$	q^4-1	
		$\beta s_2 s_3 s_1$		q^4+1	
$[4A_1]''$	H_2	1	$\lvert A_1(q)\rvert^4$	$(q-1)^4$	$2\mid q-1$
$\{2, 5, 7, 10\}$	S_4	α		$(q-1)^2(q^2-1)$	$2\mid q-1$
D_4	s_1: $1\to 1$	$\tau\varphi$		$(q^2-1)^2$	$2\mid q-1$
$\{1, 3, 16,$	$2\leftrightarrow 5$	$\tau\gamma$		$(q-1)(q^3-1)$	$2\mid q-1$
$-17\}$	$3\to 3$	$\tau\varphi\gamma$		q^4-1	$2\mid q-1$
	$7\to 7$	$\alpha\tau\varphi\omega$		$(q+1)^4$	$2\mid q-1$
	$10\to 10$	$\tau\alpha\varphi$		$(q+1)^2(q^2-1)$	$2\mid q-1$
	$16\leftrightarrow -17$	$\gamma\varphi\tau\alpha$		$(q+1)(q^3+1)$	$2\mid q-1$
	s_2: $1\to 16\to$	$\tau\varphi\gamma\kappa$		$(q^2+1)^2$	$2\mid q-1$
	$-17\to 1$	s_1	$\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert^2$	$(q-1)^2(q^2-1)$	$2\mid q-1$
	$2\to 7\to 5\to 2$	τs_1		$(q-1)^2(q^2+1)$	$2\mid q-1$
	$10\to 10$	ξs_1		$(q^2-1)^2$	$2\mid q-1$
	s_3: $1\to 1$	$\gamma\xi s_1$		$(q+1)^2(q^2-1)$	$2\mid q-1$
	$2\leftrightarrow 5$	$\gamma\tau s_1$		$(q-1)(q^3+1)$	$2\mid q-1$
	$3\to 3$	$\alpha\gamma s_1$		$(q+1)(q^3-1)$	$2\mid q-1$
	$7\leftrightarrow 10$	$\alpha\tau s_1$		q^4-1	$2\mid q-1$
	$16\to 16$	$\alpha\kappa\varphi s_1$		q^4+1	$2\mid q-1$
	$17\to -17$	$\alpha\omega\varphi s_1$		$(q+1)^2(q^2+1)$	$2\mid q-1$
		s_2	$\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^3-1)$	$2\mid q-1$
		αs_2		$(q-1)(q^3+1)$	$2\mid q-1$
		$\alpha\omega s_2$		$(q+1)(q^3-1)$	$2\mid q-1$
		$\alpha\tau\omega\varphi s_2$		$(q+1)(q^3+1)$	$2\mid q-1$
		$\gamma\tau s_2$		(q^4-q^2+1)	$2\mid q-1$

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Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
		$\xi\varphi s_2$		$(q^2-q+1)^2$	$2\mid q-1$
		$\pi\omega s_2$		$(q^2+q+1)^2$	$2\mid q-1$
		s_3	$\lvert A_1(q^2)\rvert^2$	$(q-1)^4$	$2\mid q-1$
		αs_3		$(q-1)^2(q^2-1)$	$2\mid q-1$
		$\tau\varphi s_3$		$(q^2-1)^2$	$2\mid q-1$
		$\alpha\varphi s_3$		$(q^2-1)^2$	$2\mid q-1$
		$\tau\Upsilon s_3$		$(q-1)(q^3-1)$	$2\mid q-1$
		$\tau\varphi\Upsilon s_3$		q^4-1	$2\mid q-1$
		$\alpha\Upsilon\pi s_3$		q^4-1	$2\mid q-1$
		$\alpha\tau\varphi\omega s_3$		$(q+1)^4$	$2\mid q-1$
		$\tau\alpha\varphi s_3$		$(q+1)^2(q^2-1)$	$2\mid q-1$
		$\Upsilon\varphi\tau\alpha s_3$		$(q+1)(q^3+1)$	$2\mid q-1$
		$\tau\varphi\Upsilon\kappa s_3$		$(q^2+1)^2$	$2\mid q-1$
		$s_2s_1s_3$	$\lvert A_1(q^4)\rvert$	$(q-1)^2(q^2-1)$	$2\mid q-1$
		$\tau s_2s_1s_3$		$(q-1)^2(q^2+1)$	$2\mid q-1$
		$\kappa s_2s_1s_3$		$(q^2-1)^2$	$2\mid q-1$
		$\Upsilon\kappa s_2s_1s_3$		$(q+1)^2(q^2-1)$	$2\mid q-1$
		$\Upsilon\tau s_2s_1s_3$		$(q-1)(q^3+1)$	$2\mid q-1$
		$\varphi\Upsilon s_2s_1s_3$		$(q+1)(q^3-1)$	$2\mid q-1$
		$\varphi\tau s_2s_1s_3$		q^4-1	$2\mid q-1$
		$\varphi\xi\alpha s_2s_1s_3$		q^4+1	$2\mid q-1$
		$\varphi\Upsilon\alpha s_2s_1s_3$		$(q+1)^2(q^2+1)$	$2\mid q-1$
A_2+2A_1	$S_4\times(\boldsymbol{Z}_2)^2$	1	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^4$	
$\{2, 3, 5, 6\}$	$(\boldsymbol{Z}_2)^2$	λ		$(q-1)^2(q^2-1)$	
A_3	$s_1: 2\leftrightarrow 3$	$\lambda\kappa$		$(q^2-1)^2$	
$\{8, -9, 10\}$	$4\to 4$	$\lambda\vartheta$		$(q-1)(q^3-1)$	
	$5\to 5$	$\lambda\vartheta\kappa$		q^4-1	
	$6\to 6$	s_1	$\lvert A_2(q)\rvert\lvert A_1(q)^2\rvert$	$(q+1)^2(q^2-1)$	
	$8\to -8$	λs_1		$(q^2-1)^2$	
	$9\to -9$	$\lambda\kappa s_1$		$(q-1)^2(q^2-1)$	

Δ_J	Ω_J	$[w]$	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
	$10\to -10$	$\lambda\vartheta s_1$		$(q-1)(q^3+1)$	
	$s_2: 2\to 2$	$\lambda\vartheta\kappa s_1$		$(q-1)^2(q^2+1)$	
	$3\to 3$	s_2	$\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^4$	
	$4\to -(\varepsilon_2+\varepsilon_5)$	λs_2		$(q+1)^2(q^2-1)$	
	$5\leftrightarrow 6$	$\lambda\kappa s_2$		$(q^2-1)^2$	
	$8\to -8$	$\lambda\vartheta s_2$		$(q+1)(q^3+1)$	
	$9\to -9$	$\lambda\vartheta\kappa s_2$		q^4-1	
	$10\to -10$	s_1s_2	$\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q^2)\rvert$	$(q-1)^2(q^2-1)$	
		λs_1s_2		$(q^2-1)^2$	
		$\lambda\kappa s_1s_2$		$(q+1)^2(q^2-1)$	
		$\lambda\vartheta s_1s_2$		$(q+1)(q^3-1)$	
		$\lambda\vartheta\kappa s_1s_2$		$(q+1)^2(q^2+1)$	
$2A_2$	$(S_3\times\boldsymbol{Z}_2)\wr\boldsymbol{Z}_2$	1	$\lvert A_2(q)\rvert^2$	$(q-1)^4$	
$\{5, 6, 8, 10\}$	$W(B_2)$	β		$(q-1)^2(q^2-1)$	
$2A_2$	$s_1: 1\to -1$	$\mu\beta$		$(q-1)(q^3-1)$	
$\{1, 2, 3, 11\}$	$2\to 2$	$\beta\alpha$		$(q^2-1)^2$	
	$3\to -3$	$\mu\beta\alpha$		$(q+1)(q^3-1)$	
	$5\leftrightarrow 10$	$\mu\beta\Upsilon\alpha$		$(q^2+q+1)^2$	
	$6\leftrightarrow 8$	s_1	$\lvert A_2(q^2)\rvert$	$(q^2-1)^2$	
	$11\to 11$	βs_1		$(q+1)^2(q^2-1)$	
		$\mu\beta s_1$		$(q+1)^2(q^2+q+1)$	
		αs_1		$(q-1)^2(q^2-1)$	
		$\Upsilon\alpha s_1$		$(q-1)^2(q^2-q+1)$	
		$\beta\alpha s_1$		$(q^2-1)^2$	
		$\beta\mu\alpha s_1$		$(q+1)(q^3-1)$	
		$\beta\Upsilon\alpha s_1$		$(q-1)(q^3+1)$	
		$\beta\mu\Upsilon\alpha s_1$		q^4+q+1	
		s_2^2	$\lvert {}^2A_2(q^2)\rvert^2$	$(q+1)^4$	
		βs_2^2		$(q+1)^2(q^2-1)$	

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Δ_J	Ω_J	$[w]$	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrence
		$\beta\mu s_2^2$		$(q+1)(q^3+1)$	
		$\beta\alpha s_2^2$		$(q^2-1)^2$	
		$\mu\beta\alpha s_2^2$		$(q-1)(q^3+1)$	
		$\mu\beta\gamma\alpha s_2^2$		$(q^2-q+1)^2$	
		s_1s_2	$\lvert{}^2A_2(q^2)\rvert\lvert A_2(q)\rvert$	$(q^2-1)^2$	
		βs_1s_2		q^4-1	
		$\beta\mu s_1s_2$		q^4+q^2+1	
		$\beta\mu\alpha s_1s_2$		q^4-1	
		s_2	$\lvert{}^2A_2(q^4)\rvert$	$(q^2+1)^2$	
		βs_2		q^4-1	
		$\beta\mu s_2$		q^4-q^2+1	
		$\beta\mu\alpha s_2$		q^4-1	
A_3+A_1	$S_4\times(\boldsymbol{Z}_2)^2$	1	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^4$	
$\{2, 5, 6, 7\}$	$\boldsymbol{Z}_2$	α		$(q-1)^2(q^2-1)$	
A_3+A_1	s: $1\to1$	$\gamma\kappa$		$(q^2-1)^2$	
$\{1, 3, 9, 10\}$	$2\to2$	$\gamma\alpha$		$(q-1)(q^3-1)$	
	$3\to-3$	$\kappa\gamma\alpha$		q^4-1	
	$5\leftrightarrow7$	λ		$(q-1)^2(q^2-1)$	
	$6\to6$	$\alpha\lambda$		$(q^2-1)^2$	
	$9\to-9$	$\gamma\kappa\lambda$		$(q+1)^2(q^2-1)$	
	$10\to10$	$\gamma\alpha\lambda$		$(q+1)(q^3-1)$	
		$\kappa\gamma\alpha\lambda$		$(q+1)^2(q^2+1)$	
		s	$\lvert{}^2A_3(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2(q^2-1)$	
		αs		$(q^2-1)^2$	
		$\gamma\kappa s$		$(q-1)^2(q^2-1)$	
		$\gamma\alpha s$		$(q-1)(q^3+1)$	
		$\kappa\gamma s$		$(q-1)$ $\times(q^3-q^2+q-1)$	
		λs		$(q+1)^4$	
		$\alpha\lambda s$		$(q+1)^2(q^2-1)$	
		$\gamma\kappa\lambda s$		$(q^2-1)^2$	
		$\gamma\alpha\lambda s$		$(q+1)(q^3+1)$	
		$\kappa\gamma\alpha\lambda s$		q^4-1	
A_4	$S_5\times\boldsymbol{Z}_2$	1	$\lvert A_4(q)\rvert$	$(q-1)^4$	
$\{6, 7, 8, 10\}$	$\boldsymbol{Z}_2$	α		$(q^2-1)(q-1)^2$	
A_4	s: $1\to-1$	$\alpha\delta$		$(q^2-1)^2$	
$\{1, 2, 3, 4\}$	$2\to-2$	$\alpha\gamma$		$(q-1)(q^3-1)$	
	$3\to-3$	$\alpha\gamma\beta$		$(q+1)(q^3-1)$	
	$4\to-4$	$\alpha\gamma\delta$		q^4-1	
	$6\leftrightarrow10$	$\alpha\delta\beta\gamma$		$q^4+q^3+q^2$ $+q+1$	
	$7\leftrightarrow8$	s	$\lvert{}^2A_4(q^2)\rvert$	$(q+1)^4$	
		αs		$(q+1)^2(q^2-1)$	
		$\alpha\delta s$		$(q^2-1)^2$	
		$\alpha\gamma s$		$(q+1)(q^3+1)$	
		$\alpha\gamma\beta s$		$(q-1)(q^3+1)$	
		$\alpha\gamma\delta s$		q^4-1	
		$\alpha\delta\beta\gamma s$		$q^4-q^3+q^2$ $-q+1$	
D_4	$W(F_4)$	1	$\lvert D_4(q)\rvert$	$(q-1)^4$	
$\{2, 3, 4, 5\}$	S_3	η		$(q-1)^2(q^2-1)$	
D_4	s_1: $2\to2$	$\eta\kappa$		$(q^2-1)^2$	
$\{7, 8, -9,$	$3\leftrightarrow5$	$\eta\vartheta$		$(q-1)(q^3-1)$	
$10\}$	$4\to4$	$\eta\kappa\vartheta$		q^4-1	
	$7\leftrightarrow-9$	$\eta\pi\kappa\lambda$		$(q+1)^4$	
	$8\leftrightarrow8$	$\eta\pi\kappa$		$(q+1)^2(q^2-1)$	
	$10\to10$	$\eta\pi\kappa\vartheta$		$(q+1)(q^3+1)$	
	s_2: $2\to3\to5$	$\eta\kappa\vartheta x$		$(q^2+1)^2$	
	$\to2$	s_1	$\lvert{}^2D_4(q^2)\rvert$	$(q^2-1)(q-1)^2$	
	$4\to4$	ηs_1		$(q^2+1)(q-1)^2$	

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Δ_J	Ω_J	$[w]$	$\lvert(M^{\sigma})_{\sigma}\rvert$	$\lvert(S^{\sigma})_{\sigma}\rvert$	Condition for occurrence
	$7\to-9\to$	ys_1		$(q^2-1)^2$	
	$-14\to7$	ϑys_1		$(q^2-1)(q+1)^2$	
	$8\to8$	$\vartheta\eta s_1$		$(q-1)(q^3+1)$	
	$10\to10$	$\pi\vartheta s_1$		$(q+1)(q^3-1)$	
		$\pi\eta s_1$		q^4-1	
		$\pi x\kappa s_1$		q^4+1	
		$\pi\eta\kappa s_1$		$(q^2+1)(q+1)^2$	
		s_2	$\lvert{}^3D_4(q^3)\rvert$	$(q-1)(q^3-1)$	
		πs_2		$(q-1)(q^3+1)$	
		$\pi\lambda s_2$		$(q+1)(q^3-1)$	
		$\pi\eta\lambda\kappa s_2$		$(q+1)(q^3+1)$	
		$\vartheta\eta s_2$		q^4-q^2+1	
		$y\kappa s_2$		$(q^2-q+1)^2$	
		$z\lambda s_2$		$(q^2+q+1)^2$	
$5A_1$	H_1	1	$\lvert A_1(q)\rvert^5$	$(q-1)^3$	$2\mid q-1$
$\{2,3,5,7,$	S_4	ξ		$(q-1)(q^2-1)$	$2\mid q-1$
$10\}$	s_1: $2\to2$	$\xi\pi$		$(q+1)(q^2-1)$	$2\mid q-1$
$3A_1$	$3\to3$	$\kappa\xi\pi$		$(q+1)^3$	$2\mid q-1$
$\{9,13,14\}$	$5\to5$	s_1	$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^3$	$(q-1)(q^2-1)$	$2\mid q-1$
	$7\leftrightarrow10$	ξs_1		$+1)(q^2-(q1)$	$2\mid q-1$
	$9\leftrightarrow14$	πs_1		$(q-1)(q^2+1)$	$2\mid q-1$
	$13\to13$	$\xi\pi s_1$		$(q+1)(q^2+1)$	$2\mid q-1$
	s_2: $2\to2$	s_1s_2	$\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert^2$	q^3-1	$2\mid q-1$
	$3\to3$	ξs_1s_2		q^3+1	$2\mid q-1$
	$5\leftrightarrow7$	$s_1s_2s_3$	$\lvert A_1(q^4)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	$2\mid q-1$
	$9\to9$	$\xi s_1s_2s_3$		$(q-1)(q^2+1)$	$2\mid q-1$
	$10\to10$	$\pi s_1s_2s_3$		$(q+1)(q^2-1)$	$2\mid q-1$
	$13\leftrightarrow14$	$\xi\pi s_1s_2s_3$		$(q+1)(q^2+1)$	$2\mid q-1$
	s_3: $2\leftrightarrow5$	s_1s_3	$\lvert A_1(q^2)\rvert^2\lvert A_1(q)\rvert$	$(q-1)^3$	$2\mid q-1$
	$3\to3$	ξs_1s_3		$(q-1)(q^2-1)$	$2\mid q-1$

Δ_J	Ω_J	$[w]$	$\lvert(M^{\sigma})_{\sigma}\rvert$	$\lvert(S^{\sigma})_{\sigma}\rvert$	Condition for occurrence
	$7\to7$	πs_1s_3		$(q-1)(q^2-1)$	$2\mid q-1$
	$9\leftrightarrow14$	$\pi\kappa s_1s_3$		$(q+1)(q^2-1)$	$2\mid q-1$
	$10\to10$	$\xi\pi s_1s_3$		$(q+1)(q^2-1)$	$2\mid q-1$
	$13\to13$	$\xi\pi\kappa s_1s_3$		$(q+1)^3$	$2\mid q-1$
A_2+3A_1	$S_3\times(\mathbf{Z}_2)^2$		$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^3$	$(q-1)^3$	
				$(q-1)(q^2-1)$	
			$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
				$(q+1)(q^2-1)$	
			$\lvert A_2(q)\rvert\lvert A_1(q^3)\rvert$	q^3-1	
				$(q+1)(q^2+q+1)$	
			$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^3$	$(q+1)(q^2-1)$	
				$(q+1)^3$	
			$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
				$(q+1)(q^2-1)$	
			$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^3)\rvert$	q^3+1	
				$(q-1)(q^2-q+1)$	
$2A_2+A_1$	$S_3\times(\mathbf{Z}_2)^2$		$\lvert A_2(q)\rvert^2\lvert A_1(q)\rvert$	$(q-1)^3$	
				$(q-1)(q^2-1)$	
				q^3-1	
			$\lvert A_2(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
				$(q+1)(q^2-1)$	
				$(q-1)(q^2-q+1)$	
			$\lvert{}^2A_2(q^2)\rvert^2\lvert A_1(q)\rvert$	$(q+1)^3$	
				$(q+1)(q^2-1)$	
				q^3+1	
			$\lvert A_2(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
				$(q+1)(q^2-1)$	
				$(q+1)(q^2+q+1)$	

(Continued)

Δ_J	Ω_J	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrence
$[A_3+2A_1]'$	$W(B_2)\times \boldsymbol{Z}_2$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)^3$	
		$\lvert A_3(q)\rvert\lvert A_1(q^2)\rvert$	$(q+1)(q^2-1)$	
			$(q-1)(q^2+1)$	
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q+1)(q^2-1)$	
			$(q+1)(q^2+1)$	
$[A_3+2A_1]''$	$S_4\times(\boldsymbol{Z}_2)^2$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^3$	$2\mid q-1$
			$(q-1)(q^2-1)$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$
			q^3-1	$2\mid q-1$
			$(q+1)(q^2+1)$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^3$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$
			$(q-1)(q^2-1)$	$2\mid q-1$
			q^3+1	$2\mid q-1$
			$(q-1)(q^2+1)$	$2\mid q-1$
		$\lvert A_3(q)\rvert\lvert A_1(q^2)\rvert$	$(q+1)^3$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$
			$(q-1)(q^2-1)$	$2\mid q-1$
			q^3+1	$2\mid q-1$
			$(q-1)(q^2+1)$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q-1)^3$	$2\mid q-1$
			$(q-1)(q^2-1)$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$
			q^3-1	$2\mid q-1$
			$(q+1)(q^2+1)$	$2\mid q-1$

Δ_J	Ω_J	$\lvert(M^q)_\sigma\rvert$	$\lvert(S^q)_\sigma\rvert$	Condition for occurrence
A_3+A_2	$W(B_2)\times \boldsymbol{Z}_2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
		$\lvert {}^2A_3(q^2)\rvert\lvert A_2(q)\rvert$	$(q-1)(q^2-1)$	
			$(q-1)(q^2+1)$	
		$\lvert A_3(q)\rvert\lvert {}^2A_2(q^2)\rvert$	$(q+1)(q^2-1)$	
			$(q+1)(q^2+1)$	
		$\lvert {}^2A_3(q^2)\rvert \times\lvert {}^2A_2(q^2)\rvert$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)^3$	
A_4+A_1	$S_3\times \boldsymbol{Z}_2$	$\lvert A_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q)\rvert$	q^3-1	
			$(q+1)^3$	
			$(q+1)(q^2-1)$	
			q^3+1	
A_5	$S_3\times(\boldsymbol{Z}_2)^2$	$\lvert A_5(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			q^3-1	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)(q^2+q+1)$	
		$\lvert {}^2A_5(q^2)\rvert$	$(q+1)(q^2-1)$	
			$(q-1)(q^2-1)$	
			$(q-1)(q^2-q+1)$	
			$(q+1)^3$	
			$(q+1)(q^2-1)$	
			q^3+1	
D_4+A_1	$W(B_3)$	$\lvert D_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$	

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Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)^3$	
		$\lvert{}^2D_4(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
			$(q-1)(q^2+1)$	
			$(q+1)(q^2-1)$	
			$(q+1)(q^2+1)$	
		$\lvert{}^3D_4(q^3)\rvert\lvert A_1(q)\rvert$	q^3-1	
			q^3+1	
D_5	$S_4\times Z_2$	$\lvert D_5(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			q^3-1	
			$(q+1)(q^2+1)$	
		$\lvert{}^2D_5(q^2)\rvert$	$(q+1)^3$	
			$(q+1)(q^2-1)$	
			$(q-1)(q^2-1)$	
			q^3+1	
			$(q-1)(q^2+1)$	
A_2+4A_1	$S_4\times Z_2$	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^4$	$(q-1)^2$	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert^2$	q^2-1	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^3)\rvert \times\lvert A_1(q)\rvert$	q^2+q+1	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert^2$	$(q-1)^2$	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^4)\rvert$	q^2-1	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^4$	$(q+1)^2$	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert^2$	q^2-1	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^3)\rvert \times\lvert A_1(q)\rvert$	q^2-q+1	$2\mid q-1$

Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert^2$	$(q+1)^2$	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^4)\rvert$	q^2+q+1	$2\mid q-1$
$2A_2+2A_1$	$W(B_2)$	$\lvert A_2(q)\rvert^2\lvert A_1(q)\rvert^2$	$(q-1)^2$	
		$\lvert A_2(q)\rvert\lvert{}^2A_2(q^2)\rvert \times\lvert A_1(q^2)\rvert$	q^2-1	
		$\lvert A_2(q^2)\rvert\lvert A_1(q)\rvert^2$	q^2-1	
		$\lvert{}^2A_2(q^4)\rvert\lvert A_1(q^2)\rvert$	q^2+1	
		$\lvert{}^2A_2(q^2)\rvert^2\lvert A_1(q)\rvert^2$	$(q+1)^2$	
$3A_2$	$(S_3)^2\times Z_2$	$\lvert A_2(q)\rvert^3$	$(q-1)^2$	$3\mid q-1$
			q^2-1	$3\mid q-1$
			q^2+q+1	$3\mid q-1$
		$\lvert A_2(q^2)\rvert\lvert{}^2A_2(q^2)\rvert$	$(q-1)^2$	$3\mid q+1$
			q^2-1	$3\mid q+1$
			q^2+q+1	$3\mid q+1$
		$\lvert{}^2A_2(q^2)\rvert^3$	$(q+1)^2$	$3\mid q+1$
			q^2-1	$3\mid q+1$
			q^2-q+1	$3\mid q+1$
		$\lvert A_2(q^2)\rvert\lvert A_2(q)\rvert$	$(q+1)^2$	$3\mid q-1$
			q^2-1	$3\mid q-1$
			q^2-q+1	$3\mid q-1$
		$\lvert A_2(q^3)\rvert$	$(q-1)^2$	$3\mid q-1$
			q^2-1	$3\mid q-1$
			q^2+q+1	$3\mid q-1$
		$\lvert{}^2A_2(q^6)\rvert$	$(q+1)^2$	$3\mid q+1$
			q^2-1	$3\mid q+1$
			q^2-q+1	$3\mid q+1$
A_3+3A_1	$(Z_2)^3$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^3$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
		$\lvert{}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^3$	$(q+1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$

(Continued)

Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
		$\lvert A_3(q)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert$	$(q+1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q^2)\rvert \times\lvert A_1(q)\rvert^2$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
$A_3+A_2+A_1$	$(\boldsymbol{Z}_2)^2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert \times\lvert A_1(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_3(q^2)\rvert\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q)\rvert$	$(q+1)^2$	
			q^2-1	
$[2A_3]'$	$W(B_2)$	$\lvert A_3(q)\rvert^2$	$(q-1)^2$	
		$\lvert {}^2A_3(q^2)\rvert\lvert A_3(q)\rvert$	q^2-1	
		$\lvert A_3(q^2)\rvert$	q^2-1	
		$\lvert {}^2A_3(q^2)\rvert^2$	q^2-1	
		$\lvert {}^2A_3(q^4)\rvert$	$(q+1)^2$	
$[2A_3]''$	$(\boldsymbol{Z}_2)^2 \wr \boldsymbol{Z}_2$	$\lvert A_3(q)\rvert^2$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			$(q+1)^2$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_3(q)\rvert$	q^2-1	$2\mid q-1$
			q^2+1	$2\mid q-1$
		$\lvert A_3(q^2)\rvert$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			q^2-1	$2\mid q-1$
			$(q+1)^2$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert^2$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			$(q+1)^2$	$2\mid q-1$
		$\lvert {}^2A_3(q^4)\rvert$	q^2-1	$2\mid q-1$
			q^2+1	$2\mid q-1$

Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	Condition for occurrence
A_4+2A_1	$(\boldsymbol{Z}_2)^2$	$\lvert A_4(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$	
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q^2)\rvert$	q^2-1	
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2$	
		$\lvert A_4(q)\rvert\lvert A_1(q^2)\rvert$	q^2-1	
A_4+A_2	$(\boldsymbol{Z}_2)^2$	$\lvert A_4(q)\rvert\lvert A_2(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_4(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	$(q+1)^2$	
			q^2-1	
$[A_5+A_1]'$	$(\boldsymbol{Z}_2)^2$	$\lvert A_5(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_5(q^2)\rvert\lvert A_1(q)\rvert$	q^2-1	
			$(q+1)^2$	
$[A_5+A_1]''$	$S_3\times\boldsymbol{Z}_2$	$\lvert A_5(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			q^2+q+1	$2\mid q-1$
		$\lvert {}^2A_5(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			q^2-q+1	$2\mid q-1$
A_6	$(\boldsymbol{Z}_2)^2$	$\lvert A_6(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_6(q^2)\rvert$	$(q+1)^2$	
			q^2-1	
D_4+2A_1	$W(B_2)\times\boldsymbol{Z}_2$	$\lvert D_4(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			q^2+q+1	$2\mid q-1$
		$\lvert D_4(q)\rvert\lvert A_1(q^2)\rvert$	q^2-1	$2\mid q-1$
			q^2+1	$2\mid q-1$
		$\lvert {}^2D_4(q^2)\rvert\lvert A_1(q)\rvert^2$	q^2-1	$2\mid q-1$
			q^2+1	$2\mid q-1$
		$\lvert {}^2D_4(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q+1)^2$	$2\mid q-1$

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Δ_J	Ω_J	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence
			q^2-1	$2\mid q-1$
			$(q-1)^2$	$2\mid q-1$
D_4+A_2	$S_3\times Z_2$	$\lvert D_4(q)\rvert\lvert A_2(q)\rvert$	$(q-1)^2$	
		$\lvert {}^2D_4(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	q^2-1	
		$\lvert {}^3D_4(q^3)\rvert\lvert A_2(q)\rvert$	q^2+q+1	
		$\lvert D_4(q)\rvert\lvert {}^2A_2(q^2)\rvert$	$(q+1)^2$	
		$\lvert {}^2D_4(q^2)\rvert\lvert A_2(q)\rvert$	q^2-1	
		$\lvert {}^3D_4(q^3)\rvert\lvert {}^2A_2(q^2)\rvert$	q^2-q+1	
D_5+A_1	$(Z_2)^2$	$\lvert D_5(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2D_5(q^2)\rvert\lvert A_1(q)\rvert$	q^2-1	
			$(q+1)^2$	
D_6	$W(B_2)$	$\lvert D_6(q)\rvert$	$(q-1)^2$	
			q^2-1	
			$(q+1)^2$	
		$\lvert {}^2D_6(q^2)\rvert$	q^2-1	
			q^2+1	
E_6	$S_3\times Z_2$	$\lvert E_6(q)\rvert$	$(q-1)^2$	
			q^2-1	
			q^2+q+1	
		$\lvert {}^2E_6(q^2)\rvert$	$(q+1)^2$	
			q^2-1	
			q^2-q+1	
$3A_2+A_1$	$S_3\times Z_2$	$\lvert A_2(q)\rvert^3\lvert A_1(q)\rvert$	$q-1$	$3\mid q-1$
		$\lvert A_2(q^2)\rvert\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q)\rvert$	$q-1$	$3\mid q+1$
		$\lvert {}^2A_2(q^2)\rvert^3\lvert A_1(q)\rvert$	$q+1$	$3\mid q+1$
		$\lvert A_2(q^2)\rvert\lvert A_2(q)\rvert \times\lvert A_1(q)\rvert$	$q+1$	$3\mid q-1$
		$\lvert A_2(q^3)\rvert\lvert A_1(q)\rvert$	$q-1$	$3\mid q-1$
		$\lvert {}^2A_2(q^6)\rvert\lvert A_1(q)\rvert$	$q+1$	$3\mid q+1$

Δ_J	Ω_J	$\lvert(M^\sigma)_\sigma\rvert$	$\lvert(S^\sigma)_\sigma\rvert$	Condition for occurrence
A_3+A_2 $+2A_1$	$(Z_2)^2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert \times\lvert A_1(q)\rvert^2$	$q-1$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q)\rvert^2$	$q+1$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_2(q)\rvert \times\lvert A_1(q^2)\rvert$	$q-1$	$2\mid q-1$
		$\lvert A_3(q)\rvert\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q^2)\rvert$	$q+1$	$2\mid q-1$
$2A_3+A_1$	$(Z_2)^3$	$\lvert A_3(q)\rvert^2\lvert A_1(q)\rvert$	$q-1$	$4\mid q-1$
			$q+1$	$4\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert^2\lvert A_1(q)\rvert$	$q-1$	$4\mid q+1$
			$q+1$	$4\mid q+1$
		$\lvert A_3(q^2)\rvert\lvert A_1(q)\rvert$	$q-1$	$4\mid q+1$
			$q+1$	$4\mid q+1$
			$q-1$	$4\mid q-1$
			$q+1$	$4\mid q-1$
A_4+A_2 $+A_1$	Z_2	$\lvert A_4(q)\rvert\lvert A_2(q)\rvert \times\lvert A_1(q)\rvert$	$q-1$	
		$\lvert {}^2A_4(q^2)\rvert\lvert {}^2A_2(q^2)\rvert \times\lvert A_1(q)\rvert$	$q+1$	
A_4+A_3	Z_2	$\lvert A_4(q)\rvert\lvert A_3(q)\rvert$	$q-1$	
		$\lvert {}^2A_4(q^2)\rvert\lvert {}^2A_3(q^2)\rvert$	$q+1$	
A_5+2A_1	Z_2	$\lvert A_5(q)\rvert\lvert A_1(q)\rvert^2$	$q-1$	$2\mid q-1$
		$\lvert {}^2A_5(q^2)\rvert\lvert A_1(q)\rvert^2$	$q+1$	$2\mid q-1$
A_5+A_2	$(Z_2)^2$	$\lvert A_5(q)\rvert\lvert A_2(q)\rvert$	$q-1$	$3\mid q-1$
			$q+1$	$3\mid q-1$
		$\lvert {}^2A_5(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	$q-1$	$3\mid q+1$
			$q+1$	$3\mid q+1$
A_6+A_1	Z_2	$\lvert A_6(q)\rvert\lvert A_1(q)\rvert$	$q-1$	
		$\lvert {}^2A_6(q^2)\rvert\lvert A_1(q)\rvert$	$q+1$	
$[A_7]'$	Z_2	$\lvert A_7(q)\rvert$	$q-1$	
		$\lvert {}^2A_7(q^2)\rvert$	$q+1$	

(Continued)

Δ_J	Ω_J	$\lvert (M^\sigma)_\sigma \rvert$	$\lvert (S^\sigma)_\sigma \rvert$	Condition for occurrence
$[A_7]''$	$(Z_2)^2$	$\lvert A_7(q) \rvert$	$q-1$	$2 \mid q-1$
			$q+1$	$2 \mid q-1$
		$\lvert {}^2A_7(q^2) \rvert$	$q-1$	$2 \mid q-1$
			$q+1$	$2 \mid q-1$
D_4+A_3	$(Z_2)^2$	$\lvert D_4(q) \rvert \lvert A_3(q) \rvert$	$q-1$	$2 \mid q-1$
		$\lvert {}^2D_4(q^2) \rvert \lvert {}^2A_3(q^2) \rvert$	$q-1$	$2 \mid q-1$
		$\lvert {}^2D_4(q^2) \rvert \lvert A_3(q) \rvert$	$q+1$	$2 \mid q-1$
		$\lvert D_4(q) \rvert \lvert {}^2A_3(q^2) \rvert$	$q+1$	$2 \mid q-1$
D_5+2A_1	$(Z_2)^2$	$\lvert D_5(q) \rvert \lvert A_1(q) \rvert^2$	$q-1$	$2 \mid q-1$
		$\lvert {}^2D_5(q^2) \rvert \lvert A_1(q^2) \rvert$	$q-1$	$2 \mid q-1$
		$\lvert {}^2D_5(q^2) \rvert \lvert A_1(q) \rvert^2$	$q+1$	$2 \mid q-1$
		$\lvert D_5(q) \rvert \lvert A_1(q^2) \rvert$	$q+1$	$2 \mid q-1$
D_5+A_2	Z_2	$\lvert D_5(q) \rvert \lvert A_2(q) \rvert$	$q-1$	
		$\lvert {}^2D_5(q^2) \rvert \lvert {}^2A_2(q^2) \rvert$	$q+1$	
D_6+A_1	Z_2	$\lvert D_6(q) \rvert \lvert A_1(q) \rvert$	$q-1$	$2 \mid q-1$
			$q+1$	$2 \mid q-1$
E_6+A_1	Z_2	$\lvert E_6(q) \rvert \lvert A_1(q) \rvert$	$q-1$	
		$\lvert {}^2E_6(q^2) \rvert \lvert A_1(q) \rvert$	$q+1$	
D_7	Z_2	$\lvert D_7(q) \rvert$	$q-1$	
		$\lvert {}^2D_7(q^2) \rvert$	$q+1$	

Δ_J	Ω_J	$\lvert (M^\sigma)_\sigma \rvert$	$\lvert (S^\sigma)_\sigma \rvert$	Condition for occurrence
E_7	Z_2	$\lvert E_7(q) \rvert$	$q-1$	
			$q+1$	
$2A_4$	Z_4	$\lvert A_4(q) \rvert^2$	1	$5 \mid q-1$
		$\lvert {}^2A_4(q^2) \rvert^2$	1	$5 \mid q-4$
		$\lvert {}^2A_4(q^4) \rvert$	1	$5 \mid q-3$
			1	$5 \mid q-2$
$A_5+A_2+A_1$	Z_2	$\lvert A_5(q) \rvert \lvert A_2(q) \rvert \times \lvert A_1(q) \rvert$	1	$6 \mid q-1$
		$\lvert {}^2A_5(q^2) \rvert \lvert {}^2A_2(q^2) \rvert \times \lvert A_1(q) \rvert$	1	$6 \mid q+1$
A_7+A_1	Z_2	$\lvert A_7(q) \rvert \lvert A_1(q) \rvert$	1	$4 \mid q-1$
		$\lvert {}^2A_7(q^2) \rvert \lvert A_1(q) \rvert$	1	$4 \mid q+1$
A_8	Z_2	$\lvert A_8(q) \rvert$	1	$3 \mid q-1$
		$\lvert {}^2A_8(q^2) \rvert$	1	$3 \mid q+1$
D_5+A_3	Z_2	$\lvert D_5(q) \rvert \lvert A_3(q) \rvert$	1	$4 \mid q-1$
		$\lvert {}^2D_5(q^2) \rvert \lvert {}^2A_3(q^2) \rvert$	1	$4 \mid q+1$
D_8	1	$\lvert D_8(q) \rvert$	1	$2 \mid q-1$
E_6+A_2	Z_2	$\lvert E_6(q) \rvert \lvert A_2(q) \rvert$	1	$3 \mid q-1$
		$\lvert {}^2E_6(q^2) \rvert \lvert {}^2A_2(q^2) \rvert$	1	$3 \mid q+1$
E_7+A_1	1	$\lvert E_7(q) \rvert \lvert A_1(q) \rvert$	1	$2 \mid q-1$
E_8	1	$\lvert E_8(q) \rvert$	1	

the tori which are obtained by twisting the maximal split torus T_0 by the elements of Ω_ϕ, where here Ω_ϕ is the whole Weyl group W. The conjugacy classes of W are known [3], therefore, the reader can have a complete list of the tori $(T_w)_\sigma$, $w \in W$, and their orders from the material of [3]. Thus we have not included in our tables the cases $J=\varnothing$.

We note that from the above tables one can obtain the degrees of Deligne-Lusztig [7] representations of the groups E_7 and E_8 of adjoint type. In fact, these degrees are the p'-parts of $|G_\sigma|/|C_{G_\sigma}(x)|$, where G is a simply connected group E_7 or E_8 and $C_{G_\sigma}(x)$ are the centralizers in G_σ of semisimple elements in G_σ.

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