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Configurations and Invariant Gauss-Manin Connections for Integrals II

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In this note, by using certain canonical invariant 1-forms, we shall compute explicitly the formulae of Gauss-Manin connections for the integrals which have been investigated in [1] and prove [3] Theorem 2 in the hyperquadric case (see Theorems 1, 2 and 3). We shall follow the terminologies used in [1].

§1. Conformal case.

First, we are going to find out an explicit representation of invariant Gauss-Manin connection for the following integral which admits conformal transformations:

$$(1.1)\qquad \hat{\varphi}(\phi)(=\hat{\varphi}(\lambda_0, \lambda_1, \cdots, \lambda_m; \phi)) \\ =\int (x_1^2+\cdots+x_n^2+1)^{\lambda_0}\hat{f}_1(x)^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}(x)dx_1\wedge dx_2\wedge\cdots\wedge dx_n\,,$$

where $\hat{f}_1, \cdots, \hat{f}_m$ denote general linear functions: $\hat{f}_j=\sum_{\nu=1}^n u_{j\nu}x_\nu + u_{j0}$. Here we use the variables $\sqrt{-1}\,x_\nu$ instead of x_ν in [1] (J, III_0).

As in [1], we define the symmetric matrix of $(m+1)$ order $A=((a_{ij}))$ $0\leqq i, j\leqq m$ by putting $a_{ij}=\sum_{\nu=0}^n u_{i\nu}u_{j\nu}$ and $a_{i0}=a_{0i}=u_{i0}$, where we normalize $a_{ii}=1$. We denote by $A\binom{I}{J}$ the minor determinant of the $i_1, \cdots, i_p$-th rows and the $j_1, \cdots, j_p$-th columns for $I=(i_1, \cdots, i_p)$ and $J=(j_1, \cdots, j_p)$, and write $A(I)$ for $A\binom{I}{I}$.

It has been stated in [1] Lemma 3.3 that the above integrals have a basis of functions:

$$(1.2)\qquad \hat{\varphi}(i_1\cdots i_p)=\hat{T}_{i_1}^{-1}\cdots\hat{T}_{i_p}^{-1}\hat{\varphi}(\phi)=\int \hat{U}(\lambda)\frac{dx_1\wedge\cdots\wedge dx_n}{\hat{f}_{i_1}\cdots\hat{f}_{i_p}}$$

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for $1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$, where $\hat{U}(\lambda)$ denotes $(x_1^2+\cdots+x_n^2+1)^{\lambda_0}\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}$ and $\hat{T}_k^{\pm 1}$ the k-th difference operator with respect the variables $\lambda_1, \cdots, \lambda_m$. So we have only to find out the variation formula for $\hat{\varphi}(i_1\cdots i_p)$ as a linear combination of them. As in [1] (3.2), we define the mapping ρ by:

$$(1.3)\qquad \tilde{\varphi}(\phi)(=\tilde{\varphi}(\mu_0, \lambda_1, \cdots, \lambda_m; \phi))=\Gamma(-\lambda_0)\cdot 2^{-\lambda_0-1}\hat{\varphi}(\phi)=\rho\hat{\varphi}(\phi)$$
$$=\int \exp\left[\frac{-(x_0^2+\cdots+x_n^2)}{2}\right]\hat{f}_1\left(\frac{x_i}{x_0}\right)^{\lambda_1}\cdots\hat{f}_m\left(\frac{x_i}{x_0}\right)^{\lambda_m}\cdot x_0^{\mu_0+\lambda_1+\cdots+\lambda_m}$$
$$\times dx_0\wedge dx_1\wedge\cdots\wedge dx_n$$

for $\mu_0=-2\lambda_0-\sum_1^m\lambda_j-n-1$.

Then, according to [1] (3.4), we have, for $I=(i_1, \cdots, i_p)$,

$$(1.4)\qquad \begin{aligned} A(I)d\hat{\varphi}(I)&=A(I)d[\hat{T}_{i_1}^{-1}\cdots\hat{T}_{i_p}^{-1}\hat{\varphi}(\phi)] \\ &=A(I)d[\hat{T}_{i_1}^{-1}\cdots\hat{T}_{i_p}^{-1}\rho^{-1}\tilde{\varphi}(\phi)] \\ &=A(I)d[\rho^{-1}T_0^pT_{i_1}^{-1}\cdots T_{i_1}^{-1}\tilde{\varphi}(\phi)] \\ &=A(I)d[\rho^{-1}T_0^p\tilde{\varphi}(I)] \\ &=A(I)\rho^{-1}d[T_0^p\tilde{\varphi}(I)]\,. \end{aligned}$$

We denote by $\partial_\nu I$ the sequence $(i_1, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_p)$ for $I=(i_1, \cdots, i_p)$. According to [1] Proposition 2.4 the right hand side is equal to the following, where the set of indices $\{0, 1, 2, \cdots, m\}$ here are used instead of $\{1, 2, \cdots, m\}$:

$$(1.5)\qquad \rho^{-1}A(I)dT_0^p\tilde{\varphi}(I)$$
$$=\frac{1}{2}\sum_{j\neq k\notin I}\underbrace{\left\{dA\begin{pmatrix}I, j\\ I, k\end{pmatrix}-\frac{1}{2}d\log A(I, j)A\begin{pmatrix}I, j\\ I, k\end{pmatrix}\right.}_{(\alpha)}$$
$$\underbrace{\left.-\frac{1}{2}d\log A(I, k)A\begin{pmatrix}I, j\\ I, k\end{pmatrix}\right\}}_{(\alpha)}\cdot\lambda_j\lambda_k\rho^{-1}T_0^p\tilde{\varphi}(I, j, k)$$
$$+\sum_{k\notin I}\underbrace{\left\{dA\begin{pmatrix}I, 0\\ I, k\end{pmatrix}-\frac{1}{2}d\log A(I, 0)\cdot A\begin{pmatrix}I, 0\\ I, k\end{pmatrix}\right.}_{(\beta)}$$
$$\underbrace{\left.-\frac{1}{2}d\log A(I, k)\cdot A\begin{pmatrix}I & 0\\ I & k\end{pmatrix}\right\}}_{(\beta)}\lambda_j\cdot(\mu_0+p)\rho^{-1}\cdot T_0^{p-1}\tilde{\varphi}(I, k)$$
$$+\frac{A(I)}{2}\underbrace{\left\{-d\log A(I)-\sum_{\nu=1}^{|I|}\lambda_{i_\nu}d\log\left(\frac{A(\partial_\nu I)}{A(I)}\right)\right.}_{(\gamma)}$$

$$\underbrace{+\sum_{k\notin I}\lambda_k d\log\left(\frac{A(I,k)}{A(I)}\right)}_{(\gamma)}\Bigg\}\rho^{-1}\cdot T_0^p\tilde{\varphi}(I)$$

$$+\frac{1}{2}\sum_{\substack{\mu\neq\nu\\ 1\leq\mu,\nu\leq|I|}}(-1)^{\mu+\nu}\Bigg\{\underbrace{-dA\begin{pmatrix}\partial_\mu I\\ \partial_\nu I\end{pmatrix}+\frac{1}{2}A\begin{pmatrix}\partial_\mu I\\ \partial_\nu I\end{pmatrix}d\log A(\partial_\nu I)}_{(\delta)}$$

$$\underbrace{+\frac{1}{2}A\begin{pmatrix}\partial_\mu I\\ \partial_\nu I\end{pmatrix}d\log A(\partial_\mu I)}_{(\delta)}\Bigg\}\rho^{-1}\cdot T_0^p\tilde{\varphi}(\partial_\mu\partial_\nu I)$$

$$+\sum_{\substack{k\notin I\\ 1\leq\nu\leq|I|}}(-1)^{\nu-1}\lambda_k\Bigg\{\underbrace{dA\begin{pmatrix}k,\partial_\nu I\\ I\end{pmatrix}-\frac{1}{2}A\begin{pmatrix}k,\partial_\nu I\\ I\end{pmatrix}d\log A(\partial_\nu I)}_{(\varepsilon)}$$

$$\underbrace{-\frac{1}{2}A\begin{pmatrix}k,\partial_\nu I\\ I\end{pmatrix}d\log A(k,I)}_{(\varepsilon)}\Bigg\}\rho^{-1}\cdot T_0^p\tilde{\varphi}(k,\partial_\nu I)$$

$$+\sum_{\nu=1}^{|I|}(-1)^{\nu-1}(\mu_0+p)\Bigg\{\underbrace{dA\begin{pmatrix}0,\partial_\nu I\\ I\end{pmatrix}-\frac{1}{2}A\begin{pmatrix}0,\partial_\nu I\\ I\end{pmatrix}d\log A(\partial_\nu I)}_{(\zeta)}$$

$$\underbrace{-\frac{1}{2}A\begin{pmatrix}0,\partial_\nu I\\ I\end{pmatrix}d\log A(0,I)}_{(\zeta)}\Bigg\}\rho^{-1}T_0^{p-1}\tilde{\varphi}(\partial_\nu I)\,.$$

In view of [1] (3.4), the right hand side is equal to

$$(1.6)\qquad \frac{1}{2}\sum_{j\neq k;\,j,k\notin I}\{(\alpha)\}\lambda_j\lambda_k\left(\frac{\hat{T}_0\hat{\varphi}(I,j,k)}{-2(\lambda_0+1)}\right)$$
$$+\sum_{k\notin I}\{(\beta)\}\lambda_j(\mu_0+p)\left(\frac{\hat{T}_0\hat{\varphi}(I,k)}{-2(\lambda_0+1)}\right)$$
$$+\frac{A(I)}{2}\{(\gamma)\}\hat{\varphi}(I)$$
$$+\frac{1}{2}\sum_{1\leq\mu\neq\nu\leq|I|}(-1)^{\mu+\nu}\{(\delta)\}(-2\lambda_0\hat{T}_0^{-1}\hat{\varphi}(\partial_\mu\partial_\nu I))$$
$$+\sum_{k\notin I}\sum_{1\leq\nu\leq|I|}(-1)^{\nu-1}\lambda_k\{(\varepsilon)\}\tilde{\varphi}(k,\partial_\nu I)$$
$$+\sum_{1\leq\nu\leq|I|}(-1)^{\nu-1}(\mu_0+p)\{(\zeta)\}\hat{\varphi}(\partial_\nu I)\,.$$

Now we apply successively the formulae [1] (D, III_p) $0\leq p\leq n+1$ to $\hat{T}_0\hat{\varphi}(I,j,k)$ and $\hat{T}_0\hat{\varphi}(I,k)$, and apply [1] ($D$, III_p^*) to $-2\lambda_0\hat{T}_0^{-1}\hat{\varphi}(\partial_\mu\partial_\nu I)$. Then in the same way as [1] (E, III_0) we have the following expression:

PROPOSITION 1. *For* $I\subset\{1, 2, \cdots, m\}$, $|I|\leqq n$,

$$
\begin{aligned}
(1.7)\quad & A(I)d\hat{\varphi}(I) \\
&= \sum_{s=1}^{n-|I|} \sum_{k_\sigma\notin I, k_\sigma\neq k_\tau \text{ for } \sigma\neq\tau} \frac{\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_s}}{s!} \frac{\theta\begin{pmatrix} I \\ I, k_1, \cdots, k_s\end{pmatrix}}{(\mu_0+|I|+1)\cdots(\mu_0+|I|+s-1)} \\
&\times\hat{\varphi}_*(I, k_1, \cdots, k_s) \\
&+\sum \frac{\lambda_{k_1}\cdots\lambda_{k_{n+1-|I|}}}{(n+1-|I|)!(\mu_0+|I|+1)\cdots(\mu_0+n)} \\
&\times\theta\begin{pmatrix} I \\ I, k_1, \cdots, k_{n+1-|I|}\end{pmatrix} \frac{\hat{T}_0\hat{\varphi}_*(I, k_1, \cdots, k_{n+1-|I|})}{-2(\lambda_0+1)} \\
&+ \sum_{|J|\leqq|I|-2, J\subseteqq I, k\notin J} \lambda_k\theta\begin{pmatrix} I \\ J\end{pmatrix} \cdot \frac{A\begin{pmatrix} 0, J \\ k, J\end{pmatrix}}{A(J)}\hat{\varphi}(J, k) \\
&+ \sum_{J\subseteqq I, |J|\leqq|I|-2} (\mu_0+|J|+1)\theta\begin{pmatrix} I \\ J\end{pmatrix}\frac{A(0, J)}{A(J)}\hat{\varphi}(J) \\
&+\frac{1}{2}A(I)\left\{-d\log A(I) - \sum_{\nu=1}^{|I|}\lambda_{i_\nu} d\log\left(\frac{A(\partial_\nu I)}{A(I)}\right)\right. \\
&\left.+\sum_{k\notin I}\lambda_k d\log\left(\frac{A(I, k)}{A(I)}\right)\right\}\hat{\varphi}(I) \\
&+\sum_{k\notin I}\sum_{\nu=1}^{|I|}\lambda_k\theta\begin{pmatrix} I \\ \partial_\nu I, k\end{pmatrix}\hat{\varphi}(\partial_\nu I, k) + \sum_{\nu=1}^{|I|}(\mu_0+|I|)\theta\begin{pmatrix} I \\ \partial_\nu I\end{pmatrix}\hat{\varphi}(\partial_\nu I)\,,
\end{aligned}
$$

where

$$
(1.8)\qquad \hat{\varphi}_*(I) = \hat{\varphi}(I) + \sum_{\nu=1}^{p}(-1)^\nu \frac{A\begin{pmatrix} I \\ 0, \partial_\nu I\end{pmatrix}}{A(I)}\hat{\varphi}(\partial_\nu I)\,,
$$

$$
\begin{aligned}
(1.9)\quad \theta\begin{pmatrix} I \\ I, j\end{pmatrix}\left(=\frac{A(I, j)}{A(0, I, j)}(\beta)\right) &= \left\{dA\begin{pmatrix} I, 0 \\ I, j\end{pmatrix} - \frac{1}{2}d\log A(I, 0)\cdot A\begin{pmatrix} I, 0 \\ I, j\end{pmatrix}\right. \\
&\left. -\frac{1}{2}d\log A(I, j)\cdot A\begin{pmatrix} I, 0 \\ I, j\end{pmatrix}\right\} \cdot \frac{A(I, j)}{A(0, I, j)}\,,
\end{aligned}
$$

$$
\begin{aligned}
(1.10)\quad \theta\begin{pmatrix} I \\ I, j, k\end{pmatrix} &= \left\{dA\begin{pmatrix} I, j \\ I, k\end{pmatrix} - \frac{1}{2}d\log A(I, j)A\begin{pmatrix} I, j \\ I, k\end{pmatrix} - \frac{1}{2}d\log A(I, k)A\begin{pmatrix} I, j \\ I, k\end{pmatrix}\right. \\
&\left. -\theta\begin{pmatrix} I \\ I, k\end{pmatrix}\frac{A\begin{pmatrix} 0, k, I \\ j, k, I\end{pmatrix}}{A(k, I)} + \theta\begin{pmatrix} I \\ I, j\end{pmatrix}\frac{A\begin{pmatrix} 0, j, I \\ j, k, I\end{pmatrix}}{A(j, I)}\right\}\,,
\end{aligned}
$$

and $\theta\binom{I}{I,\, i_1,\, \cdots,\, i_s}$ $(s\geqq 3)$, *are defined by the recurrence equations, which concide with* [1] (3.12) *for* $I=\phi$.

$$(1.11)\qquad \theta\binom{I}{I,\, k_1\cdots k_s}=\sum_{\nu=1}^{s}(-1)^{\nu}\theta\binom{I}{I,\, k_1,\, \cdots,\, \hat{k}_\nu,\, \cdots,\, k_s} \times\frac{A\binom{0,\, k_1,\, \cdots,\, \hat{k}_\nu,\, \cdots,\, k_s,\, I}{k_1,\, \cdots,\, k_s,\, I}}{A(k_1,\, \cdots,\, k_\nu,\, \cdots,\, k_s,\, I)}\,.$$

On the other hand,

$$(1.12)\qquad \theta\binom{I}{\partial_\mu\partial_\nu I}\Big(=\frac{1}{2}(-1)^{\mu+\nu}\cdot(\delta)\Big)=\frac{1}{2}(-1)^{\mu+\nu}\Big\{-dA\binom{\partial_\mu I}{\partial_\nu I} +\frac{1}{2}A\binom{\partial_\mu I}{\partial_\nu I}d\log A(\partial_\nu I)+\frac{1}{2}A\binom{\partial_\mu I}{\partial_\nu I}d\log A(\partial_\mu I)\Big\}\,,$$

while, $\theta\binom{I}{J}$ *for* $J\subset I$, $|J|\leqq|I|-3$ *are determined inductively:*

$$(1.13)\qquad \theta\binom{I}{J}=\sum_{j\in I-J}\frac{A\binom{j,\, J}{0,\, J}}{A(j,\, J)}\operatorname{sgn}(j,\, J)\cdot\theta\binom{I}{j,\, J}$$

for $|J|\leqq|I|-3$. *Finally,*

$$(1.14)\qquad \theta\binom{I}{\partial_\nu I,\, k}(=(\varepsilon)(-1)^{\nu-1})=(-1)^{\nu-1}\Big\{dA\binom{k,\, \partial_\nu I}{I} -\frac{1}{2}A\binom{k,\, \partial_\nu I}{I}d\log A(\partial_\nu I)-\frac{1}{2}A\binom{k,\, \partial_\nu I}{I}d\log A(k,\, I)\Big\}$$

and

$$(1.15)\qquad \theta\binom{I}{\partial_\nu I}(=(-1)^{\nu-1}\cdot(\zeta))=(-1)^{\nu-1}\Big\{dA\binom{0,\, \partial_\nu I}{I} -\frac{1}{2}A\binom{0,\, \partial_\nu I}{I}\cdot d\log A(\partial_\nu I)-\frac{1}{2}A\binom{0,\, \partial_\nu I}{I}d\log A(0,\, I)\Big\}\,.$$

If $|I|=n+1$, *then according to* (3.14) *in* [1], $\hat{T}_0\hat{\varphi}(I)$ *is represented as follows:*

$$(1.16)\qquad \hat{T}_0\hat{\varphi}(I)=\sum_{1\leq\mu\neq\nu\leq n+1}(-1)^{\mu+\nu}\frac{A\binom{\partial_\mu I}{\partial_\nu I}}{A(I)}\hat{\varphi}(\partial_\mu\partial_\nu I)\,.$$

The equations (1.7)$\sim$(1.15) *give the explicit Gauss-Manin connection for the integral* $\hat{\varphi}(\phi)$, *which implies* [1] *Theorem* 6.

§2. Determination of the invariant 1-forms $\theta^{(1)}\binom{\phi}{i_1, i_2, \cdots, i_p}$, $1 \leqq i_1 < \cdots < i_p \leqq m$.

In [1] we have considered the following integral on the hyperquadric: $\hat{S}_0$: $x_1^2 + \cdots + x_{n+1}^2 + 1 = 0$,

$$\hat{\varphi}_*^{(1)}(\phi) = \hat{\varphi}^{(1)}(\phi) = \int \hat{f}_1^{\lambda_1} \cdots \hat{f}_m^{\lambda_m} \tau \tag{2.1}$$

for linear functions $\hat{f}_j = \sum_{\nu=1}^{n+1} u_{j\nu} x_\nu + u_{j0}$ and the n-form τ

$$\tau = \sum_{j=1}^{n+1} (-1)^j x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1} \tag{2.2}$$

and proved the following theorem:

THEOREM 7 [1]. *There exists a system of* 1*-forms* $\theta^{(1)}\binom{\phi}{i_1, \cdots, i_p}$ *such that*

$$d\hat{\varphi}_*^{(1)}(\phi) = \sum_{p=1}^{n+1} \frac{1}{p!} \sum \frac{\lambda_{i_1} \cdots \lambda_{i_p} (-1)^{p-1}}{(\lambda_\infty + n - 1) \cdots (\lambda_\infty + n - p + 1)} \times \theta^{(1)}\binom{\phi}{i_1, \cdots, i_p} \hat{\varphi}_*^{(1)}(i_1, \cdots, i_p) \tag{2.3}$$

where $\hat{\varphi}_*^{(1)}(i_1 \cdots i_p)$ *is defined as follows*:

$$\hat{\varphi}_*^{(1)}(I) = \hat{\varphi}^{(1)}(I) + \sum_{\nu=1}^{p} (-1)^\nu \frac{A\binom{I}{0, \partial_\nu I}}{A(I)} \hat{\varphi}^{(1)}(\partial_\nu I) \tag{2.4}$$

$$\hat{\varphi}^{(1)}(I) = \int \hat{U}_0(x) \frac{\tau}{f_{i_1} f_{i_2} \cdots f_{i_p}}, \qquad \text{for } I = (i_1, \cdots, i_p),$$

and the 1*-form* $\theta^{(1)}\binom{\phi}{i_1 \cdots i_p}$ *does not depend either on* n *or on* $\lambda_1, \cdots, \lambda_m$. *It depends only on* $f_{i_1}, \cdots, f_{i_p}$. (*We have denoted by* $\hat{U}_0$ *the function* $f_1^{\lambda_1} \cdots f_m^{\lambda_m}$).

LEMMA 1.

$$\begin{cases} \theta^{(1)}\binom{\phi}{i} = \theta\binom{\phi}{i} = \dfrac{da_{0i}}{A(0, i)}, & p = 1 \\ \theta^{(1)}\binom{\phi}{i_1 \cdots i_p} = \theta\binom{\phi}{i_1 \cdots i_p} \dfrac{A(i_1 \cdots i_p)}{A(0, i_1 \cdots i_p)} & \end{cases} \tag{2.5}$$

for $n+1 \geqq p \geqq 2$. *Therefore* $\theta^{(1)}\binom{\phi}{i_1 \cdots i_p}$ *are successively determined by* (1.9) *or* [1] (3.12).

PROOF. In [1] Lemma 4.7, the first equality has been proved. So we want to prove the second one. *Since* $\theta^{(1)}\binom{\phi}{i_1\cdots i_s}$ *depends only on* $f_{i_1},\cdots,f_{i_s}$ *and does not depend on* $\lambda_1,\cdots,\lambda_m$ *or* m, *we have only to prove it in case where* $m=n+1$ *and* $(i_1,\cdots,i_p)\subset(1,2,\cdots,n+1)$. Then, according to [1] Proposition 3.4, we have

$$(2.6)\qquad d\hat{\varphi}(\phi)=\sum_{s=1}^{n}\frac{1}{s!}\sum\frac{\lambda_{i_1}\cdots\lambda_{i_s}}{(\mu_0+1)\cdots(\mu_0+s-1)}\theta\begin{pmatrix}\phi\\ i_1\cdots i_s\end{pmatrix}\frac{A(i_1\cdots i_s)}{A(0\,i_1\cdots i_s)}\hat{\varphi}_*(i_1\cdots i_s)$$
$$+\frac{1}{(n+1)!}\frac{\lambda_1\lambda_2\cdots\lambda_{n+1}}{(\mu_0+1)\cdots(\mu_0+n-1)}\theta\begin{pmatrix}\phi\\ 1\,2\cdots n+1\end{pmatrix}\frac{\hat{T}_0\hat{\varphi}(1\,2\cdots n+1)}{-2(\lambda_0+1)}\,.$$

On the other hand, [1] $(D,\ \mathrm{III}_{n+1})$ implies

$$(2.7)\qquad (\mu_0+n)\hat{T}_0\hat{\varphi}(1\,2\cdots n+1)=-2(\lambda_0+1)\frac{A(1\,2\cdots n+1)}{A(0\,1\,2\cdots n+1)}\hat{\varphi}_*(12\cdots n+1)\,,$$

so that $d\hat{\varphi}(\phi)$ is equal to

$$(2.8)\qquad \sum_{s=1}^{n}\frac{1}{s!}\sum\frac{\lambda_{i_1}\cdots\lambda_{i_s}}{(\mu_0+1)\cdots(\mu_0+s-1)}\theta\begin{pmatrix}\phi\\ i_1\cdots i_s\end{pmatrix}\frac{A(i_1\cdots i_s)}{A(0\,i_1\cdots i_s)}\hat{\varphi}_*(i_1\cdots i_s)$$
$$+\frac{1}{(n+1)!}\frac{\lambda_1\lambda_2\cdots\lambda_{n+1}}{(\mu_0+1)\cdots(\mu_0+n-1)}$$
$$\times\theta\begin{pmatrix}\phi\\ 1\,2\cdots n+1\end{pmatrix}\frac{A(1\,2\cdots n+1)}{A(0\,1\,2\cdots n+1)}\hat{\varphi}_*(1\,2\cdots n+1)\,.$$

Therefore when $\lambda_0\to-1$, μ_0 being equal to $-\lambda_\infty-n=-\sum_1^\infty\lambda_j-n$, we have

$$(2.9)\qquad d\hat{\varphi}^{(1)}(\phi)=d\hat{\varphi}_*^{(1)}(\phi)=\widehat{\lim_{\lambda_0\to-1}}-2(\lambda_0+1)d\hat{\varphi}(\phi)$$
$$=\sum_{s=1}^{n}\sum_{(i_1\cdots,i_s)}\frac{1}{s!}\frac{\lambda_{i_1}\cdots\lambda_{i_s}}{(-\lambda_\infty-n+1)\cdots(-\lambda_\infty+s-1-n)}\theta\begin{pmatrix}\phi\\ i_1\cdots i_s\end{pmatrix}$$
$$\times\frac{A(i_1\cdots i_s)}{A(0\,i_1\cdots i_s)}\hat{\varphi}_*^{(1)}(i_1\cdots i_s)$$
$$+\frac{1}{(n+1)!}\frac{\lambda_1\lambda_2\cdots\lambda_{n+1}}{(-\lambda_\infty-n+1)\cdots(-\lambda_\infty)}\theta\begin{pmatrix}\phi\\ 1\,2\cdots n+1\end{pmatrix}$$
$$\times\frac{A(1\,2\cdots n+1)}{A(0\,1\,2\cdots n+1)}\cdot\hat{\varphi}_*^{(1)}(1\,2\cdots n+1)\,;$$

In view of [1] (4.11), the forms $\hat{\varphi}_*^{(1)}(i_1\cdots i_s)$, $1\leqq i_1<\cdots<i_s\leqq n+1$ become a system of generators of the cohomology $H^n(Y,\nabla_0)$ on $Y=\hat{S}_0-\bigcup_{j=1}^{n+1}(\hat{f}_j=0)$ with respect to the covariant differentiation $\nabla_0=d+$

$\sum_{j=1}^{n+1} d\log \hat{f}_j \wedge$. Therefore the above expression is unique, which implies the Lemma.

We have proved, from (2.3) and (2.5),

THEOREM 1.

$$d\hat{\varphi}^{(1)}(\phi)=\sum_{s=1}^{n}\frac{1}{s!}\sum\frac{\lambda_{i_1}\cdots\lambda_{i_s}\theta\begin{pmatrix}\phi\\ i_1\cdots i_s\end{pmatrix}}{(-\lambda_\infty-n+1)\cdots(-\lambda_\infty-n+s-1)}\hat{\varphi}_*^{(1)}(i_1\cdots i_s)\times\frac{A(i_1\cdots i_s)}{A(0,i_1\cdots i_s)}+\sum\frac{\hat{\varphi}_*^{(1)}(i_1\cdots i_{n+1})}{(-\lambda_\infty-n+1)\cdots(-\lambda_\infty)}\lambda_{i_1}\cdots\lambda_{i_{n+1}}\times\theta\begin{pmatrix}\phi\\ i_1\cdots i_{n+1}\end{pmatrix}\cdot\frac{A(i_1,\cdots,i_{n+1})}{A(0,i_1\cdots i_{n+1})}. \tag{2.10}$$

More generally, by using the formulae (1.7) and $\lim_{\lambda_0\to -1} -2(\lambda_0+1)\hat{\varphi}(I)=\hat{\varphi}^{(1)}(I)$, we have

THEOREM 1′. *For* $|I|\leqq n+1$,

$$\begin{aligned}d\hat{\varphi}^{(1)}(I)=&\sum_{s=1}^{n-|I|}\sum\frac{\lambda_{k_1}\cdots\lambda_{k_s}}{s!\,(-\lambda_\infty-n+|I|+1)\cdots(-\lambda_\infty-n+|I|+s-1)}\\&\times\theta\begin{pmatrix}I\\ I,k_1\cdots k_s\end{pmatrix}\cdot\hat{\varphi}_*^{(1)}(I,k_1,\cdots,k_s)\\&+\sum\frac{\lambda_{k_1}\cdots\lambda_{k_{n+1-|I|}}}{(n+1-|I|)!\,(-\lambda_\infty-n+|I|+1)\cdots(-\lambda_\infty)}\\&\times\theta\begin{pmatrix}I\\ I,k_1\cdots k_{n+1-|I|}\end{pmatrix}\cdot\hat{\varphi}_*^{(1)}(I,k_1,\cdots,k_{n+1-|I|})\\&+\sum_{|J|\leqq|I|-2,J\subseteqq I,J\not\ni k}\lambda_k\theta\begin{pmatrix}I\\ J\end{pmatrix}\frac{A\begin{pmatrix}0,J\\ k,J\end{pmatrix}}{A(J)}\hat{\varphi}^{(1)}(J,k)\\&+\sum_{|J|\leqq|I|-2,J\subseteqq I}(-\lambda_\infty-n+|J|+1)\theta\begin{pmatrix}I\\ J\end{pmatrix}\frac{A(0,J)}{A(J)}\hat{\varphi}^{(1)}(J)\\&+\frac{1}{2}A(I)\Big\{-d\log A(I)+\sum_{\nu=1}^{|I|}\lambda_{i_\nu}d\log\Big(\frac{A(\partial_\nu I)}{A(I)}\Big)\\&+\sum_{k\notin I}\lambda_k d\log\Big(\frac{A(I,k)}{A(I)}\Big)\Big\}\hat{\varphi}^{(1)}(I)\\&+\sum_{k\notin I}\sum_{\nu=1}^{|I|}\lambda_k\theta\begin{pmatrix}I\\ \partial_\nu I,k\end{pmatrix}\cdot\hat{\varphi}^{(1)}(k,\partial_\nu I)\\&+\sum_{\nu=1}^{|I|}(\mu_0+|I|)\theta\begin{pmatrix}I\\ \partial_\nu I\end{pmatrix}\hat{\varphi}^{(1)}(\partial_\nu I)\,.\end{aligned}\tag{2.11}$$

Here we have the fundamental relation for $\hat{\varphi}^{(1)}(J)$, $|J|\leqq n+1$:

$$\frac{1}{2}\sum_{\substack{\alpha,\beta\in I\\ \alpha\neq\beta}}(-1)^{\alpha+\beta}\frac{A(0,\,\partial_\alpha\partial_\beta I)}{A\binom{0,\,\partial_\alpha I}{0,\,\partial_\beta I}}\hat{\varphi}^{(1)}(\partial_\alpha\partial_\beta I)+\sum_{\alpha\in I}(-1)^{\alpha}\frac{A(\partial_\alpha I)}{A\binom{0,\,\partial_\alpha I}{I}}\hat{\varphi}^{(1)}(\partial_\alpha I)=0 \tag{2.12}$$

for any sequence of indices $I=(i_1, i_2, \cdots, i_{n+2})$ (*See* [1] (4.9).), *while*

$$\hat{\varphi}^{(1)}(I)[i_1, \cdots, i_{n+2}] = \sum_{\nu=1}^{n+2}\hat{\varphi}^{(1)}(\partial_\nu I)(-1)^{\nu-1}[i_1, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_{n+2}]\,. \tag{2.13}$$

Consequently (2.11) *is a maximally overdetermined system of order*

$$\sum_{s=0}^{n}\binom{m}{s}+\binom{m-1}{n}\,.$$

§3. Orthogonal inhomogeneous case.

The following integral has been investigated in [1] §1:

$$\tilde{\varphi}(\phi)=\tilde{\varphi}(\lambda_1\cdots\lambda_m;\,\phi) = \int \exp\,[f_0(x)]f_1^{\lambda_1}(x)\cdots f_m^{\lambda_m}(x)dx_1\wedge\cdots\wedge dx_n \tag{3.1}$$

for $f_0=-(1/2)(x_1^2+\cdots+x_n^2)$ and general *inhomogeneous* linear functions $f_j=\sum_{\nu=1}^{n}u_{j\nu}x_\nu+u_{j0}$. $U(\lambda)$ and ω will denote respectively $\exp\,[f_0]f_1^{\lambda_1}\cdots f_m^{\lambda_m}$ and $d\log U(\lambda)$. ∇_ω denotes the covariant differentiation defined by $\nabla_\omega\psi=d\psi+\omega\wedge\psi$ in the space C^n-S, where $S=\bigcup_{j=1}^{m}(f_j=0)$.

In [1] Proposition 1.1p and 1.2p we have proved the following fact which is reproduced now.

PROPOSITION 2. *Define the difference operators* $T_k^{\pm1}$:

$$T_k^{\pm1}\tilde{\varphi}(\lambda_1, \cdots, \lambda_m;\,\phi)=\tilde{\varphi}(\lambda_1, \cdots, \lambda_k\pm1, \cdots, \lambda_m;\,\phi)\,. \tag{3.2}$$

Then $\tilde{\varphi}$ *being regarded as an analytic function of* $\lambda_1, \cdots, \lambda_m$, *the following relation are fundamental among* $T_k^{\pm1}$ *in the commutative algebra* $C[T_1, T_1^{-1}, \cdots, T_m, T_m^{-1}, a_{jk}(0\leqq j, k\leqq m)]$ *generated by* $T_k^{\pm1}$ *and* a_{jk}.

$$T_k=a_{0k}+\sum_{j=1}^{m}\lambda_j a_{jk}T_j^{-1}\,,\qquad 1\leqq k\leqq m \tag{3.3}$$

$$(\lambda_k-1)T_k^{-2}=-a_{0k}-\sum_{j=1}^{m}\lambda_j a_{kj}T_j^{-1}\,,\qquad 1\leqq k\leqq m \tag{3.4}$$

and

$$\sum_{\nu=1}^{n+1}(-1)^{\nu-1}[i_1, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_{n+1}]T_{i_\nu}=[i_1, i_2, \cdots, i_{n+1}] \tag{3.5}$$

for any sequence of induces $I=(i_1, i_2, \cdots, i_{n+1})$.

PROOF. The crucial point is that, for any $\lambda_1, \lambda_2, \cdots, \lambda_m$, (3.3) or (3.4) are equivalent to the relations

$$\nabla_\omega(\psi)\sim 0\ , \tag{3.6}$$

where ψ is a linear combination of $(n-1)$-forms $\psi_\nu=(-1)^{\nu-1}dx_1\wedge\cdots\wedge dx_{\nu-1}\wedge dx_{\nu+1}\wedge\cdots\wedge dx_n$, $1\leqq\nu\leqq n$. On the other hand, an arbitrary $(n-1)$ form $\psi\in\Omega^{n-1}(*S, \nabla_\omega)$ is spanned by $f_1^{\sigma_1}\cdots f_m^{\sigma_m}\psi_\nu$ for $\sigma_1, \cdots, \sigma_m\in \boldsymbol{Z}$. But $\nabla(f_1^{\sigma_1}\cdots f_m^{\sigma_m}\psi_\nu)$ just gives the relations (3.3) or (3.4), replacing $\lambda_1, \cdots, \lambda_m$ by $\lambda_1+\sigma_1, \cdots, \lambda_m+\sigma_m$ respectively. The identity (3.5) immediately follows from partial fractions. Q.E.D.

As a result of these, the above integral has a basis of functions

$$\begin{aligned}\tilde{\varphi}(i_1\cdots i_p)&=T_{i_1}^{-1}\cdots T_{i_p}^{-1}\tilde{\varphi}(\phi)\\ &=\int U(\lambda)\frac{dx_1\wedge\cdots\wedge dx_n}{f_{i_1}\cdots f_{i_p}}\end{aligned}\tag{3.7}$$

for $1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$, (see [1] Lemma 1.1). By using these basis, the maximally overdetermined linear difference system for $\tilde{\varphi}(\phi)$ has been described in [1] Proposition 1.1p and [1] Proposition 1.2p. As in [1] §1, we define the symmetric matrix $A=((a_{ij}))_{0\leq i,j\leq m}$ of $(m+1)$ order by $a_{ij}=\sum_{\nu=1}^n u_{i\nu}u_{j\nu}$ and $a_{i0}=a_{0i}=u_{i0}$.

Then, according to [1] Theorem 2, we have the following explicit formula for the Gauss-Manin connection of (3.1) and (3.7):

PROPOSITION 3. *For* $0\leqq|I|\leqq n$,

$$\begin{aligned}\mathrm{A}(I)d\tilde{\varphi}(I)=&\frac{1}{2}\sum_{j\neq k\notin I}\left\{dA\binom{I,\,j}{I,\,k}-\frac{1}{2}A\binom{I,\,j}{I,\,k}\right.\\ &\left.\times d\log A(I,\,j)-\frac{1}{2}A\binom{I,\,j}{I,\,k}d\log A(I,\,k)\right\}\lambda_j\lambda_k\tilde{\varphi}(I,\,j,\,k)\\ &+\frac{1}{2}A(I)\left\{-d\log A(I)-\sum_{\nu=1}^{|I|}\lambda_{i_\nu}d\log\left(\frac{A(\partial_\nu I)}{A(I)}\right)\right.\\ &\left.+\sum_{k\notin I}\lambda_k d\log\left(\frac{A(I,\,k)}{A(I)}\right)\right\}\tilde{\varphi}(I)+\frac{1}{2}\sum_{1\leq\mu\neq\nu\leq|I|}(-1)^{\mu+\nu}\left\{-dA\binom{\partial_\mu I}{\partial_\nu I}\right.\\ &\left.+\frac{1}{2}A\binom{\partial_\mu I}{\partial_\nu I}d\log A(\partial_\nu I)+\frac{1}{2}A\binom{\partial_\mu I}{\partial_\nu I}d\log A(\partial_\mu I)\right\}\tilde{\varphi}(\partial_\mu\partial_\nu I)\\ &+\sum_{k\notin I}\sum_{\nu=1}^{|I|}(-1)^{\nu-1}\lambda_k\left\{dA\binom{k,\,\partial_\nu I}{I}-\frac{1}{2}A\binom{k,\,\partial_\nu I}{I}d\log A(\partial_\nu I)\right.\end{aligned}\tag{3.8}$$

$$-\frac{1}{2}A\begin{pmatrix}k, \partial_\nu I\\ I\end{pmatrix}d\log A(k, I)\Bigg\}\tilde{\varphi}(k, \partial_\nu I)$$

$$+\sum_{\nu=1}^{|I|}\frac{(-1)^\nu}{2}\left\{A\begin{pmatrix}I\\ 0, \partial_\nu I\end{pmatrix}d\log(A(\partial_\nu I)A(I))-2dA\begin{pmatrix}I\\ 0, \partial_\nu I\end{pmatrix}\right\}\tilde{\varphi}(\partial_\nu I)$$

$$+\sum_{k\notin I}\frac{\lambda_k}{2}\left\{2dA\begin{pmatrix}0, I\\ k, I\end{pmatrix}-A\begin{pmatrix}0, I\\ k, I\end{pmatrix}d\log(A(I)A(k, I))\right\}\tilde{\varphi}(k, I)\ ,$$

with the fundamental relations

$$\sum_{\nu=1}^{n+1}(-1)^{\nu-1}[i_1\cdots i_{\nu-1}, i_{\nu+1}\cdots i_{n+1}]\tilde{\varphi}(\partial_\nu I)=[i_1, \cdots, i_{n+1}]\tilde{\varphi}(I) \tag{3.9}$$

for $I=(i_1, \cdots, i_{n+1})$.

PROOF. In fact

$$d\tilde{\varphi}(I)=dT_{i_1}^{-1}\cdots T_{i_p}^{-1}\tilde{\varphi}(\phi) \tag{3.10}$$
$$=T_{i_1}^{-1}\cdots T_{i_p}^{-1}d\tilde{\varphi}(\phi)$$

$$=T_{i_1}^{-1}\cdots T_{i_p}^{-1}\left\{\sum_j da_{j0}\lambda_j\tilde{\varphi}(j)+\frac{1}{2}\sum_{j,k}da_{jk}\lambda_j\lambda_k\tilde{\varphi}(jk)\right\}\ , \tag{3.11}$$

([1] Proposition (1.3))

$$=\sum_{j\notin I}da_{j0}\lambda_j\tilde{\varphi}(j, I)+\sum_{\nu=1}^{p}da_{i_\nu 0}(\lambda_{i_\nu}-1)T_{i_\nu}^{-1}\tilde{\varphi}(I)$$
$$+\frac{1}{2}\sum_{j,k\notin I}da_{jk}\lambda_j\lambda_k\tilde{\varphi}(I, j, k)+\sum_{\nu=1}^{|I|}\sum_{k\notin I}da_{i_\nu k}(\lambda_{i_\nu}-1)\lambda_k T_{i_\nu}^{-1}\tilde{\varphi}(k, I)$$
$$+\frac{1}{2}\sum_{1\le\mu,\nu\le|I|}da_{i_\mu i_\nu}(\lambda_{i_\mu}-1)(\lambda_{i_\nu}-1)T_{i_\mu}^{-1}T_{i_\nu}^{-1}\tilde{\varphi}(I)\ .$$

Then we have only to apply the formulae [1] (D, I_p^*) and [1] (D, I_n^*) to the right-hand side.

§4. General bi-linear case.

We want to describe in a slightly more general form the Gauss-Manin connection of the integral which has been considered in [1] §5:

$$\tilde{\varphi}(\phi)=\int\exp\left[-\sum_{i=1}^{n}x_iy_i\right]f_1(x)^{\lambda_1}\cdots f_s(x)^{\lambda_s}f_{s+1}(y)^{\lambda_{s+1}} \tag{4.1}$$
$$\times\cdots f_{s+t}(y)^{\lambda_{t+s}}dx_1\cdots dx_n dy_1\cdots dy_n\ ,\qquad (m=t+s)\ ,$$

for general linear functions $f_1, \cdots, f_s, f_{s+1}, \cdots, f_{s+t}$, where

$$f_j(x)=\sum_{\nu=1}^{n}u_{j\nu}x_\nu+u_{j0}\quad\text{and}\quad f_{s+j}(y)=\sum_{\nu=1}^{n}u_{j+s,\nu}y_\nu+u_{j+s,0}\ .$$

We shall assume $s, t\geqq n$.

Let ∇ be the covariant differentiation defined by $\nabla\psi=d\psi+\omega\wedge\psi$ where ω denotes

$$(4.2)\qquad -\sum_{i=1}^{n}(x_i dy_i+y_i dx_i)+\sum_{j=1}^{s}\lambda_j d\log f_j(x)+\sum_{j=s+1}^{s+t}\lambda_j d\log f_j(y)\,.$$

Let $\mathfrak{A}$ be the symmetric matrix of $((\mathfrak{A}_{ij}))$ $0\leqq i, j\leqq m$ of $(m+1)$ order, such that

$$(4.3)\qquad \begin{cases}\mathfrak{A}_{i,j}=0 \quad \text{for} \quad 1\leqq i\leqq s, 1\leqq j\leqq s \quad \text{or} \quad s+1\leqq i\leqq m, s+1\leqq j\leqq m\\ \mathfrak{A}_{i,j}=a_{i,j-s},\ 1\leqq i\leqq s,\ s+1\leqq j\leqq m\\ \mathfrak{A}_{i,j}=a_{j,i-s},\ s+1\leqq i\leqq m,\ 1\leqq j\leqq s\\ \mathfrak{A}_{i,0}=\mathfrak{A}_{0,i}=a_{i,0},\ 1\leqq i\leqq m\\ \mathfrak{A}_{0,0}=1\end{cases}$$

and $A=((a_{ij'}))$ $0\leqq i\leqq s$, $0\leqq j'\leqq t$ be the matrix of $(s+t)\times(t+1)$ type, where we put $a_{ij'}=\sum_{\nu=1}^{n}u_{i,\nu}u_{j+s,\nu}$, $u_{i0}=a_{i0}$ and $a_{i0'}=u_{s+i,0}$.

We denote by S_j the hyperplane $f_j(x)=0$ for $1\leqq j\leqq s$ and $f_j(y)=0$ for $s+1\leqq j\leqq m$.

We shall abbreviate f_{j+s}, λ_{j+s} or S_{j+s} by $f_{j'}$, $\lambda_{j'}$ or $S_{j'}$.

LEMMA 2. *The cohomology $H^{2n}(C^n\times C^n-\bigcup_{j=1}^{s+t}S_j, \nabla_\omega)$ is generated by the forms*

$$(4.4)\qquad \varphi(i_1\cdots i_p;\ j_1'\cdots j_p')=\frac{\tau}{f_{i_1}\cdots f_{i_p}f_{j_1'}\cdots f_{j_p'}}$$

for $0\leqq p\leqq n$, where τ denotes $dx_1\wedge\cdots\wedge dx_n\wedge dy_1\wedge\cdots\wedge dy_n$. Consequently its dimension is at most equal to $\sum_{p=0}^{n}\binom{s}{p}\binom{t}{p}$.

It will be proved later that the dimension of the above cohomology is just equal to $\sum_{p=0}^{n}\binom{s}{p}\binom{t}{p}$ (see Proposition 4).

We denote by $\tilde{\varphi}(I, J')$ the integral of the form $\varphi(I, J')$

$$(4.5)\qquad \tilde{\varphi}(I, j')=\int U(\lambda)\varphi(I, J')$$

for

$$U(\lambda)=\exp\Big[-\sum_{i=1}^{n}x_i y_i\Big]f_1^{\lambda_1}(x)\cdots f_s^{\lambda_s}(x)f_{s+1}^{\lambda_{s+1}}(y)\cdots f_{s+t}^{\lambda_{s+t}}(y)\,.$$

First we remark that for any $f_{i_1},\cdots,f_{i_{n+1}}$ (or $f_{j_1'},\cdots,f_{j_{n+1}'}$) the equality holds:

$$(4.6)\qquad \sum_{\nu=1}^{n+1}(-1)^{\nu-1}\frac{[i_1\cdots i_{\nu-1}i_{\nu+1}\cdots i_{n+1}]}{f_{i_1}\cdots f_{i_{\nu-1}}f_{i_{\nu+1}}\cdots f_{i_{n+1}}}=\frac{[i_1\cdots i_{n+1}]}{f_{i_1}f_{i_2}\cdots f_{i_{n+1}}},$$

where $[j_1\cdots j_n]$ and $[j_1, \cdots, j_{n+1}]$ denotes the determinant $\det((u_{j_\mu\nu}))_{1\leq\mu,\nu\leq n}$ and $\det((u_{j_\mu\nu}))_{0\leq\mu,\nu\leq n}$, respectively.

By a result obtained in [1], [2], the above cohomology has a system of generators,

$$(4.7)\qquad \varphi(I; J')=\frac{\tau}{f_{i_1}\cdots f_{i_p}f_{j'_1}\cdots f_{j'_q}}$$

for $I=(i_1\cdots i_p)$, $J'=(j'_1\cdots j'_q)$, $0\leq p\leq n$, $0\leq q\leq n$. We must find out the fundamental relations between them. In order to do this, we put, for $|I|=|J'|$,

$$(4.8)\qquad \tilde{\tilde{\varphi}}(I; J')=A\begin{pmatrix} I \\ J'\end{pmatrix}\tilde{\varphi}(I; J')$$

and for $|I|+1=|J'|$ (or $|I|=|J'|+1$)

$$(4.9)\qquad \tilde{\tilde{\varphi}}(I; J')=A\begin{pmatrix} 0, I \\ J'\end{pmatrix}\tilde{\varphi}(I; J')$$

$$\left(\text{or } \tilde{\tilde{\varphi}}(I; J')=A\begin{pmatrix} I \\ 0, J'\end{pmatrix}\tilde{\varphi}(I; J')\right).$$

Then

Lemma 3. *For* $|I|+1=|J'|$,

$$(4.10)\qquad 0=\tilde{\tilde{\varphi}}(I; J')+\sum_{j\notin I}\lambda_j\tilde{\tilde{\varphi}}(j, I; J')+\sum_{\nu=1}^{|J'|}(-1)^\nu\tilde{\tilde{\varphi}}(I; \partial_\nu J').$$

For $|I|=|J'|+1$,

$$(4.11)\qquad 0=\tilde{\tilde{\varphi}}(I; J')+\sum_{k'\notin J'}\lambda_{k'}\tilde{\tilde{\varphi}}(I; k'J')+\sum_{\nu=1}^{|I|}(-1)^\nu\tilde{\tilde{\varphi}}(\partial_\nu I; J').$$

Lemma 4. *For* $p=|I|=|J'|$, $k\notin I$ *and* $k'\notin J'$, *the following difference system holds:*

$$(D,\ I_{p,p})\qquad T_k\tilde{\tilde{\varphi}}(I; J')=-\sum_{\mu=1}^{|I|}\frac{A\begin{pmatrix} k, \partial_\mu I \\ J'\end{pmatrix}}{A\begin{pmatrix} 0, \partial_\mu I \\ J'\end{pmatrix}}(-1)^\mu\tilde{\tilde{\varphi}}(\partial_\mu I; J')$$

$$+\frac{A\begin{pmatrix} k, I \\ 0, J'\end{pmatrix}}{A\begin{pmatrix} I \\ J'\end{pmatrix}}\tilde{\tilde{\varphi}}(I; J')+\sum_{j'\notin J'}\lambda_{j'}\frac{A\begin{pmatrix} k, I \\ j', J'\end{pmatrix}}{A\begin{pmatrix} 0, I \\ j', J'\end{pmatrix}}\tilde{\tilde{\varphi}}(I; j'J')$$

$$(D,\ I_{p,p})\qquad T_{k'}\tilde{\tilde{\varphi}}(I;J')=-\sum_{\mu'=1}^{|J'|}\frac{A\begin{pmatrix}I\\k',\ \partial_{\mu'}J'\end{pmatrix}}{A\begin{pmatrix}I\\0,\ \partial_{\mu'}J'\end{pmatrix}}(-1)^{\mu'}\tilde{\tilde{\varphi}}(I;\partial_{\mu'}J')$$

$$+\frac{A\begin{pmatrix}0&I\\k'&J'\end{pmatrix}}{A\begin{pmatrix}I\\J'\end{pmatrix}}\tilde{\tilde{\varphi}}(I;J')+\sum_{j\notin I}\lambda_j\frac{A\begin{pmatrix}j,&I\\k',&J'\end{pmatrix}}{A\begin{pmatrix}j,&I\\0,&J'\end{pmatrix}}\tilde{\tilde{\varphi}}(jI;J')$$

LEMMA 5. *For* $p=|I|=|J'|$, $1\leqq\mu\leqq|J'|$,

$$(D,\ I^*_{p,p})\qquad (\lambda_{i_\mu}-1)T^{-1}_{i_\mu}\tilde{\tilde{\varphi}}(I;J')=\sum_{\nu=1}^{|J'|}\frac{A\begin{pmatrix}\partial_\mu,&I\\\partial_\nu,&J'\end{pmatrix}}{A\begin{pmatrix}I\\0,\ \partial_\mu J'\end{pmatrix}}(-1)^{\mu+\nu}\tilde{\tilde{\varphi}}(I;\partial_\nu J')$$

$$+\sum_{k\notin I}\lambda_k\frac{A\begin{pmatrix}k,\ \partial_\mu I\\J'\end{pmatrix}}{A\begin{pmatrix}k&I\\0&J'\end{pmatrix}}(-1)^\mu\tilde{\tilde{\varphi}}(k,I;J')+(-1)^\mu\frac{A\begin{pmatrix}0,\ \partial_\mu I\\J'\end{pmatrix}}{A\begin{pmatrix}I\\J'\end{pmatrix}}\tilde{\tilde{\varphi}}(I;J')$$

and

$$(D,\ I^{*\prime}_{p,p})\qquad (\lambda_{j'_\mu}-1)T^{-1}_{j'_\mu}\tilde{\tilde{\varphi}}(I;J')=\sum_{\nu=1}^{|I|}\frac{A\begin{pmatrix}\partial_\nu I\\\partial_\mu J'\end{pmatrix}}{A\begin{pmatrix}0,\ \partial_\nu I\\J'\end{pmatrix}}(-1)^{\mu+\nu}\tilde{\tilde{\varphi}}(\partial_\nu I;J')$$

$$+\sum_{k'\notin J'}\lambda_{k'}\frac{A\begin{pmatrix}I\\k',\ \partial_\mu J'\end{pmatrix}}{A\begin{pmatrix}0,&I\\k',&J'\end{pmatrix}}(-1)^\mu\tilde{\tilde{\varphi}}(I;k'J')$$

$$+(-1)^\mu\frac{A\begin{pmatrix}I\\0,\ \partial_\mu J'\end{pmatrix}}{A\begin{pmatrix}I\\J'\end{pmatrix}}\tilde{\tilde{\varphi}}(I;J')\ .$$

for $1\leqq\mu\leqq|J'|$.

PROOF OF LEMMA 2. It has been proved in [1] (Prop. 1, 1_p) that the integral (3, 7) satisfies the following difference equation:

$$(4.12)\qquad A(I)T_k\tilde{\varphi}(I)=\sum_{\mu=1}^{|I|}A\begin{pmatrix}I\\k,\ \partial_\mu I\end{pmatrix}(-1)^{\mu-1}\tilde{\varphi}(\partial_\mu I)$$

$$+A\begin{pmatrix}0,&I\\k,&I\end{pmatrix}\tilde{\varphi}(I)+\sum_{j\notin I}\lambda_jA\begin{pmatrix}j,&I\\k,&I\end{pmatrix}\tilde{\varphi}(j,I)\ ,$$

for $k \notin I$, so that, if $A(I)=0$, then

$$\sum_{\mu=1}^{|I|} A\begin{pmatrix} I \\ k,\ \partial_\mu I \end{pmatrix}(-1)^{\mu-1}\tilde{\varphi}(\partial_\mu I)+A\begin{pmatrix} 0,\ I \\ k,\ I \end{pmatrix}\tilde{\varphi}(I)+\sum_{j\notin I}\lambda_j A\begin{pmatrix} j,\ I \\ k,\ I \end{pmatrix}\tilde{\varphi}(j,\ I)=0\ . \tag{4.13}$$

Seeing that $\mathfrak{A}(I, J')=0$, we can apply the equalities (4.13) to the integral (4.5), and we get

$$\begin{aligned} 0=&\sum_{\nu=1}^{|J'|}\mathfrak{A}\begin{pmatrix} I,\ J' \\ k,\ I,\ \partial_\nu J' \end{pmatrix}(-1)^{|I|+1}\tilde{\varphi}(I;\ \partial_\nu J') \\ &+\mathfrak{A}\begin{pmatrix} 0,\ I,\ J' \\ j,\ I,\ J' \end{pmatrix}\tilde{\varphi}(I;\ J')+\sum_j \lambda_j\mathfrak{A}\begin{pmatrix} k,\ I,\ J' \\ j,\ I,\ J' \end{pmatrix}\tilde{\varphi}(j,\ I;\ J')\ , \end{aligned} \tag{4.14}$$

because $\mathfrak{A}\begin{pmatrix} I_1,\ J_1' \\ I_2,\ J_2' \end{pmatrix}=0$ except for the case $|I_1|=|J_2'|$, $|I_2|=|J_1'|$. On the other hand the following identities hold:

$$\mathfrak{A}\begin{pmatrix} I,\ J' \\ k,\ I,\ \partial_\nu J' \end{pmatrix}=A\begin{pmatrix} I \\ \partial_\nu J' \end{pmatrix}A\begin{pmatrix} k,\ I \\ J' \end{pmatrix} \tag{4.15}$$

$$\mathfrak{A}\begin{pmatrix} 0,\ I,\ J' \\ k,\ I,\ J' \end{pmatrix}=(-1)^{|J'|}A\begin{pmatrix} 0,\ I \\ J' \end{pmatrix}A\begin{pmatrix} k,\ I \\ J' \end{pmatrix} \tag{4.16}$$

$$\mathfrak{A}\begin{pmatrix} k,\ I,\ J' \\ j,\ I,\ J' \end{pmatrix}=(-1)^{|J'|}A\begin{pmatrix} j,\ I \\ J' \end{pmatrix}A\begin{pmatrix} k,\ I \\ J' \end{pmatrix}\ . \tag{4.17}$$

From the assumption of generality of the matrix A, we have $A\begin{pmatrix} k\ \ I \\ J' \end{pmatrix}\neq 0$, and (4.13) implies the Lemma.

LEMMA 6. *The integral $\tilde{\varphi}(I;\ J')$, $0\leqq |I|$, $|J'|\leqq n$, is a linear combination of $\tilde{\varphi}(i_1\cdots i_p;\ j_1'\cdots j_p')$ for $0\leqq p\leqq n$.*

PROOF. We assume $p>q$. Then

$$\begin{aligned} \tilde{\varphi}(i_1\cdots i_p;\ j_1'\cdots j_q') &= T_{i_{q+1}}^{-1}\cdots T_{i_p}^{-1}\tilde{\varphi}(i_1\cdots i_q;\ j_1'\cdots j_q') \\ &= T_{i_{q+2}}\cdots T_{i_p}^{-1}\tilde{\varphi}(i_1\cdots i_q i_{q+1};\ j_1'\cdots j_q') \\ &= T_{i_{q+2}}\cdots T_{i_p}^{-1}\frac{-1}{A\begin{pmatrix} i_1\cdots i_{q+1} \\ 0,\ j_1',\ j_q' \end{pmatrix}}\{\sum(-1)^\nu\tilde{\varphi}(i_1\cdots \hat{i}_\nu\cdots i_{q+1};\ j_1'\cdots j_q') \\ &\qquad +\sum\lambda_{j'}\tilde{\varphi}(i_1\cdots i_{q+1};\ J'j')\} \end{aligned} \tag{4.18}$$

owing to (4.10).

Namely $\tilde{\varphi}(i_1\cdots i_p;\ j_1'\cdots j_q')$ is a linear combination of $\tilde{\varphi}(i_1\cdots \hat{j}_\mu\cdots i_p;\ j_1'\cdots j_q')$ and $\tilde{\varphi}(i_1\cdots i_p;\ j_1'\cdots j_q' j')$. By repeating this procedure, we arrive at the conclusion of the Lemma.

Proposition 4. *The linear relations* (4.6) (4.10)~(4.11), DI_{pp}, DI'_{pp}, DI^*_{pp}, $DI^{*\prime}_{pp}$ *are fundamental among* $\tilde{\varphi}(\lambda_1, \cdots, \lambda_m; \phi)$. *Consequently,* $\tilde{\varphi}(i_1\cdots i_p; j'_1\cdots j'_p)$ $0\leqq p\leqq n$ *becomes a basis for the difference system* $DI_{p,p}$, DI'_{pp} DI^*_{pp} *and* $DI^{*\prime}_{pp}$. *Thus the number of linearly independent integrals is equal to* $\sum_{p=0}^{n}\binom{s}{p}\binom{t}{p}$.

Finally we have the variation formula for $\tilde{\tilde{\varphi}}(I; J')$, $|I|=|J'|$, which defines the Gauss-Manin connection of (4.1) and (4.5):

Theorem 2. $\tilde{\tilde{\varphi}}(I; J')$ *satisfies the following logarithmic Gauss-Manin connection:*

$$
\begin{aligned}
(4.19)\qquad d\tilde{\tilde{\varphi}}(I; J') = & \sum_{i\notin I}\sum_{j'\notin J'} \lambda_i\lambda_{j'}\, d\log\left[A\begin{pmatrix} i, I \\ j', J'\end{pmatrix} A\begin{pmatrix} I \\ J'\end{pmatrix}\right]\tilde{\tilde{\varphi}}(i, I; j', J') \\
& + \sum_{\mu=1}^{|I|}\sum_{\nu=1}^{|J'|}(-1)^{\mu+\nu-1} d\log\left[A\begin{pmatrix} \partial_\mu I \\ \partial_\nu J'\end{pmatrix} A\begin{pmatrix} I \\ J'\end{pmatrix}\right]\tilde{\tilde{\varphi}}(\partial_\mu I; \partial_\nu J') \\
& + \sum_{k\notin I}\sum_{\nu=1}^{|J'|}(-1)^{\nu-1}\lambda_k d\log\left[A\begin{pmatrix} k, \partial_\nu I \\ J'\end{pmatrix} A\begin{pmatrix} I \\ J'\end{pmatrix}\right]\tilde{\tilde{\varphi}}(k, \partial_\nu I; J') \\
& + \sum_{k'\notin J'}\sum_{\nu=1}^{|J'|}(-1)^{\nu-1}\lambda_{k'} d\log\left[A\begin{pmatrix} I \\ k', \partial_\nu J'\end{pmatrix} A\begin{pmatrix} I \\ J'\end{pmatrix}\right]\tilde{\tilde{\varphi}}(I; k', \partial_\nu J') \\
& + (\lambda_I + \lambda_{J'} - \lambda_{I^c} - \lambda_{J'^c})\, d\log A\begin{pmatrix} I \\ J'\end{pmatrix}\tilde{\tilde{\varphi}}(I; J') \\
& + \sum_{\nu=1}^{|I|}(-1)^\nu d\log A\begin{pmatrix} 0, \partial_\nu I \\ J'\end{pmatrix}\tilde{\tilde{\varphi}}(\partial_\nu I; J') \\
& + \sum_{\nu=1}^{|J'|}(-1)^\nu d\log A\begin{pmatrix} I \\ 0, \partial_\nu J'\end{pmatrix}\tilde{\tilde{\varphi}}(I; \partial_\nu J')\ ; \\
& + \sum_{j\notin I}\lambda_j d\log A\begin{pmatrix} j, I \\ 0, J'\end{pmatrix}\tilde{\tilde{\varphi}}(j, I; J') \\
& + \sum_{k'\notin J'}\lambda_{k'} d\log A\begin{pmatrix} 0, I \\ k', J'\end{pmatrix}\tilde{\tilde{\varphi}}(I; k', J')\ .
\end{aligned}
$$

where λ_I *and* λ_{I^c} *denote* $\sum_{k\in I}\lambda_k$ *and* $\sum_{k\notin I}\lambda_k$, *respectively.*

Example. (4.1) for $n=1$ is equal to

$$
(4.20)\qquad \tilde{\varphi}(\phi)=\int \exp[xy]\prod_{i=1}^{s}(x-\alpha_i)^{\lambda_i}\prod_{j=1}^{t}(y-\beta_j)^{\mu_j}dxdy\ .
$$

In this case $a_{ij'}=1$, $a_{i0}=-\alpha_i$ and $a_{0j'}=-\beta_{j'}$, so that

$$
\tilde{\tilde{\varphi}}(\phi)=\tilde{\varphi}(\phi)\ ,\qquad \tilde{\tilde{\varphi}}(i)=-\alpha_i\tilde{\varphi}(i)\ ,\qquad \tilde{\tilde{\varphi}}(j')=-\beta_{j'}\tilde{\varphi}(j')
$$

and $\tilde{\tilde{\varphi}}(ij')=\tilde{\varphi}(ij')$. (4.10) and (4.11) imply

$$\text{(4.21)}\qquad \begin{cases} 0=\tilde{\tilde{\varphi}}(j')+\sum_{k=1}^{s}\lambda_k\tilde{\tilde{\varphi}}(k;\,j')-\tilde{\tilde{\varphi}}(\phi) \\ 0=\tilde{\tilde{\varphi}}(i)+\sum_{k'=1}^{t}\lambda_{k'}\tilde{\tilde{\varphi}}(i;\,k')-\tilde{\tilde{\varphi}}(\phi) \\ 0=\tilde{\tilde{\varphi}}(i;\,j'k')-\tilde{\tilde{\varphi}}(i;\,k')+\tilde{\tilde{\varphi}}(i;\,j') \\ 0=\tilde{\tilde{\varphi}}(ij;\,k')-\tilde{\tilde{\varphi}}(j;\,k')+\tilde{\tilde{\varphi}}(i;\,k')\ . \end{cases}$$

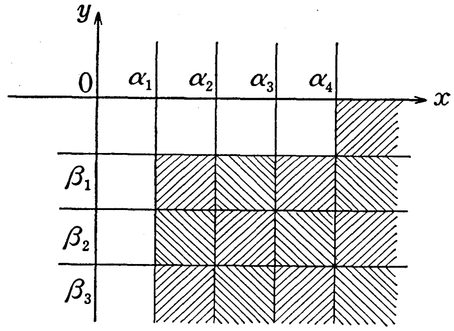
Linearly independent ones are $\tilde{\tilde{\varphi}}(\phi)$ and $\tilde{\tilde{\varphi}}(ij')$ $1\leqq i\leqq s$, $1\leqq j'\leqq t$. Then,

$$\text{(4.22)}\qquad d\tilde{\tilde{\varphi}}(\phi)=\sum\lambda_i d\log\alpha_i\cdot\tilde{\tilde{\varphi}}(i)+\sum\lambda_{j'}d\log\beta_{j'}\cdot\tilde{\tilde{\varphi}}(j)$$

$$\begin{aligned}\text{(4.23)}\qquad d\tilde{\tilde{\varphi}}(ij')&=-d\log\alpha_i\cdot\tilde{\tilde{\varphi}}(j')-d\log\beta_{j'}\cdot\tilde{\tilde{\varphi}}(i)\\ &\quad+\sum\lambda_k d\log(\alpha_i-\alpha_k)\cdot\tilde{\tilde{\varphi}}(ki;\,j')\\ &\quad+\sum\lambda_{k'}d\log(\beta_{j'}-\beta_{k'})\cdot\tilde{\tilde{\varphi}}(i;\,k'j')\\ &=-d\log\alpha_i\cdot\tilde{\tilde{\varphi}}(j')-d\log\beta_{j'}\cdot\tilde{\tilde{\varphi}}(i)\\ &\quad+\sum_{k\neq i}\lambda_k d\log(\alpha_i-\alpha_k)\{\tilde{\tilde{\varphi}}(i;\,j')-\tilde{\tilde{\varphi}}(k;\,j')\}\\ &\quad+\sum_{k'\neq j'}\lambda_{k'}d\log(\beta_{j'}-\beta_{k'})\{\tilde{\tilde{\varphi}}(i;\,j')-\tilde{\tilde{\varphi}}(i;\,k')\}\ .\end{aligned}$$

If $0<\alpha_1<\cdots<\alpha_s$ and $\beta_t<\beta_{t-1}<\cdots<\beta_1<0$, then $(st+1)$ linearly independent domains of integration (4.20) can be chosen as the real rectangular ones in the figure.

$s=4,\ t=3,\ st+1=13$



§5. Hyperlogarithms attached to the configuration of hyperplane sections in hyperquadrics.

Let V be the hyperquadric defined by the equation in CP^n:

$$\text{(5.1)}\qquad \xi_0^2+\xi_1^2+\cdots+\xi_{n+1}^2=0\ .$$

Let $S_1, S_2, \cdots, S_m, S_{m+1}$ be hyperplanes lying in general position to each other and V. Let us take S_{m+1} to be $\xi_0=0$. Then $V'=V-S_{m+1}$, $S'_1=S_1-S_{m+1}, \cdots, S'_m=S_m-S_{m+1}$ are described as quadratic and linear equations in C^n respectively:

$$(5.2)\qquad \begin{cases} V': x_1^2+x_2^2+\cdots+x_{n+1}^2+1=0 \\ S'_j: f_j\left(=\sum_{\nu=1}^{n+1} u_{j\nu}x_\nu+u_{j0}\right)=0\,, \qquad 1\leqq j\leqq m\,. \end{cases}$$

We denote by τ the canonical n-form $\sum_{j=1}^{n+1}(-1)^j x_j dx_1\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\cdots\wedge dx_{n+1}$ on V' which is invariant with respect to the natural action of $SO(n+1, C)$. W' denotes $\bigcup_{j=1}^m S'_j$.

First we prove the following

PROPOSITION 5. *$H^n(V'-W', C)$ is spanned by a basis consisting of the forms*

$$(5.3)\qquad \varphi^{(1)}(i_1\cdots i_p)=\frac{\tau}{f_{i_1}f_{i_2}\cdots f_{i_p}}\,, \quad 1\leqq p\leqq n\,, \qquad 1\leqq i_1<\cdots<i_p\leqq m$$

$$(5.3)'\qquad \varphi_+^{(1)}(i_1\cdots i_n)=\frac{\{1, f_{i_1}\cdots f_{i_n}\}^\perp}{f_{i_1}f_{i_2}\cdots f_{i_n}}\,, \quad \tau 1\leqq i_1<\cdots<i_n\leqq m\,,$$

where $\{1, f_{i_1}, \cdots, f_{i_n}\}^\perp$ denotes the linear function g such that

$$(g, 1)=(g, f_{i_1})=\cdots=(g, f_{i_n})=0$$

and here (a, b) denotes the inner product $\sum_{j=0}^{n+1}\alpha_j\beta_j$ for $a=\sum_{j=1}^{n+1}\alpha_j x_j+\alpha_0$ and $b=\sum_{j=1}^{n+1}\beta_j x_j+\beta_0$. In the sequel we shall assume $(f_j, f_j)=1$ for $-n\leqq j\leqq m$, then

$$(5.4)\qquad \frac{\{f_{i_1}, \cdots, f_{i_n}\}^\perp\tau}{f_{i_1}f_{i_2}\cdots f_{i_n}}=\frac{1}{\sqrt{A(0, i_1, \cdots, i_n)}}d\log f_{i_1}\wedge d\log f_{i_2}\wedge\cdots\wedge d\log f_{i_n}$$

in $V'-W'$.

PROOF. This proposition immediately follows from the following four Lemmas.

LEMMA 5.1. *For any $1\leqq j\leqq n+1$,*

$$(5.5)\qquad (-1)^j dx_1\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\cdots\wedge dx_{n+1}\equiv x_j\tau \mod (f_0, df_0)\,.$$

LEMMA 5.2. *For $I=\{i_1, i_2, \cdots, i_{n+1}\}$, we have the identity*

$$(5.6)\qquad \frac{\tau}{f_{i_1}f_{i_2}\cdots f_{i_{n+1}}}=-\sum_{\nu=1}^{|I|}\frac{A(\partial_\nu I)}{A(I)}\varphi_+^{(1)}(\partial_\nu I)-2\sum_{\nu=1}^{|I|}\frac{A\begin{pmatrix} I \\ 0, \partial_\nu I\end{pmatrix}}{A(I)}(-1)^\nu\varphi^{(1)}(\partial_\nu I)\,.$$

LEMMA 5.3. *An arbitrary element of the space $\Omega^n(V', {}^*W')$, of rational n-forms in V' which are holomorphic in $V'-W'$, is cohomologous to a form*

$$\frac{\varphi(x)}{f_1^{k_1}\cdots f_n^{k_n}}\tau \tag{5.7}$$

for some $k_1, \cdots, k_n \in \boldsymbol{Z}^+$ and $\varphi \in \boldsymbol{C}[x_1, \cdots, x_{n+1}]$.

LEMMA 5.4. *If $k_\nu > 1$ in* (5.7), *then there exists a $\varphi' \in \boldsymbol{C}[x_1, \cdots, x_{n+1}]$ such that* (5.7) *is cohomologous to*

$$\frac{\varphi'}{f_1^{k_1}\cdots f_\nu^{k_\nu-1}\cdots f_n^{k_n}} \tag{5.8}$$

The proofs of the Lemma 5.1~5.4 are easily done by a direct computation. See [2] or [4].

Let $S_{-n}, S_{-n+1}, \cdots, S_{-1}$ be the hyperplane sections of V' defined by linear functions $f_j = \sum_{\nu=1}^{n+1} u_{j\nu}x_\nu + u_{j0}$, $-n-1 \leqq j \leqq -1$. We shall assume that $S_{-n-1}, S_n, \cdots, S_{-1}, S_1, \cdots, S_m$ are in general position. Then these $(m+n+1)$ hyperplane sections define a point of the configuration space of hyperplane sections $\mathscr{L}$ which is parametrized by the $(m+n+1)$ order matrix $A=((a_{ij}))$, $-n \leqq i, j \leqq m$, such that $a_{ii}=1$, $A(I)=0$ for $|I| \geqq n+3$ and $A(I) \neq 0$ for $|I| \leqq n+2$. Here we put $a_{jk} = \sum_{\nu=0}^{n} u_{j\nu}u_{k\nu}$ for $j, k \neq 0$ and $a_{j0} = u_{j0}$.

$\mathscr{L}$ is therefore equal to the complement of $\mathscr{Y}$ in $\mathscr{X}$ where $\mathscr{X}$ denotes *the determinantal variety* consisting of symmetric matrices A such that $a_{ii}=1$, $-n-1 \leqq i \leqq m$ and $A(I)=0$ for $|I| \geqq n+3$, and $\mathscr{Y}$ denotes the subvariety of $\mathscr{X}$ consisting of A such that $A(I)=0$, for some I, $|I| \leqq n+2$.

DEFINITION. (See [3]) By a fundamental n-simplex Δ we mean an n-simplex satisfying the following conditions:

i) $\Delta \subset V'$

ii) $\partial \Delta_{i_1\cdots i_p} = \bigcup_{j \notin \{i_1\cdots i_p\}} \Delta_{j i_1\cdots i_p}$, where $\Delta_{i_1\cdots i_p}$ denotes $\Delta \cap S_{i_1} \cap \cdots \cap S_{i_p}$. The vertices of Δ consist of the $(n+1)$ points $S_{-n-1} \cap \cdots \cap S_{\nu-1} \cap S_{\nu+1} \cap \cdots \cap S_0$, $-n-1 \leqq \nu \leqq -1$. $\Delta_{i_1\cdots i_p}$ becomes a fundamental simplex in $V'_{i_1\cdots i_p} = V' \cap S_{i_1} \cap \cdots \cap S_{i_p}$. For $(k_1, \cdots, k_p) \subset (-n-1, -n, \cdots, -1)$, we denote by $\tau_{k_1\cdots k_p}$ the canonical $(n-p)$-form on $V'_{k_1\cdots k_p}$, which is invariant with respect to the action of the group $SO(n+1-p)$. Then it is easily shown that

$$\left.\frac{\tau}{df_{k_1}\wedge\cdots\wedge df_{k_p}}\right|_{V'_{k_1\cdots k_p}} = (-1)^q \frac{A(k_1, \cdots, k_q)^{(n-p-1)/2}}{A(0\, k_1\cdots k_q)^{(n-p)/2}} \cdot \tau_{k_1\cdots k_p}\,. \tag{5.9}$$

We fix J for $J\subset\{-n-1, \cdots, -1\}$, $|J|\leqq n$. Let $\bar{f}_k$ $(k\notin J)$ be the restriction of f_k to V_J, and $\bar{A}$ be the symmetric matrix of $(m+n+1)-|J|$ order consisting of the (k, k') elements $\bar{a}_{kk'}=(\bar{f}_k, \bar{f}_{k'})$, the canonical inner product of $\bar{f}_k$ and $\bar{f}_{k'}$ invariant with respect to $SO(n+1-p, \boldsymbol{C})$. Then we have the equality

$$\bar{A}\begin{pmatrix} K \\ K' \end{pmatrix}=A\begin{pmatrix} J, K \\ J, K' \end{pmatrix}\Big/A(J)\,. \tag{5.10}$$

Consider the integral

$$F_n(\nu_1, \cdots, \nu_n)=\int_\Delta f_1^{\nu_1}\cdots f_m^{\nu_m}\tau \tag{5.11}$$

for $\nu_1, \cdots, \nu_m\in\boldsymbol{Z}$. According to the Proposition 5, the form $f_1^{\nu_1}\cdots f_m^{\nu_m}\tau$ is cohomologous to a linear combination of $\varphi^{(1)}(J)$ and $\varphi_+^{(1)}(J)$:

$$f_1^{\nu_1}\cdots f_m^{\nu_m}\tau=(\text{a linear combination of } \varphi^{(1)}(J) \text{ and } \varphi_+^{(1)}(J))+d\psi\,, \tag{5.12}$$

for a suitable $(n-1)$-form $\psi\in\Omega^{n-1}(V', {}^*W')$. Therefore, by Stokes formula,

$$\hat{\varphi}=(\text{a linear combination of } \hat{\varphi}^{(1)}(J) \text{ and } \hat{\varphi}_+^{(1)}(J))+\int_{\partial\Delta}\psi\,, \tag{5.13}$$

where $\partial\Delta\subset\bigcup_{-n\leqq j\leqq 0}S_j$, and $\hat{\varphi}^{(1)}(J)$ and $\hat{\varphi}_+^{(1)}(J)$ are defined as follows

$$\hat{\varphi}^{(1)}(J)=\int_\Delta\varphi^{(1)}(J) \tag{5.14}$$

$$\hat{\varphi}_+^{(1)}(J)=\int_\Delta\varphi_+^{(1)}(J)\,. \tag{5.15}$$

Repeating this procedure for the second term in the right-hand side of (5.13) we have

LEMMA 5.5. *The integral $F_n(\nu_1\cdots\nu_m)$ is a linear combination of the integrals*

$$\hat{\varphi}_{i_1\cdots i_p}^{(1)}(J)=\int_{\Delta_{i_1\cdots i_p}}\varphi_{i_1\cdots i_p}^{(1)}(J) \tag{5.16}$$

$$\hat{\varphi}_{+,i_1\cdots i_p}^{(1)}(J)=\int_{\Delta_{i_1\cdots i_p}}\varphi_{+,i_1\cdots i_p}^{(1)}(J) \tag{5.16$'$}$$

for $0\leqq p\leqq n$, and $-n\leqq i_1<\cdots<i_p\leqq 0$, $J=\{j_1\cdots j_q\}\subset\{1\,2\cdots m\}$, where $\varphi_{i_1\cdots i_p}^{(1)}(J)$ and $\varphi_{+,i_1\cdots i_p}^{(1)}(J)$ denote $(n-p)$ forms of the same type as (5.3) and (5.3)$'$ on $V'_{i_1i_2\cdots i_p}$:

(5.17) $$\varphi^{(1)}_{i_1\cdots i_p}(J)=\frac{\tau_{i_1}\cdots\tau_{i_p}}{f_{j_1}f_{j_2}\cdots f_{j_q}}\qquad q\leqq n-p$$

(5.17)′ $$\varphi^{(1)}_{+,i_1\cdots i_p}(J)=\frac{\{1, f_{j_1}, \cdots, f_{j_{n-p}}\}^{\perp}\tau_{i_1\cdots i_p}}{f_{j_1}\cdots f_{j_{n-p}}}\,.$$

Now we want to study the structure of the Gauss-Manin connection of the integrals $\hat{\varphi}^{(1)}(J)$ and $\hat{\varphi}^{(1)}_{+}(J)$. For this purpose, we can use Theorem 1 and Theorem 1′. In fact let us consider the integrals $\hat{\varphi}^{(1)}(J;\lambda)$ and $\hat{\varphi}^{(1)}_{+}(J,\lambda)$ on V': for $\lambda=(\lambda_{-n}, \lambda_{-n+1}, \cdots, \lambda_m)$

(5.18) $$\hat{\varphi}^{(1)}(J;\lambda)=\int\hat{U}_0(\lambda)\varphi^{(1)}(J)\quad |I|\leqq n$$

(5.18)′ $$\hat{\varphi}^{(1)}_{+}(J;\lambda)=\int\hat{U}_0(\lambda)d\log f_{j_1}\wedge\cdots\wedge d\log f_{j_n}$$

for $|J|=n$, which is equal to

(5.18)″ $$\sqrt{A(0, J)}\int\hat{U}_0(\lambda)\varphi^{(1)}_{+}(J)\,,$$

owing to (5.4). Here we have put $\hat{U}_0(\lambda)=f_{-n}^{\lambda_{-n}}f_{-n+1}^{\lambda_{-n+1}}\cdots f_0^{\lambda_0}f_1^{\lambda_1}\cdots f_m^{\lambda_m}$. Then the functions $\hat{\varphi}^{(1)}(J;\lambda)$ satisfy the Gauss-Manin connection (2.11). Remark that

(5.19) $$\lim_{\substack{\lambda_k\to 0\\ -n\leqq k\leqq m}}\hat{\varphi}^{(1)}(J;\lambda)=\hat{\varphi}^{(1)}(J)$$

if $J\subset\{1, 2, \cdots, m\}$. By taking $\lambda_{-n}, \lambda_{-n+1}, \cdots, \lambda_0\mapsto 0$, we have further

(5.20) $$\lim\lambda_{k_1}\cdots\lambda_{k_p}\hat{\varphi}^{(1)}(J;\lambda)=0\,,$$

except for the case $(k_1, \cdots, k_p)\subset(-n, -n+1, \cdots, -1, 0)$, in which we have

(5.21) $$\lim\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_p}\hat{\varphi}^{(1)}(J;\lambda)=(-1)^p\frac{A(k_1\cdots k_p)^{(n-p-1)/2}}{A(0\,k_1\cdots k_p)^{(n-p)/2}}\varphi^{(1)}_{k_1\cdots k_p}(J)$$

for $0\leqq p\leqq n+1$, in view of (5.9).

Consequently, as a result of these limiting procedures, we have:

Proposition 6. *For* $|I|\leqq n$.

(5.22) $$\frac{1}{\sqrt{A(I)}}d[\sqrt{A(I)}\,\hat{\varphi}^{(1)}(I)]$$

$$=d\hat{\varphi}^{(1)}(I)+\frac{1}{2}dA(I)\cdot\hat{\varphi}^{(1)}(I)$$

$$=\sum_{s=1}^{n-|I|}\frac{1}{s!\,(-n+|I|+1)\cdots(-n+|I|+s-1)}\theta\begin{pmatrix} I \\ I\,k_1\cdots k_s\end{pmatrix}$$

$$\times\hat{\varphi}^{(1)}_{*k_1\cdots k_s}(I)(-1)^s\frac{A(k_1\cdots k_s)^{(n-s-1)/2}}{A(0\,k_1\cdots k_s)^{(n-s)/2}}$$

$$+\sum\frac{1}{(n+1-|I|)!\,(-n+|I|+1)\cdots(-1)}\cdot\theta\begin{pmatrix} I \\ I\,k_1\cdots k_{n+1-|I|}\end{pmatrix}$$

$$\times\hat{\varphi}^{(1)}_{*k_1\cdots k_{n+1-|I|}}(I)\cdot(-1)^{n+1-|I|}\frac{A(k_1\cdots k_{n+1-|I|})^{(|I|-2)/2}}{A(0\,k_1\cdots k_{n+1-|I|})^{(|I|-1)/2}}$$

$$+\sum_{J\subsetneq I}\theta\begin{pmatrix} I \\ J\end{pmatrix}\frac{A\begin{pmatrix}0, J\\ k, J\end{pmatrix}}{A(J)}\hat{\varphi}^{(1)}_k(J)(-1)\frac{1}{A(0\,k)^{(n-1)/2}}$$

$$+\sum_{J\subsetneq I}(-n+|J|+1)\theta\begin{pmatrix} I \\ J\end{pmatrix}\frac{A(0, J)}{A(J)}\hat{\varphi}^{(1)}(J)\ ;$$

where $\hat{\varphi}^{(1)}_{*K}(I)$ *are related to* $\hat{\varphi}^{(1)}_K(I)$ *as follows:*

$$\hat{\varphi}^{(1)}_{*K}(I)=\hat{\varphi}^{(1)}_K(I)+\sum_{\nu=1}^{|I|}(-1)^{\nu}\frac{A\begin{pmatrix}K, I\\ 0, K, \partial_\nu I\end{pmatrix}}{A(K, I)}\hat{\varphi}^{(1)}_K(\partial_\nu I)\ . \tag{5.23}$$

We shall denote by $O=O(\mathscr{X}, {}^*\mathscr{Y})$ and $\Theta=\Theta(\mathscr{X}, {}^*\mathscr{Y})$ the set of all rational functions (and all rational vector fields respectively) on $\mathscr{X}$ which are holomorphic on $\mathscr{X}-\mathscr{Y}$. Let $\tilde{O}$ and $\tilde{\Theta}$ be the minimal extensions of $O(\mathscr{X}, {}^*\mathscr{Y})$ and $\Theta(\mathscr{X}, {}^*\mathscr{Y})$ respectively as O-module including the functions $A(I)^{\pm 1/2}$ for $I\subset\{0, 1, 2, \cdots, m\}$. Let $\mathscr{E}^q_p(p\geqq q)$ be the $\tilde{O}$-module generated by the functions $\hat{\varphi}^{(1)}_J(I)$, $|I|\leqq q$, $|J|\geqq n-p$ and $|J|+|I|\leqq n$ and by the functions $\hat{\varphi}^{(1)}_{+J}(I)$ for $|I|\leqq q$, $|J|+|I|=n$. For simplicity we put $\mathscr{E}^q_p=0$ for $q>p$. If $p\leqq p'$ and $q\leqq q'$, then $\mathscr{E}^q_p\subset\mathscr{E}^{q'}_{p'}$. $\mathscr{E}^0_0$ is $\tilde{O}$ itself. Clearly $\hat{\varphi}^{(1)}_J(I)\sqrt{A(I\,J)/A(J)}\in\mathscr{E}^{|I|}_{n-|J|}$.

PROPOSITION 7. *For an arbitrary* $X\in\tilde{\Theta}$,

$$X\Big(\sqrt{\frac{A(J\,I)}{A(J)}}\hat{\varphi}^{(1)}_J(I)\Big)\equiv 0 \bmod \mathscr{E}^{|I|-1}_{n-|J|}\oplus\mathscr{E}^{|I|}_{n-|J|-1} \tag{5.24}$$

for $|J|+|I|\leqq n$.

PROOF. This proposition is an immediate consequence of the formula (5.22), considering it in V'_J instead of V' and replacing $\sqrt{A(I)}\,\hat{\varphi}^{(1)}(I)$ by $\sqrt{A(J, I)/A(J)}\,\hat{\varphi}^{(1)}_J(I)$. Therefore, we have only to show (5.24) in the case of $J=\phi$. In this case the right-hand side of (5.24) is contained in $\mathscr{E}_{n-1}^{|I|}\oplus\mathscr{E}_{n}^{|I|-1}$, in view of (5.6) and (5.23). The proof is finished.

Let $\tilde{\mathscr{X}}$ the minimal fininite abelian covering of $\mathscr{X}$ which uniformize all the functions $\sqrt{A(I)}$ for $I\subset\{-n, -n+1, \cdots, -1, 0, 1, \cdots, m\}$. Then from Proposition 7, the following holds:

THEOREM 3. *$\hat{\varphi}^{(1)}(I)$ has a unipotent monodromy on $\tilde{\mathscr{X}}$ and therefore can be represented as a linear combination of rational functions multiplied by hyperlogarithms on $\tilde{\mathscr{X}}$, which are holomorphic outside $\mathscr{Y}$, namely as an iterated integral of the rational 1-forms on $\tilde{\mathscr{X}}$, whose poles are located over $\mathscr{Y}$. The exact representation is obtained by solving the differential equations* (5.22) *iteratively. In the case of hyper-quadrics, the above proposition shows Theorem* 2 *which has been stated in* [3].

REMARK. So far, we have not treated *the cases where $f_0, f_1, \cdots, f_m$ are not necessarily in general position.* In such degenerate cases, the formulae for Gauss-Manin connections must be modified, *by the continuity method,* from the ones obtained in this article. Typical integrals appear in mathematical physics:

i) Correlation functions for Random matrices

$$\int \exp\left[-\frac{1}{2}\left(\sum_{j=1}^{n} x_j^2\right)\right]\prod_{1\leq i<j\leq n} |x_i-x_j|^{\beta} dx_{l+1}\cdots dx_n$$

or

ii) Correlation functions for vortex models

$$\int \exp\left[-\frac{1}{2}\sum_{i=1}^{n} |z_i|^2\right]\prod_{1\leq i<j\leq n} |z_i-z_j|^{\beta} dz_{l+1}d\bar{z}_{l+1}\cdots dz_n d\bar{z}_n$$

for $0\leqq l<n$.

In fact, these two cases are degenerate ones of the integrals (3.1) and (4.1) respectively. It seems to be interesting to compute the Gauss-Manin connections for them and to give reasonable asymptotes for $n\to\infty$, l being fixed.

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