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On Regular Fréchet-Lie Groups V.

Several Basic Properties.

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Introduction.

In the previous paper [7], we have defined the concept of regular Fréchet-Lie groups, and given two fundamental theorems, which correspond to the fundamental theorem of calculus.

A group G will be called an *FL-group*, if G is a C^∞ Fréchet manifold modeled on a Fréchet space $\mathfrak{G}$ (we always assume for a Fréchet space to be locally convex) and the group operations are C^∞. (Cf. Introduction and § 2 of [7] for the reason why we hesitate to use the word "Fréchet-Lie group".) Roughly speaking, a *regular Fréchet-Lie group* is an *FL*-group on which product integrals are well-defined. We shall repeat briefly what is a product integral on an *FL*-group G.

Let $\Delta=\{t_0, t_1, \cdots, t_m\}$ be a division of an interval $J=[a, b]$ such that $a=t_0$, $b=t_m$. For a division Δ of J, we denote by $|\Delta|$ the maximum of $|t_{j+1}-t_j|$. We call a pair (h, Δ) a *step function* on $[0, \varepsilon]\times J$ ($|\Delta|<\varepsilon$), if h is a mapping of $[0, \varepsilon]\times J$ into G such that

(i) $h(0, t)\equiv e$ for all $t\in[a, b]$ and $h(s, t)$ is C^1 in s for each fixed t.

(ii) $h(s, t)=h(s, t_j)$ for $(s, t)\in[0, \varepsilon]\times[t_j, t_{j+1})$.

For a step function (h, Δ), we define the *product integral* $\prod_a^t (h, \Delta)$ by

$$\prod_a^t (h, \Delta)=h(t-t_k, t_k)h(t_k-t_{k-1}, t_{k-1}) \cdots h(t_1-t_0, t_0) ,$$

where k is the integer such that $t\in[t_k, t_{k+1})$. Now, let $\{(h_n, \Delta_n)\}$ be a sequence of step functions such that $\lim_{n\to\infty}|\Delta_n|=0$, and $\{h_n\}$ converges uniformly to a C^1-hair h at e with their partial derivatives $\partial h_n/\partial s$ (cf. [7], (9)), where $h: [0, \varepsilon]\times J\to G$ is called a *C^1-hair* at e, if h is continuous, $h(s, t)$ is C^1 with respect to s and $(\partial h/\partial s)(s, t)$ is continuous. We call an

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FL-group G a regular *Fréchet-Lie group*, if $\lim_{n\to\infty} \prod_a^t (h_n, \Delta_n)$ converges uniformly on J for every sequence of step functions converging to a C^1-hair h at e in the above sense. We denote the limit by $\prod_a^t (h, d\tau)$, and call it the *product integral* of h.

Set $u(t)=(\partial h/\partial s)(0, t)$. Then, u is a continuous mapping of J into $\mathfrak{G}$.

The first fundamental theorem (cf. [7], Theorem 4.1): *The product integral* $\prod_a^t (h, d\tau)$ *is* C^1 *with respect to* t, *and*

$$(1) \qquad \frac{d}{dt} \prod_a^t (h, d\tau)=u(t)\cdot\prod_a^t (h, d\tau) , \qquad \prod_a^a (h, d\tau)=e .$$

Thus, by the uniqueness theorem of the solution of the differential equation (cf. Lemma 2.5 [7]),

$$\frac{d}{dt}g(t)=u(t)\cdot g(t) , \qquad g(a)=e ,$$

we see that $\prod_a^t (h, d\tau)$ depends only on $u(t)$. Therefore, we often denote $\prod_a^t (h, d\tau)$ by $\prod_a^t (1+u(\tau))d\tau$. Let $C^0(J, \mathfrak{G})$ be the Fréchet space of all continuous mappings of J into the Lie algebra $\mathfrak{G}$ of G, and let $C_e^1(J, G)$ be the C^∞ Fréchet manifold of all C^1 mappings $c: J\to G$ such that $c(a)=e$.

The second fundamental theorem (cf. [7] Theorem 5.1): *Let* $\mathscr{I}: C^0(J, \mathfrak{G})\to C_e^1(J, G)$ *be a mapping defined by* $\mathscr{I}(u)(t)=\prod_a^t (1+u(\tau))d\tau$. *Then,* $\mathscr{I}$ *is a* C^∞*-diffeomorphism.*

For every $u\in\mathfrak{G}$, we denote $\prod_0^t (1+u)d\tau$ by $\exp tu$. Then, it is easy to see that $\{\exp tu; t\in \boldsymbol{R}\}$ is a C^∞ one parameter subgroup of G, and the mapping $\exp: \mathfrak{G}\to G$, called the *exponential mapping*, is a C^∞ mapping.

In this paper, we shall prove several basic properties of regular Fréchet-Lie groups by using first and second fundamental theorems. The main result of this paper is that *the universal covering group of a regular Fréchet-Lie group is determined uniquely by its Lie algebra.* Hence, one can get a lot of information about a regular Fréchet-Lie group G by investigating its Lie algebra $\mathfrak{G}$. Hence, regular Fréchet-Lie groups satisfy all requests of "Lie groups" imposed in the introduction of [7].

We shall also investigate several methods to make a new regular Fréchet-Lie group starting from two regular Fréchet-Lie groups, most of which are strong ILB-Lie groups which are fundamental examples of regular Fréchet-Lie groups (cf. [7]). These methods will be useful to prove that the infinite dimensional group of all invertible Fourier integral operators of order 0 is a regular Fréchet-Lie group, which is one of the main purposes of this series. (cf. [5], [6]).

§ 1. Exponential mappings and their properties.

Let G be an FL-group and $\mathfrak{G}$ be the tangent space of G at the identity e. $\mathfrak{G}$ is naturally identified with the model space of G and hence a Fréchet space. For every $u \in \mathfrak{G}$, we denote by u^* the right-invariant C^∞ vector field on G, i.e.,

$$u^*(g)=u\cdot g \quad (=(dR_g)u) , \qquad u^*(e)=u ,$$

where dR_g is the differential of the right translation $R_g: G \to G$, $R_g h=h\cdot g$ at the identity. For every $v \in \mathfrak{G}$, define the adjoint map $\mathrm{Ad}(g)$, $g \in G$ on $\mathfrak{G}$ by

$$\mathrm{Ad}\,(g)v=\frac{d}{dt}\Big|_{t=0} gc(t)g^{-1} ,$$

where $c(t)$ is a smooth curve in G such that $c(0)=e$, $\dot{c}(0)=v$. Then, $\mathrm{Ad}\,(g)\colon \mathfrak{G} \to \mathfrak{G}$ is an isomorphism of $\mathfrak{G}$ and the map $\mathrm{Ad}\colon G\times\mathfrak{G} \to \mathfrak{G}$ defined by

$$\mathrm{Ad}\,(g,\, u)=\mathrm{Ad}\,(g)u , \quad g \in G , \quad u \in \mathfrak{G} ,$$

is a C^∞ mapping. (cf. [7], Lemma 2.4).

We define the bracket of $\mathfrak{G}$ by

$$(2) \qquad [u,\, v]=\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}\,(g(t))v , \quad u,\, v \in \mathfrak{G} ,$$

where $g(t)$ is a C^∞ curve in G such that $g(0)=e$, $\dot{g}(0)=u$. Alternatively, (2) can be written by

$$[u,\, v]=(du^*)_e v-(dv^*)_e u ,$$

for every $u, v \in \mathfrak{G}$ (cf. [7], (8)). With the above bracket (2), $\mathfrak{G}$ is a Fréchet-Lie algebra, and called the *Lie algebra of G*.

Now, let G be a regular Fréchet-Lie group. As we mentioned in the introduction, the exponential mapping $\exp\colon \mathfrak{G} \to G$ is a C^∞ mapping. In this section, we shall investigate several properties of the exponential mapping.

LEMMA 1.1. *For a C^1 curve $c(t)$ in G such that $c(0)=e$, $\{c(t/n)^n\}$ converges uniformly on* $[0, 1]$ *to a one parameter subgroup* $\exp tu$, *where* $u=\dot{c}(0)$.

PROOF. Set $h_a(s, t)=c(as)$, $a \in [0, 1]$, and h_a is a C^1-hair for small s.

Let Δ_n be a division $0<1/n<2/n<\cdots<(n-1)/n<1$ of $I=[0,1]$. Let $\{(h_a, \Delta_n)\}$ be a series of step functions such that $\lim_{n\to\infty}|\Delta_n|=0$ and $\lim_{n\to\infty} h_a=h_a$ with its derivative in s. Note that $\prod_0^1(h_a, \Delta_n)=c(a/n)^n$. Thus, we see $\prod_0^1(h_a, d\tau)=\lim_{n\to\infty} c(a/n)^n$. Remark that $\{\prod_0^1(h_a, \Delta_n)\}$ converges uniformly in a (cf. [7], Corollary 3.4).

To prove $\lim_{n\to\infty} c(a/n)^n=\exp au$, remark that by (1), $g_a(t)=\prod_0^t(h_a, d\tau)$ satisfies $(d/dt)g_a=au\cdot g_a$, $g_a(0)=e$. Hence by the uniqueness theorem ([7], Lemma 2.5) we have

$$\prod_0^1 (h_a, d\tau)=\prod_0^a (h_1, d\tau)=\exp au . \qquad \square$$

The following lemmas are useful throughout this paper.

LEMMA 1.2. *For any* $u, v\in\mathfrak{G}$,

$$\exp s\,\mathrm{Ad}\,(\exp tu)v=\exp tu \exp sv \exp -tu .$$

PROOF. Put $g(s)=\exp tu \exp sv \exp -tu$. Then, $g(s)$ satisfies

$$\frac{d}{ds}g(s)=\mathrm{Ad}\,(\exp tu)v\cdot g(s)$$

and $g(0)=e$. Using the first fundamental theorem and the uniqueness of the above differential equation, we get Lemma 1.2. $\square$

LEMMA 1.3. *Let G be a regular Fréchet-Lie group with the Lie algebra $\mathfrak{G}$. For every continuous mapping u: $[0,1]\to\mathfrak{G}$, the equation*

$$(*) \qquad \frac{dw}{dt}=[u(t), w(t)]\,, \qquad w(0)=w\in\mathfrak{G}$$

has a unique solution $\mathrm{Ad}\,(g(t))w$, *where* $g(t)=\prod_0^t(1+u(\tau))d\tau$. *Moreover, if $\mathfrak{H}$ is a finite codimensional closed Lie subalgebra of $\mathfrak{G}$ and $u(t)$ is a C^∞ curve in $\mathfrak{H}$, then the differential equation* $(*)$ *with* $w(0)=w\in\mathfrak{H}$ *has a unique solution in* $\mathfrak{H}$.

PROOF. By Lemma 2.3 in [7], we see easily that $\mathrm{Ad}\,(g(t))w$ is a solution of $(*)$. Suppose $w'(t)$ be another solution. Then by the same lemma, we see that $(d/dt)\,\mathrm{Ad}\,(g(t)^{-1})w'(t)=0$, and hence $\mathrm{Ad}\,(g(t)^{-1})w'(t)=w'(0)=w$.

Assume that $u(t)$ is a C^∞ curve in $\mathfrak{H}$. We have only to show $w(t)=\mathrm{Ad}\,(g(t))w\in\mathfrak{H}$. Set $\mathrm{ad}\,(u(t))=[u(t),\cdot]$. Then $\mathrm{ad}\,(u(t))\mathfrak{H}\subset\mathfrak{H}$. Thus $\mathrm{ad}\,(u(t))$ induces a linear mapping $\widetilde{\mathrm{ad}}\,(u(t))$ of $\mathfrak{G}/\mathfrak{H}$ into itself such that $p\,\mathrm{ad}\,(u(t))v=\widetilde{\mathrm{ad}}\,(u(t))pv$ where p: $\mathfrak{G}\to\mathfrak{G}/\mathfrak{H}$ is the projection. Thus $(d/dt)pw(t)=\widetilde{\mathrm{ad}}\,(u(t))pw(t)$, $pw(0)=0$. Since $\dim\mathfrak{G}/\mathfrak{H}<\infty$, we see $pw(t)=0$ by the uniqueness theorem of linear differential equations. Hence $w(t) \in \mathfrak{H}$. □

Now, we consider a subgroup H, possibly non-closed, of a regular Fréchet-Lie group G. Though one can not conclude H is an FL-group even if H is closed, one can define the Lie algebra $\mathfrak{H}$ of H, if H satisfies a certain condition. For a neighborhood $\tilde{U}$ of e in G, denote by $\tilde{U}_0(H)$ the subset of $\tilde{U}$ consisting of all points x which can be connected to e by piecewise smooth curves contained in $\tilde{U} \cap H$. We denote the closure of $\tilde{U}_0(H)$ by $\tilde{U}_0(H)^-$, and denote by $\mathfrak{H}$ the totality of $u \in \mathfrak{G}$ such that $\exp tu \in H$ for every $t \in \boldsymbol{R}$.

LEMMA 1.4. *Notations and assumptions being as above, suppose there is an open neighborhood $\tilde{U}$ of e such that $\tilde{U}_0(H)^- \subset H$. Then, $\mathfrak{H}$ is a closed Lie subalgebra of $\mathfrak{G}$.*

PROOF. It is obvious that if $u \in \mathfrak{H}$, then $au \in \mathfrak{H}$ for every $a \in \boldsymbol{R}$. Let $u, v \in \mathfrak{H}$. Then $g(t) = \exp tu \exp tv$ is a C^∞ curve in G, contained in H. By Lemma 2.1 in [7], we have $(dg/dt)(0) = u + v$. By Lemma 1.1, we see $\lim_{n\to\infty} (\exp (t/n)u \exp (t/n)v)^n = \exp t(u+v)$, and the convergence is uniform on the unit interval $I = [0, 1]$.

Let $[0, \varepsilon_n)$ be the maximal interval such that $g(t/n)^n \in \tilde{U}$ for $t \in [0, \varepsilon_n)$. By definition of $\tilde{U}_0(H)$, we see $g(t/n)^n \in \tilde{U}_0(H)$ for $t \in [0, \varepsilon_n)$. Suppose for a while that $\liminf \varepsilon_n = 0$. Then choosing a suitable subsequence, if necessary, one may assume that $\lim_{n\to\infty} \varepsilon_n = 0$. Since $g(\varepsilon_n/n)^n \notin \tilde{U}$, $\{g(\varepsilon_n/n)^n\}$ can not converge to e. However, since $\lim_{n\to\infty} g(s/n)^n = \exp s(u+v)$ uniformly on I, $\lim_{n\to\infty} g(\varepsilon_n/n)^n = e$ with $\lim_{n\to\infty} \varepsilon_n = 0$. This contradicts the above argument, and hence there is an $\varepsilon_0 > 0$ such that $\varepsilon_n \geqq \varepsilon_0$.

Since $g(t/n)^n \in \tilde{U}_0(H)$ for $t \in [0, \varepsilon_0)$ and $\tilde{U}_0(H)^- \subset H$, we have $\lim_{n\to\infty} g(t/n)^n \in H$ for $t \in [0, \varepsilon_0)$. This implies $\exp t(u+v) \in H$. Thus, $u+v \in \mathfrak{H}$, hence $\mathfrak{H}$ is a linear subspace of $\mathfrak{G}$.

Let $\{u_n\}$ be a sequence in $\mathfrak{H}$ converging to $u \in \mathfrak{G}$. Denote by $[0, \varepsilon_n)$ the maximal interval such that $\exp tu_n \in \tilde{U}$ for every $t \in [0, \varepsilon_n)$. Then $\exp tu_n \in \tilde{U}_0(H)$ for every $t \in [0, \varepsilon_n)$. Since $\exp tu$ is a continuous mapping of $\boldsymbol{R} \times \mathfrak{G}$ into G, there is $\varepsilon_0 > 0$ such that $\varepsilon_n \geqq \varepsilon_0$ for all n. Hence, $\lim_{n\to\infty} \exp tu_n \in \tilde{U}_0(H)^- \subset H$ for all $t \in [0, \varepsilon_0)$. Therefore, $\exp tu \in H$ for all $t \in \boldsymbol{R}$ and hence $u \in \mathfrak{H}$. $\mathfrak{H}$ is therefore a closed subspace of $\mathfrak{G}$.

By using Lemma 1.2, we get $\mathrm{Ad}(\exp tu)v \in \mathfrak{H}$ for every $u, v \in \mathfrak{H}$. It follows $[u, v] \in \mathfrak{H}$, because of (2). □

§ 2. Covering groups and subgroups.

First of all, we shall give the following.

PROPOSITION 2.1. *Let G be a regular Fréchet-Lie group. Then a covering group G' of G is a regular Fréchet-Lie group.*

PROOF. It is not hard to see that G' is an FL-group. Thus we have only to show the convergence of product integrals. Denote by pr: $G' \to G$ the natural projection, which is obviously a C^∞ homomorphism. Let $\tilde{U}$ be a connected neighborhood of e in G which is evenly covered by pr, i.e., pr is a C^∞-diffeomorphism of each connected component of $\mathrm{pr}^{-1}\tilde{U}$ onto $\tilde{U}$. We denote by $\tilde{U}'$ the identity component of $\mathrm{pr}^{-1}\tilde{U}$. The inverse mapping of pr: $\tilde{U}' \to \tilde{U}$ will be denoted by pr^{-1}.

Let $h: [0, \varepsilon] \times J \to G'$ be a C^1-hair at e, and let $\{(h_n, \Delta_n)\}$ be a sequence of step functions such that $\lim_{n\to\infty} |\Delta_n| = 0$ and $\{h_n\}$ converges uniformly to h with their partial derivatives $\{\partial h_n/\partial s\}$. pr$(h)$: $[0, \varepsilon] \times J \to G$ is a C^1-hair at e and $\{(\mathrm{pr}(h_n), \Delta_n)\}$ is a sequence of step functions in G with the same convergence property as $\{(h_n, \Delta_n)\}$. Since G is a regular Fréchet-Lie group, $\prod_a^t (\mathrm{pr}(h_n), \Delta_n)$ converges uniformly on J to a C^1 curve $\prod_a^t (\mathrm{pr}(h), d\tau)$. Hence, there is a number $\delta > 0$ such that if $t \in [a, a+\delta]$ then $\prod_a^t (\mathrm{pr}(h), d\tau)$, $\prod_a^t (\mathrm{pr}(h_n), \Delta_n) \in \tilde{U}$ for all n. Since pr^{-1} is a local isomorphism, we have $\mathrm{pr}^{-1}(\prod_a^t (\mathrm{pr}(h_n), \Delta_n)) = \prod_a^t (h_n, \Delta_n)$. Thus, $\lim_{n\to\infty} \prod_a^t (h_n, \Delta_n)$ converges uniformly on $[a, a+\delta]$.

The same proof as above shows also that for every $c \in J$ there is a compact neighborhood K_c of c such that $\lim_{n\to\infty} \prod_c^t (h_n, \Delta_n)$ converges uniformly on K_c, where if $t < c$ we define $\prod_c^t (h_n, \Delta_n)$ by $(\prod_t^c (h_n, \Delta_n))^{-1}$. Take a finite covering $\bigcup K_{c_i}$ of J, and use the property

$$\lim_{n\to\infty} \prod_c^e (h_n, \Delta_n) = \lim_{n\to\infty} \prod_d^e (h_n, \Delta_n) \lim_{n\to\infty} \prod_c^d (h_n, \Delta_n) .$$

Then, we obtain that $\lim_{n\to\infty} \prod_a^t (h_n, \Delta_n)$ converges uniformly on J. □

REMARK. The convergence of product integrals is a local property, as we saw in the argument.

We return to the stage of FL-groups and define a subgroup in the category of FL-structure.

Let G be an FL-group with Lie algebra $\mathfrak{G}$ and H a subgroup of G, possibly non-closed. For a neighborhood $\tilde{U}$ of e in G, denote by $\tilde{U}_0(H)$ the subset of $\tilde{U}$ consisting of all points $x \in \tilde{U} \cap H$ such that x and e can be connected by a piecewise smooth curve which is contained in $\tilde{U} \cap H$.

DEFINITION 2.2. Let G be an FL-group with Lie algebra $\mathfrak{G}$. A subgroup H of G is called an *FL-subgroup*, if there is a C^∞ local coordinate system $\zeta: U \to G$ at e such that $\zeta(0) = e$ where U is an open convex neighborhood of 0 in $\mathfrak{G}$, and there is a closed subspace $\mathfrak{H}$ of $\mathfrak{G}$ such that ζ and $\mathfrak{H}$ satisfy the following:

(*FLS*) ζ maps $U \cap \mathfrak{H}$ homeomorphically onto $\tilde{U}_0(H)$, where $\tilde{U} = \zeta(U)$.

Next, we induce a topology on the *FL*-subgroup H of *FL*-group G as follows: Notations being as above, let $\mathfrak{N}$ be the set of the open convex neighborhoods of 0 contained in U. Since $\tilde{W}_0(H) = \zeta(W \cap \mathfrak{H})$ for every $W \in \mathfrak{N}$, we see that $\{\zeta(W \cap \mathfrak{H}) : W \in \mathfrak{N}\}$ gives the topology under which H is an *FL*-group. Note that this topology is in general stronger than the relative topology (The above topology has been called LPSAC-*topology* in the category of strong ILB-Lie groups. See [4], I.4.). Thus, we shall indicate by (H, FL) (resp. (H, rel)) if H has the above topology (resp. the relative topology).

LEMMA 2.3. *Suppose H is an FL-subgroup of an FL-group G. Let X be an open connected neighborhood of* 0 *of a Fréchet space E.*

(i) *Suppose $\{\kappa_n\}$ is a sequence of continuous mappings κ_n of X into* (H, rel) *such that $\kappa_n(0) = e$ and $\{\kappa_n\}$ converges uniformly to a mapping $\kappa: X \to G$. Then, $\{\kappa_n\}$ converges uniformly in (H, FL), and $\kappa(X) \subset H$.*

(ii) *Let $\kappa: X \to (H, FL)$ be a continuous mapping such that $\kappa: X \to G$ is C^r* $(0 \leqq r \leqq \infty)$. *Then $\kappa: X \to (H, FL)$ is C^r.*

PROOF. (i) Let W be an arbitrary element in $\mathfrak{N}$ and set $\tilde{W} = \zeta(W)$, where these notations are as in Definition 2.2. As $\{\kappa_n\}$ is a Cauchy sequence in the uniform topology of mappings of X into (H, rel), there is a number n_0 such that if $n, m \geqq n_0$ then $\tilde{\kappa}_{n,m}(x) = \kappa_n(x)\kappa_m^{-1}(x) \in \tilde{W}$ for any $x \in X$. Since $\tilde{\kappa}_{n,m}(0) = e$ and $\tilde{\kappa}_{n,m}(X)$ is arcwise connected, we see that $\tilde{\kappa}_{n,m}(X) \subset \tilde{W}_0(H) = \zeta(W \cap \mathfrak{H})$. Hence $\{\kappa_n\}$ is a Cauchy sequence in the space of mappings of X to (H, FL) with the uniform topology. Note that $\lim_{n\to\infty} \kappa_n(x) = \kappa(x) \in G$. At this stage $\kappa(x)$ is only an element of G. However, since one may assume that $\tilde{W}_0(H)^- \subset H$ by a suitable choice of $W \in \mathfrak{N}$, where $\tilde{W}_0(H)^-$ is the closure of $\tilde{W}_0(H)$, we see that $\lim_{n\to\infty} \kappa_n(x)\kappa_{n_0}(x)^{-1} \in H$. Hence, $\kappa(x) \in H$ for $\kappa_{n_0}(x) \in H$. Consequently, $\{\kappa_n\}$ converges uniformly to κ in (H, FL).

(ii) Now, suppose $\kappa: X \to (H, FL)$ is a continuous mapping which can be regarded as C^r mapping of X into G. For an arbitrarily fixed $x_0 \in X$, $\tilde{\kappa}(x) = \kappa(x)\kappa(x_0)^{-1}$ is a continuous mapping of X into (H, FL) such that $\tilde{\kappa}(x_0) = e$. Fix $W \in \mathfrak{N}$ arbitrarily, and there is a neighborhood V of x_0 such that $\tilde{\kappa}(V) \subset \tilde{W}_0(H) = \zeta(W \cap \mathfrak{H})$. Note that $\tilde{\kappa}: V \to \zeta(W)$ is a C^r mapping, hence so is $\zeta^{-1}\tilde{\kappa}: V \to W$. However, $\zeta^{-1}\tilde{\kappa}(V) \subset W \cap \mathfrak{H}$. This implies that $\zeta^{-1}\tilde{\kappa}: V \to W \cap \mathfrak{H}$ is C^r, and hence so is $\tilde{\kappa}: V \to \zeta(W \cap \mathfrak{H})$. Thus, $\tilde{\kappa}$ is C^r on a neighborhood of x_0. Since $\kappa(x)=\tilde{\kappa}(x)\kappa(x_0)$, we get the desired result. □

Now, we are ready to prove the following:

PROPOSITION 2.4. *Every FL-subgroup of a regular Fréchet-Lie group is a regular Fréchet-Lie group.*

PROOF. Suppose H is an FL-subgroup of a regular Fréchet-Lie group G. We have only to show the convergence of product integrals. Let $h: [0, \varepsilon]\times J \to (H, FL)$ be a C^1-hair at e, where $J=[a, b]$ and let $\{(h_n, \Delta_n)\}$ be a sequence of step functions such that $\lim_{n\to\infty} |\Delta_n|=0$ and $\{h_n\}$ converges uniformly to h with their partial derivatives $\partial h_n/\partial s$. Set $\kappa_n(t)=\prod_a^t (h_n, \Delta_n)$. Then, regarding κ_n as a continuous mapping of J into G, $\{\kappa_n\}$ converges uniformly to the mapping $\kappa(t)=\prod_a^t (h, d\tau)$. Thus by (i) in Lemma 2.3, we see that $\{\kappa_n\}$ converges uniformly on J under the topology of (H, FL). Hence (H, FL) is a regular Fréchet-Lie group. □

§ 3. Lie algebra homomorphism.

Let G, H be FL-groups and $\mathfrak{G}$, $\mathfrak{H}$ their Lie algebras respectively. Suppose there is a C^∞ homomorphism Φ of G into H. Then the derivative $(d\Phi)_e$ at e gives a continuous linear mapping of $\mathfrak{G}$ into $\mathfrak{H}$.

LEMMA 3.1. *$(d\Phi)_e: \mathfrak{G}\to\mathfrak{H}$ is a Lie algebra homomorphism.*

PROOF. Let $g(t)$, $k(s)$ be C^∞ curves in G such that $g(0)=k(0)=e$ and $\dot{g}(0)=u$, $\dot{k}(0)=v$. Remark that $\Phi(g(t))$, $\Phi(k(s))$ are C^∞ curves in H such that $\Phi(g(0))=\Phi(k(0))=$identity and $d(\Phi(g(t)))/dt|_{t=0}=(d\Phi)_e u$, $d(\Phi(k(s)))/ds|_{s=0}=(d\Phi)_e v$. Then, we have by (2),

$$\begin{aligned}[(d\Phi)_e u, (d\Phi)_e v] &= \frac{\partial^2}{\partial t\partial s}\Big|_{\substack{t=0\\ s=0}} \Phi(g(t))\Phi(k(s))\Phi(g(t))^{-1} \\ &= \frac{\partial^2}{\partial t\partial s}\Big|_{\substack{t=0\\ s=0}} \Phi(g(t)k(s)g(t)^{-1}) \\ &= (d\Phi)_e[u, v] .\end{aligned}$$

So, we get the lemma. □

For regular Fréchet-Lie groups, we obtain a sort of converse of the above fact as follows, which is the goal of this section.

THEOREM 3.2. *Let G be a connected, simply connected regular Fréchet-Lie group, and H a regular Fréchet-Lie group. Let $\mathfrak{G}$, $\mathfrak{H}$ be the Lie algebras of G, H respectively. If there is a continuous homomorphism*

φ of $\mathfrak{G}$ into $\mathfrak{H}$, then there is a C^∞ homomorphism Φ of G into H such that $(d\Phi)_e=\varphi$.

REMARK. The above result shows that the local structures of regular Fréchet-Lie groups can be determined by their Lie algebras, for an isomorphism between two Lie algebras yields a C^∞ isomorphism between two universal covering groups. (cf. Proposition 2.1.)

We need several lemmas to prove Theorem 3.2.

For a Lie homomorphism φ of $\mathfrak{G}$ into $\mathfrak{H}$, let Γ be the subset of the product Lie algebra $\mathfrak{G}\times\mathfrak{H}$ composed of all elements of the form $(u, \varphi u)$, $u\in\mathfrak{G}$. Since φ is a continuous homomorphism, we see easily that Γ is a closed subalgebra of $\mathfrak{G}\times\mathfrak{H}$. Let Γ^* be the left-invariant distribution on $G\times H$ defined by

$$\Gamma^*(g, h)=dL_{(g,h)}\Gamma=\{(dL_g u, dL_h\varphi(u)); u\in\mathfrak{G}\} ,$$

where $L_{(g,h)}$, L_g, L_h mean the left translations, Γ^* may be regarded as an involutive distribution on $G\times H$, for Γ is a closed subalgebra. What we are going to make is the maximal integral submanifold $\mathscr{E}$ through the identity. If it were done, then $\mathscr{E}$ should be an FL-subgroup of $G\times H$ and $\mathscr{E}$ should give the graph of Φ.

REMARK. It is not known whether the Frobenius theorem holds in general in the category of regular Fréchet-Lie groups.

Let ξ: $U\to G$ be a C^∞ local coordinate system of G at e, where U is an open convex neighborhood of 0 in $\mathfrak{G}$ and $\xi(0)=e$. Also, we may assume that $(d\xi)_0=\mathrm{id}$. For each $u\in U$, we define a C^∞ mapping $\mu(u)$: $[0, 1]\to\mathfrak{G}$ by $\mu(u)(t)=((d/dt)\xi(tu))\xi(tu)^{-1}$. By the homomorphism φ of $\mathfrak{G}$ into $\mathfrak{H}$, we see that $\varphi\mu(u)$ is also a C^∞ mapping of $[0, 1]$ into $\mathfrak{H}$. Solve the equation $(d/dt)h(t)=\varphi\mu(u)h(t)$, $h(0)=e$. As H is a regular Fréchet-Lie group, $h(t)$ is given by the product integral. Thus, we set

$$\theta(t, u)=\prod_0^t (1+(\varphi\mu(u))(\tau))d\tau . \tag{3}$$

By the second fundamental theorem of product integrals, it is easy to see that θ: $[0, 1]\times U\to H$ is a C^∞ mapping such that $\theta(0, u)=e$.

LEMMA 3.3.

$$\varphi \operatorname{Ad}(\xi(tu))w=\operatorname{Ad}(\theta(t, u))\varphi w , \quad w\in\mathfrak{G} .$$

PROOF. By the same computation as in the proof of Lemma 1.3, we see

$$\frac{d}{dt}\varphi \operatorname{Ad}(\xi(tu))w=\varphi[\mu(u), \operatorname{Ad}(\xi(tu))w] \quad \text{(cf. [7], Lemma 2.3.)}$$
$$=[\varphi\mu(u), \varphi \operatorname{Ad}(\xi(tu))w] .$$

On the other hand,

$$\frac{d}{dt}\operatorname{Ad}(\theta(t, u))\varphi w=[\varphi\mu(u)(t), \operatorname{Ad}(\theta(t, u))\varphi w] ,$$

and if $t=0$ then two quantities coincide, hence by Lemma 1.3, we get the desired one. □

Now, consider a C^∞ Fréchet submanifold S of $G\times H$ given by

$$S=\{(\xi(u), \theta(1, u)); u \in U\} .$$

It is obvious that the tangent space of S at $(e, e)\in G\times H$ is given by Υ.

LEMMA 3.4. *The tangent space of S at $(g, h)\in S$ is $\Upsilon^*(g, h)$, i.e., S is an integral submanifold of the left-invariant distribution Υ^*.*

PROOF. Let $(g, h)=(\xi(u), \theta(1, u))$. The tangent space of S at (g, h) is given by $\{((d\xi)_u v, (d\theta_1)_u v); v \in \mathfrak{G}\}$, where $\theta_1(u)=\theta(1, u)$. Since $\Upsilon^*(g, h)=\{(dL_{\xi(u)}v, dL_{\theta_1(u)}\varphi v); v \in \mathfrak{G}\}$, we have only to show that

$$\{(dL^{-1}_{\xi(u)}(d\xi)_u v, dL^{-1}_{\theta_1(u)}(d\theta_1)_u v); v \in \mathfrak{G}\}=\Upsilon .$$

Since $dL^{-1}_{\xi(u)}(d\xi)_u\colon \mathfrak{G}\to\mathfrak{G}$ is a linear isomorphism, it is enough to show that

$$(4) \qquad \varphi dL^{-1}_{\xi(u)}(d\xi)_u v=dL^{-1}_{\theta_1(u)}(d\theta_1)_u v ,$$

for every $v \in \mathfrak{G}$. Note that $(\partial/\partial s)|_{s=0}\theta(t, u)^{-1}\theta(t, u+sv)$ is a C^∞ curve in $\mathfrak{H}$, which is 0 if $t=0$. Now, putting this fact in mind we compute as follows

$$(5) \qquad dL^{-1}_{\theta_1(u)}(d\theta_1)_u v$$
$$=\frac{d}{ds}\bigg|_{s=0}\theta_1(u)^{-1}\theta_1(u+sv)$$
$$=\int_0^1 \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial s}\bigg|_{s=0}\theta(t, u)^{-1}\theta(t, u+sv)\right\}dt$$
$$=\int_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} dL^{-1}_{\theta(t,u)}\varphi(\mu(u+sv)-\mu(u))(t)\cdot\theta(t, u+sv)dt \quad \text{(by (3))}$$
$$=\int_0^1 dL^{-1}_{\theta(t,u)}\varphi(d\mu)_u v\cdot\theta(t, u)dt$$
$$=\int_0^1 \operatorname{Ad}(\theta(t, u))^{-1}\varphi(d\mu)_u v dt .$$

Hence, by Lemma 3.3, we have

$$dL^{-1}_{\theta_1(u)}(d\theta_1)_u v=\int_0^1 \varphi \operatorname{Ad}(\xi(tu))^{-1}(d\mu)_u v dt$$
$$=\varphi \int_0^1 \operatorname{Ad}(\xi(tu))^{-1}(d\mu)_u v dt .$$

On the other hand, replacing $\theta(t, u)$ by $\xi(tu)$ in the above computation (5), we obtain also

$$dL^{-1}_{\xi(u)}(d\xi)_u v=\int_0^1 \operatorname{Ad}(\xi(tu))^{-1}(d\mu)_u v dt .$$

Thus, we get the desired equality (4). □

The above lemma shows that S is an integral submanifold of Υ^* through $(e, e)\in G\times H$. Since Υ^* is left-invariant, $L_{(g,h)}S$ is also an integral submanifold of Υ^* through $(g, h)\in G\times H$.

LEMMA 3.5. *Let $(g(s), h(s))$, $s\in[0, \infty)$, be a C^∞ curve in $G\times H$ such that $(g(0), h(0))\in S$ and $(d/ds)(g(s), h(s))\in\Upsilon^*(g(s), h(s))$ for every s. Let δ be the maximal number such that $g(s)\in\tilde{U}=\xi(U)$ for every $s\in[0, \delta)$. Then, $(g(s), h(s))\in S$ for every $s\in[0, \delta)$.*

PROOF. For $s\in[0, \delta)$, $g(s)$ can be written in the form $\xi(u(s))$. Let $\mu(u)(s)=dL^{-1}_{\xi(u(s))}\cdot(d/ds)\xi(u(s))$. Consider a C^∞ curve $(\xi(u(s)), \theta_1(u(s)))$ in $G\times H$. By definition, it is contained in S. Therefore, by Lemma 3.4,

$$\frac{d}{ds}(\xi(u(s)), \theta_1(u(s)))\in\Upsilon^*(\xi(u(s)), \theta_1(u(s))) .$$

This implies

$$\varphi\mu(u)(s)=dL^{-1}_{\theta_1((u(s))}\frac{d}{ds}\theta_1(u(s)) .$$

On the other hand, by the assumption $(g(s), h(s))$, we see

$$\left(dL^{-1}_{g(s)}\frac{d}{ds}g(s),\ dL^{-1}_{h(s)}\frac{d}{ds}h(s)\right)\in\Upsilon .$$

Therefore, $\varphi\mu(u)(s)=dL^{-1}_{h(s)}(d/ds)h(s)$ and hence $(d/ds)h(s)\theta_1(u(s))^{-1}=dL_{h(s)}\varphi\mu(u)(s)\cdot\theta_1(u(s))^{-1}-dL_{h(s)}\varphi\mu(u)(s)\cdot\theta_1(u(s))^{-1}\equiv 0$. Since $h(0)\theta_1(u(0))^{-1}=e$, we get $h(s)=\theta_1(u(s))$ for $s\in[0, \delta)$. This implies $(g(s), h(s))\in S$. □

Suppose $L_{(g,h)}S\cap L_{(g',h')}S\neq\emptyset$ for some (g, h), $(g', h')\in G\times H$. Let $\varepsilon(t)$ be any C^∞ curve in $L_{(g',h')}S$ such that $\varepsilon(0)\in L_{(g,h)}S$. Then $\varepsilon'(t)=L^{-1}_{(g,h)}\varepsilon(t)$

is a C^∞ curve in $G\times H$ such that $\varepsilon'(0)\in S$ and $(d/dt)\varepsilon'(t)\in\Upsilon^*(\varepsilon'(t))$. Thus, by Lemma 3.5, $\varepsilon'(t)\in S$ whenever $\pi\varepsilon'(t)\in\tilde{U}=\xi(U)$, where $\pi\colon G\times H\to G$ is a natural projection. Therefore, $L_{(g,h)}S\cap L_{(g',h')}S$ contains the connected component of

$$L_{(g,h)}S\cap\pi^{-1}(g\tilde{U}\cap g'\tilde{U})$$

containing $\varepsilon(0)$. Thus, $L_{(g,h)}S\cap L_{(g',h')}S$ is an open subset of $L_{(g,h)}S$. Similarly, it is an open subset of $L_{(g',h')}S$. Hence, we get that $L_{(g,h)}S\cup L_{(g',h')}S$ is an integral submanifold of Υ^*.

PROOF OF THEOREM 3.2. Let $\mathcal{O}_S$ be the family of all open subsets of S, and make a topology on $G\times H$ by calling $\mathcal{O}_{G\times H}=\bigcup\{L_{(g,h)}\mathcal{O}_S;\,(g,h)\in G\times H\}$ a generator of open subsets. Let $\mathscr{E}$ be the connected component containing the identity (e, e) under the above topology. Since every $L_{(g,h)}S$ is an open subset of $\mathscr{E}$ for every $(g,h)\in\mathscr{E}$, we see that $\mathscr{E}$ is an integral submanifold of Υ^*.

It is obvious that $\mathscr{E}$ is a connected subgroup of $G\times H$ because $L_{(g,h)}\mathscr{E}=\mathscr{E}$ for any $(g,h)\in\mathscr{E}$. As $\pi\colon L_{(g,h)}S\to g\tilde{U}$ is a C^∞-diffeomorphism, $\pi\colon\mathscr{E}\to G$ is a C^∞ isomorphism, for G is connected and simply connected. Thus, regarding $\mathscr{E}$ as the graph of a C^∞ homomorphism Φ of G into H, we get $(d\Phi)_e=\varphi$. □

REMARK. The proof of Theorem 3.2 works well under the assumption that G is an FL-group.

§ 4. Frobenius theorem for finite codimensional closed Lie subalgebras.

Let G be an FL-group, and H an FL-subgroup of G.

DEFINITION 4.1. H is called to be *locally flat*, if $\{\mathfrak{H}, \zeta, U\}$ given in the definition (Definition 2.2) of FL-subgroups satisfy the following:

(a) $\zeta\colon U\to G$ satisfies (FLS) in Definition 2.2.

(b) There is a complementary subspace $\mathfrak{M}$ of $\mathfrak{H}$ in $\mathfrak{G}$.

(c) There are neighborhoods V, W of zeros of $\mathfrak{H}$, $\mathfrak{M}$ respectively such that $U=V\oplus W$ and $\zeta(V\oplus m)=\zeta(V)\cdot\zeta(m)$ for every $m\in W$.

In this section, we consider a finite codimensional closed Lie subalgebra $\mathfrak{H}$ of the Lie algebra $\mathfrak{G}$ of a regular Fréchet-Lie group G. As $\dim\mathfrak{G}/\mathfrak{H}=\dim\mathfrak{M}<\infty$, there is a finite dimensional complementary subspace $\mathfrak{M}$ of $\mathfrak{H}$, that is,

$$\mathfrak{G}=\mathfrak{H}\oplus\mathfrak{M}\,. \tag{6}$$

Let $p: \mathfrak{G} \to \mathfrak{G}/\mathfrak{H} = \mathfrak{M}$ be the projection associated with (6). The goal of this section is the following:

THEOREM 4.2. *Let $\mathfrak{H}$ be a finite codimensional closed Lie subalgebra of $\mathfrak{G}$ of a regular Fréchet-Lie group G. Then, there is a locally flat FL-subgroup H of G having $\mathfrak{H}$ as its Lie algebra.*

REMARK. This theorem has been already known by Leslie [3] under slightly stronger conditions. We shall give here another proof, for the method and lemmas used here will be useful in the future coming papers in this series.

To prove the above theorem, we consider a right-invariant distribution $\tilde{\mathfrak{H}}$ on G given by

$$\tilde{\mathfrak{H}}(g) = dR_g \mathfrak{H} = \{dR_g u;\ u \in \mathfrak{H}\}\ , \tag{7}$$

and make the maximal integral submanifold of $\tilde{\mathfrak{H}}$ through the identity. This will be done by proving the *Frobenius theorem* for the distribution $\tilde{\mathfrak{H}}$.

Let $\xi: U = V \oplus W \to G$ be a C^∞ local coordinate system at e with the same property as in (a), (b) of Definition 4.1 where V, W are open neighborhoods of zeros of $\mathfrak{H}$, $\mathfrak{M}$ respectively, and $\xi(0+0) = e$. For every $v+w \in V \oplus W$, define a linear map $j(v+w)$ by

$$j(v+w) = dR^{-1}_{\xi(v+w)} (d\xi)_{v+w}\ .$$

Since $\dim \mathfrak{M} < \infty$, one may assume that

$$pj(v+w): \mathfrak{M} \longrightarrow \mathfrak{M}$$

is an isomorphism for every $v+w \in V \oplus W$, whose inverse will be denoted by $I(v+w)$. Hence I is a C^∞ mapping of $V \oplus W$ into $GL(\mathfrak{M})$. Set

$$\begin{cases} \Phi(v+w)u = u - j(v+w)\cdot I(v+w)pu \\ \Psi(v+w)u = j(v+w)I(v+w)pu\ , \end{cases}$$

for $u \in \mathfrak{G}$. We see easily that $\Phi(v+w)u \in \mathfrak{H}$, $\Psi(v+w)u \in j(v+w)\mathfrak{M}$. Φ and Ψ are C^∞ and give a direct decomposition of $\mathfrak{G}$ other than (6). Therefore, we have

$$\operatorname{codim} \mathfrak{H} = \operatorname{codim} j(v+w)\mathfrak{H}\ . \tag{9}$$

Define the mapping $\varphi(v+w)$, $v+w \in V \oplus W$ by

$$\varphi(v+w)u = pj(v+w)^{-1}\Phi(v+w)j(v+w)u\ , \quad u \in \mathfrak{G}\ . \tag{10}$$

LEMMA 4.3. *φ is a C^∞ mapping of $(V \oplus W) \times \mathfrak{H}$ into $\mathfrak{M}$ and*

$$(d\xi)_{v+w}\{u+\varphi(v+w)u;\ u\in\mathfrak{H}\}=\tilde{\mathfrak{H}}(\xi(v+w))\ .$$

PROOF. Remark that for every $u\in\mathfrak{G}$,

$$j(v+w)u=\Phi(v+w)j(v+w)u+\Psi(v+w)j(v+w)u\ .$$

Then, we get

$$u=(1-p)j(v+w)^{-1}\Phi(v+w)j(v+w)u+\varphi(v+w)u$$
$$+j(v+w)^{-1}\Psi(v+w)j(v+w)u\ .$$

Note that the last two terms are contained in $\mathfrak{M}$. Thus, if $u\in\mathfrak{H}$, then

$$u=(1-p)j(v+w)^{-1}\Phi(v+w)j(v+w)u\ ,$$

and hence

$$u+\varphi(v+w)u=j(v+w)^{-1}\Phi(v+w)j(v+w)u\in j(v+w)^{-1}\mathfrak{H}\ .$$

Therefore, $\{u+\varphi(v+w)u;\ u\in\mathfrak{H}\}\subset j(v+w)^{-1}\mathfrak{H}$. To prove the equality, remark that $\operatorname{codim}\{u+\varphi(v+w)u;\ u\in\mathfrak{H}\}=\operatorname{codim}\mathfrak{H}$ and use (9). Then, we get the desired equality. □

Let V_1, W_1 be convex neighborhoods of zeros in V, W respectively such that $\bar{V}_1\subset V$, $\bar{W}_1\subset W$ and $\bar{W}_1$ is compact. Let $\mathscr{Y}$ be a Fréchet space consisting of all C^∞ mappings of $\bar{W}_1\times[0, 1]$ into $\mathfrak{M}$. Define a mapping χ: $V_1\to\mathscr{Y}$ by

$$\chi(v)(w, t)=\varphi(tv+w)v\ .$$

Then, χ is a C^∞ mapping such that $\chi(0)=0$.

The following lemma is well-known as the uniqueness and the regularity of solutions of ordinary differential equations (cf. [3]):

LEMMA 4.4. *There is an open neighborhood N of 0 in $\mathscr{Y}$ and an open neighborhood W_2 of 0 in W_1 satisfying the following*:

(i) *For any $X\in N$, the differential equation*

$$\frac{d}{dt}\omega(t)=X(\omega(t), t)\ ,\qquad \omega(0)=w\in W_2$$

can be solved uniquely for $t\in[0, 1]$ and $\omega(t)\in W_1$.

(ii) *Let $\psi(X, w, t)$ be the solution of the above equation. Then, ψ: $N\times W_2\times[0, 1]\to W_1$ is a C^∞ mapping.*

Since $\chi(0)=0$ and χ: $V_1\to\mathscr{Y}$ is C^∞, there is an open convex neighborhood V_2 of 0 in V_1 such that $\bar{V}_2\subset V_1$ and $\chi(\bar{V}_2)\subset N$. Now, consider the equation

$$\frac{d}{dt}\omega(t)=\varphi(tv+\omega(t))v\ , \qquad \omega(0)=w\in W_2\ , \tag{11}$$

for every $v\in V_2$. Let $\psi(v, w, t)$ be the solution of (11) and set $\psi_1(v, w)=\psi(v, w, 1)$. Then, ψ_1: $V_2\oplus W_2\to W_1$ is a C^∞ mapping by virture of the Lemma 4.4, and it is easy to see $\psi_1(tv, w)=\psi(v, w, t)$.

For every fixed $w\in W_2$, $\{(v, \psi_1(v, w)); v\in V_2\}$ forms a C^∞ Fréchet-submanifold S_w of $V_2\oplus W_1$ of finite codimension. What we are going to show is that $\xi(S_w)$ is an integral submanifold of $\tilde{\mathfrak{H}}$.

Let $(v, \psi_1(v, w))$ be a point of S_w. The tangent space of S_w at this point is given by $\{v_1+(d\psi_1)_{(v,w)}v_1; v_1\in\mathfrak{H}\}$. Thus, we have only to show that this equals $j(v+\psi_1(v, w))^{-1}\mathfrak{H}$. Hence, it is enough to show

$$j(v+\psi_1(v, w))\{v_1+(d\psi_1)_{(v,w)}v_1;\ v_1\in\mathfrak{H}\}\subset\mathfrak{H}\ ,$$

because if so, then we have the equality by considering codimensions. Set

$$\mathscr{L}(v_1)=pj(v+\psi_1(v, w))(v_1+(d\psi_1)_{(v,w)}v_1)\ ,$$

then we have only to show that $\mathscr{L}(v_1)=0$ for every $v_1\in\mathfrak{H}$. Indeed, we get

$$\begin{aligned}\mathscr{L}(v_1)&=p\cdot\frac{d}{ds}\Big|_{s=0}\xi(v+sv_1+\psi_1(v+sv_1, w))\xi(v+\psi_1(v, w))^{-1}\\ &=p\int_0^1\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{s=0}k(t, s)k(t, 0)^{-1}dt\\ &=pd\xi\int_0^1\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{s=0}\xi^{-1}(k(t, s)k(t, 0)^{-1})dt\ ,\end{aligned}$$

where $k(t, s)=\xi(tv+tsv_1+\psi_1(tv+tsv_1, w))$.

Hence, we have

$$\mathscr{L}(v_1)=pd\xi\int_0^1\Big[\frac{\partial}{\partial s}\Big|_{s=0}(d\xi)^{-1}\frac{\partial}{\partial t}k(t, s)k(t, 0)^{-1}\Big]dt\ .$$

Set $(\partial/\partial t)\xi(tv'+\psi_1(tv', w))=dL_{\xi(tv'+\psi_1(tv',w))}\Xi(tv')$. Then, we get

$$\frac{\partial}{\partial t}k(t, s)k(t, 0)^{-1}=dL_{k(t,s)}dR^{-1}_{k(t,0)}(\Xi(tv+tsv_1)-\Xi(tv))\ .$$

Therefore, we see

$$\mathscr{L}(v_1)=p\int_0^1\mathrm{Ad}\,(\xi(tv+\psi_1(tv, w)))\frac{\partial}{\partial s}\Big|_{s=0}\Xi(tv+tsv_1)dt\ . \tag{12}$$

LEMMA 4.5. *Notations and assumptions being as above, we have*

$$\Xi(tv') \in \mathrm{Ad}\,(\xi(w))^{-1}\mathfrak{H}\ , \quad \textit{for every} \quad v' \in \mathfrak{H}\ .$$

PROOF. Recall that $\Xi(tv')$ is defined by $(d/dt)\xi(tv'+\psi_1(tv', w))=dL_{\xi(tv'+\psi_1(tv',w))}\Xi(tv')$. For simplicity we denote $k(t)=\xi(tv'+\psi_1(tv', w))$, and we set $(d/dt)k(t)=u(t)\cdot k(t)$. Then by the definition of ψ_1 we see that $u(t)\in\mathfrak{H}$ and $k(0)=\xi(w)$. Moreover, we see easily $\Xi(tv')=\mathrm{Ad}\,(k(t)^{-1})u(t)$. Thus, $\Xi(tv')\in\mathrm{Ad}\,(k(t)^{-1})\mathfrak{H}$. Set $h(t)=k(t)k(0)^{-1}$. Then $(d/dt)h(t)=u(t)\cdot h(t)$ and $h(0)=e$. Hence by Lemma 1.3 we have $\mathrm{Ad}\,(h(t))\mathfrak{H}=\mathfrak{H}$. Therefore, $\mathrm{Ad}\,(k(t)^{-1})\mathfrak{H}=\mathrm{Ad}\,(\xi(w))^{-1}\mathfrak{H}$. □

By the above lemma, we see that $(\partial/\partial s)|_{s=0}\Xi(t(v+sv_1))\in\mathrm{Ad}\,(\xi(w)^{-1})\mathfrak{H}$, for $\mathrm{Ad}\,(\xi(w)^{-1})\mathfrak{H}$ is a closed subspace of $\mathfrak{G}$. Therefore, we have

$$\mathrm{Ad}\,(\xi(tv+\psi_1(tv, w)))\frac{\partial}{\partial s}\Big|_{s=0}\Xi(tv+tsv_1)\in\mathrm{Ad}\,(\xi(tv+\psi_1(tv, w))\xi(w)^{-1})\mathfrak{H}\ .$$

Recall that $\xi(tv+\psi_1(tv, w))\xi(w)^{-1}=h(t)$ in the proof of Lemma 4.5. Hence by the above lemma again, we see that $\mathrm{Ad}\,(h(t))\mathfrak{H}=\mathfrak{H}$. By (12) we get $\mathscr{L}(v_1)=0$, and we obtain the following:

LEMMA 4.6. *$\xi(S_w)$ is an integral submanifold of $\tilde{\mathfrak{H}}$ for every $w\in W_2$.*

As $\tilde{\mathfrak{H}}$ is right-invariant, $\xi(S_w)g$ is also an integral submanifold of $\tilde{\mathfrak{H}}$ for every $g\in G$.

LEMMA 4.7. *Let $g(t)$ be a C^∞ curve in G such that $g(0)\in\xi(S_w)$ and $(d/dt)g(t)\in\tilde{\mathfrak{H}}(g(t))$. Then as far as $g(t)$ is contained in $\xi(V_2\oplus W_1)$, $g(t)$ is in fact contained in $\xi(S_w)$.*

PROOF. Put $\xi^{-1}(g(t))=v(t)+w(t)$, $v(t)\in V_2$, $w(t)\in W_1$. Then, $v(t)+\psi_1(v(t), w)$ is a C^∞ curve in S_w such that $\xi(v(0)+\psi_1(v(0), w))=g(0)\in\xi(S_w)$. Evidently,

$$\frac{d}{dt}(v(t)+\psi_1(v(t), w))=\frac{d}{dt}v(t)+\varphi(v(t)+\psi_1(v(t), w))\frac{d}{dt}v(t)\ ,$$

because S_w is an integral submanifold of $(d\xi)^{-1}\tilde{\mathfrak{H}}$. On the other hand, we have

$$\frac{d}{dt}(v(t)+w(t))=\frac{d}{dt}v(t)+\varphi(v(t)+w(t))\frac{d}{dt}v(t)$$

because the left hand side of the above equation must be contained in $(d\xi)^{-1}\mathfrak{H}^*(\xi(v(t)+w(t)))$ and this is given by $\{u+\varphi(v(t)+w(t))u;\ u\in\mathfrak{H}\}$.

Therefore, we see that $\psi_1(v(t), w)$ and $w(t)$ satisfy the same differential equation with the same initial condition. Since W_1 is finite dimensional, we get $w(t)=\psi_1(v(t), w)$ by the uniqueness theorem. $\square$

PROOF OF THEOREM 4.2. Let $\mathcal{O}(S_w)$ be the family of all open subsets of $\xi(S_w)$. We denote by $\mathcal{O}(G)$ the family $\bigcup\{\mathcal{O}(S_w)g; g \in G\}$, where $\mathcal{O}(S_w)\cdot g = \{U\cdot g; U \in \mathcal{O}(S_w)\}$. If $U_1\cdot g_1 \cap \cdots \cap U_N\cdot g_N \neq \emptyset$ for some $U_i \in \mathcal{O}(S_w)$ and $g_i \in G$, $i=1, 2, \cdots, N$, then this is an element of $\mathcal{O}(S_w)\cdot g_1$ by virture of Lemma 4.7. Hence regarding $\mathcal{O}(G)$ as a basis of open sets of G, $\mathcal{O}(G)$ defines a topology on G. This topology makes G a locally connected topological group. Let H be the identity component. Then, H is a subgroup of G. Since $\xi(S_0)$ is an open subset of H, H is an FL-group with the Lie algebra $\mathfrak{H}$.

Consider a C^∞ mapping $\P(v, w)=\xi^{-1}(\xi(v+\psi_1(v, 0))\xi(w))$. It is well-defined on a neighborhood $V_3 \oplus W_3$ of $\mathfrak{G}$ such that $\bar{V}_3 \subset V_2$, $\bar{W}_3 \subset W_2$, and it satisfies $\P(0, 0)=0$. Since $\P(V_3, w) \subset S_w$ for sufficiently small V_3, $\P(v, w)$ can be written in the form $(v', \psi_1(v', w))$ for some $v' \in V_2$. Remark that $(D_2\psi_1)_{(0,0)}: \mathfrak{M} \to \mathfrak{M}$ is the identity. Thus, by the implicit function theorem which will be proved below, the equation $w'=\psi_1(v', w)$ can be smoothly solved with respect to w if w' is sufficiently close to 0. By $w=\alpha(v', w')$ we denote the solution, i.e., $w' \equiv \psi_1(v', \alpha(v', w'))$.

Now, remark that $\xi(v'+\psi_1(v', w))\xi(w)^{-1} \in H$. Therefore, if v' and w are sufficiently small, then we see easily that

$$\xi(v'+\psi_1(v', w))\xi(w)^{-1}=\xi(v+\psi_1(v, 0)) .$$

This shows that v depends smoothly on v' and $w=\alpha(v', w')$. Therefore, $\P$ is a C^∞ diffeomorphism of a neighborhood $V_4 \oplus W_4$ of zero of $\mathfrak{G}$ onto a neighborhood $\P(V_4 \oplus W_4)$ of zero. Obviously, $\xi\P(v, w)=\xi(v+\psi_1(v, 0))\xi(w)=\xi\P(v, 0)\xi\P(0, w)$. It follows that H is a locally flat FL-subgroup of G. $\square$

So, nothing remains but to prove an implicit function theorem for ψ_1, which is stated as follows:

LEMMA 4.8 (An implicit function theorem). *Let U_1 be an open neighborhood of 0 in a Fréchet space $\boldsymbol{E}$, and V_1 an open neighborhood of 0 in $\boldsymbol{R}^n$. Suppose there is a C^∞ mapping f of $U_1 \oplus V_1$ into $\boldsymbol{R}^n$ such that $f(0, 0)=0$. If $(D_2f)_{(0,0)}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is invertible, then there are neighborhoods V_2 of 0 in V_1 and U_2 of 0 in U_1 and C^∞ mapping $\varphi: U_2 \oplus V_2 \to V_1$ such that $f(u, \varphi(u, v)) \equiv v$.*

PROOF. Without loss of generality, one may assume that $(D_2f)_{(0,0)}$

is the identity. Set $f_u(v)=f(u, v)$. There is a neighborhood U_1' of 0 in U_1 such that $(Df_u)_0$ is invertible for every $u\in U_1'$. Moreover, there are an ε-neighborhood $V(\varepsilon)$ of 0 in R^n and an open neighborhood U_2' such that $|f_u(0)|<\varepsilon/4$ and $|\mathrm{Id}-(Df_u)_v|<1/2$ for every $u\in U_2'$, $v\in V(\varepsilon)$. Obviously, $|f_u(v)-f_u(0)|\geqq|v|-\int_0^1|(\mathrm{Id}-(Df_u)_{tv})v|dt\geqq(1/2)|v|$. Hence, $f_u(V(\varepsilon))\supset f_u(0)+V(\varepsilon/2)$. Thus $f_u(V(\varepsilon))\supset V(\varepsilon/4)$ for every $u\in U_2'$. Since f_u is a C^∞ diffeomorphism of $V(\varepsilon)$ into R^n and f_u is C^∞ with respect to $u\in U_2'$, so is $f_u^{-1}\colon V(\varepsilon/4)\to R^n$. Set $\varphi(u, v)=f_u^{-1}(v)$. Then, we get $f(u, \varphi(u, v))\equiv v$. □

§ 5. Extensions of regular Fréchet-Lie groups.

Suppose we have an abstract group G and a normal subgroup N. In this section, we assume the following:

(Ext. 0) N and G/N are FL-groups under certain topologies, respectively.

Therefore, we have an exact sequence

$$1 \longrightarrow N \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } G/N \longrightarrow 1 \tag{13}$$

where i, π denote the natural inclusion and projection respectively. We shall consider at first when G becomes an FL-group.

DEFINITION 5.1. We call the exact sequence (13) an *FL-extension of* G/N if (Ext. 0) is satisfied and also if the following conditions are satisfied:

(Ext. 1) There is an open neighborhood $\hat{V}$ of the identity $\hat{e}$ in G/N, and a local cross-section $\gamma\colon \hat{V}\to G$ such that $\gamma(\hat{e})=e$ (the identity), namely, $\pi\gamma=\mathrm{id.}$ on $\hat{V}$ and the mapping $(\hat{g}, n)\to\gamma(\hat{g})n$ gives a one-to-one correspondence of $\hat{V}\times N$ onto $\pi^{-1}(\hat{V})$.

(Ext. 2) The mapping $r_\gamma\colon \hat{V}_1\times\hat{V}_1\to N$ defined by

$$r_\gamma(\hat{g}, \hat{h})=\gamma(\hat{g}\hat{h})^{-1}\gamma(\hat{g})\gamma(\hat{h}) \tag{14}$$

is C^∞, where $\hat{V}_1$ is a neighborhood of $\hat{e}$ in $\hat{V}$ such that $\hat{V}_1^2\subset\hat{V}$, $\hat{V}_1^{-1}=\hat{V}_1$.

(Ext. 3) The mapping $\alpha_\gamma\colon \hat{V}_1\times N\to N$ defined by

$$\alpha_\gamma(\hat{g}, m)=\gamma(\hat{g})^{-1}m\gamma(\hat{g}) \quad (=A(\gamma(\hat{g})^{-1}, m))\ , \tag{15}$$

is C^∞, where $A(n, m)=nmn^{-1}$, $n, m\in G$. We often denote $\alpha_\gamma(\hat{g}, m)$ by $\alpha_\gamma(\hat{g})m$.

PROPOSITION 5.2. *For an FL-extension of G/N, if G is generated by $\pi^{-1}(\hat{V}_1)$, then G has a natural structure of an FL-group such that N*

is a locally flat FL-subgroup of G and $\gamma: \hat{V}_1 \to G$ *is a* C^∞ *local section and* $\pi: G \to G/N$ *is* C^∞.

PROOF. By (Ext. 1) every element of $\pi^{-1}(\hat{V}_1)$ can be written as $\gamma(\hat{g})m$. Using (14), (15), we easily have

$$\text{(16)}\qquad \begin{cases} \gamma(\hat{g})m\gamma(\hat{h})n = \gamma(\hat{g}\hat{h})r_\gamma(\hat{g}, \hat{h})\alpha_\gamma(\hat{h}, m)n \ , \\ (\gamma(\hat{g})m)^{-1} = \gamma(\hat{g}^{-1})\alpha_\gamma(\hat{g}^{-1}, m)^{-1}r_\gamma(\hat{g}, \hat{g}^{-1})^{-1} \ . \end{cases}$$

Hence, it is not hard to see that $\pi^{-1}(\hat{V}_1)$ has a structure of a local FL-group, regarding $(\hat{g}, m) \to \gamma(\hat{g})m$ as a local coordinate system. It is obvious that $\gamma: \hat{V}_1 \to \pi^{-1}(\hat{V}_1)$ and $\pi: \pi^{-1}(\hat{V}_1) \to \hat{V}_1$ are C^∞ mappings. Since G is generated by $\pi^{-1}(\hat{V}_1)$, we see that G has a structure of an FL-group. Since $\pi^{-1}(\hat{V}_1)$ is C^∞ diffeomorphic to $N \times \hat{V}_1$ through the mapping $(n, \hat{g}) \to \gamma(\hat{g})n = A(\gamma(\hat{g}), n)\gamma(\hat{g})$ N is a locally flat FL-subgroup of G. □

Remark that by (16), the associative law on G is equivalent to the equality:

$$r_\gamma(\hat{g}, \hat{h}\hat{k})r_\gamma(\hat{h}, \hat{k}) = r_\gamma(\hat{g}\hat{h}, \hat{k})\alpha_\gamma(\hat{k}, r_\gamma(\hat{g}, \hat{h})) \ .$$

Let G be an FL-group and assume for G to have the FL-extension (13). Now, we shall compute the Lie algebra of G. Let $\mathfrak{K}$, $\mathfrak{N}$ and $\mathfrak{G}$ be the Lie algebras of $K = G/N$, N and G respectively. Notations being as above, set $\Gamma = (d\gamma)_{\hat{e}}$ and $\Gamma: \mathfrak{K} \to \mathfrak{G}$ is a continuous linear mapping such that $(d\pi)_{\hat{e}}\Gamma = \mathrm{id}$, and $\mathfrak{G} = \Gamma\mathfrak{K} \oplus \mathfrak{N}$. Define R_Γ and a_Γ by

$$\text{(17)}\qquad \begin{cases} R_\Gamma(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X, Y]) \ , & X, Y \in \mathfrak{K} \ , \\ a_\Gamma(X)v = -[\Gamma(X), v] \ , & X \in \mathfrak{K} \ , \quad v \in \mathfrak{N} \ . \end{cases}$$

To get the structure of $\mathfrak{G}$ is to compute R_Γ and a_Γ, because the Lie algebra structure of $\mathfrak{G}$ is given by

$$[\Gamma(X) + v, \Gamma(Y) + w] = \Gamma([X, Y]) + R_\Gamma(X, Y) - a_\Gamma(X)w + a_\Gamma(Y)v + [v, w] \ , \quad \text{for} \quad X, Y \in \mathfrak{K} \ , \quad v, w \in \mathfrak{N} \ .$$

LEMMA 5.3. (i) $R_\Gamma(X, Y) = (D_1D_2r_\gamma)_{(\hat{e},\hat{e})}(X, Y) - (D_1D_2r_\gamma)_{(\hat{e},\hat{e})}(Y, X)$.

(ii) *Let* $\hat{g}(t)$, $h(s)$ *be* C^∞ *curves in* K, N *respectively such that* $(d/dt)|_{t=0}\,\hat{g}(t) = X$, $(d/ds)|_{s=0}h(s) = v$. *Then,*

$$a_\Gamma(X)v = \frac{\partial^2}{\partial t \partial s}\Big|_{\substack{s=0\\t=0}} \alpha_\gamma(\hat{g}(t))h(s) \ .$$

(iii) *The Jacobi identity is equivalent to the following:*

$$\begin{cases} \underset{X,Y,Z}{\mathfrak{S}} \{a_\Gamma(X)R_\Gamma(Y, Z) - R_\Gamma([X, Y], Z)\} = 0 & \text{(2nd Bianchi identity)}, \\ \{[a_\Gamma(X), a_\Gamma(Y)] - a_\Gamma([X, Y])\}w = [R_\Gamma(X, Y), w], \end{cases}$$

for $X, Y, Z \in \mathfrak{K}$ *and* $w \in \mathfrak{N}$, *where* $\mathfrak{S}$ *means the cyclic summation.*

PROOF. The equality in (ii) is obvious by Lemma 2.3 in [7].

To prove (i), let $\hat{g}(t)$, $\hat{h}(s)$ be C^∞ curves in K such that $\hat{g}(0) = \hat{h}(0) = \hat{e}$, $(d/dt)|_{t=0}\hat{g} = X$ and $(d/ds)|_{s=0}\hat{h} = Y$. Note that

$$\begin{aligned}[][\Gamma(X), \Gamma(Y)] &= \frac{\partial^2}{\partial t \partial s}\bigg|_{\substack{s=0\\t=0}} \gamma(\hat{g}(t))\gamma(\hat{h}(s))\gamma(\hat{g}(t))^{-1} \\ &= \frac{\partial^2}{\partial t \partial s}\bigg|_{\substack{s=0\\t=0}} \gamma(\hat{g}(t))\gamma(\hat{h}(s))\gamma(\hat{g}(t)^{-1})r_\gamma(\hat{g}(t), \hat{g}(t)^{-1})^{-1} \\ &= \frac{\partial^2}{\partial t \partial s}\bigg|_{\substack{s=0\\t=0}} \{\gamma(\hat{g}(t)\hat{h}(s)\hat{g}(t)^{-1})r_\gamma(\hat{g}(t), \hat{h}(s)\hat{g}(t)^{-1}) \\ &\qquad \times r_\gamma(\hat{h}(s), \hat{g}(t)^{-1})r_\gamma(\hat{g}(t), \hat{g}(t)^{-1})^{-1}\}. \end{aligned}$$

Since $r_\gamma(\hat{g}, \hat{e}) = r_\gamma(\hat{e}, \hat{h}) = \hat{e}$, we see

$$\begin{aligned}[][\Gamma(X), \Gamma(Y)] &= \frac{d}{dt}\bigg|_{t=0} \Big\{\Gamma(\mathrm{Ad}\,(\hat{g}(t))Y) \\ &\qquad + \frac{\partial}{\partial s}\bigg|_{s=0} r_\gamma(\hat{g}(t), \hat{h}(s)\hat{g}(t)^{-1})r_\gamma(\hat{g}(t), \hat{g}(t)^{-1})^{-1} \\ &\qquad + \mathrm{Ad}\,(r_\gamma(\hat{g}(t), \hat{g}(t)^{-1}))(D_1 r_\gamma)_{(\hat{e}, \hat{g}(t)^{-1})}(Y)\Big\} \\ &= \Gamma([X, Y]) + \frac{\partial^2}{\partial t \partial s}\bigg|_{\substack{s=0\\t=0}} r_\gamma(\hat{g}(t), \hat{h}(s)\hat{g}(t)^{-1})r_\gamma(\hat{g}(t), \hat{g}(t)^{-1})^{-1} \\ &\qquad - (D_2 D_1 r_\gamma)_{(\hat{e}, \hat{e})}(Y, X), \end{aligned}$$

because of $(D_1 r_\gamma)_{(\hat{e}, \hat{e})}(Y) = 0$. Since

$$\begin{aligned} \frac{\partial}{\partial t}\bigg|_{t=0} r_\gamma(\hat{g}(t), \hat{h}(s)\hat{g}(t)^{-1}) &= \frac{\partial}{\partial t}\bigg|_{t=0} r_\gamma(\hat{g}(t), \hat{h}(s)) + \frac{\partial}{\partial t}\bigg|_{t=0} r_\gamma(\hat{e}, \hat{h}(s)\hat{g}(t)^{-1}) \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} r_\gamma(\hat{g}(t), \hat{h}(s)) = (D_1 r_\gamma)_{(\hat{e}, \hat{h}(s))}(X) \end{aligned}$$

and

$$\frac{\partial}{\partial t}\bigg|_{t=0} r_\gamma(\hat{g}(t), \hat{g}(t)^{-1}) = \frac{d}{dt}\bigg|_{t=0} r_\gamma(\hat{g}(t), \hat{e}) + \frac{d}{dt}\bigg|_{t=0} r_\gamma(\hat{e}, \hat{g}(t)^{-1}) = 0,$$

we get

$$R_\Gamma(X, Y) = -(D_1 D_2 r_\gamma)_{(\hat{e},\hat{e})}(Y, X) + \frac{d}{ds}\Big|_{s=0} (D_1 r_\gamma)_{(\hat{e},\hat{h}(s))}(X) .$$

Therefore, by (17), we obtain (i).

Equalities of (iii) are obtained by direct computations of

$$\mathfrak{S}[[\Gamma(X)+u', \Gamma(Y)+v'], \Gamma(Z)+w'] \equiv 0 . \qquad \square$$

Now, the purpose of this section is to prove the following:

THEOREM 5.4. *Let G be an FL-group and let N be a closed normal subgroup of G. Suppose N is a regular Fréchet-Lie group under the relative topology and G/N is a regular Fréchet-Lie group such that the natural projection $\pi: G \to G/N$ is C^∞. If there is a C^∞ local cross-section $\gamma: \hat{V} \to G$, where $\hat{V}$ is an open neighborhood of $\hat{e}$ in G/N, then G is a regular Fréchet-Lie group.*

Although the proof will be given in several lemmas below, we remark at first that *if in an FL-extension* (13), *N and G/N are regular Fréchet-Lie group, then the subgroup of G generated by $\pi^{-1}(\hat{V}_1)$ is a regular Fréchet-Lie group.*

Let $\{(h_n, \Delta_n)\}$ be a sequence of step functions in G defined on $[0, \varepsilon]\times[a, b]$ such that $\lim_{n\to\infty}|\Delta_n|=0$ and $\{h_n\}$ converges uniformly to a C^1-hair h at e with their partial derivatives $\partial h_n/\partial s$ (cf. the notation and the definition in [7]). Since $h_n(0, t)\equiv h(0, t)\equiv e$, one may assume $h([0, \varepsilon]\times[a, b])\subset\pi^{-1}(\hat{V})$ and $h_n([0, \varepsilon]\times[a, b])\subset\pi^{-1}(\hat{V})$ for every n, taking ε small if necessary.

Set $\hat{h}_n(s, t)=\pi h_n(s, t)$, $\hat{h}(s, t)=\pi h(s, t)$. As $\pi: G\to G/N$ is a C^∞ homomorphism, $\{(\hat{h}_n, \Delta_n)\}$ is a sequence of step functions in $K=G/N$ such that $\{\hat{h}_n\}$ converges uniformly to $\hat{h}$. For simplicity, we shall denote $\hat{g}_n(t)=\prod_a^t(\hat{h}_n, \Delta_n)$ and $\hat{g}(t)=\prod_a^t(\hat{h}, d\tau)$.

Using the continuity of $\hat{g}$ and the uniformity of the convergence of $\{\hat{g}_n\}$, we get easily the following:

LEMMA 5.5. *There is a number c, $a<c$, independent of n, such that if $a\leqq t\leqq c$ then $\hat{g}_n(t)$, $\hat{g}(t)\in\hat{V}_1$, where $\hat{V}_1$ is a neighborhood of $\hat{e}$ such that $\hat{V}_1^2\subset\hat{V}$, $\hat{V}_1^{-1}=\hat{V}_1$.*

For the given division Δ_n: $a=t_0<t_1<\cdots<t_n=b$, we denote by $\hat{g}_{\Delta_n}(t)$ the mapping defined by $\hat{g}_{\Delta_n}(t)=\hat{g}_n(t_k)$ for $t\in[t_k, t_{k+1})$.

LEMMA 5.6. *$\{\hat{g}_{\Delta_n}(t)\}$ converges uniformly to $\hat{g}(t)$ on $[a, b]$.*

PROOF. Note that $\{\hat{g}_n(t)\}$ converges uniformly to $\hat{g}(t)$. Since $\hat{g}(t)$ is uniformly continuous on $[a, b]$, for any $\delta>0$, there is $\delta'>0$ such that if

$|t-t'|<\delta'$, then $\rho(\hat{g}(t), \hat{g}(t'))<\delta/2$ where ρ is an invariant metric on K. For a sufficiently large n, we have $|\Delta_n|<\delta'$. Hence for any $t\in[a, b]$, there is $t_j\in\Delta_n$ such that $|t-t_j|<\delta'$. Thus, $\rho(\hat{g}(t_j), \hat{g}(t))<\delta/2$. Since $\{\hat{g}_n\}$ converges uniformly to $\hat{g}$, we see $\rho(\hat{g}(t_j), \hat{g}_n(t_j))<\delta/2$ for sufficiently large n. Thus, remarking $\hat{g}_{\Delta_n}(t)=\hat{g}_n(t_j)$, we get $\rho(\hat{g}_{\Delta_n}(t), \hat{g}(t))<\delta$. □

Define $\mu_n(s, t)$, $\mu(s, t)$ by

$$h_n(s, t)=\gamma(\hat{h}_n(s, t))\mu_n(s, t)\ , \qquad h(t, s)=\gamma(\hat{h}(s, t))\mu(s, t)\ ,$$

respectively. Obviously, $\mu_n(s, t)$, $\mu(s, t)\in N$ and it is easy to see the following:

Lemma 5.7. *$\{(\gamma(\hat{h}_n), \Delta_n)\}$ is a sequence of step functions in G such that $\lim_{n\to\infty}\gamma(\hat{h}_n(s, t))$ converges uniformly to a C^1-hair $\gamma(h(s, t))$ at e in G with their partial derivatives $\{(\partial/\partial s)\gamma(\hat{h}_n)\}$. Moreover, $\{(\mu_n, \Delta_n)\}$ is a sequence of step functions in N such that $\lim_{n\to\infty}\mu_n(s, t)$ converges uniformly to a C^1-hair μ at e in N with partial derivatives $\{\partial\mu_n/\partial s\}$.*

Now, set for $(s, t)\in[0, \varepsilon]\times[a, c]$,

$$\begin{cases}\lambda_n(s, t)=r_\gamma(\hat{h}_n(s, t), \hat{g}_{\Delta_n}(t))\alpha_\gamma(\hat{g}_{\Delta_n}(t), \mu_n(s, t))\\ \lambda(s, t)=r_\gamma(\hat{h}(s, t), \hat{g}(t))\alpha_\gamma(\hat{g}(t), \mu(s, t))\ ,\end{cases} \tag{18}$$

where $\alpha_\gamma(\hat{g}, n)=\gamma(\hat{g})^{-1}n\gamma(\hat{g})$, which are well-defined if ε is sufficiently small. Obviously, (λ_n, Δ_n) is a step function in N defined on $[0, \varepsilon]\times[a, c]$, where c is given in Lemma 5.5.

Lemma 5.8. *$\{\lambda_n\}$ converges uniformly to a C^1-hair $\lambda(s, t)$ at e in N with their partial derivatives $\{\partial\lambda_n/\partial s\}$ by taking $\lim_{n\to\infty}|\Delta_n|=0$.*

Proof. By Lemma 5.6, we see that $\gamma(\hat{g}_{\Delta_n}(t))$, $\gamma(\hat{g}_{\Delta_n}(t))^{-1}$ converges uniformly to $\gamma(\hat{g}(t))$, $\gamma(\hat{g}(t))^{-1}$ respectively. Thus, we see easily that $\{\lambda_n\}$ converges to λ. Taking the derivative of (15) with respect to s and using the fact that r_γ and α_γ are C^∞, we get that $\{\partial\lambda_n/\partial s\}$ converges uniformly to $\partial\lambda/\partial s$ because of Lemma 5.7. □

Proof of Theorem 5.4. By Lemma 5.8 and the assumption that N is a regular Fréchet-Lie group, we see that $\prod_a^t(\lambda_n, \Delta_n)$ converges uniformly to $\prod_a^t(\lambda, d\tau)$ for $t\in[a, c]$. Since $\{\gamma(\hat{g}_{\Delta_n}(t))\}$ converges uniformly to $\gamma(\hat{g}(t))$, $\gamma(\prod_a^t(\hat{h}_n, \Delta_n))\prod_a^t(\lambda_n, \Delta_n)$ converges uniformly to $\gamma(\prod_a^t(\hat{h}, d\tau))\prod_a^t(\lambda, d\tau)$ on $[a, c]$.

On the other hand, setting $A_n(t)=\prod_a^t(\lambda_n, \Delta_n)$, we get

$$\begin{aligned}\gamma\Big(\prod_a^t (\hat{h}_n, \Delta_n)\Big) \prod_a^t (\lambda_n, \Delta_n)\\ &= \gamma(\hat{g}_n(t))\Lambda_n(t)\\ &= \gamma(\hat{h}_n(t-t_k, t_k)\hat{g}_n(t_k))\lambda_n(t-t_k, t_k)\Lambda_n(t_k)\\ &= \gamma(\hat{h}_n(t-t_k, t_k))\gamma(\hat{g}_n(t_k))r_\Gamma(\hat{h}_n(t-t_k, t_k), \hat{g}_n(t_k))^{-1}\lambda_n(t-t_k, t_k)\Lambda_n(t_k)\\ &= \gamma(\hat{h}_n(t-t_k, t_k))\mu_n(t-t_k, t_k)\gamma(\hat{g}_n(t_k))\Lambda_n(t_k)\\ &= h_n(t-t_k, t_k)\gamma(\hat{g}_n(t_k))\Lambda_n(t_k).\end{aligned}$$

Thus, repeating this computation, we get

$$\gamma\Big(\prod_a^t (\hat{h}_n, \Delta_n)\Big) \prod_a^t (\lambda_n, \Delta_n) = \prod_a^t (h_n, \Delta_n) .$$

Therefore, $\{\prod_a^t (h_n, \Delta_n)\}$ converges uniformly to $\gamma(\prod_a^t (\hat{h}, d\tau)) \prod_a^t (\lambda, d\tau)$ for $t \in [a, c]$.

Remark that $\prod_a^t (h, d\tau) = \prod_s^t (h, d\tau) \prod_a^s (h, d\tau)$. By the same reason as in the proof of Proposition 2.1, the above convergence for $t \in [a, c]$ yields the uniform convergence on $[a, b]$. Thus, G is a regular Fréchet-Lie group. □

§ 6. Several examples of regular Fréchet-Lie groups.

As we proved in [7], every strong ILB-Lie group is a regular Fréchet-Lie group. In particular, every finite dimensional Lie group is a regular Fréchet-Lie group. Now, suppose we have a regular Fréchet-Lie group G and a C^∞ compact manifold M with or without boundary. By $\pi: \mathscr{E} \to M$, we denote a C^∞ fibre bundle over M with fibre G and with the automorphism group of G as the transformation group, where the transition function $g_{\alpha\beta}$ from a trivialization $U_\alpha \times G$ into $U_\beta \times G$ satisfies that

$$g_{\alpha\beta}: (U_\alpha \cap U_\beta) \times G \to G$$

is C^∞. Remark that we do not give any topology on the group of automorphisms of G. Let $C^k(\mathscr{E})$, $0 \leqq k \leqq \infty$, be the group of all C^k sections of $\mathscr{E}$, where the group-structure is given by fibrewise product. It is known that $C^k(\mathscr{E})$ is an FL-group for every k (cf. Lemmas 1.2 and 5.2 in [7] for an idea of the proof and also see [1]).

LEMMA 6.1. *Notations and assumptions being as above, $C^0(\mathscr{E})$ is a regular Fréchet-Lie group.*

PROOF. Let $\{(h_n(s, t), \Delta_n)\}$ be a sequence of step functions in $C^0(\mathscr{E})$ defined on $[0, \varepsilon] \times [a, b]$ such that $\lim_{n\to\infty} |\Delta_n| = 0$ and $\{h_n(s, t)\}$ converges uniformly to a C^1-hair $h: [0, \varepsilon]\times[a, b]\to C^0(\mathscr{E})$ with their partial derivatives $\{\partial h_n/\partial s\}$. For each $x\in M$, $\{h_n(s, t)(x)\}$ converges uniformly to $h(s, t)(x)$. Since each fibre of $\mathscr{E}$ is a regular Fréchet-Lie group, $\prod_a^t (h_n(x), \Delta_n)$ converges uniformly to $\prod_a^t (h(x), d\tau)$ with respect to $t\in[a, b]$. By the compactness of M, and by the same reasoning as in the proof of Lemma 3.7 of [7], we see that the above convergence is also uniform with respect to $(t, x)\in[a, b]\times M$. Thus, $\prod_a^t (h_n, \Delta_n)$ converges uniformly in $C^0(\mathscr{E})$, hence $C^0(\mathscr{E})$ is a regular Fréchet-Lie group. □

DEFINITION 6.2. Let G be an FL-group and $\boldsymbol{F}$ a Fréchet space. A mapping $\rho: G\times\boldsymbol{F}\to\boldsymbol{F}$ is called a *smooth representation*, if ρ satisfies the following:

(i) ρ is C^∞ and $\rho(g, u)$ is linear with respect to $u\in\boldsymbol{F}$, $\rho(g, u)$ will be denoted sometimes by $\rho_g(u)$ or gu.

(ii) $\rho(g, \rho(h, u))=\rho(gh, u)$.

LEMMA 6.3. *Suppose G is a regular Fréchet-Lie group and $\rho: G\times\boldsymbol{F}\to\boldsymbol{F}$ is a smooth representation. Then the group $\boldsymbol{F}*G$ is a regular Fréchet-Lie group, where $\boldsymbol{F}*G$ is the direct product with the following group-operation*:

$$(u, g)*(v, h)=(u+\rho_g(v), gh) .$$

PROOF. Obviously, $\boldsymbol{F}*G$ is an FL-group, and there is a C^∞ section $\gamma: G\to\boldsymbol{F}*G$, $\gamma(g)=(0, g)$. Evidently, r_γ in the previous section (cf. Definition 5.1, (Ext. 2)) is zero and $A_\gamma(g)(u, e)=(\rho_g(u), e)$. Note that $\boldsymbol{F}$ is a regular Fréchet-Lie group (cf. Lemma 3.9, [7]). By Theorem 5.4, we get the desired one. □

Note that the tangent bundle T_G of G possesses an FL-group structure, isomorphic to $\mathfrak{G}*G$ (cf. Lemma 2.2 in [7]). Hence, we have the following:

COROLLARY 6.4. *The tangent bundle T_G identified with $\mathfrak{G}*G$ is a regular Fréchet-Lie group if and only if G is a regular Fréchet-Lie group.*

Let G be a regular Fréchet-Lie group with the Lie algebra $\mathfrak{G}$, and let V be a finite dimensional vector space over $\boldsymbol{R}$. It is obvious that the space $L(V, \mathfrak{G})$ of all linear mappings of V into $\mathfrak{G}$ is a Fréchet space. The adjoint action of G on $\mathfrak{G}$ can be naturally extended to a smooth representation of G on $L(V, \mathfrak{G})$, which will be denoted again by $\mathrm{Ad}: G\times L(V, \mathfrak{G})\to L(V, \mathfrak{G})$. Hence, by the above lemma, we see that $L(V, \mathfrak{G})*G$

is a regular Fréchet-Lie group.

Let J_0^k $(\boldsymbol{R}^n, G)$ be the space of k-jets at $0 \in \boldsymbol{R}^n$ of local C^k mappings of $\boldsymbol{R}^n$ into G (cf. [1], p. 19). $J_0^k(\boldsymbol{R}^n, G)$ is a group by the pointwise product. Now, the following is not hard to prove:

LEMMA 6.5. *Let V_k be a linear space of all $\boldsymbol{R}$-valued polynomials of n-variables of degree $\leqq k$ without constant terms. Then, $J_0^k(\boldsymbol{R}^n, G)$ is isomorphic to $L(V_k, \mathfrak{G}) * G$, and hence $J_0^k(\boldsymbol{R}^n, G)$ is a regular Fréchet-Lie group. If we define $J_0^\infty(\boldsymbol{R}^n, G)$ by the projective limit of $J_0^k(\boldsymbol{R}^n, G)$ then $J_0^\infty(\boldsymbol{R}^n, G)$ is also isomorphic to $L(V_\infty, \mathfrak{G}) * G$, hence it is a regular Fréchet-Lie group, where $L(V_\infty, \mathfrak{G})$ is the Fréchet space defined by the projective limit of $\{L(V_k, \mathfrak{G})\}$.*

Now, suppose $\mathscr{E}$ is a C^∞ fibre bundle over M with a regular Fréchet-Lie group as a fibre, and with the group of automorphisms of G as the transformation group. Suppose M is a compact C^∞ manifold with or without boundary. For each $x \in M$, let $J_x^k(\mathscr{E})$ be the set of all k-jets of local C^k sections of $\mathscr{E}$ at x. Obviously, $J_x^k(\mathscr{E})$ is isomorphic to $J_0^k(\boldsymbol{R}^n, G)$, hence $J_x^k(\mathscr{E})$ is regular Fréchet-Lie group.

Set $J^k(\mathscr{E}) = \bigcup_{x \in M} J_x^k(\mathscr{E})$. It is easy to see that $J^k(\mathscr{E})$ is a C^∞ fibre bundle over M for every k, $0 \leqq k \leqq \infty$. Hence by Lemma 6.1, we see that $C^0(J^k(\mathscr{E}))$ is a regular Fréchet-Lie group.

Recall that $C^k(\mathscr{E})$ is an FL-group. Taking the derivatives up to the order k, there is a natural monomorphism of $C^k(\mathscr{E})$ into $C^0(J^k(\mathscr{E}))$, and the image is a closed subgroup. Remark that the topology of $C^k(\mathscr{E})$ is given by the relative topology in $C^0(J^k(\mathscr{E}))$. Therefore, by a similar proof as in the proof of Proposition 2.4, we obtain the following:

PROPOSITION 6.6. *Notations and assumptions being as above, $C^k(\mathscr{E})$ is a regular Fréchet-Lie group for every k, $0 \leqq k \leqq \infty$.*

REMARK. If we take $\mathscr{E}$ by $I \times G$, $I = [0, 1]$ and $M = I$, then we get the fact that $C_e^1(I, G)$ is a regular Fréchet-Lie group (cf. [7], § 5, Lemma 5.2 and Remark.)

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