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Remarks on Stability for Semiproper Exceptional Leaves

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Introduction

A leaf of a codimension one foliation of a closed manifold is called *stable* if it has a saturated tubular neighborhood foliated as a product. About 1950, G. Reeb [12] (See also A. Haefliger [6].) showed that a compact leaf is stable if and only if it has a trivial holonomy group. It seems reasonable to conjecture that a proper leaf with a finitely generated fundamental group will be stable if it has a trivial holonomy group. (Note that the fundamental groups of compact leaves are always finitely generated and see also T. Inaba [11].) In fact, in 1976, T. Inaba [9], [10] extended Reeb's original theorem for proper leaves with finitely generated fundamental groups of codimension one foliations of closed three-manifolds. But this result is false if the fundamental groups of the leaves are not finitely generated. (See H. Imanishi [8].) In this paper, we extend Inaba's result for semiproper leaves and show that this extension is also false for leaves with infinitely generated fundamental groups by constructing a counterexample explicitly.

Section 1 gives basic definitions and fundamental properties of holonomy. Section 2 shows that Inaba's result is valid for semiproper leaves as well. Section 3 summarizes the result of G. Hector [7] for use in Section 4. Section 4 is devoted to constructing an example of unstable semiproper exceptional leaves without holonomy.

I would like to express my gratitude to T. Inaba for his valuable advice and hearty encouragement during the preparation of this paper.

§ 1. Introduction to the techniques.

First of all we recall some definitions and basic notions. Throughout this paper, $\mathscr{F}$ will denote a transversely orientable $C^r(0\leqq r\leqq\infty)$ codimension one foliation with C^∞ leaves of a closed manifold M and $\mathscr{L}$ will

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denote a fixed one-dimensional C^∞ foliation transverse to $\mathscr{F}$. (Such a transverse foliation $\mathscr{L}$ always exists if $r \geqq 1$, while if $r=0$ we will only treat the case in which such an $\mathscr{L}$ exists, say, the case in which every leaf of $\mathscr{F}$ is integral to a C^0 hyperplane field.)

A leaf L of $\mathscr{F}$ can be *locally dense* (i.e. $\operatorname{int} \bar{L} \neq \emptyset$), *proper* (locally closed, hence a regular submanifold of M), or *exceptional* (all other cases). An *$\mathscr{F}$-saturated* set is a subset of M which is a union of leaves of $\mathscr{F}$. The *$\mathscr{F}$-saturation* of a subset X of M is the smallest $\mathscr{F}$-saturated set containing X and is denoted by $\operatorname{sat}_{\mathscr{F}} X$. An injective immersion $f: (M, \mathscr{F}) \to (M', \mathscr{F}')$ of a foliated manifold $(M, \mathscr{F})$ into another foliated manifold $(M', \mathscr{F}')$ is *foliation-preserving* if f maps each leaf of $\mathscr{F}$ onto a leaf of $\mathscr{F}'$.

DEFINITION 1. (See T. Inaba [11].) A proper leaf L of $\mathscr{F}$ is *stable* if there exist an open $\mathscr{F}$-saturated neighborhood U of L in M and a foliation-preserving diffeomorphism

$$\varphi: (L \times]-1, 1[, \{L \times \{t\}\}_{t \in]-1,1[}) \longrightarrow (U, \mathscr{F}|U)$$

such that $\varphi(L \times \{0\}) = L$. Otherwise, L is called *unstable*.

Sides of leaves of $\mathscr{F}$ are the leaves of $q^*\mathscr{F}$, where $q: \tilde{M} \to M$ is the unit tangent bundle to $\mathscr{L}$. A side $\tilde{L}$ of a leaf $L = q(\tilde{L})$ of $\mathscr{F}$ is *proper* if a transversal $\tau: [0, 1] \to M$ starting from L in the direction $\tilde{L}$ satisfies $\tau(]0, \varepsilon[) \cap L = \emptyset$ for some $\varepsilon > 0$. A leaf of $\mathscr{F}$ is *semiproper* if it has a proper side. Note that semiproper leaves are always nowhere dense.

DEFINITION 1′. A semiproper leaf L with a proper side $\tilde{L}$ of $\mathscr{F}$ is *stable on $\tilde{L}$* if there exists a foliation-preserving injective immersion

$$\varphi: (L \times [0, 1[, \{L \times \{t\}\}_{t \in [0,1[}) \longrightarrow (M, \mathscr{F})$$

such that $\varphi(x, 0) = x$ and $d\varphi_{(x,0)}(\partial/\partial t)$ points in the direction $\tilde{L}$ for all $x \in L$. Otherwise, L is called *unstable on $\tilde{L}$*.

For all $x \in M$, we let L_x and T_x denote the leaves of $\mathscr{F}$ and $\mathscr{L}$ which contain x respectively. Let L be a leaf of $\mathscr{F}$ and $l: ([0, 1], \{0, 1\}) \to (L, x)$ a loop in L at $x \in L$. By the standard argument for the foliated structures, we can construct a *fence* F at x such that the following conditions are satisfied:

(1) $F: [0, 1] \times V \to M$ is a continuous map, where V is a neighborhood of 0 in $\boldsymbol{R}$.

(2) $F(\cdot, 0) = l: [0, 1] \longrightarrow L$.

(3) $\quad F(t, s) \in L_{F(0,s)} \cap T_{F(t,0)}$ for all $(t, s) \in [0, 1] \times V$.

(4) $\quad F(0, \cdot): V \longrightarrow M$ is a C^r embedding.

Similarly, we can define a *fence* F at x *on* a side $\tilde{L}$ of L. In this situation, we require that V is a neighborhood of 0 in $[0, \infty[$ and $dF_{(t,0)}(\partial/\partial s)$ points in the direction $\tilde{L}$ for all $t \in [0, 1]$.

For each fence F at x, the local diffeomorphism

$$\begin{array}{ccc} \gamma_F: F(\{0\} \times V) & \longrightarrow & F(\{1\} \times V) \\ \cup & & \cup \\ F(0, s) & \longmapsto & F(1, s) \end{array}$$

at x is defined. The pseudogroup of all γ_F's is called the *holonomy pseudogroup* of L at x and is denoted by $\mathscr{HP}(L, x)$. The set of germs of elements of $\mathscr{HP}(L, x)$ (called *holonomies*) forms a group called the *holonomy group* of L at x and is denoted by $\mathscr{H}(L, x)$. The isomorphism class of $\mathscr{H}(L, x)$ is independent of the choice of the base point x, therefore we will sometimes omit it. The well-defined map

$$\begin{array}{ccc} \Psi: \pi_1(L, x) & \longrightarrow & \mathscr{H}(L, x) \\ \cup & & \cup \\ [F(\cdot, 0)] & \longmapsto & \text{the germ of } \gamma_F \text{ at } x \end{array}$$

is a surjective homomorphism called the *holonomy homomorphism*, where F is a fence at x and $[F(\cdot, 0)]$ is the homotopy class represented by $F(\cdot, 0)$. Similarly, by using fences at x on a side $\tilde{L}$ of L, the *holonomy pseudogroup* $\mathscr{HP}_{\tilde{L}}(L, x)$ of L at x *on* $\tilde{L}$ and the *holonomy group* $\mathscr{H}_{\tilde{L}}(L, x)$ at x *on* $\tilde{L}$ are defined.

DEFINITION 2. (See T. Inaba [11] and R. Sacksteder and A. J. Schwartz [13].) Let L be a leaf of $\mathscr{F}$, $x \in L$, $\tilde{L}$ a side of L, and $\tau: ([0, 1], 0) \rightarrow (T_x, x)$ a transversal starting in the direction $\tilde{L}$. Then $\mathscr{HP}(L, x)$ [resp. $\mathscr{HP}_{\tilde{L}}(L, x)$] is *locally trivial* if there exists a neighborhood N_x of x in T_x [resp. $\tau([0, 1]) \cap T_x$] such that the restriction to N_x of every element of $\mathscr{HP}(L, x)$ [resp. $\mathscr{HP}_{\tilde{L}}(L, x)$] is the identity. Otherwise, $\mathscr{HP}(L, x)$ [resp. $\mathscr{HP}_{\tilde{L}}(L, x)$] is called *locally infinite*.

We let K denote the interval $I = [-1, 1]$ or the circle S^1.

DEFINITION 3. (See A. Haefliger [6, 1.8].) $(\xi; \mathscr{F}) = (p, E, B; \mathscr{F})$ is called a *foliated K-bundle* over B if E is the total space of a K-bundle ξ over B, $p: E \rightarrow B$ is the bundle projection, and $\mathscr{F}$ is a codimension one foliation of E such that each fiber of ξ is transverse to $\mathscr{F}$.

Given a manifold B with base point x and a homomorphism

$\varphi: \pi_1(B, x) \to \mathrm{Diff}^r K$, where $\mathrm{Diff}^r K$ is the group of C^r diffeomorphisms of K, $\pi_1(B, x)$ acts on the universal covering space $\tilde{B}$ of B by covering transformations. It also acts on K via φ, and on $\tilde{B} \times K$ by acting on each factor:

$$\begin{array}{ccc} \pi_1(B, x) \times (\tilde{B} \times K) & \longrightarrow & \tilde{B} \times K . \\ \cup\!\!\!\!\!\!\omega & & \cup\!\!\!\!\!\!\omega \\ (\omega, (y, t)) & \longmapsto & (y \cdot \omega, \varphi(\omega^{-1})(t)) \end{array}$$

A foliated K-bundle $(\xi; \mathscr{F}) = (p, E, B; \mathscr{F}(\varphi))$ is defined so that the total space of ξ is a foliated manifold $(E, \mathscr{F}(\varphi)) = (\tilde{B} \times K, \{\tilde{B} \times \{t\}\}_{t \in K})/\pi_1(B, x)$ and the bundle projection $p: E = (\tilde{B} \times K)/\pi_1(B, x) \to \tilde{B}/\pi_1(B, x) = B$ is the natural map between orbit spaces. Conversely given a foliated K-bundle $(\xi; \mathscr{F}) = (p, E, B; \mathscr{F})$ and $x \in B$, leaves of $\mathscr{F}$ are covering spaces of B and a loop $l: ([0, 1], \{0, 1\}) \to (B, x)$ at x determines a diffeomorphism $\tilde{l}(0) \mapsto \tilde{l}(1)$ of the fiber at x, where $\tilde{l}: [0, 1] \to L_{\tilde{l}(0)}$ is the unique path with initial point $\tilde{l}(0)$ which covers l. It is clear that this diffeomorphism depends only on the homotopy class of l and this procedure gives a homomorphism $\varphi: \pi_1(B, x) \to \mathrm{Diff}^r K$ such that $\mathscr{F} = \mathscr{F}(\varphi)$. We call φ the *total holonomy homomorphism* for $(\xi; \mathscr{F})$ and $\mathscr{T}\mathscr{H}(\mathscr{F}) = \varphi(\pi_1(B, x))$ the *total holonomy group* for $(\xi; \mathscr{F})$. The foliation $\mathscr{F} = \mathscr{F}(\varphi)$ has properties analogous to those of the orbit space of the action of $\mathscr{T}\mathscr{H}(\mathscr{F})$ on K:

$$\begin{array}{ccc} \Gamma: \mathscr{T}\mathscr{H}(\mathscr{F}) \times K & \longrightarrow & K . \\ \cup\!\!\!\!\!\!\omega & & \cup\!\!\!\!\!\!\omega \\ (f, t) & \longmapsto & f(t) \end{array}$$

Since we assumed in this paper that $\mathscr{F}$ is transversely orientable, $\mathscr{T}\mathscr{H}(\mathscr{F})$ is a subgroup of the group $\mathrm{Diff}^r_+ K$ of orientation-preserving C^r diffeomorphisms of K and ξ is orientable. Especially, if $K = I$, ξ is trivial. If $K = S^1$, however, ξ is not always trivial. (See J. W. Wood [16, Theorem 1.1].) Such orientable S^1-bundles are classified by their Euler class $\chi(\xi) \in H^2(B; Z)$. Fortunately both foliated S^1-bundles $(p', E', \Sigma_3; \mathscr{F}(\chi'))$ and $(p, E, \Sigma_3; \mathscr{F}(\chi))$ over the compact orientable surface Σ_3 of genus three which we will construct in Sections 3 and 4 are trivial as S^1-bundles by the following criterion:

PROPOSITION 1. *Let $(\xi; \mathscr{F}) = (p, E, \Sigma_g; \mathscr{F})$ be a C^r foliated (orientable) S^1-bundle over a compact orientable surface Σ_g of genus $g \geqq 1$ with base point x and $\varphi: \pi_1(\Sigma_g, x) \to \mathrm{Diff}^r_+ S^1$ the total holonomy homomorphism for $(\xi; \mathscr{F})$. Then ξ is trivial if φ factors through a free group F_n on n generators, that is, there exist two homomorphisms ψ and h such that the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(\Sigma_g, x) & \xrightarrow{\quad h \quad} & F_n \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathrm{Diff}^r_+ S^1 \, . & \end{array}$$

PROOF. Let $(\eta; \mathscr{G})=(q, X, BF_n; \mathscr{G})$ be the foliated S^1-bundle over $BF_n=K(F_n, 1)=S^1\vee\cdots\vee S^1$ (n-times) for which ψ is the total holonomy homomorphism. Take a map $g: \Sigma_g=K(\pi_1(\Sigma_g, x), 1)\to BF_n=K(F_n, 1)$ such that $g_\sharp=h$. (g is a classifying map for the principal F_n-bundle over Σ_g determined by h.) Then $g^*\eta=\xi$. $H^2(K(F_n, 1); \boldsymbol{Z})=0$, especially $\chi(\eta)=0$ and η is trivial. $\chi(\xi)\in g^*(H^2(K(F_n, 1); \boldsymbol{Z}))\subset H^2(\Sigma_g; \boldsymbol{Z})\cong\boldsymbol{Z}$. Hence $\chi(\xi)=0$ and ξ is trivial. q.e.d.

§ 2. Reeb stability for semiproper leaves.

Various authors have investigated stability for proper leaves of codimension one foliations (e.g., G. Reeb [12], T. Inaba [9], [10], [11], P. R. Dippolito [4], [5], J. Cantwell and L. Conlon [1], [2], etc.). Our starting point is the following fundamental theorem:

THEOREM 1. (See G. Reeb [12] and A. Haefliger [6, p. 381].) *Let L be a compact leaf of $\mathscr{F}$. Then L is stable if and only if L has a trivial holonomy group.*

In 1976, T. Inaba succeeded in generalizing Theorem 1 for proper leaves of codimension one foliations of closed manifolds as follows:

THEOREM 2. (See T. Inaba [9], [10].) *Suppose that M is a closed three-manifold and let L be a proper leaf of $\mathscr{F}$ such that the fundamental group of L is finitely generated. Then L is stable if and only if L has a trivial holonomy group.*

Theorem 2 is a direct corollary of following two theorems. (See T. Inaba [9], [10].)

THEOREM 3. *Let L be a proper leaf of $\mathscr{F}$. Then L is stable if and only if L has a locally trivial holonomy pseudogroup.*

THEOREM 4. *Let M be a compact three-manifold (possibly with boundary) and L a leaf of $\mathscr{F}$ such that the fundamental group of L is finitely generated. $\mathscr{F}$ is supposed to be tangent to ∂M if $\partial M\neq\varnothing$. Then L has a trivial holonomy group if and only if L has a locally trivial holonomy pseudogroup.*

We generarize Theorem 2 as follows:

THEOREM A. *Suppose that M is a closed three-manifold and let L be a semiproper leaf of $\mathscr{F}$ with a proper side $\tilde{L}$ such that the fundamental group of L is finitely generated. Then L is stable on $\tilde{L}$ if and only if the holonomy group of L on $\tilde{L}$ is trivial.*

Theorem A is a direct consequence from Theorem 4 and the following "proper-side-version" of Theorem 3.

THEOREM B. *Let L be a semiproper leaf of $\mathscr{F}$ with a proper side $\tilde{L}$. Then L is stable on $\tilde{L}$ if and only if the holonomy pseudogroup of L on $\tilde{L}$ is locally trivial.*

PROOF. For each point x of L, let τ^x: $([0, 1], 0)\to(T_x, x)$ be a transversal starting in the direction $\tilde{L}$ and U a component of $M\backslash\bar{L}$ containing $\tau^x(]0, \varepsilon[)$ for some $\varepsilon>0$. On the completion $\hat{U}$ of U in the metric induced from a Riemannian metric of M, the pullback $\hat{i}^*\mathscr{F}$ has a boundary leaf $L_0\subset\partial\hat{U}$ containing the limit of $\tau^x(t)$ as $t\searrow 0$, where $\hat{i}$; $\hat{U}\to M$ is the isometric immersion induced from the inclusion map $i: U\hookrightarrow M$. (See P. R. Dippolito [4].) L_0 covers L and is diffeomorphic to $\tilde{L}$:

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\cong} & L_0\,. \\ \upin & & \upin \\ d\tau_0^x(d/dt)/\|d\tau_0^x(d/dt)\| & \longmapsto & \lim\limits_{t\searrow 0}\tau^x(t) \end{array}$$

Therefore the local triviality of $\mathscr{HP}_{\tilde{L}}(L, x)$ is equivalent to the local triviality of $\mathscr{HP}(L_0, x_0)$, where $x_0=\lim_{t\searrow 0}\tau^x(t)$ in $\hat{U}$. Consequently, the proof of Theorem 3 (See T. Inaba [9].) is also valid for Theorem B, via the induced isometric immersion $\hat{i}$. q.e.d.

However there are counterexamples to Theorems 2 and A if the assumption that L has a finitely generated fundamental group is got rid of. The example of H. Imanishi [8, p. 622] is the one to Theorem 2. We will construct a counterexample to Theorem A without that assumption in Section 4.

On the other hand, it seems quite natural to conjecture that Theorems 2 and A can be extended for closed manifolds of dimension greater than three. (According to T. Inaba [11], we call this conjecture the "*generalized Reeb stability conjecture*" or abbreviately the "*GRS conjecture*".) But in 1980, T. Inaba [11] has constructed a C^0 foliation of a closed manifold of dimension five or greater than five with an unstable proper leaf which has a finitely generated fundamental group and a trivial holonomy group. This *Inaba foliation* is a counterexample to the GRS conjecture for C^0 foliations of closed manifolds of dimension$\geqq 5$. The GRS conjecture for C^1 foliations or for closed four-manifolds remains an interesting but difficult open question.

§ 3. Hector's C^∞ diffeomorphisms of S^1.

Let G be a subgroup of $\mathrm{Diff}_+^\infty S^1$. A subset C of S^1 is called a *minimal* set of G if C is a nonempty closed subset invariant under G which has no proper subsets with such properties. A minimal set C of G is *exceptional* if C is neither a single closed orbit nor all of S^1.

In this section, we recall the construction of orientation-preserving C^∞ diffeomorphisms f and g of S^1 in G. Hector [7] such that the group G' generated by f and g admits an exceptional minimal set C'.

We consider S^1 as the circle obtained from the interval $[-2, 14]$ by identifying its endpoints. At first, we define f by

$$f(t)=t+4 \mod 16 \quad \text{for all} \quad t\in[-2, 14]\,.$$

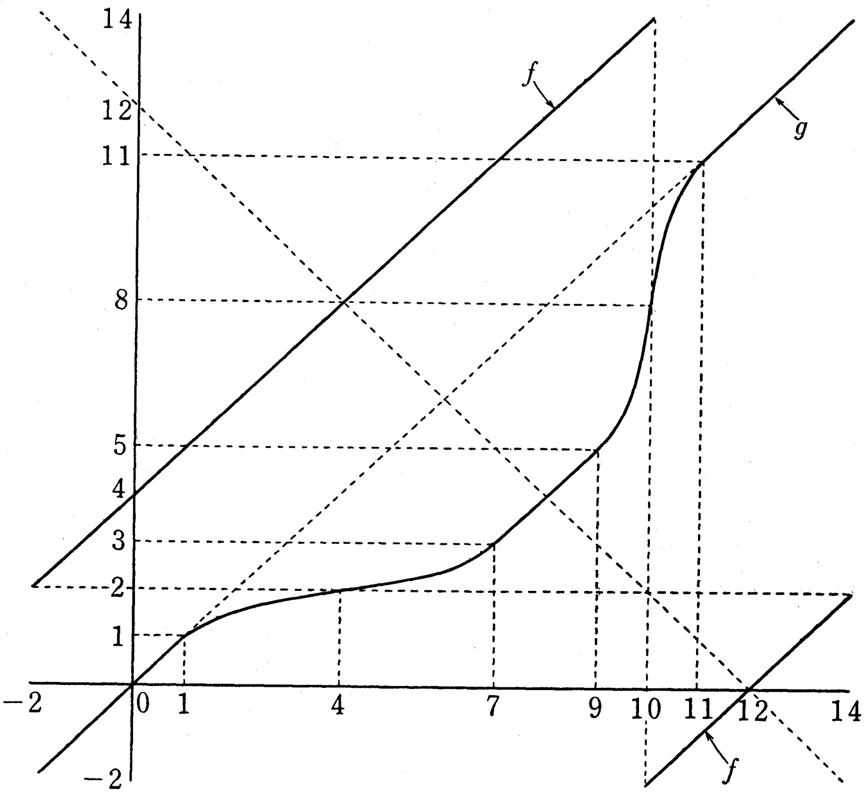

FIGURE 1. Graphs of f and g

Next define g so that

(1) $$\operatorname{supp} g=\overline{\{t\in S^1;\ g(t)\neq t\}}=[1, 11]\ ,$$

(2) the graph of g is symmetric with respect to the line $s=-t+12$,

(3) $$\begin{cases} g(t)<t & \text{for all } t\in]1, 11[\ , \\ g(t)=t+4 & \text{for all } t\in [7, 9]\ , \\ g'(t)<1 & \text{for all } t\in]1, 7[\ , \end{cases}$$

(4) $g(4)=2$ and the graph of $g|[1, 7]$ is symmetric with respect to the point $(4, 2)$.*)

Finally define the set C' by $C'=\overline{S^1\backslash\bigcup_{t\in I}\Gamma_{G'}(t)}$, where $\Gamma_{G'}\colon G'\times S^1\to S^1$ is the action of G' on S^1, $\Gamma_{G'}(t)$ is the orbit of $t\in S^1$ under $\Gamma_{G'}$, and $I=[-1, 1]$.

The following is essential to our construction in Section 4.

PROPOSITION 2. (See G. Hector [7].) *$\Gamma_{G'}$ is trivial on I and C' is an exceptional minimal set of G'.*

PROOF. See G. Hector [7].

Let Σ_3 be a compact orientable surface of genus three with base point x. The fundamental group of Σ_3 based at x is presented as follows:

$$\pi_1(\Sigma_3, x)=\langle \alpha_i, \beta_i\ (i=1, 2, 3)|\prod_{i=1}^{3}[\alpha_i, \beta_i]=e\rangle\ .$$

We define a homomorphism

$$\chi'\colon \pi_1(\Sigma_3, x)\longrightarrow \mathrm{Diff}_+^\infty S^1$$

as $\chi'(\beta_1)=f$, $\chi'(\beta_2)=g$, and $\chi'(\beta_3)=\chi'(\alpha_i)=\mathrm{id}$ for $i=1, 2, 3$. This provides a C^∞ foliated S^1-bundle $(\xi'; \mathscr{F}')=(p', E', \Sigma_3; \mathscr{F}(\chi'))$ for which χ' is the total holonomy homomorphism. Since χ' factors through a free group F_2 on two generators, ξ' is trivial by Proposition 1 and $p'\colon \Sigma_3\times S^1\to\Sigma_3$ is the projection to the first factor.

Since each $t\in\Gamma_{G'}(-1)\cup\Gamma_{G'}(1)$ is an endpoint of a gap of the exceptional minimal set C' of G', the following is a direct corollary of Proposition 2.

PROPOSITION 3. *$\mathscr{F}(\chi')|\operatorname{sat}_{\mathscr{F}(\chi')}(\{x\}\times I)$ is trivial. Both leaves $L'_{(x,-1)}$*

*) In G. Hector [7, p. 252], a confusion prevails, that is, the condition (4) we required above is used without request.

and $L'_{(x,1)}$ *of* $\mathscr{F}(\mathcal{X}')$ *are semiproper exceptional leaves. The positive side* $\tilde{L}'_{(x,-1)}$ *of* $L'_{(x,-1)}$ *and the negative side* $\tilde{L}'_{(x,1)}$ *of* $L'_{(x,1)}$ *are proper sides.*

REMARK. Especially, $L'_{(x,i)}$ is stable on $\tilde{L}'_{(x,i)}$ for $i=-1, 1$.

For making short, we let $\tilde{x}$ and L denote $(x, -1)$ and $L'_{(x,-1)}$ respectively. The definition of $\mathcal{X}'$ shows:

PROPOSITION 4. *There exists a* C^∞ *injective immersion*

$$\varphi: L\times I \longrightarrow \Sigma_3\times S^1$$

such that the following properties are satisfied:

(1) $\varphi(L\times I)=\mathrm{sat}_{\mathscr{F}(\mathcal{X}')}(\{x\}\times I)$,

(2) $\varphi: (L\times I, \{L\times\{t\}\}_{t\in I}, \{\{y\}\times I\}_{y\in L})\longrightarrow(\Sigma_3\times S^1, \mathscr{F}(\mathcal{X}'), \{\{z\}\times S^1\}_{z\in\Sigma_3})$ *is foliation-preserving,*

(3) $\varphi(\tilde{x}, t)=(x, t)$ *for all* $t\in I$,

(4) $\varphi(\cdot, -1): L\hookrightarrow\Sigma_3\times S^1$ *is the inclusion map.*

§4. Unstable semiproper exceptional leaves without holonomy.

In this section, we prove the following theorem, which is the main result of this paper.

THEOREM C. *There exist a closed* C^∞ *manifold* M *of dimension* $n\geqq 3$ *and a* C^∞ *codimension one foliation* $\mathscr{F}$ *of* M *such that* $\mathscr{F}$ *has a semiproper exceptional leaf* L *with a proper side* $\tilde{L}$ *satisfying the following properties*:

(1) $\pi_1(L)$ *is not finitely generated,*

(2) $\mathscr{H}_{\tilde{L}}(L)$ *is trivial,*

(3) $\mathscr{H}\mathscr{P}_{\tilde{L}}(L)$ *is locally infinite (hence* L *is unstable on* $\tilde{L}$).

PROOF. The proof is performed by constructing an example explicitly.

Our first job is to choose an orientation-preserving C^∞ diffeomorphism h of S^1. Again we regard S^1 as the interval $[-2, 14]$ with its endpoints identified. f and g are the same as those in Section 3. We start with a sequence $\{a_n\}_{n=0,1\cdots}$ such that

(1) $$0<a_n\nearrow 1\ .$$

Next define two sequences $\{b_n\}_{n=0,1\cdots}$ and $\{c_n\}_{n=0,1,\cdots}$:

$$b_n=\{g^n f(a_n)-g^n f(-a_n)\}/2a ,$$
$$c_n=\{g^n f(a_n)+g^n f(-a_n)\}/2b_n ,$$

where $0<a<1$. Then

(2) $$0<b_n \longrightarrow 0 .$$

Choose a C^∞ function $\lambda: \boldsymbol{R}\to\boldsymbol{R}$ such that

(3) $\overline{\{t\in\boldsymbol{R};\ \lambda(t)\neq 0\}}=[-a, a]$,

(4) $\max\{|\lambda'(t)|;\ t\in\boldsymbol{R}\}<1/\max\{b_n^{\,n-1};\ n\in\boldsymbol{Z}^+\}$,

(5) λ is C^∞ tangent to the zero function at $-a$ and a.

Next define $\mu_n: \boldsymbol{R}\longrightarrow\boldsymbol{R}$ for $n=0, 1, \cdots$:

$$\mu_n(t)=t-b_n^{\,n}\lambda(t/b_n-c_n) \quad \text{for all} \quad t\in\boldsymbol{R} .$$

Then

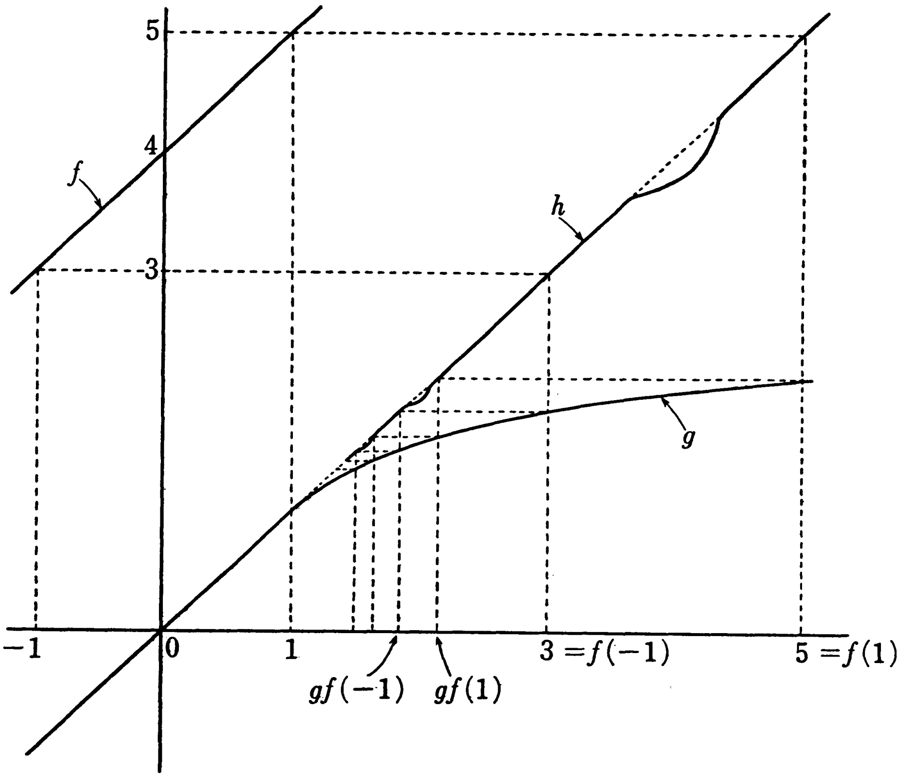

FIGURE 2. Graph of h

(6) $\operatorname{supp}\mu_n=\overline{\{t\in \boldsymbol{R};\ \mu_n(t)\neq t\}}=[g^n f(-a_n),\ g^n f(a_n)]\ \subset \operatorname{int} I_n$,
where $I_n=[g^n f(-1),\ g^n f(1)]$,

(7) $\mu_n'(t)>0$ for all $t\in \boldsymbol{R}$.

Finally define an orientation-preserving homeomorphism h of S^1 by

$$h=\begin{cases}\mu_n & \text{on } I_n \text{ for } n=0, 1, \cdots ,\\ \mathrm{id} & \text{on } S^1\backslash\bigcup_{n=0}^{\infty} I_n .\end{cases}$$

By (2), h is C^∞ tangent to the identity at 1. This completes the definition of h. Thus $h\in \mathrm{Diff}_+^\infty S^1$.

We let G denote the subgroup of $\mathrm{Diff}_+^\infty S^1$ generated by f, g and h, Γ_G the action of G on S^1, $\Gamma_G(t)$ the orbit of $t\in S^1$ under Γ_G and C the set $\overline{S^1\backslash\bigcup_{t\in I}\Gamma_G(t)}$. Let G' and C' be as in Section 3.

PROPOSITION 5. *$C=C'$ and C is an exceptional minimal set of G.*

PROOF. This follows from Proposition 2 and the definition of h.

We define a homomorphism

$$\chi\colon \pi_1(\Sigma_3, x)\longrightarrow \mathrm{Diff}_+^\infty S^1$$

as $\chi(\beta_1)=f$, $\chi(\beta_2)=g$, $\chi(\beta_3)=h$, $\chi(\alpha_i)=\mathrm{id}$ for $i=1, 2, 3$. This provides a C^∞ foliated S^1-bundle $(\xi;\ \mathscr{F})=(p,\ E,\ \Sigma_3;\ \mathscr{F}(\chi))$ for which χ is the total holonomy homomorphism. Since χ factors through a free group F_3 on three generators, ξ is trivial by Proposition 1 and $p\colon E=\Sigma_3\times S^1\rightarrow\Sigma_3$ is the projection to the first factor. Let χ', $\mathscr{F}(\chi')$, $L'_{(x,i)}$ for $i=-1, 1$, etc. be as in Section 3. Each $t\in\Gamma_G(-1)\cup\Gamma_G(1)$ is an endpoint of a gap of the exceptional minimal set C of G. So the following is a direct consequence from Proposition 5.

PROPOSITION 6. *$\mathrm{sat}_{\mathscr{F}(\chi)}(\{x\}\times I)$ coincides with $\mathrm{sat}_{\mathscr{F}(\chi')}(\{x\}\times I)$, especially $L_{(x,i)}\in\mathscr{F}(\chi)$ coincides with $L'_{(x,i)}\in\mathscr{F}(\chi')$ for $i=-1, 1$. Both leaves $L_{(x,-1)}$ and $L_{(x,1)}$ are semiproper exceptional leaves. The positive side $\tilde{L}_{(x,-1)}$ of $L_{(x,-1)}$ and the negative side $\tilde{L}_{(x,1)}$ of $L_{(x,1)}$ are proper sides.*

Choose two simple loops u and v: $([0, 1], \{0, 1\})\rightarrow(\Sigma_3, x)$ which represent α_3 and β_3 respectively so that $u([0, 1])\cap v([0, 1])=\{x\}$. For $n=0, 1, \cdots$, we can lift u [resp. v] to $\Sigma_3\times S^1$ so that the unique lift $\tilde{u}_n$ [resp. $\tilde{v}_n$] with initial point $(x, g^n f(-1))$ is a loop in L and $\tilde{u}_n([0, 1])=u([0, 1])\times\{g^n f(-1)\}$ [resp. $\tilde{v}_n([0, 1])=v([0, 1])\times\{g^n f(-1)\}$] because $g^n f(-1)$ is a fixed point of $\mathrm{id}=\chi(\alpha_3)$ [resp. $h=\chi(\beta_3)$]. These $\tilde{u}_n$'s [resp. $\tilde{v}_n$'s] are pairwise disjoint countably many embedded loops on account of the "unique-lifting property" for p. Since

$$\tilde{u}_n([0,1])\cap\tilde{v}_m([0,1])=\begin{cases}\{g^n f(-1)\} & \text{for } n=m\,,\\ \phi & \text{for } n\neq m\end{cases}$$

by the choice of u and v, $L\backslash\bigcup_{n=0}^{\infty}\tilde{v}_n([0,1])$ is connected. Hence L has countably many handles so that $\pi_1(L,\tilde{x})$ is not finitely generated.

Let $\mathcal{TH}(\mathcal{F}(\chi))_t$ be the isotropy group of $t\in S^1$ in $\mathcal{TH}(\mathcal{F}(\chi))$ and $H=\{\gamma\in\pi_1(\Sigma_3,x);\ \chi(\gamma)(-1)=-1\}=\chi^{-1}(\mathcal{TH}(\mathcal{F}(\chi))_{-1})$. Then H is a subgroup of $\pi_1(\Sigma_3,x)$ and is obviously isomorphic to $\pi_1(L,\tilde{x})$:

$$(p\circ\varphi)_\#\colon \pi_1(L,\tilde{x})\cong H\,,$$

where $\varphi\colon L\times I\to\Sigma_3\times S^1$ is the C^∞ injective immersion in Proposition 4.

$$\begin{array}{ccccc}\pi_1(L,\tilde{x}) & \xrightarrow{\ \cong\ } & H & \xrightarrow{\ \chi|H\ } & \mathrm{Diff}_+^\infty I\\ \varphi_\#\big\downarrow & & \cap & & \\ \pi_1(\Sigma_3\times S^1,\tilde{x}) & \xrightarrow[p_\#]{} & \pi_1(\Sigma_3,x) & \xrightarrow[\chi]{} & \mathrm{Diff}_+^\infty S^1\end{array}$$

H is also isomorphic to $\pi_1(L_{(x,1)},(x,1))$ because $\mathcal{TH}(\mathcal{F}(\chi))_{-1}\equiv\mathcal{TH}(\mathcal{F}(\chi))_1$ by the definition of χ. Moreover a C^∞ diffeomorphism $k\colon(L,\tilde{x})\to(L_{(x,1)},(x,1))$ is defined as follows: For each $(y,t)\in L$, $k(y,t)$ is the point at which the path on the fiber $p^{-1}(y)=\{y\}\times S^1$ starting from (y,t) in the positive direction $\tilde{L}$ meets $L_{(x,1)}$ at the first time. Note that $p|L=(p|L_{(x,1)})\circ k$.

Let $q\colon L\times I\to L$ is the projection to the first factor. Then $(\eta;\mathcal{F}(\psi))=(q,L\times I,L;\varphi^*\mathcal{F}(\chi))$ is a C^∞ foliated I-bundle for which $\psi=(\chi|H)\circ p_\#\circ\varphi_\#\colon \pi_1(L,\tilde{x})\to\mathrm{Diff}_+^\infty I$ is the total holonomy homomorphism, where $\varphi_\#\colon\pi_1(L,\tilde{x})\to\pi_1(\Sigma_3\times S^1,\tilde{x})$ and $p_\#\colon\pi_1(\Sigma_3\times S^1,\tilde{x})\to\pi_1(\Sigma_3,x)$ are the homomorphisms induced from φ and p respectively.

Hence stability of L [resp. $L_{(x,1)}$] on $\tilde{L}$ [resp. $\tilde{L}_{(x,1)}$] in $\mathcal{F}(\chi)$ is equivalent to stability of $L\times\{-1\}$ [resp. $L\times\{1\}$] in $\mathcal{F}(\psi)$. Let $\bar{\mu}_n=(g^n f|I)^{-1}(\mu_n|I_n)(g^n f|I)\in\mathrm{Diff}_+^\infty I$. By (6),

$$(\bar{6})\qquad \operatorname{supp}\bar{\mu}_n=[-a_n,a_n]\subset\operatorname{int}I\quad\text{for}\quad n=0,1,\cdots.$$

By the first part of Proposition 3, $(\bar{6})$, and the definition of ψ, $\mathcal{H}(L\times\{-1\},(\tilde{x},-1))$ [resp. $\mathcal{H}(L\times\{1\},(\tilde{x},1))$] is trivial but $\mathcal{HG}(L\times\{-1\},(\tilde{x},-1))$ [resp. $\mathcal{HG}(L\times\{1\},(\tilde{x},1))$] is locally infinite by (1) and $(\bar{6})$. Thus $\mathcal{H}_{\tilde{L}}(L,\tilde{x})$ [resp. $\mathcal{H}_{\tilde{L}_{(x,1)}}(L_{(x,1)},(x,1))$] is trivial but $\mathcal{HG}_{\tilde{L}}(L,\tilde{x})$ [resp.

$\mathscr{HP}_{\tilde{L}_{(x,1)}}(L_{(x,1)}, (x, 1))]$ is locally infinite, that is L [resp. $L_{(x,1)}$] is unstable on $\tilde{L}$ [resp. $\tilde{L}_{(x,1)}$] by Theorem B. This completes our construction if $n=3$. And if $n \geqq 4$, the foliated manifold $(S^{n-3} \times \Sigma_3 \times S^1, S^{n-3} \times \mathscr{F}(\chi))$ and the leaf $S^{n-3} \times L_{(x,i)} \in S^{n-3} \times \mathscr{F}(\chi)$ for $i=-1, 1$ suffices. q.e.d.

REMARK 1. Similarly, we can construct a C^1 (but not C^2) foliation of $\Sigma_2 \times S^1$ which has unstable semiproper exceptional leaves without holonomy by using Denjoy's C^1 (but not C^2) diffeomorphism f_D (See A. Denjoy [3] or P. A. Schweitzer [15, Appendix].) instead of by using Hector's C^∞ diffeomorphisms f and g, where Σ_2 is a compact orientable surface of genus two.

REMARK 2. R. Sacksteder [13] had already constructed a C^∞ foliation with an exceptional minimal set when G. Hector [7] constructed such a foliation. However Sacksteder's semiproper exceptional leaves have holonomy, so the Sacksteder foliation can not be used for our construction.

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